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Primitive prime divisors in the critical orbits of one-parameter families of rational polynomials

BY RUFEI REN
Department of Mathematics, Fudan University, 220 *Handan Rd., Yangpu District, Shanghai* 200433*, China.*
e-mail: rufeir@fudan.edu.cn

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Abstract

For a polynomial $f(x)\in\mathbb{Q}[x]$ and rational numbers c, u, we put $f_c(x) := f(x)+c$, and consider the Zsigmondy set $\mathcal{Z}(f_c, u)$ associated to the sequence $\{f_c^n(u)-u\}_{n\geqslant 1}$, see Definition 1·1, where f_c^n is the n-st iteration of f_c. In this paper, we prove that if u is a rational critical point of f, then there exists an $\mathbf{M}_f>0$ such that $\mathbf{M}_f \geqslant \max_{c\in\mathbb{Q}}\{\#\mathcal{Z}(f_c, u)\}$.

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1. *Introduction*

For every polynomial $f(x)\in\mathbb{Q}[x]$ and $\alpha\in\mathbb{Q}$ we put $f_\alpha(x) := f(x)+\alpha$. Therefore, f_α can be considered as a one-parameter family of polynomials. For every $u\in\mathbb{Q}$ we write

$$\mathbb{S}_{f,u} := \{c\in\mathbb{Q} \mid \{f_c^n(u)-u\}_{n\geqslant 1} \text{ is infinite}\},$$

where f_c^n is the nth iteration of f_c. In particular, if $u=0$, we put $\mathbb{S}_f := \mathbb{S}_{f,0}$.

We denote by $\mathrm{val}_p(-)$ the p-adic valuation of $\mathbb{Q}$ normalized by $\mathrm{val}_p(p)=1$. In keeping with the terminology of [5], for every polynomial $f(x)\in\mathbb{Q}[x]$, $u\in\mathbb{Q}$ and $n\geqslant 1$ we say that p is a **primitive prime divisor** of $f^n(u)-u$ if $\mathrm{val}_p(f^n(u)-u)>0$ and $\mathrm{val}_p(f^k(u)-u)\leqslant 0$ for all $1\leqslant k<n$.

Definition 1·1. The *Zsigmondy set* of the sequence $\{f^n(u)-u\}_{n\geqslant 1}$ is defined by

$$\mathcal{Z}(f,u) := \{n\geqslant 1 \mid f^n(u)-u \text{ has no primitive prime divisor}\}.$$

The primary application of bounds on the Zsigmondy set is towards understanding arboreal Galois representations associated to iteration of rational maps over number fields. It was first studied by Bang [1] and Zsigmondy [9]. Since then, there have been quite a few research papers on characterising/bounding Zsigmondy sets of various sequences in various settings, e.g., Carmichael [2], Schinzel [8], Rice [7], Ingram–Silverman [5], Doerksen–Haensch [3], Gratton–Nguyen–Tucker[4], Krieger [6], etc.

In this paper, we are interested in the size of the Zsigmondy set of a sequence obtained from the critical orbit of polynomials with rational coefficients of degree $d\geqslant 2$. We denote by Σ the set of finite primes of $\mathbb{Z}$ and reserve p for a prime number.

We first state our main theorem.

THEOREM 1·2. *For every polynomial $f \in \mathbb{Q}[x]$ of degree $d \geqslant 2$ with a critical point $u \in \mathbb{Q}$ there is a constant $\mathbf{M}_f > 0$, depending only on f (independent of $c \in \mathbb{Q}$), such that*

$$\#\mathcal{Z}(f_c, u) \leqslant \mathbf{M}_f$$

for every $c \in \mathbb{S}_{f,u}$.

It is worth mentioning that Rice [7] was the first to prove the finiteness of $\mathcal{Z}(f, 0)$ for each individual polynomial $f(x) \neq x^d$. In [3], Doerksen–Haensch prove Theorem 1·2 for the case that $f(x) = x^d$, $u = 0$ and $c \in \mathbb{Z}$, which is generalised by Krieger in [6] to every $c \in \mathbb{Q}$, see [6, theorem 1.1]. Our contribution is to prove Theorem 1·2 for general polynomials which are not necessary to be monic nor integer. As we consider polynomials f that are more complicated than x^d, we did not aim to get the sharpest uniform bound $\mathbf{M}_f$.

