

ON THE CANONICAL IDEALS OF ONE-DIMENSIONAL COHEN–MACAULAY LOCAL RINGS

JUAN ELIAS

*Departament d'Àlgebra i Geometria, Universitat de Barcelona,
Gran Via 585, 08007 Barcelona, Spain (elias@ub.edu)*

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Abstract In this paper we consider the problem of explicitly finding canonical ideals of one-dimensional Cohen–Macaulay local rings. We show that Gorenstein ideals contained in a high power of the maximal ideal are canonical ideals. In the codimension 2 case, from a Hilbert–Burch resolution, we show how to construct canonical ideals of curve singularities. Finally, we translate the problem of the analytic classification of curve singularities to the classification of local Artin Gorenstein rings with suitable length.

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1. Introduction

Let (R, \mathfrak{m}) be a one-dimensional Cohen–Macaulay local ring with maximal ideal \mathfrak{m} for which there exists a canonical module ω_R ; this is the case, for instance, if R is the quotient of a local Gorenstein ring. Recall that R possesses a canonical ideal (more precisely, the canonical module ω_R of R exists and is contained in R) if and only if the total ring of fractions of the \mathfrak{m} -adic completion of R is Gorenstein [13, Satz 6.21]. See [4, Chapter 3] for the basic properties of canonical modules and canonical ideals.

The question that motivates this paper is, can we explicitly describe canonical ideals? Recall that Boij [3] addressed this problem for projective zero-dimensional schemes. A second question that we consider is, can we use canonical ideals for the analytic classification of singularities? In this paper we study these questions in the one-dimensional case.

The contents of this paper are as follows. In §2 we show that Gorenstein ideals contained in a high power of the maximal ideal are canonical ideals (see Proposition 2.3). In the codimension 2 case, and following [3], from a Hilbert–Burch resolution we show how to construct canonical ideals for curve singularities (see Proposition 2.8). In this section we recall how to ‘explicitly’ describe the canonical module by using Rosenlicht’s regular differential forms [20, Chapter IV, §3.9]. This strategy is very useful in the case

of branches and especially in the case of monomial curve singularities. In §3 we address the problem of the analytic classification of curve singularities. We show canonical ideals $I \subset \mathcal{O}_{(X,0)}$, where X is a curve singularity, for which we compute the multiplicity and the socle degree of the Artin Gorenstein quotient $\mathcal{O}_{(X,0)}/I$ (see Proposition 3.4). In Theorem 3.5, we translate the problem of the analytic classification of curve singularities X to the classification of local Artin Gorenstein rings $\mathcal{O}_{(X,0)}/I$ of suitable length.

Notation. Let (R, \mathfrak{m}) be a one-dimensional Noetherian local ring with maximal ideal \mathfrak{m} and residue field $\mathbf{k} = R/\mathfrak{m}$. If I is a \mathfrak{m} -primary ideal of R , we denote by $\text{HF}_I^1(n) = \text{Length}_R(R/I^{n+1})$ the Hilbert–Samuel function of I . Hence, there exist integers $e_0 = e_0(I) \geq 1$ and $e_1(I)$ such that $\text{HP}_I^1(n) := e_0(I)(n+1) - e_1(I)$; this is the Hilbert–Samuel polynomial of I , i.e. $\text{HF}_I^1(n) = \text{HP}_I^1(n)$ for $n \gg 0$. The integer $e_0(I)$ is the multiplicity of I . We denote by $\text{HF}_I^0(n) = \text{length}_R(I^n/I^{n+1})$ the zeroth Hilbert–Samuel function of I . Then, $\text{HP}_I^0(n) = e_0(I)$ for $n \gg 0$ and the postulation number of I is the least integer $\text{pn}(I)$ such that $\text{HF}_I^0(n) = \text{HP}_I^0(n)$ for all $n \geq \text{pn}(I)$. We set $\text{HF}_R^i = \text{HF}_{\mathfrak{m}}^i$, $\text{HP}_R^i = \text{HP}_{\mathfrak{m}}^i$, $i = 0, 1$, and $\text{pn}(R) = \text{pn}(\mathfrak{m})$. If M is an Artin R -module, we define the socle degree $s(M)$ of M as the last integer t such that $\mathfrak{m}^t M \neq 0$. The socle of M is by definition the R -submodule $\text{soc}(M) = (0 :_M \mathfrak{m})$ of M .

Next we recall some basic facts regarding curve singularities. Let $(X, 0)$ be a reduced curve singularity of $(\mathbb{C}^n, 0) = \text{Spec}(\mathcal{O}_{(\mathbb{C}^n, 0)})$, i.e. $\mathcal{O}_{(\mathbb{C}^n, 0)} = \mathbb{C}[[x_1, \dots, x_n]]$ and $(X, 0) = \text{Spec}(\mathcal{O}_{(X,0)})$, where $\mathcal{O}_{(X,0)} = \mathcal{O}_{(\mathbb{C}^n, 0)}/I_X$ is a one-dimensional reduced ring with maximal ideal \mathfrak{m}_X . We write $\text{HF}_X^i = \text{HF}_{\mathcal{O}_{(X,0)}}^i$ and $\text{HP}_X^i = \text{HP}_{\mathcal{O}_{(X,0)}}^i$. We assume that n is the embedding dimension of $(X, 0)$, which is equivalent to saying that $\mathfrak{m}_X/\mathfrak{m}_X^2$ is isomorphic as a \mathbb{C} -vector space to the homogeneous linear forms of $P = \mathbb{C}[X_1, \dots, X_n]$. Hence, all elements $x \in \mathfrak{m}_X/\mathfrak{m}_X^2$ define an element in the quotient $\mathcal{O}_{(X,0)}$ that we will denote again by x . Let

$$\nu: \bar{X} = \text{Spec}(\overline{\mathcal{O}_{(X,0)}}) \rightarrow (X, 0)$$

be the normalization of $(X, 0)$, where $\overline{\mathcal{O}_{(X,0)}}$ is the integral closure of $\mathcal{O}_{(X,0)}$ on its full ring $\text{tot}(\mathcal{O}_{(X,0)})$ of fractions. The singularity order of $(X, 0)$ is $\delta(X) = \dim_{\mathbb{C}}(\mathcal{O}_{\bar{X}}/\mathcal{O}_{(X,0)})$. We denote by \mathcal{C} the conductor of the finite extension $\nu^*: \mathcal{O}_{(X,0)} \hookrightarrow \overline{\mathcal{O}_{(X,0)}}$ and by $c(X)$ the dimension of $\overline{\mathcal{O}_{(X,0)}}/\mathcal{C}$. Let

$$\omega_{(X,0)} = \text{Ext}_{\mathcal{O}_{(\mathbb{C}^n, 0)}}^{n-1}(\mathcal{O}_{(X,0)}, \Omega_{\mathcal{O}_{(\mathbb{C}^n, 0)}}^n)$$

be the dualizing module of $(X, 0)$. We can consider the composition morphism of $\mathcal{O}_{(X,0)}$ -modules

$$\gamma_X: \Omega_{(X,0)} \rightarrow \nu_* \Omega_{\bar{X}} \cong \nu_* \omega_{\bar{X}} \rightarrow \omega_{(X,0)}.$$

Let $d: \mathcal{O}_{(X,0)} \rightarrow \Omega_{(X,0)}$ be the universal derivation. We then have a \mathbb{C} -map $\gamma_X d$ that we also denote by $d: \mathcal{O}_{(X,0)} \rightarrow \omega_{(X,0)}$. The Milnor number of $(X, 0)$ is $\mu(X) = \dim_{\mathbb{C}}(\omega_{(X,0)}/d\mathcal{O}_{(X,0)})$ [6]. Notice that $(X, 0)$ is non-singular if and only if $\mu(X) = 0$ if and only if $\delta(X) = 0$ if and only if $c(X) = 0$.

