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Robust Analysis of Preferential Attachment Models with Fitness

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The preferential attachment network with fitness is a dynamic random graph model. New vertices are introduced consecutively and a new vertex is attached to an old vertex with probability proportional to the degree of the old one multiplied by a random fitness. We concentrate on the typical behaviour of the graph by calculating the fitness distribution of a vertex chosen proportional to its degree. For a particular variant of the model, this analysis was first carried out by Borgs, Chayes, Daskalakis and Roch. However, we present a new method, which is robust in the sense that it does not depend on the exact specification of the attachment law. In particular, we show that a peculiar phenomenon, referred to as Bose–Einstein condensation, can be observed in a wide variety of models. Finally, we also compute the joint degree and fitness distribution of a uniformly chosen vertex.

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1. Introduction

Preferential attachment models were popularized by Barabási and Albert [1] as a possible model for complex networks such as the world wide web. The authors observed that a simple mechanism can explain the occurrence of power law degree distributions in real world networks. Often networks are the result of a continuous dynamic process: new members enter social networks or new web pages are created and linked to popular old ones. In this process new vertices prefer to establish links to old vertices that are well connected. Mathematically, we consider a sequence of random graphs (*random dynamic network*), where new vertices are introduced consecutively and then connected to each old vertex with a probability proportional to the degree of the old vertex. As proved rigorously by Bollobás, Riordan, Spencer and Tusnády [5], this rather simple mechanism leads to networks with power law degree distributions and thus offers an explanation for their occurrence.

There are many variations of the classic model to address different shortcomings: see, for example, [13] for an overview. For example, a more careful analysis of the classical model shows that one can observe a 'first to market' advantage, where from a certain point onwards the vertex with maximal degree will always remain maximal: see, for example, [10]. Clearly, this is not the only possible scenario observed in real networks. One possible improvement is to model the fact that vertices have an intrinsic quality or fitness, which would allow even younger vertices to overtake old vertices in popularity.

Introducing fitness has a significant effect on the network formation. In particular, it may provoke condensation effects as indicated in [4]. The first mathematically rigorous analysis was carried out in [6] for the following variant of the model. First every (potential) vertex $i \in \mathbb{N}$ is assigned an independent identically distributed (say μ-distributed) fitness $\mathcal{F}^{(i)}$. Starting with the network $\mathcal{G}_1$ consisting of the single vertex 1 with a self-loop, the network is formed as follows. Suppose we have constructed the graph $\mathcal{G}_n$ with vertices $\{1,\ldots,n\}$. Then we obtain $\mathcal{G}_{n+1}$ by

- insertion of the vertex $n+1$ and
- insertion of a single edge linking up the new vertex to the old vertex $i \in \{1,\ldots,n\}$ with probability proportional to

$$\mathcal{F}^{(i)} \deg_{\mathcal{G}_n}(i), \tag{1.1}$$

where $\deg_{\mathcal{G}}(i)$ denotes the degree of vertex i in a graph $\mathcal{G}$.

Borgs, Chayes, Daskalakis and Roch [6] compute the asymptotic fitness distribution of a vertex chosen proportional to its degree. This limit distribution is either absolutely continuous with respect to μ ('fit-get-richer phase') or has a singular component that puts mass on the essential supremum of μ ('condensation phase' or 'Bose–Einstein phase'). In the condensation phase a positive fraction of mass is shifted towards the essential supremum of μ.

The model introduced above can be obtained from a particular branching process, a Crump–Mode–Jagers process, by a random time change: see [3]. This allows them to derive the results in [6] via a classical almost sure limit theorem of Nerman [17] combined with a domination argument. Related techniques are also essential in [6], which leads to severe restrictions on the model specifications. This is in strong contrast to the intuition of physicists, which suggests that explicit details of the model do not have an impact.

The aim of this article is to present a new robust approach that allows us to deal with general preferential attachment models with fitness. In particular we include models where new vertices connect to a random number (or a fixed number larger than one) of old vertices. In our analysis the crucial problem is to show convergence of the random normalization. In the specific model above, the normalization is obtained by summing the weights in (1.1).

In general, preferential attachment models feature a normalization which guarantees that the number of edges established by new vertices is of constant order. In most models convergence of the normalization is a direct consequence of its definition. For instance, in classical preferential attachment the normalization is deterministic: see [5]. One exception is that of preferential attachment networks with sublinear weight functions.

For a particular variant, these are again time-changed Crump–Mode–Jagers processes and the analysis can be based on the Nerman result: see [21]. Once the convergence of the normalization has been established, the remaining analysis is robust with respect to the model specification: see, for example, [12, 7, 14, 8].

Our approach is based on a bootstrapping argument. The idea is to start with a bound θ on the normalization, from which we deduce a new bound $T(\theta)$. Then, by a continuity argument, we deduce that the correct limit of the normalization is a fixed point of T. We stress that the mapping T is new and has not yet appeared in the physics literature on complex networks with fitness. This is the basis for showing convergence of the asymptotic fitness and degree distribution.

For the model with fitness, our proofs show that the condensation effect can be observed irrespective of the fine details of the model. The phenomenon of Bose–Einstein condensation seems to have a universal character; for an overview of further models see [11]. The precise analysis of the dynamics in a closely related model are carried out in [9].

2. Definitions and main results

We consider a dynamic graph model with fitness. Each vertex $i \in \mathbb{N}$ is assigned an independent μ-distributed fitness $\mathcal{F}^{(i)}$, where μ is a compactly supported distribution on the Borel sets of $(0,\infty)$. We call μ the *fitness distribution.*

We measure the importance of a vertex i in a directed graph $\mathcal{G}$ by its *impact*

$$\mathrm{imp}_{\mathcal{G}}(i) := 1 + \text{ indegree of } i \text{ in } \mathcal{G}.$$

For technical reasons, we set $\mathrm{imp}_{\mathcal{G}}(i) := 0$ if i is not a vertex of $\mathcal{G}$.

The complex network is represented by a sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of random directed multigraphs without loops that is built according to the following rules. Each graph $\mathcal{G}_n$ consists of n vertices labelled by $1,\ldots,n$. The first graph consists of the single vertex 1 and no edges. Further, given $\mathcal{G}_n$, the network $\mathcal{G}_{n+1}$ is formed by carrying out the following two steps:

- insertion of the vertex $n+1$,
- insertion of directed edges $n+1 \to i$ for each old vertex $i \in \{1,\ldots,n\}$ with intensity proportional to

$$\mathcal{F}^{(i)} \cdot \mathrm{imp}_{\mathcal{G}_n}(i). \tag{2.1}$$

Note that this is not a unique description of the network formation. We still need to clarify the explicit rule for how new vertices connect to old ones. We will do this in terms of the *impact evolutions*: for each $i \in \mathbb{N}$, we consider the process $\mathcal{I}^{(i)} = (\mathcal{I}^{(i)}_n)_{n\in\mathbb{N}}$ defined by

$$\mathcal{I}^{(i)}_n := \mathrm{imp}_{\mathcal{G}_n}(i).$$

Since all edges point from younger to older vertices and since in each step all new edges attach to the new vertex, the sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ can be recovered from the impact evolutions $(\mathcal{I}^{(i)} : i \in \mathbb{N})$. Indeed, for any $i, j, n \in \mathbb{N}$ with $i < j \leqslant n$, there are exactly

$$\Delta\mathcal{I}^{(i)}_{j-1} := \mathcal{I}^{(i)}_j - \mathcal{I}^{(i)}_{j-1}$$

links pointing from j to i in $\mathcal{G}_n$. Note that each impact evolution $\mathcal{I}^{(i)}$ is monotonically increasing, $\mathbb{N}_0$-valued and satisfies $\mathcal{I}_n^{(n)} = 1$ and $\mathcal{I}_n^{(i)} = 0$ for $i > n$. Moreover, any choice for the impact evolutions with these three properties describes uniquely a dynamic graph model. We state the assumptions in terms of the impact evolutions. For a discussion of relevant examples, we refer the reader to Examples 2.2 and 2.3 below.

Assumptions. Let $\lambda > 0$ be a parameter and define

$$\bar{\mathcal{F}}_n := \frac{1}{\lambda n} \sum_{i=1}^{n} \mathcal{F}^{(i)} \mathcal{I}_n^{(i)} = \frac{1}{\lambda n} \langle \mathcal{F}, \mathcal{I}_n \rangle,$$

where $\mathcal{I}_n := (\mathcal{I}_n^{(i)})_{i \in \mathbb{N}}$.

We assume that the following three conditions are satisfied:

(A1)

$$\mathbb{E}[\Delta \mathcal{I}_n^{(i)} | \mathcal{G}_n] = \frac{\mathcal{F}^{(i)} \mathcal{I}_n^{(i)}}{n \bar{\mathcal{F}}_n},$$

(A2) there exists a constant C^{var} such that

$$\mathrm{Var}(\Delta \mathcal{I}_n^{(i)} | \mathcal{G}_n) \leqslant C^{\mathrm{var}} \mathbb{E}[\Delta \mathcal{I}_n^{(i)} | \mathcal{G}_n],$$

(A3) conditionally on $\mathcal{G}_n$, for $i \neq j$, we assume that $\Delta \mathcal{I}_n^{(i)}$ and $\Delta \mathcal{I}_n^{(j)}$ are non-positively correlated.

By assumption the essential supremum of μ is finite and strictly positive, say s. Since the model will still satisfy assumptions (A1)–(A3) if we replace $\mathcal{F}^{(i)}$ by $\tilde{\mathcal{F}}^{(i)} = \mathcal{F}^{(i)}/s$, we can and will assume without loss of generality that

(A0)

$$\operatorname{ess\,sup}(\mu) = 1.$$

In the following, if we omit the domain of integration, we will always mean that any integral is taken over the interval $(0, 1]$. Moreover, if we condition on $\mathcal{G}_n$ as in assumptions (A1)–(A3), we always mean that the corresponding σ-algebra contains the information about the fitness $\mathcal{F}^{(i)}$ of vertices $i = 1, \ldots, n$, but not for $i > n$.