Definition 1·3. A polynomial $g(x) \in \mathbb{Q}[x]$ is called x^2-*divisible* if it has degree $d \geqslant 2$ and is of the form

$$g(x) = u_d x^d + \cdots + u_2 x^2 \in \mathbb{Q}[x].$$

At the last section we will prove that the following theorem implies Theorem 1·2.

THEOREM 1·4. *Given an x^2-divisible $g(x) \in \mathbb{Z}[x]$ of degree $d \geqslant 3$ there is a constant $\mathbf{M}_g > 0$, depending only on g, such that*

$$\#\mathcal{Z}(g_c, 0) \leqslant \mathbf{M}_g$$

for every $c \in \mathbb{S}_g$.

Note that one can give an explicit expression of the lower bound $\mathbf{M}_g$ when combining the decomposition of $\mathbb{S}_g$ in Proposition 2·2 with Propositions 3·3, 3·4, 3·7 and 3·8.

This paper is inspired by Krieger's work in [6]. We generalise her result from the special polynomial $f(x) = x^d$ to arbitrary polynomials in $\mathbb{Q}[x]$. We first address the difficulties of this generalisation as follows.

The first difficulty is from dealing with the non-monic case, in which the denominator $f_c^n(c)$ is no longer always equal to d^n's power of the denominator of c. To deal with it, we introduce a factorisation of an integer with respect to the leading term u_d of f, see (2·9), which allows us to focus on the major factor of the denominator of $f_c^n(c)$.

The second difficulty is from the critical points of large multiplicities. Due to this reason, some arguments in [6] do not work for our case. For example, Krieger uses Mahler's theorem to control $|f_c^n(c)|$ by $|f_c^{n-1}(c) - f_c^{-1}(0)|$. However, this estimation might not be enough when $f_c^{n-1}(c)$ is very close to a critical point with large multiplicity. It forces us to control $|f_c^n(c)|$ in Proposition 2·11 by $|f_c^{n-N}(c) - f_c^{-N}(0)|$ for some relatively large $N > 1$.

2. *Introduction of Proposition 2·2 and some estimates*

We split this section into two parts. In the first part, we introduce our main technical result Proposition 2·2, whose proof will be given in Section 3, and prove that it implies Theorem 1·4. In the second part, we focus on estimating $\ln|A_n|$, which appears in Proposition 2·2.

Let us first set conventions and introduce some notation.

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(1) We set $\mathbb{N} := \{1, 2, \ldots\}$ to be the set of natural numbers and for every $n \in \mathbb{N}$ we denote by $[n]$ the finite set $\{1, 2, \ldots, n\}$.

(2) We denote by Σ the set of finite primes of $\mathbb{Z}$ and reserve p for a prime number. For every $n \in \mathbb{Z}\backslash\{0\}$ the sum $\sum_{p|n}$ and the product $\prod_{p|n}$ are taken over all its distinct prime factors whose number is denoted by $\omega(n)$.

We will always write an x^2-divisible $g(x) \in \mathbb{Z}[x]$ by $g(x) = u_d x^d + \cdots + u_2 x^2 \in \mathbb{Z}[x]$, and define its **length** by

$$L_g := 1 + \sum_{i=2}^{d-1} |u_i|/|u_d|.$$

For every x^2-divisible $g(x) \in \mathbb{Z}[x]$, $c \in \mathbb{Q}$ and $n \geqslant 0$ we write the $(n+1)$th iteration $g_c^{n+1}(0)$ as

$$g_c^{n+1}(0) = g_c^n(c) := \frac{A_n}{B_n},$$

where $B_n > 0$, A_n are coprime and both depend on g and c. Clearly, we have $c = g_c(0) = A_0/B_0$.