In the following result we collect some basic results on μ and other numerical invariants that we will use later on.

Proposition 1.1. *Let $(X, 0)$ be a reduced curve singularity of embedding dimension n . Then the following statements hold.*

- (i) $\mu(X) = 2\delta(X) - r + 1$, where r is the number of branches of $(X, 0)$.
- (ii) It holds that

$$e_0(X) - 1 \leq e_1(X) \leq \delta(X) \leq \mu(X)$$

and $e_1(X) \leq \binom{e_0(X)}{2} - \binom{n-1}{2}$.

- (iii) If X is singular, then $\delta(X) + 1 \leq c(X) \leq 2\delta(X)$ and $c(X) = 2\delta(X)$ if and only if $\mathcal{O}_{(X,0)}$ is a Gorenstein ring.

Proof. (i) See [6, Proposition 1.2.1]. (ii) See [6, Proposition 1.2.4 (i)] and [9, 10, 17]. (iii) See [20, Proposition 7, p. 80] and [2]. □

2. Canonical ideals

The first aim of this section is to find conditions on an \mathfrak{m} -primary ideal I to be a canonical ideal.

Lemma 2.1. *Let (R, \mathfrak{m}) be a one-dimensional Cohen–Macaulay local ring and let I be an \mathfrak{m} -primary ideal of R . Let $x \in \mathfrak{m}$ be a parameter of R .*

- (i) If $R/x^n I$ is a Gorenstein ring for some $n \geq 1$, then R/I is a Gorenstein ring and $(x^n I :_R \mathfrak{m}) = x^n (I :_R \mathfrak{m})$.
- (ii) Assume that $I \subset xR$. If R/I is a Gorenstein ring, then $R/x^n I$ is a Gorenstein ring for all $n \geq 1$.

Proof.

- (i) The short exact sequence

$$0 \rightarrow R/I \xrightarrow{x^n} R/x^n I \rightarrow R/x^n R \rightarrow 0$$

yields the exact sequence

$$0 \rightarrow \text{Hom}_R(R/\mathfrak{m}, R/I) \xrightarrow{x^n} \text{Hom}_R(R/\mathfrak{m}, R/x^n I) \rightarrow \text{Hom}_R(R/\mathfrak{m}, R/x^n R)$$

of socles, which shows that

$$\text{Hom}_R(R/\mathfrak{m}, R/I) \cong \text{Hom}_R(R/\mathfrak{m}, R/x^n I)$$

because $\text{Hom}_R(R/\mathfrak{m}, R/I) \neq 0$ and $\text{Length}_R(\text{Hom}_R(R/\mathfrak{m}, R/x^n I)) = 1$. Hence, R/I is a Gorenstein ring.

- (ii) Let $\alpha \in (x^n I :_R \mathfrak{m})$. Then, since $x\alpha \in x^n I \subset x^{n+1}R$, we get $\alpha \in x^n R$, which shows that $\pi((x^n I :_R \mathfrak{m})/x^n I) = 0$, where $\pi: R/x^n I \rightarrow R/x^n R$ denotes the canonical epimorphism. Hence, we get the isomorphism

$$\mathrm{Hom}_R(R/\mathfrak{m}, R/I) \cong^{x^n} \mathrm{Hom}_R(R/\mathfrak{m}, R/x^n I)$$

in the exact sequence

$$0 \rightarrow \mathrm{Hom}_R(R/\mathfrak{m}, R/I) \xrightarrow{x^n} \mathrm{Hom}_R(R/\mathfrak{m}, R/x^n I) \rightarrow \mathrm{Hom}_R(R/\mathfrak{m}, R/x^n R)$$

of socles. Thus, $\mathrm{Length}_R(\mathrm{Hom}_R(R/\mathfrak{m}, R/x^n I)) = 1$, so that $R/x^n I$ is a Gorenstein ring for all $n \geq 1$.

□

Proposition 2.2. *Let I be an \mathfrak{m} -primary ideal of R . Then I is a canonical ideal of R if and only if there exists a parameter $x \in \mathfrak{m}$ of R such that R/xI is a Gorenstein ring.*

Proof. We have only to prove the *if* part. Let Q be the total ring of fractions of R . Since $H_{\mathfrak{m}}^1(I) \cong Q/I$, it suffices to see that $\mathrm{Length}_R(I :_Q \mathfrak{m})/I = 1$. Let $\alpha \in (I :_Q \mathfrak{m})$. Then, since $x\alpha \in I \subset R$ and $(x\alpha)\mathfrak{m} \subset xI$, we have

$$x\alpha \in (xI :_R \mathfrak{m}) = x(I :_R \mathfrak{m})$$

(see the proof of Lemma 2.1 (i)). Hence, $I \subsetneq (I :_Q \mathfrak{m}) \subset (I :_R \mathfrak{m})$, so that

$$\mathrm{Length}_R(I :_Q \mathfrak{m})/I = \mathrm{Length}_R(I :_R \mathfrak{m})/I = 1$$

as wanted. □

In the next result we prove that an \mathfrak{m} -primary Gorenstein ideal contained in a high power of the maximal ideal is a canonical ideal. Notice that this result cannot be extended to any \mathfrak{m} -primary Gorenstein ideal. Let R be a one-dimensional Cohen–Macaulay local ring of Cohen–Macaulay type 2 and embedding dimension $b \geq 3$, for example, $R = \mathbf{k}[[t^5, t^6, t^7]]$. Then the maximal ideal is a Gorenstein ideal minimally generated by $b \geq 3$ elements. Since the minimal number of generators of a canonical ideal is the Cohen–Macaulay type of R , \mathfrak{m} is not a canonical ideal.