Remark 2.1. Assumptions (A1)–(A3) guarantee that the total number of edges in $\mathcal{G}_n$ is of order λn: see Lemma 3.2.

Let us give two examples that satisfy our assumptions.

Example 2.2 (Poisson outdegree (M1)). The definition depends on a parameter $\lambda > 0$. In model (M1), given $\mathcal{G}_n$, the new vertex $n+1$ establishes for each old vertex $i \in \{1, \ldots, n\}$ an independent Poisson-distributed number of links $n+1 \to i$ with parameter

$$\frac{\mathcal{F}^{(i)} \mathcal{I}_n^{(i)}}{n \bar{\mathcal{F}}_n}.$$

Note that the conditional outdegree of a new vertex $n+1$, given $\mathcal{G}_n$, is Poisson-distributed with parameter λ.

Example 2.3 (fixed outdegree (M2)). The definition relies on a parameter $\lambda \in \mathbb{N}$ denoting the deterministic outdegree of new vertices. Given $\mathcal{G}_n$, the number of edges connecting $n+1$ to the individual old vertices $1,\ldots,n$ forms a multinomial random variable with parameters λ and

$$\left(\frac{\mathcal{F}^{(i)}\,\mathcal{I}_n^{(i)}}{\lambda n\,\bar{\mathcal{F}}_n}\right)_{i=1,\ldots,n}, \quad \text{where } \bar{\mathcal{F}}_n = \frac{1}{\lambda n}\sum_{i=1}^{n}\mathcal{F}^{(i)}\,\mathcal{I}_n^{(i)}.$$

The model (M2) with $\lambda = 1$ is the one analysed in [6].

We analyse a sequence of random measures $(\Gamma_n)_{n\in\mathbb{N}}$ on $(0,1]$ given by

$$\Gamma_n := \frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_n^{(i)}\,\delta_{\mathcal{F}^{(i)}},$$

the *impact distributions*. These measures describe the relative impact of fitnesses. Note also that, up to normalization, Γ_n is the distribution of the fitness of a vertex chosen proportional to its impact.

Theorem 2.4. *Suppose that assumptions* (A0)–(A3) *are satisfied. If*

$$\int \frac{f}{1-f}\,\mu(df) \geqslant \lambda,$$

we let $\theta^* \geqslant 1$ denote the unique value with

$$\int \frac{f}{\theta^* - f}\,\mu(df) = \lambda$$

and otherwise set $\theta^* := 1$. We have

$$\lim_{n\to\infty}\bar{\mathcal{F}}_n = \theta^*, \quad \textit{almost surely}$$

and we distinguish two regimes.

(i) Fit-get-richer phase. *Suppose that*

$$\int \frac{f}{1-f}\,\mu(df) \geqslant \lambda.$$

(Γ_n) *converges, almost surely, in the weak* topology to Γ, where*

$$\Gamma(df) := \frac{\theta^*}{\theta^* - f}\,\mu(df).$$

(ii) Bose–Einstein phase. *Suppose that*

$$\int \frac{f}{1-f}\,\mu(df) < \lambda.$$

(Γ_n) *converges, almost surely, in the weak* topology to Γ, where*

$$\Gamma(df) := \frac{1}{1-f}\,\mu(df) + \left(1 + \lambda - \int \frac{1}{1-f'}\,\mu(df')\right)\delta_1(df).$$

Remark 2.5. In particular, the two phases can be characterized as follows. In the *fit-get-richer phase*, that is, if

$$\int \frac{f}{1-f}\,\mu(df) \geqslant \lambda,$$

then the limit of (Γ_n) is absolutely continuous with respect to μ. However, in the *Bose–Einstein phase*, that is, if

$$\int \frac{f}{1-f}\,\mu(df) < \lambda,$$

then the limit of (Γ_n) is not absolutely continuous with respect to μ, but has an atom in 1. The explanation for this phenomenon is that a positive fraction of newly incoming edges connects to vertices with fitness that is ever closer to the essential supremum of the fitness distribution μ, which in the limit amounts to an atom at the essential supremum.

If μ itself has an atom in its essential supremum, that is, if $\mu(\{1\}) > 0$, then

$$\int \frac{f}{1-f}\mu(df) = \infty,$$

so we are always in the fit-get-richer phase and $\theta^* > 1$.

Next, we restrict our attention to vertices with a fixed impact $k \in \mathbb{N}$. For $n \in \mathbb{N}$ we consider the random measure

$$\Gamma_n^{(k)} := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k\}}\delta_{\mathcal{F}^{(i)}},$$

representing – up to normalization – the random fitness of a uniformly chosen vertex with impact k.

To prove convergence of $(\Gamma_n^{(k)})$, we need additional assumptions. Indeed, so far our assumptions admit models for which vertices are always connected by multiple edges, in which case there would be no vertices with impact 2.

We will work with the following assumptions:

(A4) For all $k \in \mathbb{N}$,

$$\sup_{i=1,\dots,n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k\}}\, n|\,\mathbb{P}(\Delta\mathcal{I}_n^{(i)} = 1|\mathcal{G}_n) - \frac{\mathcal{F}^{(i)}\mathcal{I}_n^{(i)}}{n\,\bar{\mathcal{F}}_n}| \to 0, \text{ almost surely.}$$

Further, we impose a stronger assumption on the correlation structure:

(A5) Given $\mathcal{G}_n$, the collection $\{\Delta\mathcal{I}_n^{(i)}\}_{i=1}^n$ is negatively quadrant dependent in the sense that for any $i \neq j$, and any $k, \ell \in \mathbb{N}$,

$$\mathbb{P}(\Delta\mathcal{I}_n^{(i)} \leqslant k; \Delta\mathcal{I}_n^{(j)} \leqslant \ell|\mathcal{G}_n) \leqslant \mathbb{P}(\Delta\mathcal{I}_n^{(i)} \leqslant k|\mathcal{G}_n)\mathbb{P}(\Delta\mathcal{I}_n^{(j)} \leqslant \ell|\mathcal{G}_n).$$

Remark 2.6. Note that both Examples 2.2 and 2.3 also satisfy these additional assumptions. Assumption (A4) guarantees together with assumption (A1) that the expected conditional increment of $\mathcal{I}_n^{(i)}$ is dominated by the case that exactly one new vertex connects to i. Assumption (A5) is obviously stronger than non-positive correlation (A3), but it is still satisfied in most natural model specifications.

Theorem 2.7. *Suppose that assumptions* (A0)–(A5) *are satisfied, and let* $\theta^* \in [1,\infty)$ *be defined as in Theorem 2.4. Then we have that, almost surely,* $(\Gamma_n^{(k)})$ *converges in the weak** *topology to* $\Gamma^{(k)}$, *where*

$$\Gamma^{(k)}(df) := \frac{1}{k+\frac{\theta^*}{f}} \frac{\theta^*}{f} \prod_{i=1}^{k-1} \frac{i}{i+\frac{\theta^*}{f}}\, \mu(df). \tag{2.2}$$

The theorem immediately allows us to control the number of vertices with impact $k \in \mathbb{N}$. Let

$$p_n(k) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k\}} = \Gamma_n^{(k)}((0,1]).$$

Corollary 2.8. *Under the assumptions of Theorem 2.7, we have that*

$$\lim_{n\to\infty} p_n(k) = \int \frac{1}{k+\frac{\theta^*}{f}} \frac{\theta^*}{f} \prod_{i=1}^{k-1} \frac{i}{i+\frac{\theta^*}{f}}\, \mu(df), \text{ almost surely.}$$

Outline of the article. Section 3 starts with preliminary considerations. In particular, it introduces a stochastic approximation argument which, among other applications, has also appeared in the context of generalized urn models: see, for example, the survey by Pemantle [19]. Roughly speaking, key quantities are expressed as approximations to stochastically perturbed differential equations. The perturbation is asymptotically negligible, and we obtain descriptions by differential equations that are typically referred to as *master equations*.

Section 4 is concerned with the proof of Theorem 2.4. Here the main task is to prove convergence of the random normalization $(\bar{\mathcal{F}}_n)$. This goal is achieved via a bootstrapping argument. Starting with an upper bound on $(\bar{\mathcal{F}}_n)$ of the form

$$\limsup_{n\to\infty} \bar{\mathcal{F}}_n \leqslant \theta, \text{ almost surely,}$$

we show in Lemma 4.2 that this statement remains true when replacing θ by

$$T(\theta) := 1 + \frac{1}{\lambda} \int \frac{\theta-1}{\theta-f} f\, \mu(df). \tag{2.3}$$

For the proof of Theorem 2.4, we will iterate the argument to obtain convergence of the normalization to the largest fixed point of T, which we will denote by θ^*. Moreover, we see that $\theta^* > 1$ if and only if

$$\int \frac{x}{1-x}\mu(dx) > \lambda,$$

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which corresponds to the fit-get-richer phase. In this case, if in addition $\mu(\{1\}) = 0$, T also has the fixed point 1. But one can check that only the larger fixed point θ^* is stable. However, in the condensation phase, T has only a single fixed point, that is, $\theta^* = 1$, which in that case is also stable. See also Figure 1 for an illustration.

Section 5 is concerned with the proof of Theorem 2.7. The proof is based on stochastic approximation techniques introduced in Section 3. In our setting these differential equations are non-linear because of the normalization $\bar{\mathcal{F}}_n$. However, since we can control the normalization by Theorem 2.4, in the analysis of the joint fitness and degree distribution, we arrive at linear equations (or more precisely inequalities) for the stochastic approximation. The latter then yield Theorem 2.7 via an approximation argument.