Definition 2·1. For an x^2-divisible $g(x) \in \mathbb{Z}[x]$ and a set S in $\mathbb{S}_g$ we call that g has *rapidly increasing numerators* on S if there exists an integer $N > 0$ such that for every $c \in S$ there is a finite set J_c with $\#J_c \leqslant N$ such that for every $n \notin J_c$ we have

$$\ln|A_n| > \sum_{p|n} \ln\left|A_{n/p}\right|. \tag{2·1}$$

We now state our main proposition, which is followed by the proof of Theorem 1·4.

PROPOSITION 2·2 (Main Proposition). *Every x^2-divisible $g(x) \in \mathbb{Z}[x]$ has rapidly increasing numerators on $\mathbb{S}_g$.*

Proof of Theorem 1·4 in assuming Proposition 2·2. By [6, lemma 2.3 and corollary 2.4], if $n \in \mathcal{Z}(g_c, 0)$, then $A_n \mid \prod_{p|n} A_{n/p}$ and hence

$$\ln|A_n| \leqslant \sum_{p|n} \ln\left|A_{n/p}\right|.$$

Together with Proposition 2·2, this completes the proof.

The naive idea of proving Proposition 2·2 is to give a lower bound for $\ln|A_n|$ and an upper bound for $\prod_{p|n} |A_{n/p}|$ such that the lower bound is always greater than the upper bound when n is large enough. Consider that

$$\ln|A_n| = \ln B_n + \ln|g_c^n(c)|. \tag{2·2}$$

It is sufficient for us to control $\ln B_n$ and $\ln|g_c^n(c)|$.

2·1. *Upper bounds for* $\ln B_n$ *and* $\ln|g_c^n(c)|$

LEMMA 2·3. *Given any x^2-divisible $g(x) \in \mathbb{Z}[x]$ and $c \in \mathbb{Q}$, for every $n \geqslant 0$ we have:*

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(1) $\ln B_n \leqslant d^n \ln B_0$;
(2) $\ln |g_c^n(c)| \leqslant d^n \ln \left(2|u_d| \max\left\{|c|, 4L_g\right\}\right)$.

Proof. (1) Since $g_c^n(c)$ can be written as $A_n'/B_0^{d^n}$ for some $A_n' \in \mathbb{Z}$, we have $B_n | B_0^{d^n}$.

(2) It is enough to prove

$$|g_c^n(c)| \leqslant (2|u_d|)^{\frac{d^n-1}{d-1}} \left(\max\left\{|c|, 4L_g\right\}\right)^{d^n}, \quad \text{for all } n \geqslant 0.$$

For $n = 0$, we have $|g_c^0(c)| = |c| \leqslant \max\left\{|c|, 4L_g\right\}$.

Assume that the desired inequality holds for some $n \geqslant 0$ and temporarily denote its right side by T_n. Then we have

$$\begin{aligned} |g_c^{n+1}(c)| &\leqslant |u_d||g_c^n(c)|^d + \sum_{i=2}^{d-1} |u_i||g_c^n(c)|^i + |c| \leqslant |u_d|(T_n + L_g)T_n^{d-1} \\ &\leqslant |u_d| \cdot 2T_n \cdot T_n^{d-1} = (2|u_d|)^{\frac{d^{n+1}-1}{d-1}} \left(\max\left\{|c|, 4L_g\right\}\right)^{d^{n+1}}. \end{aligned}$$

The proof follows by induction.

2·2. *A lower bound for* $\ln B_n$

Consider that

$$\frac{A_{n+1}}{B_{n+1}} = g_c^{n+1}(c) = g_c\left(\frac{A_n}{B_n}\right) = \sum_{i=2}^{d} u_i \frac{A_n^i}{B_n^i} + \frac{A_0}{B_0}. \tag{2·3}$$

LEMMA 2·4. *Given any x^2-divisible $g(x) \in \mathbb{Z}[x]$ and $c \in \mathbb{Q}$, for every $n \geqslant 0$ if $p \in I(B_n)$, then we have*

(i) *$p \in I(B_{n+1})$ and*
(ii) $\mathrm{val}_p(B_{n+1}) = d\,\mathrm{val}_p(B_n) - \mathrm{val}_p(u_d)$.