Corollary 2.3. *Let $x \in \mathfrak{m}$ and assume that $\mathfrak{m}^{r+1} = x\mathfrak{m}^r$ for some $r \geq 0$. Let I be an \mathfrak{m} -primary ideal of R such that $I \subset \mathfrak{m}^{r+1}$. If R/I is a Gorenstein ring, then I is a canonical ideal of R , whence $\mathrm{tot}(\hat{R})$ is a Gorenstein ring.*

Proof. Since we have that $I \subset \mathfrak{m}^{r+1} \subset xR$, the assertion follows from Lemma 2.1 and Proposition 2.2. □

Remark 2.4. Recall that $\mathrm{pn}(R) \leq e_0(R) - 1$ [16, Proposition 12.14]. Hence, if $I \subset \mathfrak{m}^{e_0(R)}$ is an \mathfrak{m} -primary ideal such that R/I is a Gorenstein ring, then I is a canonical ideal.

The last result points out that a basic problem in commutative algebra is to find methods to construct Gorenstein ideals. We know that complete intersection ideals are Gorenstein; by a result of Serre, in codimension 2, being Gorenstein is equivalent to being a complete intersection; in codimension 3, Gorenstein ideals are the ideals generated by the Pfaffians of skew-symmetric matrixes [5]. In Proposition 2.8 we show how to construct canonical ideals, which are Gorenstein, from a Hilbert–Burch resolution. On the other hand, notice that if I is a canonical ideal and $y \in \mathfrak{m}$ is a non-zero divisor of R , then $y^t I$, $t \geq 1$, is a canonical ideal as well, but the length of R/yI is not under control. In fact, for all $t \geq 1$ we have [16, Theorem 12.5],

$$\begin{aligned} \text{Length}_R(R/y^t I) &= \text{Length}_R(R/I) + \text{Length}_R(I/y^t I) \\ &= \text{Length}_R(R/I) + t \text{Length}_R(R/(y)) \\ &\geq \text{Length}_R(R/I) + te_0(R). \end{aligned}$$

In the next result we find canonical ideals for which we compute the multiplicity or the socle degree; in the second part we take $t \geq 4\mu(X) + 1$, where $(X, 0)$ is a reduced curve singularity, because we have to consider a large t in Theorem 3.5. See Example 2.7 for an explicit application of the next result.

Proposition 2.5. *Let $(X, 0)$ be a reduced curve singularity.*

(i) *Let z be a degree $t \geq 2\mu(X) + 1$ superficial element of $\mathcal{O}_{(X,0)}$. Then the $\mathcal{O}_{(X,0)}$ -module $z\omega_{(X,0)}$ is a canonical ideal of $(X, 0)$ such that $\mathcal{O}_{(X,0)}/z\omega_{(X,0)}$ is a Gorenstein ring of colength $te_0(X) - 2\delta(X)$.*

(ii) *For $t \geq 4\mu(X) + 1$ the socle degree of $\mathcal{O}_{(X,0)}/z\omega_{(X,0)}$ is at most*

$$e_0(X)(t - 2\mu(X) - 1) + 2\delta(X) + e_1(X) + 2(1 - r).$$

This number is bounded from above by $\delta(X)(4e_0(X) + 3)$.

(iii) *If $(X, 0)$ is Gorenstein, then $\mathcal{O}_{(X,0)}$ is a canonical ideal and for every superficial element z of degree $t \geq 1$, $z\mathcal{O}_{(X,0)}$ is a canonical ideal.*

Proof. (i) Since $\mathcal{O}_{(X,0)}$ is a one-dimensional reduced ring, we know that $\omega_{(X,0)}$ is a sub- $\mathcal{O}_{(X,0)}$ -module of $\text{tot}(\mathcal{O}_{(X,0)})$ [4, Proposition 3.3.18]. Let us consider the perfect pairing [20, Chapter IV]

$$\begin{aligned} \frac{\nu_* \mathcal{O}_{\bar{X}}}{\mathcal{O}_{(X,0)}} \times \frac{\omega_{(X,0)}}{\nu_* \Omega_{\bar{X}}} \xrightarrow{\eta} \mathbb{C} \\ F \times \alpha \rightarrow \sum_{i=1}^r \text{res}_{p_i}(F\alpha) \end{aligned}$$

and notice that for all $\lambda \in R$ it holds that

$$\eta(\lambda F, \alpha) = \sum_{i=1}^r \text{res}_{p_i}(\lambda F\alpha) = \eta(F, \lambda\alpha).$$

Hence, since $\mathfrak{m}^{c(X)}$ annihilates the quotient $\nu_*\mathcal{O}_{\bar{X}}/\mathcal{O}_{(X,0)}$, we get that $\mathfrak{m}^{c(X)}$ annihilates the quotient $\omega_{(X,0)}/\nu_*\Omega_{\bar{X}}$, too. Hence, $\mathfrak{m}^{c(X)}\omega_{(X,0)} \subset \nu_*\Omega_{\bar{X}} = \nu_*\mathcal{O}_{\bar{X}}$. Again, since $\mathfrak{m}^{c(X)}\nu_*\mathcal{O}_{\bar{X}} \subset \mathcal{O}_{(X,0)}$, we get $\mathfrak{m}^{2c(X)}\omega_{(X,0)} \subset \mathcal{O}_{(X,0)}$. On the other hand, the epimorphism of $\mathcal{O}_{(X,0)}$ -modules

$$\frac{\omega_{(X,0)}}{\mathcal{O}_{(X,0)}d\mathcal{O}_{(X,0)}} \rightarrow \frac{\omega_{(X,0)}}{\nu_*\Omega_{\bar{X}}}$$

assures us that $\mathfrak{m}^{\mu(X)}\omega_{(X,0)} \subset \nu_*\Omega_{\bar{X}} = \nu_*\mathcal{O}_{\bar{X}}$. Hence, $\mathfrak{m}^{c(X)}\nu_*\mathcal{O}_{\bar{X}} \subset \mathcal{O}_{(X,0)}$ and

$$\mathfrak{m}^{\mu(X)+c(X)}\omega_{(X,0)} \subset \mathcal{O}_{(X,0)}.$$

Let us consider the sequence

$$\mathfrak{m}^t\omega_{(X,0)} \subset \mathcal{O}_{(X,0)} \subset \nu_*\mathcal{O}_{\bar{X}} = \nu_*\Omega_{\bar{X}} \subset \omega_{(X,0)}.$$

From the perfect pairing from the beginning of the proof, we get $\dim_{\mathbb{C}}(\omega_{(X,0)}/\mathcal{O}_{(X,0)}) = 2\delta(X)$. Since z is a degree $t \geq 2\mu(X)$ superficial element of $\mathcal{O}_{(X,0)}$,

$$\dim_{\mathbb{C}}(\omega_{(X,0)}/z\omega_{(X,0)}) = te_0(X)$$

(see [16, Theorem 12.5]). From this identity and Proposition 1.1, we get the first part of the claim.