3. Preliminaries

We first recall the general idea of stochastic approximation, which goes back to [20] and can be stated, for example, for a stochastic process $(\mathbf{X}_n)_{n\geqslant 0}$ taking values in $\mathbb{R}^d$. Then, $(\mathbf{X}_n)_{n\geqslant 0}$ is known as a stochastic approximation process if it satisfies a recursion of the type

$$\mathbf{X}_{n+1} - \mathbf{X}_n = \frac{1}{n+1} F(\mathbf{X}_n) + \mathbf{R}_{n+1} - \mathbf{R}_n, \tag{3.1}$$

where F is a suitable vector field and the increment of $\mathbf{R}$ corresponds to an (often stochastic) error. In our setting, we could, for example, restrict to the case when μ is supported on finitely many values $\{f_1,\dots,f_d\} \subset (0,1]$, and let

$$X_n(k) = \frac{1}{n}\sum_{i=1}^{n} \mathcal{I}_n^{(i)} \mathbb{1}_{\{\mathcal{F}^{(i)}=f_k\}}$$

denote the proportion of vertices that have fitness f_k weighted by their impact. Then, one can easily calculate the conditional expectation of $X_{n+1}(k)$ given the graph $\mathcal{G}_n$ up to time n. Indeed, as we will see in the proof of Proposition 4.1, under our assumptions we obtain that

$$\mathbb{E}[X_{n+1}(k) - X_n(k) \,|\, \mathcal{G}_n] = \frac{1}{n+1}\left(\mu(\{f_k\}) + \frac{f_k}{\bar{\mathcal{F}}_n} X_n(k) - X_n(k)\right).$$

Therefore, we note that $\mathbf{X_n} = (X_n(k))_{k=1}^d$ satisfies

$$X_{n+1}(k) - X_n(k) = \frac{1}{n+1}\left(\mu(\{f_k\}) + \frac{f_k}{\bar{\mathcal{F}}_n} X_n(k) - X_n(k)\right) + R_{n+1}(k) - R_n(k),$$

so that $\mathbf{X}_n = (X_n(k))_{k=1}^d$ satisfies an equation of type (3.1), provided we take

$$R_{n+1}(k) - R_n(k) = X_{n+1}(k) - \mathbb{E}[X_{n+1}(k)|\mathcal{G}_n],$$

which defines a martingale, for which we can employ the standard techniques to show convergence.

Provided that the random perturbations are asymptotically negligible, it is possible to analyse the random dynamical system by the corresponding master equation

$$\dot{\mathbf{x}}_t = F(\mathbf{x}_t).$$

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There are many articles exploiting such connections, and an overview is provided by Benaïm [2]. The connection to general urn models is further explained by Pemantle [19]. In preferential attachment models, these techniques only seem to have been used directly in the work of Jordan (see, *e.g.*, [15, 16]), covering only the case of finitely many fitness values completely. More generally, the resulting differential equation is closely related to what is known as the master equation in heuristic derivations: see, for example, [18, chapter 14].

However, in our setting this method is not directly applicable. First of all, we would like to consider arbitrary fitness distributions (*i.e.*, not restricted to finitely many values), and secondly, the resulting equation is not linear, because of the appearance of the normalization $\bar{\mathcal{F}}_n$. The latter problem is addressed by using a bootstrapping method (as described in the Introduction). However, this leads to an inequality on the increment, rather than an equality as in (3.1). Fortunately, the resulting vector field F has a very simple structure and so we can deduce the long-term behaviour of $(\mathbf{X}_n)_{n\geqslant 0}$ by elementary means; the corresponding technical result is Lemma 3.1. By using inequalities, we also gain the flexibility to approximate arbitrary fitness distribution by discretization.

In order to keep our proofs self-contained, we will first state and prove an easy special case of the technique adapted to our setting.

Lemma 3.1. *Let $(X_n)_{n\geqslant 0}$ be a non-negative stochastic process. We suppose that the following estimate holds:*

$$X_{n+1} - X_n \leqslant \frac{1}{n+1}(A_n - B_n X_n) + R_{n+1} - R_n, \tag{3.2}$$

where

(i) *(A_n) and (B_n) are almost surely convergent stochastic processes with deterministic limits $A, B > 0$,*
(ii) *(R_n) is an almost surely convergent stochastic process.*

Then we have that, almost surely,

$$\limsup_{n\to\infty} X_n \leqslant \frac{A}{B}.$$

Similarly, if instead, under the same conditions (i) *and* (ii)*,*

$$X_{n+1} - X_n \geqslant \frac{1}{n+1}(A_n - B_n X_n) + R_{n+1} - R_n,$$

then almost surely

$$\liminf_{n\to\infty} X_n \geqslant \frac{A}{B}.$$

Proof. This is a slight adaptation of Lemma 2.6 in [19]. Fix $\delta \in (0,1)$. By our assumptions, almost surely, we can find n_0 such that, for all $m, n \geqslant n_0$,

$$A_n \leqslant (1+\delta)A, \quad B_n \geqslant (1-\delta)B, \quad |R_m - R_n| \leqslant \delta.$$

Then, by (3.2), we have that, for any $m > n \geqslant n_0$,

$$X_m - X_n \leqslant \sum_{j=n}^{m-1} \frac{1}{j+1}(A_j - B_j X_j) + |R_m - R_n| \\ \leqslant \sum_{j=n}^{m-1} \underbrace{\frac{1}{j+1}((1+\delta)A - (1-\delta)BX_j))}_{=:Y_j} + \delta. \tag{3.3}$$

Let

$$C := \frac{(1+\delta)A}{(1-\delta)B}.$$

For each index $j \geqslant n_0$ with $X_j \geqslant C + \delta$, we have that

$$Y_j \leqslant -B(1-\delta)\delta/(j+1).$$

Since the harmonic series diverges, by (3.3) there exists $m_0 \geqslant n_0$ with $X_{m_0} \leqslant C + \delta$.

Next, we prove that for any $m \geqslant m_0$ we have $X_m \leqslant C + 3\delta$ provided that n_0 is chosen sufficiently large (*i.e.*, $\frac{1}{n_0+1}(1+\delta)A \leqslant \delta$). Suppose that $X_m > C + \delta$. We choose m_1 to be the largest index smaller than m with $X_{m_1} \leqslant C + \delta$. Clearly, $m_1 \geqslant m_0$ and an application of estimate (3.3) gives

$$X_m \leqslant X_{m_1} + Y_{m_1} + \delta \leqslant C + 2\delta + \frac{1}{m_1+1}(1+\delta)A \leqslant C + 3\delta = \frac{(1+\delta)A}{(1-\delta)B} + 3\delta.$$

Since $\delta \in (0,1)$ is arbitrary, we get that, almost surely,

$$\limsup_{n\to\infty} X_n \leqslant \frac{A}{B}.$$

The argument for the reverse inequality works analogously. □

As a first application of Lemma 3.1, we can show that the total number of edges converges if properly normalized.

Lemma 3.2. *Almost surely, we have that*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathcal{I}_n^{(i)} = 1 + \lambda.$$

Proof. Define

$$Y_n := \frac{1}{n}\sum_{i=1}^{n} \mathcal{I}_n^{(i)}.$$

Then we calculate the conditional expectation of Y_{n+1} given $\mathcal{G}_n$, using that $\mathcal{I}_{n+1}^{n+1} = 1$ by definition, that is,

$$\begin{aligned}\mathbb{E}[Y_{n+1}|\mathcal{G}_n] &= \frac{1}{n+1}\Big(\sum_{i=1}^n \mathbb{E}[\mathcal{I}_{n+1}^{(i)}|\mathcal{G}_n] + 1\Big)\\ &= Y_n + \frac{1}{n+1}\Big(1 + \sum_{i=1}^n \mathbb{E}[\Delta\mathcal{I}_n^{(i)}|\mathcal{G}_n] - Y_n\Big)\\ &= Y_n + \frac{1}{n+1}(1+\lambda - Y_n),\end{aligned}$$

where we used assumption (A1) on the conditional mean of $\Delta\mathcal{I}_n^{(i)}$ and the definition of $\bar{\mathcal{F}}_n$. Thus we can write

$$Y_{n+1} - Y_n = \frac{1}{n+1}(1+\lambda-Y_n) + R_{n+1} - R_n, \tag{3.4}$$

where we define $R_0 = 0$ and

$$\Delta R_n := R_{n+1} - R_n = Y_{n+1} - \mathbb{E}[Y_{n+1}|\mathcal{G}_n].$$

Therefore, R_n is a martingale and R_n converges almost surely if we can show that $\mathbb{E}[(\Delta R_n)^2|\mathcal{G}_n]$ is summable. Indeed, first using (A3), which states that impact evolutions of distinct vertices are non-positively correlated, we can deduce that

$$\begin{aligned}\mathbb{E}[(\Delta R_n)^2|\mathcal{G}_n] &\leqslant \frac{1}{(n+1)^2}\Big(\sum_{i=1}^n \mathbb{E}[(\Delta\mathcal{I}_n^{(i)} - \mathbb{E}[\Delta\mathcal{I}_n^{(i)}|\mathcal{G}_n])^2|\mathcal{G}_n] + 1\Big)\\ &\leqslant \frac{1}{(n+1)^2}\Big(C^{\mathrm{var}}\sum_{i=1}^n \mathbb{E}[\Delta\mathcal{I}_n^{(i)}|\mathcal{G}_n] + 1\Big)\\ &\leqslant \frac{1}{(n+1)^2}(C^{\mathrm{var}}\lambda + 1),\end{aligned}$$

which is summable.

Hence, we can apply both parts of Lemma 3.1 together with the convergence of (R_n) to obtain the almost sure convergence $\lim_{n\to\infty} Y_n = 1+\lambda$. □

Later on, we will need some *a priori* bounds on the normalization sequence.