Proof. Note that for every $n \geqslant 0$ we have

$$\mathrm{val}_p\left(\frac{u_d A_n^d}{B_n^d}\right) = \mathrm{val}_p(u_d) + d\,\mathrm{val}_p(A_n) - d\,\mathrm{val}_p(B_n), \tag{2·4}$$

$$\mathrm{val}_p\left(\frac{u_i A_n^i}{B_n^i}\right) \geqslant (1-d)\mathrm{val}_p(B_n), \quad i = 2, \ldots, d-1, \tag{2·5}$$

$$\mathrm{val}_p(\frac{A_0}{B_0}) \geqslant -\mathrm{val}_p(B_0). \tag{2·6}$$

Therefore, if $p \in I(B_n)$, then we have $\mathrm{val}_p(A_n) = 0$ and

$$\mathrm{val}_p\left(\frac{u_d A_n^d}{B_n^d}\right) < (1-d)\mathrm{val}_p(B_n) \leqslant \mathrm{val}_p\left(\frac{u_i A_n^i}{B_n^i}\right) \tag{2·7}$$

for every $2 \leqslant i \leqslant d-1$. Note that the term on the right-hand side of this inequality does not exist for the case $d = 2$.

We now prove this lemma by induction.

For $n=0$ we have $\mathrm{val}_p(B_0) > \mathrm{val}_p(u_d)$. Combined with (2·3) and (2·7), this implies

$$\mathrm{val}_p\left(\frac{A_1}{B_1}\right) = \mathrm{val}_p\left(\frac{u_d A_0^d}{B_0^d}\right) = \mathrm{val}_p(u_d) - d\,\mathrm{val}_p(B_0) < -\mathrm{val}_p(u_d)$$

and hence $p \in I(B_1)$.

Now we assume that this lemma holds for every $0 \leqslant k \leqslant n$.

(1) If $p \notin I(B_0)$, we have

$$\mathrm{val}_p(B_0) \leqslant \mathrm{val}_p(u_d) < \mathrm{val}_p(B_n).$$

Combined with (2·3) and (2·7), this implies

$$\mathrm{val}_p\left(\frac{A_{n+1}}{B_{n+1}}\right) = \mathrm{val}_p\left(\frac{u_d A_n^d}{B_n^d}\right) = \mathrm{val}_p(u_d) - d\,\mathrm{val}_p(B_n) < -\mathrm{val}_p(u_d) \tag{2·8}$$

and hence $p \in I(B_{n+1})$.

(2) If $p \in I(B_0)$, then by induction, we have $p \in I(B_n)$ and

$$\mathrm{val}_p(B_n) = d^n \mathrm{val}_p(B_0) - \mathrm{val}_p(u_d) \sum_{i=0}^{n-1} d^i \geqslant \mathrm{val}_p(B_0).$$

Combining (2·3) with (2·7), we also obtain (2·8). This completes the induction.

For every $a \in \mathbb{Z}$ we denote

$$I(a) := \{p \in \Sigma \mid \mathrm{val}_p(a) > \mathrm{val}_p(u_d)\},$$

and put

$$\widehat{a} := \prod_{p \in I(a)} p^{\mathrm{val}_p(a)}. \tag{2·9}$$

When $I(a)$ is empty, we put $\widehat{a} := 1$. Note that we always have

$$|a| \leqslant |u_d \widehat{a}|. \tag{2·10}$$

LEMMA 2·5. *Given any x^2-divisible $g(x) \in \mathbb{Z}[x]$ of degree $d \geqslant 3$ and $c \in \mathbb{Q}$, for every $0 \leqslant n' \leqslant n$ we have*

$$\ln B_n \geqslant \frac{d^{n-n'}}{3} \ln \widehat{B_{n'}}.$$

Proof. If $\widehat{B_{n'}} = 1$, it is trivial.