(ii) We have the following inequality for $t \geq 4\mu(X) + 1$:

$$\begin{aligned} s\left(\frac{\mathcal{O}_{(X,0)}}{z\omega_{(X,0)}}\right) &= s\left(\frac{\mathfrak{m}^{2\mu(X)+1}}{z\omega_{(X,0)}}\right) + 2\mu(X) + 1 \\ &\leq \text{Length}\left(\frac{\mathfrak{m}^{2\mu(X)+1}}{z\omega_{(X,0)}}\right) + 2\mu(X) \\ &= \text{Length}\left(\frac{\mathcal{O}_{(X,0)}}{z\omega_{(X,0)}}\right) - \text{Length}\left(\frac{\mathcal{O}_{(X,0)}}{\mathfrak{m}^{2\mu(X)+1}}\right) + 2\mu(X). \end{aligned}$$

From Proposition 1.1 and the first part of this result, we obtain

$$\begin{aligned} s\left(\frac{\mathcal{O}_{(X,0)}}{z\omega_{(X,0)}}\right) &\leq (te_0(X) - 2\delta(X)) - (e_0(2\mu(X) + 1) - e_1(X)) + 2\mu(X) \\ &= e_0(X)(t - 2\mu(X) - 1) + 2\delta(X) + e_1(X) + 2(1 - r). \end{aligned}$$

From Proposition 1.1 we obtain that the socle degree is bounded from above by $\delta(X)(4e_0(X) + 3)$.

(iii) Since any superficial element is a non-zero divisor, we get the claim. □

Recall that it is possible to give an ‘explicit’ description of $\omega_{(X,0)}$ by using Rosenlicht’s regular differential forms (see [20, Chapter IV, § 3.9] and [6, § 1]). This strategy is very useful in the case of branches, especially in the case of monomial curve singularities.

We denote by $\Omega_{\bar{X}}(p.)$ the set of meromorphic forms in \bar{X} with at most a single pole in the set $\{p_1, \dots, p_r\}$. Then Rosenlicht’s differential forms are defined as follows: $\omega_{(X,0)}^R$ is the set of $\nu_*(\alpha)$, $\alpha \in \Omega_{\bar{X}}(p.)$, such that for all $F \in \mathcal{O}_{(X,0)}$,

$$\sum_{i=1}^r \text{res}_{p_i}(F\alpha) = 0.$$

Notice that we have a mapping that we also denote by

$$d_R: \mathcal{O}_{(X,0)} \rightarrow \omega_{(X,0)} \rightarrow \nu_*\Omega_{\bar{X}} \hookrightarrow \omega_{(X,0)}^R.$$

In [1, Chapter VIII, § 1] it is proved that $\omega_{(X,0)} \cong \omega_{(X,0)}^R$ and $d_R = \phi d$, where $d: \mathcal{O}_{(X,0)} \rightarrow \omega_{(X,0)}$. From now on we assume that $(X, 0)$ is a branch, i.e. $r = 1$. Let $t \in \text{tot}(\mathcal{O}_{(X,0)})$ be a uniformizing parameter of $(X, 0)$; this means that $\overline{\mathcal{O}_{(X,0)}} \cong \mathbb{C}[[t]]$. We can consider $\mathcal{O}_{(X,0)}$ as a sub- \mathbb{C} -algebra of $\mathbb{C}[[t]]$ and then we may assume that there exists a parametrization of $(X, 0)$,

$$\begin{aligned} x_1 &= t^{n_1}, \\ x_i &= f_i(t), \quad i = 2, \dots, n, \end{aligned}$$

with $n_1 = e_0(X)$ and $\text{val}_t(f_i) \geq n_1$, $i = 2, \dots, n$. Here, val_t denotes the valuation with respect to t that is defined in $\text{tot}(\mathcal{O}_{(X,0)})$. Notice that the conductor \mathcal{C} of the extension $\mathcal{O}_{(X,0)} \subset \overline{\mathcal{O}_{(X,0)}}$ is, in particular, an ideal of $\overline{\mathcal{O}_{(X,0)}} \cong \mathbb{C}[[t]]$, so it is generated by t^c .

For any subset N of $\text{tot}(\mathcal{O}_{(X,0)})$ we denote by Γ_N the set of rational numbers $\text{val}_t(a)$ for all $a \in N \setminus \{0\}$. We assume that Γ_N contains the zero element.

The following result is well known. We include it here for the reader’s convenience (see [11, Example 2.1.9]).

Proposition 2.6. *Let $(X, 0)$ be a branch. It holds that $\Gamma_{\omega_{(X,0)}} \subset \mathbb{Z} \setminus (-\Gamma_X - 1)$ and, in particular, $\Gamma_{\omega_{(X,0)}} \subset -c + \mathbb{N}$ and there exists $\alpha \in \omega_{(X,0)}$ such that $\text{val}_t(\alpha) = -c$. If $(X, 0)$ is a monomial curve singularity, then $\Gamma_{\omega_{(X,0)}} = \mathbb{Z} \setminus (-\Gamma_X - 1)$.*

Proof. Every $\alpha \in \omega_{(X,0)}$ can be written $\alpha = t^n g(t) dt$ with $n \in \mathbb{Z}$ and $g(t) \in \mathbb{C}[[t]]$ an invertible series. Then, for all $F = t^m g(t) \in \mathcal{O}_{(X,0)}$ with $g(t) \in \mathbb{C}[[t]]$ an invertible series, we get that $\text{res}_0(\alpha F) = 0$. In particular, this implies that $n+m \neq -1$, so $n \in \mathbb{Z} \setminus (-\Gamma_X - 1)$. Let us consider an element $\alpha = t^n g(t) dt \in \omega_{(X,0)}$ with $g(t) \in \mathbb{C}[[t]]$ an invertible series with $n \leq -c - 1$. Since $-n - 1 \geq c$, we get $t^{-n-1} \in \mathcal{O}_{(X,0)}$ and then a contradiction: $\text{val}_t(\alpha t^{-n-1}) = -1$. The differential $\alpha = t^{-c} dt$ belongs to $\omega_{(X,0)}$ and $\text{val}_t(\alpha) = -c$.

Let us assume now that $(X, 0)$ is a monomial curve singularity and let n be an integer of $\mathbb{Z} \setminus (-\Gamma_X - 1)$. Consider the differential $\alpha = t^n dt$: we only have to prove that $\alpha \in \omega_{(X,0)}$. Let us consider $F = \sum_{i \geq 0} a_i t^i$, a power series with coefficients $a_i \in \mathbb{C}$, $i \geq 0$. Since $(X, 0)$ is monomial, we get that $F \in \mathcal{O}_{(X,0)}$ if and only if for all $a_i \neq 0$ it holds that $i \in \Gamma_X$. If $\alpha \notin \omega_{(X,0)}$, then there exists $F \in \mathcal{O}_{(X,0)}$ such that $\text{res}_0(\alpha F) = a_{-n-1} \neq 0$. This implies that $-n - 1 \in \Gamma_X$, which is in contradiction to the hypothesis $n \notin -\Gamma_X - 1$. \square

Example 2.7. Let us consider the monomial curve X with parametrization $x_1 = t^4$, $x_2 = t^7$, $x_3 = t^9$. Then, $e_0 = 4$, $c = 11$ and $\delta = 6$. Then, ω_X is the \mathbb{C} -vector space spanned by $t^{-11}, t^{-7}, t^{-6}, t^{-4}, t^{-3}, t^{-2}, t^n, n \geq 0$, and the quotient $\omega_X/d\mathcal{O}_{(X,0)}$ admits as \mathbb{C} -vector space base the cosets defined by $t^{-11}, t^{-7}, t^{-6}, t^{-4}, t^{-3}, t^{-2}, 1, t, t^2, t^4, t^5, t^9$. Notice that $\mu(X) = 12 = 2\delta(X)$. From Proposition 2.5 we obtain that $x_1^a \omega_X$ is a canonical ideal of X , $a \geq 101$. On the other hand, $t^{15} \omega_X = (x_1, x_3)$ is a canonical ideal as well.