Lemma 3.3. *Almost surely, we have that*

$$\frac{1}{\lambda}\int x\,\mu(dx) \leqslant \liminf_{n\to\infty} \bar{\mathcal{F}}_n \leqslant \limsup_{n\to\infty} \bar{\mathcal{F}}_n \leqslant \frac{1+\lambda}{\lambda}.$$

Proof. For the lower bound, notice that by definition $\mathcal{I}_n^{(i)} \geqslant 1$, and therefore

$$\liminf_{n\to\infty} \bar{\mathcal{F}}_n = \liminf_{n\to\infty} \frac{1}{\lambda n}\sum_{i=1}^n \mathcal{F}^{(i)}\mathcal{I}_n^{(i)} \geqslant \liminf_{n\to\infty} \frac{1}{\lambda n}\sum_{i=1}^n \mathcal{F}^{(i)} = \frac{1}{\lambda}\int x\,\mu(dx).$$

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For the upper bound, one can use that the $\mathcal{F}^{(i)} \leqslant 1$ and combine with

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathcal{I}_n^{(i)} = 1 + \lambda,$$

which we proved in Lemma 3.2. □

4. Proof of Theorem 2.4

The central bootstrap argument is carried out at the end of this section. It is based on Lemma 4.2. Before we state and prove Lemma 4.2, we prove a technical proposition which will be crucial in the proof of the lemma.

Proposition 4.1.

(i) *Let* $\theta \geqslant 1$. *If*

$$\limsup_{n\to\infty} \bar{\mathcal{F}}_n \leqslant \theta, \text{ almost surely,}$$

then for any $0 \leqslant a < b \leqslant 1$ *we have*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{\mathcal{F}^{(i)} \in (a,b]\}} \mathcal{I}_n^{(i)} \geqslant \int_{(a,b]} \frac{\theta}{\theta - f}\, \mu(df),$$

almost surely.

(ii) *Let* $\theta > 0$. *If*

$$\liminf_{n\to\infty} \bar{\mathcal{F}}_n \geqslant \theta, \text{ almost surely,}$$

then for any $0 \leqslant a < b < \theta \wedge 1$ *we have*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{\mathcal{F}^{(i)} \in (a,b]\}} \mathcal{I}_n^{(i)} \leqslant \int_{(a,b]} \frac{\theta}{\theta - f}\, \mu(df),$$

almost surely.

□

Proof. **(i)** First we prove that, under the assumptions of (i), for $0 \leqslant f < f' \leqslant 1$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{\mathcal{F}^{(i)} \in (f,f']\}} \mathcal{I}_n^{(i)} \geqslant \frac{\theta}{\theta - f}\, \mu((f,f']), \text{ almost surely.} \tag{4.1}$$

Fix $0 \leqslant f < f' \leqslant 1$ and let

$$X_n := \Gamma_n((f,f']) = \frac{1}{n} \sum_{i \in \mathbb{I}_n} \mathcal{I}_n^{(i)},$$

where for notational convenience we let

$$\mathbb{I}_n := \mathbb{I}_n((f,f']) := \{i \in \{1,\ldots,n\} : \mathcal{F}^{(i)} \in (f,f']\}.$$

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We will show (4.1) with the help of the stochastic approximation argument explained in Section 3, and stated formally in Lemma 3.1. We need to provide a lower bound for the increment $X_{n+1} - X_n$. Using assumption (A1), we can calculate the conditional expectation of X_{n+1}:

$$\begin{aligned}\mathbb{E}[X_{n+1}|\mathcal{G}_n] &= \frac{1}{n+1}\sum_{i\in\mathbb{I}_n}\mathbb{E}[\mathcal{I}^{(i)}_{n+1}|\mathcal{G}_n] + \frac{1}{n+1}\mathbb{P}(\mathcal{F}^{(n+1)}\in(f,f'])\\ &= X_n + \frac{1}{n+1}\Big(\sum_{i\in\mathbb{I}_n}\mathbb{E}[\Delta\mathcal{I}^{(i)}_n|\mathcal{G}_n] - X_n + \mu((f,f'])\Big)\\ &= X_n + \frac{1}{n+1}\Big(\sum_{i\in\mathbb{I}_n}\frac{\mathcal{F}^{(i)}\mathcal{I}^{(i)}_n}{n\bar{\mathcal{F}}_n} - X_n + \mu((f,f'])\Big).\end{aligned}$$

Hence, rearranging yields

$$\mathbb{E}[X_{n+1}|\mathcal{G}_n] - X_n \geqslant \frac{1}{n+1}\Big(\mu((f,f']) - \Big(1 - \frac{f}{\sup_{m\geqslant n}\bar{\mathcal{F}}_m}\Big)X_n\Big).$$

Thus, we can write

$$X_{n+1} - X_n \geqslant \frac{1}{n+1}\Big(\mu((f,f']) - \Big(1 - \frac{f}{\sup_{m\geqslant n}\bar{\mathcal{F}}_m}\Big)X_n\Big) + R_{n+1} - R_n,$$

where R_n is a martingale defined via $R_0 = 0$ and

$$\Delta R_n := R_{n+1} - R_n = X_{n+1} - \mathbb{E}[X_{n+1}|\mathcal{G}_n].$$

If we can show that R_n converges almost surely, then Lemma 3.1 together with the assumption that $\limsup_{n\to\infty}\bar{\mathcal{F}}_n \leqslant \theta$ shows that

$$\liminf_{n\to\infty} X_n \geqslant \frac{\theta}{\theta - f}\mu((f,f']),$$

which is the required bound (4.1).

The martingale convergence follows if we show that $\mathbb{E}[(\Delta R_n)^2|\mathcal{G}_n]$ is summable. Indeed,

$$\Delta R_n = \frac{1}{n+1}\sum_{i\in\mathbb{I}_n}\big(\mathcal{I}^{(i)}_{n+1} - \mathbb{E}[\mathcal{I}^{(i)}_{n+1}|\mathcal{G}_n]\big) + \frac{1}{n+1}\big(\mathbb{1}_{\{\mathcal{F}^{(n+1)}\in(f,f']\}} - \mu((f,f'])\big).$$

The second moment of the last expression is clearly bounded by

$$\frac{1}{(n+1)^2}\mu((f,f']),$$

which is summable, so we can concentrate on the first term. Now, we can use (A3), the non-positive correlation of $\Delta\mathcal{I}^{(i)}_n$, and then (A2), (A1) and the definition of $\bar{\mathcal{F}}_n$ to estimate

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the conditional variance to deduce that

$$\frac{1}{(n+1)^2}\mathbb{E}\Big[\Big(\sum_{i\in\mathbb{I}_n}(\mathcal{I}^{(i)}_{n+1}-\mathbb{E}[\mathcal{I}^{(i)}_{n+1}|\mathcal{G}_n])\Big)^2\Big|\mathcal{G}_n\Big]$$
$$=\frac{1}{(n+1)^2}\mathbb{E}\Big[\Big(\sum_{i\in\mathbb{I}_n}(\Delta\mathcal{I}^{(i)}_{n}-\mathbb{E}[\Delta\mathcal{I}^{(i)}_{n}|\mathcal{G}_n])\Big)^2\Big|\mathcal{G}_n\Big]$$
$$\leqslant \frac{1}{(n+1)^2}\sum_{i\in\mathbb{I}_n}\mathrm{Var}(\Delta\mathcal{I}^{(i)}_n|\mathcal{G}_n)$$
$$\leqslant \frac{1}{(n+1)^2}C^{\mathrm{var}}\sum_{i\in\mathbb{I}_n}\frac{\mathcal{F}^{(i)}\mathcal{I}^{(i)}_n}{n\bar{\mathcal{F}}_n}\leqslant \frac{1}{(n+1)^2}C^{\mathrm{var}}\lambda.$$

The latter is obviously summable, so R_n converges almost surely.

Note that the assertion (i) follows from (4.1) by a Riemann approximation. We partition $(a,b]$ via $a=f_0<\cdots<f_\ell=b$ with an arbitrary $\ell\in\mathbb{N}$. Then it follows from (4.1) that

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{\mathcal{F}^{(i)}\in(a,b]\}}\mathcal{I}^{(i)}_n\geqslant\sum_{k=0}^{\ell-1}\frac{\theta}{\theta-f_k}\,\mu((f_k,f_{k+1}]),\ \text{almost surely,}$$

and the right-hand side approximates the integral up to an arbitrary small constant.

(ii) It suffices to prove that for $0\leqslant f<f'<\theta\wedge 1$ we have

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{\mathcal{F}^{(i)}\in(f,f']\}}\,\mathcal{I}^{(i)}_n\leqslant\frac{\theta}{\theta-f'}\,\mu((f,f']),\ \text{almost surely.}\tag{4.2}$$

This is completely analogous to part (i) using Lemma 3.1. Then the statement (ii) follows as above by a Riemann approximation. □

The next lemma takes the lower bound on the impact distribution obtained in Proposition 4.1 to produce a new upper bound on the normalization. For $\theta>1$ we set

$$T(\theta):=1+\frac{1}{\lambda}\int\frac{\theta-1}{\theta-f}\,f\,\mu(df).\tag{4.3}$$

Lemma 4.2.

(i) *Let $\theta>1$. If*

$$\limsup_{n\to\infty}\bar{\mathcal{F}}_n\leqslant\theta,\ \textit{almost surely,}\tag{4.4}$$

then

$$\limsup_{n\to\infty}\bar{\mathcal{F}}_n\leqslant T(\theta),\ \textit{almost surely.}$$

(ii) *Let $\theta>0$ and suppose that*

$$\liminf_{n\to\infty}\bar{\mathcal{F}}_n\geqslant\theta,\ \textit{almost surely.}\tag{4.5}$$

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We have, almost surely,

$$\liminf_{n\to\infty} \bar{\mathcal{F}}_n \geqslant \begin{cases} T(\theta) & \text{if } \theta > 1, \\ \theta + \frac{\theta}{\lambda}\big(1 - \mu((0,\theta))\big) & \text{if } \theta \in (0,1]. \end{cases}$$

Remark 4.3. If we define

$$T(1) := 1 + \frac{1}{\lambda}\mu(\{1\}),$$

then it follows by dominated convergence that

$$\begin{aligned} \lim_{\theta\downarrow 1} T(\theta) &= \lim_{\theta\downarrow 1}\Big(1 + \frac{1}{\lambda}\int_{(0,1]} \frac{\theta-1}{\theta-f}\, f\,\mu(df)\Big) \\ &= 1 + \lim_{\theta\downarrow 1}\frac{1}{\lambda}\int_{(0,1)} \frac{\theta-1}{\theta-f}\, f\,\mu(df) + \frac{1}{\lambda}\mu(\{1\}) = T(1). \end{aligned}$$

In particular, the lower bound in part (ii) of Lemma 4.2 is continuous at $\theta = 1$.