Now we assume that $\widehat{B_{n'}} \geqslant 2$. It is enough to show that every prime $p \in I(B_{n'})$ satisfies

$$\mathrm{val}_p(B_n) \geqslant \frac{d^{n-n'}}{3} \mathrm{val}_p\left(B_{n'}\right). \tag{2·11}$$

Using Lemma 2·4 inductively, we have

$$\mathrm{val}_p\left(B_n\right) = d^{n-n'} \mathrm{val}_p(B_{n'}) - \mathrm{val}_p(u_d) \sum_{i=0}^{n-n'-1} d^i \geqslant \left(d^{n-n'} - \sum_{i=0}^{n-n'-1} d^i\right) \mathrm{val}_p(B_{n'}).$$

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From our assumption that $d \geqslant 3$, we have

$$d^{n-n'} - \sum_{i=0}^{n-n'-1} d^i \geqslant d^{n-n'} - 2d^{n-n'-1} \geqslant \frac{1}{3} d^{n-n'},$$

which completes the proof.

2·3. *The lower bound for* $\ln |g_c^n(c)|$

LEMMA 2·6. *Given any x^2-divisible $g(x) \in \mathbb{Z}[x]$ and $c \in \mathbb{Q}$, if $|g_c^{n'}(c)| \geqslant \max\left\{4L_g, |c|\right\}$ for some $n' \geqslant 0$, then for every $n \geqslant n'$ we have*

$$|g_c^n(c)| \geqslant 2^{-\frac{d^{n-n'}-1}{d-1}} \cdot |g_c^{n'}(c)|^{d^{n-n'}}.$$

Clearly, in this case c is in the basin of infinity for g_c.

Proof. The proof follows from induction. For $n = n'$ this lemma is trivial.

Assume that this lemma holds for some $n \geqslant n'$. Then we have

$$|g_c^n(c)| \geqslant 2^{-\frac{d^{n-n'}-1}{d-1}} \cdot |g_c^{n'}(c)|^{d^{n-n'}} \geqslant \max\{4L_g, |c|\} \cdot \left|4L_g/2\right|^{\frac{d^{n-n'}-1}{d-1}} \geqslant \max\{4L_g, |c|\},$$

and hence

$$\begin{aligned} |g_c^{n+1}(c)| &\geqslant |u_d||g_c^n(c)|^d - \sum_{i=2}^{d} |u_i||g_c^n(c)|^i - |c| \\ &\geqslant |u_d|\Big(|g_c^n(c)|^d - L_g|g_c^n(c)|^{d-1}\Big) = |g_c^n(c)|^{d-1}|u_d|\Big(|g_c^n(c)| - L_g\Big) \\ &\geqslant |g_c^n(c)|^d/2 \geqslant 2^{-\frac{d^{n-n'+1}-1}{d-1}} \cdot |g_c^{n'}(c)|^{d^{n-n'+1}}. \end{aligned} \tag{2·12}$$

COROLLARY 2·7. *Given any x^2-divisible $g(x) \in \mathbb{Z}[x]$ and $c \in \mathbb{Q}$, if $|g_c^{n'}(c)| \geqslant \max\left\{4L_g, |c|\right\}$ for some $n' \geqslant 0$, then for every $n \geqslant n'$ we have*

$$\ln |g_c^n(c)| \geqslant d^{n-n'} \ln |g_c^{n'}(c)/2|.$$

Proof. It follows directly from Lemma 2·6.

Given an algebraic number $\gamma \in \mathbb{C}$ of degree ℓ with conjugates $\gamma_1 := \gamma, \gamma_2, \ldots, \gamma_\ell$ over $\mathbb{Q}$, let a_0 be an integer such that the coefficients of the polynomial $g(X) = a_0 \prod_{i=1}^{\ell}(X - \gamma_i)$ are integers of gcd 1, then we define the Mahler measure of γ by

$$M(\gamma) := |a_0| \prod_{i=1}^{\ell} \max(1, |\gamma_i|).$$

Notation 2·8. For every $r \geqslant 1$ and $\delta > 0$ we put

$$W(r, \delta) := 2 \times 10^7 \delta^{-4} \cdot \ln 4r \cdot \ln\ln 4r.$$

THEOREM 2·9 ([10, theorem 1]). *Let $0<\delta<1$. Then for every algebraic number γ of degree $r\geqslant 1$, there are at most $W(r,\delta)$ solutions $a/b\in\mathbb{Q}$ to*

$$|a/b-\gamma|<M(a/b)^{-2-\delta} \tag{2·13}$$

with $M(a/b)\geqslant\max\{4^{2/\delta}, M(\gamma)\}$.