The next step is to find explicit canonical ideals from the resolution of $\mathcal{O}_{(X,0)}$ when X is a reduced curve singularity of $(\mathbb{C}^3, 0)$. By the Hilbert–Burch theorem, we know that there exists a minimal free resolution of $\mathcal{O}_{(X,0)}$ as an $\mathcal{O}_{(\mathbb{C}^3,0)}$ -module [7],

$$0 \rightarrow \mathcal{O}_{(\mathbb{C}^3,0)}^{v-1} \xrightarrow{M} \mathcal{O}_{(\mathbb{C}^3,0)}^v \rightarrow \mathcal{O}_{(\mathbb{C}^3,0)} \rightarrow \mathcal{O}_{(X,0)} \rightarrow 0,$$

where M is a $v \times (v - 1)$ matrix with entries belonging to the maximal ideal $\mathcal{O}_{(\mathbb{C}^3,0)}$ and I_X is minimally generated by the maximal minors of M . The canonical module of $\mathcal{O}_{(X,0)}$ is minimally generated by $v - 1$ elements. Following [3,15], we consider a $(2v - 1) \times (2v - 1)$ block-matrix

$$M_A = \left(\begin{array}{c|c} A & M \\ \hline -M^\tau & 0 \end{array} \right),$$

where $A = (a_{i,j})_{i,j=1,\dots,v}$ is a $v \times v$ skew-symmetric matrix. Notice that M_A is also a skew-symmetric matrix and, by the main result of [5], we have a complex

$$0 \rightarrow \mathcal{O}_{(\mathbb{C}^3,0)} \xrightarrow{\text{Pf}(M_A)^\tau} \mathcal{O}_{(\mathbb{C}^3,0)}^{2v-1} \xrightarrow{M_A} \mathcal{O}_{(\mathbb{C}^3,0)}^{2v-1} \xrightarrow{\text{Pf}(M_A)} \mathcal{O}_{(\mathbb{C}^3,0)} \rightarrow \mathcal{O}_A = \mathcal{O}_{(\mathbb{C}^3,0)}/I_A \rightarrow 0,$$

where I_A is the ideal generated by the Pfaffians $\text{Pf}(M_A)$ of M_A . This complex is exact if and only if I_Z is a height 3 ideal and if this is the case, \mathcal{O}_A is a Gorenstein ring [5].

Proposition 2.8.

- (i) Let A be a $v \times v$ skew-symmetric matrix such that I_A/I_X is an \mathfrak{m}_X -primary ideal of $\mathcal{O}_{(X,0)}$ and $I_A/I_X \subset \mathfrak{m}_X^{\text{pn}(\mathcal{O}_{(X,0)})+1}$. Then I_A/I_X is a canonical ideal of $(X, 0)$.
- (ii) For all $t \geq \text{pn}(\mathcal{O}_{(X,0)})$ there exists a $v \times v$ skew-symmetric matrix A with entries of order at least t such that I_A/I_X is a canonical ideal of $\mathcal{O}_{(X,0)}$.

Proof. (i) This is a consequence of [5] and Corollary 2.3.

(ii) The ideal I_A is generated by the Pfaffians of M_A and these elements take the following form (see [15]).

- (1) $F_i = (-1)^{i+(v-1)(v-2)/2} \det(M_i)$, $i = 1, \dots, v$, where M_i is the matrix removing the i th row of M .
- (2) $F_k = \sum_{1 \leq i < j \leq v} (-1)^{i+j+k+(v-1)(v-4)/2} a_{i,j} \det(M_{i,j,k})$, $v + 1 \leq k \leq 2v - 1$, $M_{i,j,k}$ is the matrix removing the i, j rows of M and the k column of M .

Notice that F_1, \dots, F_v is a system of generators of I_X . Hence, I_A/I_X is generated by the cosets of F_k in $\mathcal{O}_{(X,0)}$, $v + 1 \leq k \leq 2v - 1$.

Let $\pi: \mathcal{O}_{(\mathbb{C}^3,0)} \rightarrow \mathcal{O}_{(X,0)}$ be the natural projection. Let J_X be the ideal generated by the 2×2 minors of the $v \times 3$ matrix Jac_X whose i th row is the gradient vector $\nabla F_i = (\partial F_i/\partial x_1, \partial F_i/\partial x_2, \partial F_i/\partial x_3)$, $i = 1, \dots, v$. The image $\pi(J_X)$ is the Jacobian ideal of X . Since the $(X, 0)$ is an isolated curve singularity, we have that $\pi(J_X)$ is \mathfrak{m}_X -primary, so $\pi^{-1}\pi(J_X) = I_X + J_X$ is (x_1, x_2, x_3) -primary.

On the other hand, it is easy to prove that

$$I_X + J_X \subset K = (F_i; 1 \leq i \leq v; \det(M_{i,j,k}); 1 \leq i < j \leq v; v + 1 \leq k \leq 2v - 1),$$

so K is an (x_1, x_2, x_3) -primary ideal.

Assume that $\det(M_{i,j,k})$ is a zero divisor of $\mathcal{O}_{(X,0)}$ for all $1 \leq i \leq v$, $1 \leq i < j \leq v$, $v + 1 \leq k \leq 2v - 1$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the set of minimal primes of $\mathcal{O}_{(X,0)}$. Then $\mathfrak{p}_i \neq \mathfrak{m}_X$, $i = 1, \dots, m$, and $K/I_X \subset \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_m$. Since K/I_X is an \mathfrak{m} -primary ideal of $\mathcal{O}_{(X,0)}$, there is an integer w such that

$$(x_1, x_2, x_3)^w \subset \frac{K}{I_X} \subset \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_m.$$

Hence, every element of $(x_1, x_2, x_3)^w$ is a zero divisor of $\mathcal{O}_{(X,0)}$, but this is not possible because $\mathcal{O}_{(X,0)}$ is a Cohen–Macaulay ring. We have proved that there exist integers $1 \leq i \leq v$, $1 \leq i < j \leq v$, $v + 1 \leq k \leq 2v - 1$ such that $\det(M_{i,j,k})$ is a non-zero divisor in $\mathcal{O}_{(X,0)}$.