Proof of Lemma 4.2. **(i)** Fix $\theta > 1$ and assume that (4.4) holds. Define a measure ν on $(0,1]$ via

$$\nu(df) := \frac{\theta}{\theta - f}\,\mu(df).$$

Further, set $\nu' := \nu + ((1+\lambda) - \nu((0,1]))\delta_1$. Using first that, by Lemma 3.2,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_n^{(i)} = 1 + \lambda, \text{ almost surely,}$$

and secondly the assumption (4.4) so that we can apply Proposition 4.1(i), we deduce that, for every $t \in (0,1)$, almost surely,

$$\begin{aligned} \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{\mathcal{F}^{(i)}\in(t,1]\}}\mathcal{I}_n^{(i)} &= 1 + \lambda - \liminf_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{\mathcal{F}^{(i)}\in(0,t]\}}\mathcal{I}_n^{(i)} \\ &\leqslant 1 + \lambda - \nu((0,t]) = \nu'((t,1]). \end{aligned} \tag{4.6}$$

This allows us to compute a new asymptotic upper bound for $(\bar{\mathcal{F}}_n)$. Letting $m \in \mathbb{N}$, observe that, almost surely,

$$\begin{aligned} \bar{\mathcal{F}}_n &= \frac{1}{\lambda n}\sum_{i=1}^{n}\mathcal{F}^{(i)}\,\mathcal{I}_n^{(i)} \leqslant \frac{1}{\lambda n}\sum_{i=1}^{n}\frac{1}{m}\sum_{j=0}^{m-1}\mathbb{1}_{\{\mathcal{F}^{(i)}>j/m\}}\,\mathcal{I}_n^{(i)} \\ &= \frac{1}{\lambda m}\sum_{j=0}^{m-1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{\mathcal{F}^{(i)}>j/m\}}\,\mathcal{I}_n^{(i)}, \end{aligned}$$

so that by (4.6)

$$\limsup_{n\to\infty}\bar{\mathcal{F}}_n \leqslant \frac{1}{\lambda m}\sum_{j=0}^{m-1}\nu'((j/m,1]), \text{ almost surely.}$$

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The latter expression tends with $m \to \infty$ to the integral

$$\frac{1}{\lambda}\int f\, \nu'(df)$$

and we finally get that, almost surely,

$$\limsup_{n\to\infty} \bar{\mathcal{F}}_n \leqslant \frac{1}{\lambda}\int f\, \nu'(df).$$

It remains to show that the latter integral is equal to $T(\theta)$ as defined in (4.3). Indeed, using that μ is a probability measure so that

$$\int \frac{\theta}{\theta - f}\,\mu(df) = \int \frac{f}{\theta - f}\,\mu(df) + \int \frac{\theta - f}{\theta - f}\,\mu(df) = \int \frac{f}{\theta - f}\,\mu(df) + 1,$$

it follows from the definition of ν' that

$$\begin{aligned}\frac{1}{\lambda}\int f\,\nu'(df) &= \frac{1}{\lambda}\Big(1+\lambda - \int \frac{\theta}{\theta - f}\,\mu(df) + \theta\int \frac{f}{\theta - f}\,\mu(df)\Big)\\ &= 1 + \frac{1}{\lambda}\int \frac{\theta - 1}{\theta - f} f\,\mu(df) = T(\theta),\end{aligned}$$

as claimed.

(ii) Fix $\theta > 0$ and assume that (4.5) holds. Let $\theta' \in (0,\theta)$ if $\theta \leqslant 1$ and set $\theta' = 1$ if $\theta > 1$. Then, consider the (signed) measures $\nu_{\theta'}$ and $\nu'_{\theta'}$ defined by

$$\nu_{\theta'}(df) := \frac{\theta}{\theta - f}\mathbb{1}_{\{f\in(0,\theta']\}}\,\mu(df)$$

and

$$\nu'_{\theta'} := \nu_{\theta'} + (1+\lambda - \nu_{\theta'}((0,1]))\delta_{\theta'}.$$

As above we use the assumption (4.5) and conclude with Proposition 4.1 that for $t < \theta'$, almost surely,

$$\liminf_{n\to\infty}\sum_{i=1}^{n}\mathbb{1}_{\{\mathcal{F}^{(i)}\in(t,1]\}}\mathcal{I}_n^{(i)} \geqslant 1 + \lambda - \nu_{\theta'}((0,t]) = \nu'_{\theta'}((t,1]). \tag{4.7}$$

We proceed as above and observe that, for any $m \in \mathbb{N}$,

$$\begin{aligned}\bar{\mathcal{F}}_n = \frac{1}{\lambda n}\sum_{i=1}^{n}\mathcal{F}^{(i)}\mathcal{I}_n^{(i)} &\geqslant \frac{1}{\lambda n}\sum_{i=1}^{n}\frac{\theta'}{m}\sum_{j=1}^{m-1}\mathbb{1}_{\{\mathcal{F}^{(i)}>\frac{j}{m}\theta'\}}\mathcal{I}_n^{(i)}\\ &= \frac{\theta'}{\lambda m}\sum_{j=1}^{m-1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}_{\{\mathcal{F}^{(i)}>\frac{j}{m}\theta'\}}\mathcal{I}_n^{(i)}\end{aligned}$$

which by (4.7) yields that, almost surely,

$$\liminf_{n\to\infty}\bar{\mathcal{F}}_n \geqslant \frac{\theta'}{\lambda m}\sum_{j=1}^{m}\nu'_{\theta'}((\tfrac{j}{m}\theta', 1]).$$

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Since $m \in \mathbb{N}$ is arbitrary, we get that, almost surely,

$$\liminf_{n\to\infty} \bar{\mathcal{F}}_n \geqslant \frac{1}{\lambda}\int_{(0,\theta']} f\, \nu'_{\theta'}(df) = \frac{1}{\lambda}\Big(\theta'(1+\lambda) - \theta\int_{(0,\theta']} \frac{\theta'-f}{\theta-f}\,\mu(df)\Big). \tag{4.8}$$

We distinguish two cases. If $\theta \leqslant 1$, we use that the latter integral is dominated by $\mu((0,\theta'])$ and let $\theta' \uparrow \theta$ to deduce from (4.8) that

$$\liminf_{n\to\infty} \bar{\mathcal{F}}_n \geqslant \theta + \frac{\theta}{\lambda}\big(1-\mu((0,\theta))\big), \text{ almost surely.}$$

If $\theta > 1$, we have defined $\theta' = 1$, so that $\nu_{\theta'} = \nu$ and $\nu'_{\theta'} = \nu'$ (for ν, ν' defined in part (i)) and, by the same calculation as at the end of part (i), the right-hand side of (4.8) is equal to $T(\theta)$. □

Finally, we can prove Theorem 2.4, where we first show that the normalization converges using a bootstrap argument based on Lemma 4.2. Then, we use the bound on the impact distribution obtained in Proposition 4.1 to show convergence of the impact distribution.

Proof of Theorem 2.4. **(i)** *Fit-get-richer phase.* Assume that $\theta^* \geqslant 1$ is the unique value such that

$$\int \frac{f}{\theta^*-f}\mu(df) = \lambda.$$

We will first show that $\limsup_{n\to\infty} \bar{\mathcal{F}}_n \leqslant \theta^*$ using a bootstrap argument and then show that the impact distributions converge. Finally, we will deduce the full convergence of $\bar{\mathcal{F}}_n$ from the convergence of the impact distributions.

For the asymptotic upper bound on the normalization $\bar{\mathcal{F}}_n$, suppose that θ^{**} is the smallest value in $[\theta^*, \infty)$ with

$$\limsup_{n\to\infty} \bar{\mathcal{F}}_n \leqslant \theta^{**}, \text{ almost surely.} \tag{4.9}$$

Such a value exists due to Lemma 3.3.

We prove that $\theta^{**} = \theta^*$ by contradiction. Suppose that $\theta^{**} > \theta^*$. We apply Lemma 4.2 and get that

$$\limsup_{n\to\infty} \bar{\mathcal{F}}_n \leqslant T(\theta^{**}), \text{ almost surely,} \tag{4.10}$$

where we recall that T is defined in (4.3) as

$$T(\theta) = 1 + \frac{1}{\lambda}\int \frac{\theta-1}{\theta-f} f\,\mu(df), \qquad \text{for } \theta > 1,$$

and we set

$$T(1) := 1 + \frac{1}{\lambda}\mu(\{1\}).$$

Now we note that T is continuous on $[\theta^*, \theta^{**}]$. For the case $\theta^* = 1$, we refer to Remark 4.3 and observe that then by definition of θ^* necessarily $\mu(\{1\}) = 0$, so that

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$T(1) = 1$. Moreover, T is differentiable on (θ^*, θ^{**}) with derivative

$$T'(\theta) = \frac{1}{\lambda} \int \frac{f(1-f)}{(\theta - f)^2}\, \mu(df), \quad \text{for } \theta \in (\theta^*, \theta^{**}).$$

For $\theta > \theta^* \geqslant 1$, we thus obtain

$$T'(\theta) \leqslant \frac{1}{\lambda} \int \frac{f}{\theta - f}\, \mu(df) < 1, \tag{4.11}$$

where for the last inequality we used that $\theta > \theta^*$ and θ^* satisfies

$$\int \frac{f}{\theta^* - f}\, \mu(df) = \lambda$$

by definition. Also, by definition θ^* is a fixed point of T. Therefore, by the mean value theorem,

$$T(\theta^{**}) = T(\theta^*) + T'(\theta)(\theta^{**} - \theta^*) < \theta^{**}$$

for an appropriate $\theta \in (\theta^*, \theta^{**})$. Together with (4.10), this contradicts the minimality of θ^{**}.