Theorem 2·9 implies the following result.

COROLLARY 2·10. *Given any x^2-divisible $g(x)\in\mathbb{Z}[x]$, for every $L>0$, $N\in\mathbb{N}$, $c\in[-L,L]\cap\mathbb{Q}$ and $\gamma\in\mathbb{C}$ such that $g_c^N(\gamma)=0$, there is an integer $D>0$, independent of c, such that*

$$|a/b-\gamma|<\frac{1}{(2bD)^3} \tag{2·14}$$

has at most $W(d^N,1/10)$ rational solutions a/b with $b\geqslant\max\left\{4^{20},(|u_d|B_0D)^{d^N}\right\}$.

Proof. Let $\gamma_{c,1},\ldots,\gamma_{c,d^N}$ be the roots of $g_c^N(x)=0$ in $\mathbb{C}$ which are not necessary to be distinct. Since $g_c^N(x)$ is continuous as a function of x and c, there exists an integer $D>1$ such that for every $c\in[-L,L]\cap\mathbb{Q}$ and $1\leqslant i\leqslant d^N$ we have

$$|\gamma_{c,i}|<D. \tag{2·15}$$

Without loss of generality, we put $\gamma:=\gamma_{c,1}$ and $h(x):=a_0\prod_{i=1}^{\ell}(x-\gamma_{c,i})$ to be the minimal polynomial of γ with integer coefficients of gcd 1.

Since $B_0^{d^N}g_c^N(\gamma)=0$ and $B_0^{d^N}g_c^N(x)$ is a polynomial with integer coefficients, we have $h(x)|B_0^{d^N}g_c^N(x)$. Combined with Gauss's lemma, this implies

$$a_0|(u_dB_0)^{d^N}. \tag{2·16}$$

Combining (2·15) with (2·16), we have

$$M(\gamma)<(|u_d|B_0)^{d^N}D^{\ell}. \tag{2·17}$$

On the other hand, for every rational number a/b in the lowest terms such that $|a/b|\leqslant 2D$ we have

$$b\leqslant M(a/b)\leqslant 2bD. \tag{2·18}$$

Note that Theorem 2·9 still holds when we do the following modifications.

(1) Restricting this theorem to a set of algebraic numbers and changing $M(\gamma)$ to a function of γ which is larger than $M(\gamma)$ for every γ in this set.

(2) Changing the right-hand side of (2·13) to a function of a/b which is less than $M(a/b)^{-2-\delta}$ for every rational number a/b.

(3) Changing the second $M(a/b)$ in Theorem 2·9 to a function of a/b which is less than $M(a/b)$ for every rational number a/b.

Therefore, combined with (2·17) and (2·18), Theorem 2·9 implies that there are at most $W(\ell,\delta)$ rational solutions a/b to

$$|a/b-\gamma|<(2bD)^{-2-\delta} \tag{2·19}$$

such that $|a/b|\leqslant 2D$ and $b\geqslant\max\{4^{2/\delta},(|u_d|B_0)^{d^N}D^{\ell}\}$.

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For rational number a/b such that $|a/b| \geqslant 2D$ we have

$$|a/b - \gamma| \geqslant D > 1 > (2bD)^{-2-\delta}.$$

Together with (2·19), this shows that there at most $W(\ell, \delta)$ rational solutions $a/b \in \mathbb{Q}$ to

$$|a/b - \gamma| < (2bD)^{-2-\delta} \tag{2·20}$$

with $b \geqslant \max\{4^{2/\delta}, (|u_d|B_0)^{d^N} D^\ell\}$.