Let $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ be a non-zero divisor of $\mathcal{O}_{(X,0)}$. Let us consider the skew-symmetric matrix such that

$$\begin{aligned} a_{i,j} &= x^{\text{pn}(\mathcal{O}_{(X,0)})}, \\ a_{\alpha,\beta} &= 0, \quad \alpha < \beta, \quad (\alpha, \beta) \neq (i, j). \end{aligned}$$

Then $F_k = \pm x^{\text{pn}(\mathcal{O}_{(X,0)})} \det(M_{i,j,k})$ is a non-zero divisor of $\mathcal{O}_{(X,0)}$. Hence, I_A/I_X is an \mathfrak{m} -primary ideal of $\mathcal{O}_{(X,0)}$ contained in $\mathfrak{m}^{\text{pn}(\mathcal{O}_{(X,0)})+1}$. By (i), we obtain that I_A/I_X is a canonical ideal of $\mathcal{O}_{(X,0)}$. \square

Example 2.9. Let us consider a monomial curve singularity X with parametrization $(t^{n_1}, t^{n_2}, t^{n_3})$ such that $n_1 < n_2 < n_3$ and $\text{gcd}(n_1, n_2, n_3) = 1$. Then, $e_0(X) = n_1$ and

$$\mathcal{O}_{(X,0)} = \mathbb{C}[[t^{n_1}, t^{n_2}, t^{n_3}]] \subset \overline{\mathcal{O}_{(X,0)}} = \mathbb{C}[[t]].$$

The ideal I_X is generated by

$$F_1 = x_1^{r_{3,1}} x_2^{r_{3,2}} - x_3^{c_3}, \quad F_2 = x_2^{r_{1,2}} x_3^{r_{1,3}} - x_1^{c_1} \quad \text{and} \quad F_3 = x_1^{r_{2,1}} x_3^{r_{2,3}} - x_2^{c_2},$$

where $r_{i,j} \geq 0$ and $c_i > 0$ is the least integer such that $c_i n_i = \sum_{i \neq j} r_{i,j} n_j$, $i = 1, 2, 3$ [12]. We assume that X is not a complete intersection, so $r_{i,j} > 0$. We then have that

$c_1 = r_{2,1} + r_{3,1}$, $c_2 = r_{1,2} + r_{3,2}$, $c_3 = r_{1,3} + r_{2,3}$ and F_1, F_2, F_3 are the maximal minors of a matrix

$$M = \begin{pmatrix} x_1^{r_{2,1}} & x_2^{r_{1,2}} \\ x_2^{r_{3,2}} & x_3^{r_{2,3}} \\ x_3^{r_{1,3}} & x_1^{r_{3,1}} \end{pmatrix}$$

(see [18]). Then the matrix M_A takes the form

$$M_A = C = \left(\begin{array}{ccc|cc} 0 & a_{1,2} & -a_{1,3} & x_1^{r_{2,1}} & x_2^{r_{1,2}} \\ -a_{1,2} & 0 & a_{2,3} & x_2^{r_{3,2}} & x_3^{r_{2,3}} \\ a_{1,3} & -a_{3,2} & 0 & x_3^{r_{1,3}} & x_1^{r_{3,1}} \\ \hline -x_1^{r_{2,1}} & -x_2^{r_{3,2}} & -x_3^{r_{1,3}} & 0 & 0 \\ -x_2^{r_{1,2}} & -x_3^{r_{2,3}} & -x_1^{r_{3,1}} & 0 & 0 \end{array} \right)$$

with $a_{i,j} \in \mathfrak{m}^{\text{pn}(\mathcal{O}_{(X,0)})}$. Then I_A/I_X is a canonical ideal generated by the cosets in $\mathcal{O}_{(X,0)}$ of $a_{2,3}m_{1,i} + a_{1,3}m_{2,i} + a_{1,2}m_{3,i}$, $i = 1, 2$, if one of these two elements is a non-zero divisor of $\mathcal{O}_{(X,0)}$ (see Proposition 2.8 (ii)). For instance, we can take $a_{2,3} = x_1^{\text{pn}(X)}$ and $a_{i,j} = 0$ for $(i, j) \neq (2, 3)$, and so $(x_1^{r_{2,1} + \text{pn}(X)}, x_1^{\text{pn}(X)}x_2^{r_{1,2}})$ is a canonical ideal of X .

3. Canonical ideals and the classification of curve singularities

In this section we translate the classification of curve singularities to the classification of local Artin Gorenstein rings by means of the quotients with canonical ideals.

First we have to define what generic means in our context. We denote by S_t the \mathbb{C} -vector space of forms of degree t of $\mathbb{C}[[x_1, \dots, x_n]]$.

Proposition 3.1.

(i) For all $t \geq 2\mu(X) + 1$ there exists a non-empty Zariski open set $U_t(X)$ of S_t such that the following hold.

- (1) For all $z \in U_t(X)$, $\bar{z} \in \mathcal{O}_{(X,0)}$ is a degree t superficial element and non-zero divisor.
- (2) For all $z_1, z_2 \in U_t(X)$,

$$\text{HF}_{\mathcal{O}_{(X,0)}/\bar{z}_1\omega_{(X,0)}}^1 = \text{HF}_{\mathcal{O}_{(X,0)}/\bar{z}_2\omega_{(X,0)}}^1 .$$

(3) For every degree t superficial element $y \in \mathcal{O}_{(X,0)}$ and for every $z \in U_t(X)$, it holds that for all $n \geq 0$,

$$\text{HF}_{\mathcal{O}_{(X,0)}/\bar{z}_1\omega_{(X,0)}}^1(n) \leq \text{HF}_{\mathcal{O}_{(X,0)}/y\omega_{(X,0)}}^1(n).$$

(ii) For all $z \in U_t(X)$ and $t \geq 4\mu(X) + 1$, the socle degree and the length of $\mathcal{O}_{(X,0)}/\bar{z}\omega_{(X,0)}$ are constant and analytic invariants of $(X, 0)$ and satisfy

$$s(\mathcal{O}_{(X,0)}/\bar{z}\omega_{(X,0)}) \leq e_0(X)(t - 2\mu(X) - 1) + 2\delta(X) + e_1(X) + 2(1 - r),$$

$$\text{Length}(\mathcal{O}_{(X,0)}/\bar{z}\omega_{(X,0)}) = e_0(X)t - 2\delta(X).$$

Proof. (i) From [19, Proposition 3.2], there exists a non-empty Zariski open set $W_0 \subset S_t$ such that $\bar{z} \in \mathcal{O}_{(X,0)}$ is a degree t superficial element and non-zero divisor. From Proposition 2.5 (i) we know that for all $z \in W_0$ we have $\text{Length}(\mathcal{O}_{(X,0)}/\bar{z}\omega_{(X,0)}) = \ell$ with $\ell = te_0(X) - 2\delta(X)$, so $\text{HF}_{\mathcal{O}_{(X,0)}/\bar{z}\omega_{(X,0)}}^1(n) = \ell$ for all $n \geq \ell$. Hence, we only have to consider $n \leq \ell$. Let us consider the upper semi-continuous function

$$\begin{aligned} \sigma_n : S_t &\rightarrow \mathbb{N} \\ z &\mapsto \text{HF}_{\mathcal{O}_{(X,0)}/\bar{z}\omega_{(X,0)}}^1(n). \end{aligned}$$

For each $n = 1, \dots, \ell$, let W_n be a non-empty Zariski open set $W_n \subset W_0$ such that $\sigma_n(z) = \min\{\sigma_n\}$ for all $z \in W_n$. We set $U_t(X) = W_0 \cap \dots \cap W_\ell$. From the definition of $U_t(X)$, it is easy to get (2) and (3).