We now turn to the convergence of the measures Γ_n, which we recall are defined by

$$\Gamma_n = \frac{1}{n} \sum_{i=1}^{n} \mathcal{I}_n^{(i)} \delta_{\mathcal{F}^{(i)}}.$$

Note that the measure Γ defined by

$$\Gamma(df) := \frac{\theta^*}{\theta^* - f}\, \mu(df)$$

has total mass $1 + \lambda$, since by definition of θ^* and the fact that μ is a probability measure,

$$\Gamma((0,1]) = \int \frac{\theta^*}{\theta^* - f} \mu(df) = \int \frac{\theta^* - f}{\theta^* - f}\, \mu(df) + \int \frac{f}{\theta^* - f}\, \mu(df) = 1 + \lambda.$$

Here, we have implicitly used that if $\theta^* = 1$, by definition

$$\int \frac{f}{1 - f}\, \mu(df) < \infty,$$

so that all occurring integrals are well-defined. By Lemma 3.2,

$$\Gamma_n((0,1]) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{I}_n^{(i)}$$

tends to $1 + \lambda$, almost surely, so that one can apply the portmanteau theorem to prove convergence of (Γ_n). Let

$$\mathcal{D} = \bigcup_{n \in \mathbb{N}} 2^{-n}\mathbb{Z} \cap [0,1]$$

denote the dyadic numbers on $[0,1]$. We remark that the number of dyadic intervals $(a,b]$ with endpoints $a, b \in \mathcal{D}$ is countable so that, by Proposition 4.1, there exists an almost sure event Ω_0, such that, for all dyadic intervals $(a,b]$,

$$\liminf_{n \to \infty} \Gamma_n((a,b]) \geqslant \Gamma((a,b]) \quad \text{on} \quad \Omega_0.$$

Now let $U \subset (0,1)$ be an arbitrary open set. We approximate U monotonically from within by a sequence of sets $(U_m)_{m\in\mathbb{N}}$, with each U_m being a union of finitely many pairwise disjoint dyadic intervals as above. Then, for any $m \in \mathbb{N}$, we have

$$\liminf_{n\to\infty} \Gamma_n(U) \geqslant \liminf_{n\to\infty} \Gamma_n(U_m) \geqslant \Gamma(U_m) \quad \text{on} \quad \Omega_0,$$

and by monotone convergence it follows that $\liminf_{n\to\infty} \Gamma_n(U) \geqslant \Gamma(U)$ on Ω_0. As noted above, the total masses also converge, that is, $\lim_{n\to\infty} \Gamma_n((0,1]) = \Gamma((0,1])$ almost surely, so that we can conclude from the portmanteau theorem for finite measures that

$$\Gamma_n \Rightarrow \Gamma, \text{ almost surely.}$$

Since

$$\bar{\mathcal{F}}_n = \frac{1}{\lambda} \int f \, \Gamma_n(df),$$

we deduce that, almost surely,

$$\lim_{n\to\infty} \bar{\mathcal{F}}_n = \frac{1}{\lambda} \int f \, \Gamma(df) = \theta^*.$$

(ii) *Bose–Einstein phase*. Suppose that

$$\int \frac{f}{1-f} \mu(df) < \lambda,$$

so that by definition $\theta^* = 1$. We start as in (i). Let θ^{**} denote the smallest value in $[1,\infty)$ with

$$\limsup_{n\to\infty} \bar{\mathcal{F}}_n \leqslant \theta^{**}, \text{ almost surely.}$$

As in the first part of (i), a proof by contradiction proves that $\theta^{**} = 1$. The only modification is that we use that

$$\int \frac{f}{\theta^* - f} \, \mu(df) < \lambda$$

in the justification of (4.11). Also, we use that in this case necessarily $\mu(\{1\}) = 0$, and therefore by Remark 4.3 with $T(1) := 1$ the mapping T is continuous on $[1,\infty)$.

Next, let θ_{**} denote the largest real in $(0,1]$ with

$$\liminf_{n\to\infty} \bar{\mathcal{F}}_n \geqslant \theta_{**}, \text{ almost surely.} \tag{4.12}$$

By Lemma 3.3, such a θ_{**} exists and we assume that $\theta_{**} < 1$. By Lemma 4.2, the inequality (4.12) remains valid for

$$\theta_{**} + \frac{\theta_{**}}{\lambda}\big(1 - \mu([0,\theta_{**}))\big).$$

Since $\operatorname{ess\,sup} \mu = 1$, this expression is strictly greater than θ_{**}, contradicting the maximality of θ_{**}. Hence,

$$\lim_{n\to\infty} \bar{\mathcal{F}}_n = 1, \text{ almost surely.}$$

Therefore, by Proposition 4.1 we have for $0 \leqslant a < b < 1$

$$\lim_{n\to\infty} \Gamma_n((a,b]) = \int_{(a,b]} \frac{1}{1-f}\,\mu(df) = \Gamma((a,b]),\ \text{almost surely,}$$

and, hence by Lemma 3.2 for $0 \leqslant a < 1$,

$$\lim_{n\to\infty} \Gamma_n((a,1]) = 1 + \lambda - \lim_{n\to\infty} \Gamma_n((0,a]) = \Gamma((a,1]),\ \text{almost surely.}$$

The rest of the proof is in line with the proof of (i). □

5. Proof of Theorem 2.7

The proof is achieved via a stochastic approximation technique as discussed in Section 3.

Proof of Theorem 2.7. We prove Theorem 2.7, which claims that, for each $k = 1, 2, \ldots,$ the measure

$$\Gamma_n^{(k)} := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k\}}\delta_{\mathcal{F}^{(i)}}$$

converges almost surely in the weak* topology to $\Gamma^{(k)}$ defined by

$$\Gamma^{(k)}(df) := \frac{1}{k+\frac{\theta^*}{f}}\,\frac{\theta^*}{f}\prod_{i=1}^{k-1}\frac{i}{i+\frac{\theta^*}{f}}\,\mu(df).$$

We prove the statement via induction over k. The proof of the initial statement ($k = 1$) is similar to the proof of the induction step, and we will come back to it at the end.

Induction step. Let $k \in \{2, 3, \ldots\}$, and suppose that the statement is true when replacing k by any value in $1, \ldots, k-1$. We fix $f, f' \in [0, 1]$ with either $\mu(\{f, f'\}) = 0$ or $f' = 1$ and $\mu(\{f\}) = 0$. Suppose that $\mu((f, f']) > 0$, and consider the random variables

$$X_n := \Gamma_n^{(k)}((f, f'])$$

for $n \in \mathbb{N}$. We claim that

$$\liminf_{n\to\infty} \Gamma_n^{(k)}((f, f']) = \liminf_{n\to\infty} X_n \geqslant \frac{(k-1)\int_{(f,f']} x\,\Gamma^{(k-1)}(dx)}{\theta^* + kf'}, \tag{5.1}$$

which we will show by applying Lemma 3.1.

In the first step we derive a lower bound for the increments of (X_n) that is suitable for the application of Lemma 3.1.

We restrict our attention to vertices with fitness in $(f, f']$ and let

$$\mathbb{I}_n := \mathbb{I}_n((f, f']) := \{i \in \{1, \ldots, n\} : \mathcal{F}^{(i)} \in (f, f']\}.$$

Note that since by definition $\mathcal{I}_{n+1}^{(n+1)} = 1 < k$, we can ignore the corresponding term in the definition of X_{n+1} and thus obtain

$$\begin{aligned}
\mathbb{E}[X_{n+1}|\mathcal{G}_n] &= \frac{1}{n+1}\sum_{i\in\mathbb{I}_n}\sum_{\ell=1}^{k} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=\ell\}}\mathbb{P}(\Delta\mathcal{I}_n^{(i)} = k-\ell|\mathcal{G}_n) \\
&= X_n + \frac{1}{n+1}\sum_{i\in\mathbb{I}_n}\Bigg(\sum_{\ell=1}^{k-1} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=\ell\}}\mathbb{P}(\Delta\mathcal{I}_n^{(i)} = k-\ell|\mathcal{G}_n) \\
&\qquad - \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k\}}\mathbb{P}(\Delta\mathcal{I}_n^{(i)} \neq 0|\mathcal{G}_n)\Bigg) - \frac{X_n}{n+1} \qquad (5.2)\\
&\geqslant X_n + \frac{1}{n+1}\Bigg[\sum_{i\in\mathbb{I}_n}\Bigg(\sum_{\ell=1}^{k-1} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=\ell\}}\mathbb{P}(\Delta\mathcal{I}_n^{(i)} = k-\ell|\mathcal{G}_n) \\
&\qquad - \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k\}}\Big(\mathbb{P}(\Delta\mathcal{I}_n^{(i)} \neq 0|\mathcal{G}_n) - \frac{\mathcal{F}^{(i)}k}{n\bar{\mathcal{F}}_n}\Big)\Bigg) - \Big(1+\frac{f'k}{\bar{\mathcal{F}}_n}\Big)X_n\Bigg].
\end{aligned}$$

Hence, we can write a lower bound on the increment of (X_n) as

$$X_{n+1} - X_n \geqslant \frac{1}{n+1}(A_n + B_nX_n) + R_{n+1} - R_n,$$

where we define

$$\begin{aligned}
A_n &:= \sum_{i\in\mathbb{I}_n}\Bigg(\sum_{\ell=1}^{k-1} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=\ell\}}\mathbb{P}(\Delta\mathcal{I}_n^{(i)} = k-\ell|\mathcal{G}_n) - \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k\}}\Big(\mathbb{P}(\Delta\mathcal{I}_n^{(i)} \neq 0|\mathcal{G}_n) - \frac{\mathcal{F}^{(i)}k}{n\bar{\mathcal{F}}_n}\Big)\Bigg),\\
B_n &:= 1 + \frac{f'k}{\bar{\mathcal{F}}_n},\\
R_n &:= \sum_{i=1}^{n}(X_i - \mathbb{E}[X_i|\mathcal{G}_{i-1}]),
\end{aligned}$$

with $\mathcal{G}_0$ the empty graph. By Lemma 3.1, the claim (5.1) follows immediately if we can show that, almost surely,

$$\begin{aligned}
&\text{(i)} \quad \lim_{n\to\infty} A_n = \frac{k-1}{\theta^*}\int_{(f,f']} x\,\Gamma^{(k-1)}(dx),\\
&\text{(ii)} \quad \lim_{n\to\infty} B_n = 1 + \frac{f'k}{\theta^*}, \qquad (5.3)\\
&\text{(iii)} \quad \lim_{n\to\infty} R_n \text{ exists.}
\end{aligned}$$

Observe that by the assumption that $\mu((f,f'])>0$, the limit of A_n is strictly positive.

(i) *Convergence of A_n.* We write

$$A_n = \sum_{\ell=1}^{k-1} A_n^{(\ell)} - A_n'$$

for

$$A_n^{(\ell)} := \sum_{i\in\mathbb{I}_n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=\ell\}} \mathbb{P}(\Delta\mathcal{I}_n^{(i)} = k-\ell|\mathcal{G}_n), \quad \ell \in \{1,\dots,k-1\},$$

$$A_n' := \sum_{i\in\mathbb{I}_n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k\}} \big(\mathbb{P}(\Delta\mathcal{I}_n^{(i)} \neq 0|\mathcal{G}_n) - \frac{\mathcal{F}^{(i)}k}{n\bar{\mathcal{F}}_n}\big),$$

and show convergence for each of the terms, where we will see that only the limit of $A_n^{(k-1)}$ will be non-zero.

Indeed, for $\ell = k-1$,

$$\begin{aligned}
&|A_n^{(k-1)} - \frac{k-1}{\bar{\mathcal{F}}_n}\int_{(f,f']} x\,\Gamma^{(k-1)}(dx)| \\
&\quad = \left|\sum_{i\in\mathbb{I}_n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k-1\}} \mathbb{P}(\Delta\mathcal{I}_n^{(i)} = 1|\mathcal{G}_n) - \frac{k-1}{\bar{\mathcal{F}}_n}\int_{(f,f']} x\,\Gamma^{(k-1)}(dx)\right| \\
&\quad \leqslant \sup_{i=1,\dots,n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k-1\}} n\left|\mathbb{P}(\Delta\mathcal{I}_n^{(i)} = 1|\mathcal{G}_n) - \frac{(k-1)\mathcal{F}^{(i)}}{n\,\bar{\mathcal{F}}_n}\right| \\
&\qquad + \frac{k-1}{\bar{\mathcal{F}}_n}\left|\int_{(f,f']} x\,\mathrm{d}\Gamma_n^{(k-1)}(dx) - \int_{(f,f']} x\,\Gamma^{(k-1)}(dx)\right|,
\end{aligned}$$

and the former term tends to zero due to assumption (A4) and the latter term tends to zero by the induction hypothesis. Here, we use the assumption hat $\Gamma^{(k-1)}$ puts no mass on f and f' or that $f' = 1$, so that we can apply the portmanteau theorem. Combined with the fact that under assumptions (A0)–(A3) Theorem 2.4 implies that $\lim_{n\to\infty} \bar{\mathcal{F}}_n = \theta^*$ almost surely, we obtain

$$\lim_{n\to\infty} A_n^{(k-1)} = \frac{k-1}{\theta^*}\int_{(f,f']} x\,\Gamma^{(k-1)}(dx), \text{ almost surely.} \tag{5.4}$$

Now, for $\ell \in \{1,\dots,k-2\}$, we have by assumption (A1) that

$$\begin{aligned}
A_n^{(\ell)} &= \sum_{i\in\mathbb{I}_n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=\ell\}} \mathbb{P}(\Delta\mathcal{I}_n^{(i)} = k-\ell|\mathcal{G}_n) \leqslant \sup_{i=1,\dots,n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=\ell\}} n\mathbb{P}(\Delta\mathcal{I}_n^{(i)} \geqslant 2|\mathcal{G}_n) \\
&\leqslant \sup_{i=1,\dots,n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=\ell\}} n\mathbb{E}[\Delta\mathcal{I}_n^{(i)} \mathbb{1}_{\{\Delta\mathcal{I}_n^{(i)}\geqslant 2\}}|\mathcal{G}_n] \\
&\overset{(A1)}{=} \sup_{i=1,\dots,n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=\ell\}} n|\tfrac{\mathcal{F}^{(i)}\mathcal{I}_n^{(i)}}{n\bar{\mathcal{F}}_n} - \mathbb{P}(\Delta\mathcal{I}_n^{(i)} = 1|\mathcal{G}_n)|,
\end{aligned} \tag{5.5}$$

which by assumption (A4) converges to 0 almost surely.

For the final term, we have that

$$\begin{aligned}
|A_n'| &\leqslant \sum_{i\in\mathbb{I}_n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k\}} \left|\mathbb{P}(\Delta\mathcal{I}_n^{(i)} \neq 0|\mathcal{G}_n) - \frac{\mathcal{F}^{(i)}k}{n\bar{\mathcal{F}}_n}\right| \\
&\leqslant \sum_{i\in\mathbb{I}_n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=k\}} \left|\mathbb{P}(\Delta\mathcal{I}_n^{(i)} = 1|\mathcal{G}_n) - \frac{\mathcal{F}^{(i)}k}{n\bar{\mathcal{F}}_n}\right| + \sup_{i=1,\dots,n} n\mathbb{P}(\Delta\mathcal{I}_n^{(i)} \geqslant 2|\mathcal{G}_n),
\end{aligned} \tag{5.6}$$

where the two terms on the right-hand side converge to 0 by assumption (A4) for the first term and by the same calculation as in (5.5) using assumptions (A1) and (A4) for the second term. Combining (5.4)-(5.6), we have shown that (A_n) converges to the expression claimed in (5.3).

(ii) The *convergence of* (B_n) follows immediately from $\lim_{n\to\infty} \bar{\mathcal{F}}_n = \theta^*$, almost surely, which holds by Theorem 2.4 under assumptions (A0)–(A3).

(iii) *Convergence of the remainder term* (R_n). We first show that we can write the process (R_n) as the sum of two martingales. Indeed, we find that for any $i \in \{1, \dots, n\}$,

$$\begin{aligned}\mathbb{1}_{\{\mathcal{I}^{(i)}_{n+1}=k\}} &= \mathbb{1}_{\{\mathcal{I}^{(i)}_{n+1}=k,\mathcal{I}^{(i)}_n\leqslant k\}} = \mathbb{1}_{\{\mathcal{I}^{(i)}_{n+1}\geqslant k,\mathcal{I}^{(i)}_n\leqslant k\}} - \mathbb{1}_{\{\mathcal{I}^{(i)}_{n+1}> k,\mathcal{I}^{(i)}_n\leqslant k\}}\\ &= \mathbb{1}_{\{\mathcal{I}^{(i)}_{n}=k\}} + \mathbb{1}_{\{\mathcal{I}^{(i)}_{n+1}\geqslant k,\mathcal{I}^{(i)}_n< k\}} - \mathbb{1}_{\{\mathcal{I}^{(i)}_{n+1}> k,\mathcal{I}^{(i)}_n\leqslant k\}}.\end{aligned}$$

Therefore, an increment of R_n is equal to

$$\begin{aligned}\Delta R_n &:= R_{n+1} - R_n = X_{n+1} - \mathbb{E}[X_{n+1}|\mathcal{G}_n]\\ &= \frac{1}{n+1}\sum_{i\in\mathbb{I}_n}\big(\mathbb{1}_{\{\mathcal{I}^{(i)}_{n+1}=k\}} - \mathbb{P}(\mathcal{I}^{(i)}_{n+1}=k|\mathcal{G}_n)\big) = \Delta M^{(1)}_n - \Delta M^{(2)}_n,\end{aligned}$$

where $M^{(1)}_n$ and $M^{(2)}_n$ are martingales defined by

$$M^{(1)}_{n+1} := M^{(1)}_n + \frac{1}{n+1}\left(\sum_{i\in\mathbb{I}_n}\mathbb{1}_{\{\mathcal{I}^{(i)}_n<k,\mathcal{I}^{(i)}_{n+1}\geqslant k\}} - \mathbb{E}\Big[\sum_{i\in\mathbb{I}_n}\mathbb{1}_{\{\mathcal{I}^{(i)}_n<k,\mathcal{I}^{(i)}_{n+1}\geqslant k\}}|\mathcal{G}_n\Big]\right),$$

and

$$M^{(2)}_{n+1} := M^{(2)}_n + \frac{1}{n+1}\left(\sum_{i\in\mathbb{I}_n}\mathbb{1}_{\{\mathcal{I}^{(i)}_n\leqslant k,\mathcal{I}^{(i)}_{n+1}> k\}} - \mathbb{E}\Big[\sum_{i\in\mathbb{I}_n}\mathbb{1}_{\{\mathcal{I}^{(i)}_n\leqslant k,\mathcal{I}^{(i)}_{n+1}> k\}}|\mathcal{G}_n\Big]\right), \tag{5.7}$$

both starting in 0. Since both martingales are the same up to a shift of parameter k, we only have to show that either converges almost surely for fixed $k \in \mathbb{N}$. We will show that $M^{(2)}$ converges by showing that its quadratic variation process converges almost surely. Indeed, we will show that $\mathbb{E}[(\Delta M^{(2)}_n)^2|\mathcal{G}_n]$ is almost surely summable, where $\Delta M^{(2)}_n := M^{(2)}_{n+1} - M^{(2)}_n$.