(ii) From (2) we deduce that the socle degree and the length of $\mathcal{O}_{(X,0)}/\bar{z}\omega_{(X,0)}$ are constants for all $t \geq 2\mu(X) + 1$ and $z \in U_t(X)$. The upper bounds come from Proposition 2.5. Let $\varphi : (X_1, 0) \rightarrow (X_2, 0)$ be an analytic isomorphism between two reduced curve singularities. Let $\varphi_t : \mathfrak{m}_{X_1}^t/\mathfrak{m}_{X_1}^{t+1} \rightarrow \mathfrak{m}_{X_2}^t/\mathfrak{m}_{X_2}^{t+1}$ be the \mathbb{C} -vector space isomorphism induced by φ . Since $\varphi_t(U_t(X_1)) \cap U_t(X_2) \neq \emptyset$, we get the last part of the claim. \square

We write $\ell(X) = e_0(X)(4\mu(X) + 1) - 2\delta(X)$. Notice that this is the length of the quotients $\mathcal{O}_{(X,0)}/\bar{z}\omega_{(X,0)}$ for $\bar{z} \in U_{4\mu(X)+1}(X)$.

Definition 3.2. The least socle degree of the quotients $\mathcal{O}_{(X,0)}/I$ is denoted by $\sigma(X)$. Here, I range over the set of canonical ideals $I \subset \mathcal{O}_{(X,0)}$ with $I \subset \mathfrak{m}^{2\mu(X)+1}$ and $\text{Length}(\mathcal{O}_{(X,0)}/I) = \ell(X)$. A canonical ideal I is called *deep* if $I \subset \mathfrak{m}^{2\mu(X)+1}$, $\text{Length}(\mathcal{O}_{(X,0)}/I) = \ell(X)$ and $s(\mathcal{O}_{(X,0)}/I) = \sigma(X)$. Notice that from Proposition 3.1, deep canonical ideals exist and that $\sigma(X)$ is an analytic invariant.

Remark 3.3. If $(X, 0)$ is non-singular, then we can take $t = 1$ in the last identity of Proposition 3.1; in fact, we have $s(\mathcal{O}_{(X,0)}/x\mathcal{O}_{(X,0)}) = s(\mathbf{k}) = 0$. If $(X, 0)$ is Gorenstein, then we can take as canonical ideal the whole ring $I = \mathcal{O}_{(X,0)}$. Then we have that $x \in \mathcal{O}_{(X,0)}$ is a degree one superficial element such that

$$s\left(\frac{\mathcal{O}_{(X,0)}}{x^t\mathcal{O}_{(X,0)}}\right) = s\left(\frac{\mathcal{O}_{(X,0)}}{x^{e_0(X)-1}\mathcal{O}_{(X,0)}}\right) + t - e_0(X) + 1$$

and $\text{Length}(\mathcal{O}_{(X,0)}/x^t\mathcal{O}_{(X,0)}) = e_0(X)t$ for all $t \geq e_0(X) - 1$.

Given a non-negative integer t , we denote by $\text{Hilb}_{(\mathbb{C}^n, 0)}^t$ the Hilbert scheme of length t subschemes Z of $(\mathbb{C}^n, 0)$. We denote by $[Z]$ the closed point of $\text{Hilb}_{(\mathbb{C}^n, 0)}^t$ defined by Z . From the universal property of $\text{Hilb}_{(\mathbb{C}^n, 0)}^t$ we deduce that any analytic isomorphism $\phi : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ induces a \mathbb{C} -scheme isomorphism $\tilde{\phi} : \text{Hilb}_{(\mathbb{C}^n, 0)}^t \rightarrow \text{Hilb}_{(\mathbb{C}^n, 0)}^t$ such that $\tilde{\phi}([Z]) = [\phi(Z)]$. Given a canonical ideal I of a reduced curve singularity $(X, 0)$, we denote by $(X, 0)_I$ the zero-dimensional scheme $\text{Spec}(\mathcal{O}_{(X,0)}/I)$. We know that $(X, 0)_I$ is an Artin Gorenstein scheme [4, Proposition 3.3.18]. It is well known that two canonical ideals are isomorphic [4, Theorem 3.3.4]. In the one-dimensional case one can prove more: if I_1 and I_2 are canonical ideals, there exist non-zero divisors $y_1, y_2 \in \mathcal{O}_{(X,0)}$ such that

$y_1 I_1 = y_2 I_2$ [16, Theorem 15.8]. Notice that from the proof of this result we obtain that for all $z \in U_t(X)$ there exists integer α and a regular element y such that $z^\alpha I = yz\omega_{(X,0)}$. The ideals I_1 and I_2 are isomorphic as $\mathcal{O}_{(X,0)}$ -modules, but, in general, they are not analytic isomorphic. Since $K_t = x^t\omega_{(X,0)}$ is a canonical ideal for all $t \geq 1$ with x a degree 1 superficial element, the Hilbert function of $\mathcal{O}_{(X,0)}/K_t$ varies with t .

Proposition 3.4. *There exists a non-empty subscheme $\text{Can}(X, 0)$ of the Hilbert scheme $\text{Hilb}_{(\mathbb{C}^n, 0)}^{\ell(X)}$ whose closed points correspond to zero-dimensional Gorenstein schemes $(X, 0)_I \subset (\mathbb{C}^n, 0)$ of length $\ell(X)$ and such that $I \subset \mathfrak{m}_X^{2\mu(X)+1}$ is a deep canonical ideal.*

Proof. Notice that by Proposition 2.5, there exist canonical ideals $I \subset \mathfrak{m}_X^{2\mu(X)+1}$ of colength $\ell(X)$. Hence, the set of closed points of $\text{Hilb}_{(\mathbb{C}^n, 0)}^{\ell(X)}$ corresponding to such a canonical ideal is non-empty. Conversely, by standard arguments on the semi-continuity of the dimension of \mathbf{k} -vector spaces, there exists a sub-scheme of $\text{Hilb}_{(\mathbb{C}^n, 0)}^{\ell(X)}$ such that its closed points correspond to the quotients $\mathcal{O}_{(X,0)}/I$, where we have that $I \subset \mathfrak{m}_X^{2\mu(X)+1}$. Since $\mathcal{O}_{(X,0)}$ is a one-dimensional Cohen–Macaulay ring, I is a faithful maximal Cohen–Macaulay $\mathcal{O}_{(X,0)}$ -module. Hence, from [4, Proposition 3.3.13] we obtain that I is a canonical ideal if and only if I is a type 1 Cohen–Macaulay $\mathcal{O}_{(X,0)}$ -module. Since the Cohen–Macaulay type is a positive upper semi-continuous function, we get the claim. \square

Theorem 3.5. *Given reduced singularities $(X_i, 0)$, $i = 1, 2$, the following conditions are equivalent.*

- (i) *There exists an analytic isomorphism $\phi: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $\phi(X_1, 0) = (X_2, 0)$.*
- (ii) *There exists an analytic isomorphism $\phi: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ inducing a \mathbb{C} -scheme isomorphism $\tilde{\phi}: \text{Can}(X_1, 0) \rightarrow \text{Can}(X_2, 0)$.*
- (iii) *There exists an analytic isomorphism $\phi: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ and $[(X_1, 0)_I] \in \text{Can}(X_1, 0)$ such that $[\phi((X_1, 0)_I)] \in \text{Can}(X_2, 0)$.*

Proof. Given an integer $s \geq 1$ we denote by $(X, 0)_s$ the Artin scheme defined by $\mathcal{O}_{(X,0)}/\mathfrak{m}_X^s$. The implications (i) \implies (ii) \implies (iii) are trivial. Assume that there exist $[(X_1, 0)_I] \in \text{Can}(X_1, 0)$ such that $[\phi((X_1, 0)_I)] \in \text{Can}(X_2, 0)$. If $\mu(X_1) \leq \mu(X_2)$, then, since $I \subset \mathfrak{m}_X^{2\mu(X)+1}$, we obtain that $(\phi(X_1, 0))_{2\mu(X_1)+1} = (X_2, 0)_{2\mu(X_1)+1}$. From the main result of [8, Theorem 6], we obtain that there exists an analytic isomorphism $\varphi: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ such that $\varphi\phi(X_1, 0) = (X_2, 0)$. If $\mu(X_2) < \mu(X_1)$, then we consider $(\phi^{-1}(X_2, 0))_{2\mu(X_2)+1} = (X_1, 0)_{2\mu(X_2)+1}$. Again, from the main result of [8, Theorem 6], we get (i). \square

An instance of Matlis duality is Macaulay’s inverse system: given a Gorenstein Artin algebra B of socle degree s , the Macaulay inverse system of B is a polynomial of degree s that encodes several algebraic properties of B (see [14]). In the next result we find canonical ideals $I \subset \mathcal{O}_{(X,0)}$, with suitable socle degree, as the first step in considering the inverse system of a curve singularity.

Proposition 3.6. For all $z \in U_{4\mu(X)+1}(X)$, $x \in U_1(X)$ and $a \geq 0$ it holds that

$$s\left(\frac{\mathcal{O}_{(X,0)}}{x^{a+1}\bar{z}\omega_{(X,0)}}\right) = 1 + s\left(\frac{\mathcal{O}_{(X,0)}}{x^a\bar{z}\omega_{(X,0)}}\right).$$

In particular, for suitable integer a we have

$$s\left(\frac{\mathcal{O}_{(X,0)}}{x^a\bar{z}\omega_{(X,0)}}\right) = \delta(X)(4e_0(X) + 3).$$

Proof. The proof of the first identity is standard. The second follows from Proposition 2.5 (ii) and Proposition 1.1. \square

Notice that from the last result we can attach to a curve singularity $(X, 0)$ an Artin Gorenstein local ring $B_X = \mathcal{O}_{(X,0)}/x^a z \omega_{(X,0)}$ with socle degree $\delta(X)(4e_0(X) + 3)$. Consequently, we can attach to a curve singularity the Macaulay’s inverse system ξ_X of B_X that is a degree $\delta(X)(4e_0(X) + 3)$ polynomial. Hence, the algebraic and geometric structure of X is encoded in the polynomial ξ_X . The development of this fact will be considered elsewhere.

Example 3.7. Let us consider the monomial ring $R = \mathbf{k}[[t^4, t^7, t^9]]$ of Example 2.7. In this case the polynomial ξ_X has degree 144. The explicit computation of this polynomial seems to be very hard, but we can consider a more friendly canonical ideal. Notice that $R = \mathbf{k}[[x_1, x_2, x_3]]/I$ with $I = (x_1^4 - x_2x_3, x_2^3 - x_1^3x_3, x_3^2 - x_1x_2^2)$, and $J = t^{15}\omega_X = (x_1, x_3)$ is a canonical ideal. Since $x_1^{24}(x_1, x_3) \subset \mathfrak{m}^{2\mu(X)+1}$, the analytic type of X is determined by the analytic type of the Artin Gorenstein algebra $B = R/x_1^{24}(x_1, x_2)$ [8]. The ring B has Hilbert function $\{1, 3, 4^{(23)}, 2, 1\}$ so that B is of multiplicity 99 and socle degree 26, and the inverse system is the degree 26 polynomial:

$$\begin{aligned} &31087081215590400x_1x_2x_3^{11} + 42744736671436800x_2^8x_3^6 + 284964911142912000x_1^2x_2^3x_3^9 \\ &+ 14248245557145600x_1x_2^{10}x_3^4 + 341957893371494400x_1^3x_2^5x_3^7 + 2849649111429120x_1^5x_3^{10} \\ &+ 647647525324800x_1^2x_2^{12}x_3^2 + 85489473342873600x_1^4x_2^7x_3^5 + 21372368335718400x_1^6x_2^2x_3^8 \\ &+ 2372335257600x_1^3x_2^{14} + 4749415185715200x_1^5x_2^9x_3^3 + 14248245557145600x_1^7x_2^4x_3^6 \\ &+ 43176501688320x_1^6x_2^{11}x_3 + 1781030694643200x_1^8x_2^6x_3^4 + 67848788367360x_1^{10}x_2x_3^7 \\ &+ 42405492729600x_1^9x_2^8x_3^2 + 43176501688320x_1^{11}x_2^3x_3^5 + 94234428288x_1^{10}x_2^{10} \\ &+ 3598041807360x_1^{12}x_2^5x_3^3 + 19769460480x_1^{14}x_3^6 + 39538920960x_1^{13}x_2^7x_3 \\ &+ 19769460480x_1^{15}x_2^2x_3^4 + 1235591280x_1^{16}x_2^4x_3^2 + 4845456x_1^{17}x_2^6 \\ &+ 1700160x_1^{19}x_2x_3^3 + 85008x_1^{20}x_2^3x_3 + 24x_1^{23}x_2^2 + x_1^{24}x_2^5. \end{aligned}$$

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