First using assumption (A5), that is, the conditional negative quadrant dependence of $\Delta\mathcal{I}^{(i)}_n$, we find that

$$\begin{aligned}&\mathbb{E}[(\Delta M^{(2)}_n)^2|\mathcal{G}_n]\\ &\quad\leqslant \frac{1}{(n+1)^2}\sum_{i\in\mathbb{I}_n}\mathbb{E}\Big[\Big(\mathbb{1}_{\{\mathcal{I}^{(i)}_n\leqslant k,\mathcal{I}^{(i)}_{n+1}>k\}} - \mathbb{P}\big(\mathcal{I}^{(i)}_n\leqslant k,\mathcal{I}^{(i)}_{n+1}>k|\mathcal{G}_n\big)\Big)^2\Big|\mathcal{G}_n\Big]\\ &\quad\leqslant \frac{1}{(n+1)^2}\sum_{i\in\mathbb{I}_n}\mathbb{1}_{\{\mathcal{I}^{(i)}_n\leqslant k\}}\mathbb{P}(\Delta\mathcal{I}^{(i)}_n\geqslant 1|\mathcal{G}_n)\\ &\quad\leqslant \frac{1}{(n+1)^2}\sup_{i=1,\dots,n} n\mathbb{1}_{\{\mathcal{I}^{(i)}_n\leqslant k\}}\left|\mathbb{P}(\Delta\mathcal{I}^{(i)}_n\geqslant 1|\mathcal{G}_n) - \frac{\mathcal{F}^{(i)}\mathcal{I}^{(i)}_n}{n\bar{\mathcal{F}}_n}\right| + \frac{\lambda}{(n+1)^2},\end{aligned}$$

where we used the definition of $\bar{\mathcal{F}}_n$ in the last step. By the same calculation as in (5.6), assumptions (A1) and (A4) guarantee that the first term converges to 0, so that the whole expression is indeed almost surely summable.

This completes the proof of (i)–(iii) in (5.3) and therefore the claim in (5.1) holds.

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Completing the induction step. We recall that by (5.1), we have shown that, for any $f, f' \in [0,1]$ with $f < f'$ and either $\mu(\{f, f'\}) = 0$ or $f' = 1$ and $\mu(\{f\}) = 0$, we have almost surely

$$\liminf_{n\to\infty} \Gamma_n^{(k)}((f, f']) \geqslant \frac{(k-1)\int_{(f,f']} x\,\Gamma^{(k-1)}(dx)}{\theta^* + kf'}, \tag{5.8}$$

We can drop the assumption that $\mu((f, f']) > 0$, since the statement holds trivially in that case. We now pick a countable subset $\mathbb{F} \subset [0,1]$ that is dense such that for each of its entries f we have $\mu(\{f\}) = 0$ or $f = 1$. Since $\mathbb{F}$ is countable there exists an almost sure set Ω_0 on which (5.8) holds for any pair $f, f' \in \mathbb{F}$ with $f < f'$. Suppose now that U is an arbitrary open set. By approximating the set U from below by unions of small disjoint intervals $(f, f']$ with $f, f' \in \mathbb{F}$, analogous reasoning to the proof of Proposition 4.1 shows that

$$\liminf_{n\to\infty} \Gamma_n^{(k)}(U) \geqslant (k-1)\int_U \frac{x}{\theta^* + kx}\,\Gamma^{(k-1)}(dx)$$

on Ω_0. The proof of the converse inequality, namely that, almost surely, for any closed A, we have

$$\limsup_{n\to\infty} \Gamma_n^{(k)}(A) \leqslant (k-1)\int_A \frac{x}{\theta^* + kx}\,\Gamma^{(k-1)}(dx),$$

is established analogously. We thus obtain that $\Gamma_n^{(k)}$ converges almost surely, in the weak* topology to $\Gamma^{(k)}$ given by

$$\Gamma^{(k)}(dx) = \frac{(k-1)x}{kx + \theta^*}\,\Gamma^{(k-1)}(dx) = \prod_{\ell=2}^{k} \frac{(\ell-1)x}{\ell x + \theta^*}\,\Gamma^{(1)}(dx).$$

Initializing the induction. To complete the argument, we still need to verify the statement for the initial choice $k = 1$. As before, we fix $f, f' \in [0,1]$ with $\mu(\{f, f'\}) = 0$, $\mu((f, f']) > 0$. Then, we define $X_n := \Gamma_n^{(1)}((f, f'])$ and we set

$$\mathbb{I}_n := \mathbb{I}_n((f, f']) := \{i \in \{1, \dots, n\} : \mathcal{F}^{(i)} \in (f, f']\}.$$

Then, it follows that, since $\mathcal{I}_{n+1}^{(n+1)} = 1$ by definition,

$$\begin{aligned}\mathbb{E}[X_{n+1}|\mathcal{G}_n] &= \frac{1}{n+1}\sum_{i\in\mathbb{I}_n} \mathbb{P}(\mathcal{I}_{n+1}^{(i)} = 1|\mathcal{G}_n) + \frac{1}{n+1}\mathbb{P}(\mathcal{F}^{(n+1)} \in (f, f']) \\ &= X_n + \frac{1}{n+1}\Big[-\sum_{i\in\mathbb{I}_n} \mathbb{1}_{\{\mathcal{I}_n^{(i)}=1\}}\mathbb{P}(\Delta\mathcal{I}_n^{(i)} \neq 0|\mathcal{G}_n) - X_n + \mu((f, f'])\Big].\end{aligned}$$

Thus, in analogy to the induction step, one can show that

$$X_{n+1} - X_n \geqslant \frac{1}{n+1}(A_n - B_n X_n) + R_{n+1} - R_n,$$

where $A_n \to \mu((f, f'])$ and $B_n \to 1 + f'/\theta^*$ and $R_{n+1} - R_n = X_{n+1} - \mathbb{E}[X_{n+1}|\mathcal{G}_n]$. The remainder term R_n can then be decomposed as $M_n^{(1)} - M_n^{(2)} + I_n$ as above, with $M_n^{(1)} = 0$, $M_n^{(2)}$

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defined as in (5.7), and the additional term defined by

$$I_n = \frac{1}{n}\big(\mathbb{1}_{\{\mathcal{F}^{(n)}\in(f,f']\}} - \mathbb{P}(\mathcal{F}^{(n)} \in (f,f'])\big).$$

We have already seen that $M^{(2)}$ converges. Moreover, an elementary martingale argument (for i.i.d. random variables) shows that $\sum_{n\in\mathbb{N}} I_n$ is finite almost surely. Therefore, by Lemma 3.1 we can deduce that

$$\liminf_{n\to\infty} X_n \geqslant \frac{\theta^*}{\theta^* + f'}\,\mu((f,f']).$$

Repeating the same approximation arguments as before, we obtain that $\Gamma^{(1)}$ converges almost surely in the weak* topology to $\Gamma^{(1)}$ given by

$$\Gamma^{(1)}(dx) = \frac{\theta^*}{\theta^* + x}\,\mu(dx),$$

which completes the proof by induction. □

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References

[1] Barabási, A.-L. and Albert, R. (1999) Emergence of scaling in random networks. *Science* **286**(5439) 509–512.

[2] Benaïm, M. (1999) Dynamics of stochastic approximation algorithms. In *Séminaire de Probabilités XXXIII*, Vol. 1709 of *Lecture Notes in Mathematics*, Springer, pp. 1–68.

[3] Bhamidi, S. (2007) Universal techniques to analyze preferential attachment trees: Global and local analysis. Preprint available at `http://www.unc.edu/~bhamidi/`.

[4] Bianconi, G. and Barabási, A.-L. (2001) Bose–Einstein condensation in complex networks. *Phys. Rev. Lett.* **86** 5632–5635.

[5] Bollobás, B., Riordan, O., Spencer, J. and Tusnády, G. (2001) The degree sequence of a scale-free random graph process. *Random Struct. Alg.* **18** 279–290.

[6] Borgs, C., Chayes, J., Daskalakis, C. and Roch, S. (2007) First to market is not everything: An analysis of preferential attachment with fitness. In *STOC'07: Proceedings of the 39th Annual ACM Symposium on Theory of Computing*, ACM, pp. 135–144.

[7] Cooper, C. and Frieze, A. (2003) A general model of web graphs. *Random Struct. Alg.* **22** 311–335.

[8] Deijfen, M., van den Esker, H., van der Hofstad, R. and Hooghiemstra, G. (2009) A preferential attachment model with random initial degrees. *Ark. Mat.* **47** 41–72.

[9] Dereich, S. (2013) Random networks with preferential attachment: Unfolding Bose–Einstein condensation. In preparation.

[10] Dereich, S. and Mörters, P. (2009) Random networks with sublinear preferential attachment: Degree evolutions. *Electron. J. Probab.* **14** 1222–1267.

[11] Dereich, S. and Mörters, P. (2013) Emergence of Condensation in Kingman's Model of Selection and Mutation. *Acta Appl. Math.* **127**(1) 17–26.

[12] Hagberg, O. and Wiuf, C. (2006) Convergence properties of the degree distribution of some growing network models. *Bull. Math. Biol.* **68** 1275–1291.

[13] van der Hofstad, R. (2013) *Random Graphs and Complex Networks*. Lecture notes in preparation, available at `http://www.win.tue.nl/~rhofstad/`.
[14] Jordan, J. (2006) The degree sequences and spectra of scale-free random graphs. *Random Struct. Alg.* **29** 226–242.
[15] Jordan, J. (2010) Degree sequences of geometric preferential attachment graphs. *Adv. Appl. Probab.* **42** 319–330.
[16] Jordan, J. (2013) Geometric preferential attachment in non-uniform metric spaces. *Electron. J. Probab.* **18** #15.
[17] Nerman, O. (1981) On the convergence of supercritical general (C-M-J) branching processes. *Z. Wahrsch. Verw. Gebiete* **57** 365–395.
[18] Newman, M. (2010) *Networks: An Introduction*, Oxford University Press.
[19] Pemantle, R. (2007) A survey of random processes with reinforcement. *Probab. Surv.* **4** 1–79.
[20] Robbins, H. and Monro, S. (1951) A stochastic approximation method. *Ann. Math. Statist.* **22** 400–407.
[21] Rudas, A., Tóth, B. and Valkó, B. (2007) Random trees and general branching processes. *Random Struct. Alg.* **31** 186–202.