Take $\delta := 1/10$. Combining $W(\ell, 1/10) \leqslant W(d^N, 1/10)$ with the modification(3) above, we can replace ℓ by d^N and $-2 - 1/10$ by -3, which completes the proof.

Now we consider $|c| \leqslant 4L_g$. Recall that $\mathbb{S}_g$ is the set that contains all rational number c such that $\{g_c^n(0)\}$ is infinite. By Corollary 2·7 with $n' = 0$, for every $c \in (-\infty, -4L_g] \cup [4L_g, \infty)$ we have $\lim_{n\to\infty} \ln |g_c^n(c)| = \infty$ and hence $\mathbb{S}_g \supset (-\infty, -4L_g] \cup [4L_g, \infty)$.

We denote by $\mathbb{U}_g^0$ the finite subset of $\mathbb{S}_g \cap [-4L_g, 4L_g]$ consisting of all the rational numbers with denominator dividing u_d and put $\mathbb{U}_g := [-4L_g, 4L_g] \cap (\mathbb{S}_g \backslash \mathbb{U}_g^0)$. The following proposition aims at dealing the case $c \in \mathbb{U}_g$. It is worth noting that $\widehat{B_0} \geqslant 2$ for all $c \in \mathbb{U}_g$.

PROPOSITION 2·11. *For an x^2-divisible $g(x) \in \mathbb{Z}[x]$ and a real number $\alpha \in [-4L_g, 4L_g]$ such that $g(x) \neq u_d x^d$ or $\alpha \neq 0$, there exists $0 < \delta < L_g$, $C > 0$ and an integer $N \geqslant 0$ such that for every $c \in (\alpha - \delta, \alpha + \delta) \cap \mathbb{S}_g$ if $\widehat{B_{n'}} \geqslant 2$ for some $n' \geqslant 0$, then there is a finite set $S_c \subset \mathbb{N}$ of bounded cardinality $N + n'$ such that for every $n \notin S_c$ we have*

$$\ln |g_c^n(c)| \geqslant \min\{(-1 + 1/d) \ln B_n + \ln C,\ \ln \delta\}.$$

Proof. Let

$$N_0 := \left\lceil \frac{2 \ln 3}{\ln(d/(d-1))} \right\rceil + 1,$$

which satisfies

$$9(d-1)^{N_0 - 1} \leqslant d^{N_0 - 1}. \tag{2·21}$$

Let $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ be the distinct roots of $g_\alpha^{N_0}(x) = 0$ of multiplicity $m_1, \ldots, m_r$, respectively. Choose an $0 < \epsilon < 1$ small enough such that for any two distinct $i, j \in [r]$ we have $|\gamma_i - \gamma_j| > 3\epsilon$.

By continuity of $g_c^N(x)$ as a function of x and c, there exists $0 < \delta < L_g$ such that for every $1 \leqslant i \leqslant r$ and $\alpha', \beta \in \mathbb{R}$ with $|\alpha' - \alpha| < \delta$ and $|\beta| < \delta$ there are exactly m_i roots of $g_{\alpha'}^{N_0}(x) - \beta = 0$ in the disk $O(\gamma_i, \epsilon) \subset \mathbb{C}$.

Now we consider an arbitrary $c \in (\alpha - \delta, \alpha + \delta) \cap \mathbb{S}_g$.

Let Γ be the multiset consisting of all the roots of $g_c^{N_0}(x) = 0$, i.e. two elements in Γ could be the same. From the argument above, for every $n \geqslant N_0$ if $|g_c^n(c)| < \delta$, then there exists $1 \leqslant i_0 \leqslant r$ such that $g_c^{n-N_0}(c) \in O(\gamma_{i_0}, \epsilon)$. We put

$$\Gamma_1 := \Gamma \backslash O(\gamma_{i_0}, \epsilon) \quad \text{and} \quad \Gamma_2 := \Gamma \cap O(\gamma_{i_0}, \epsilon).$$

Note that we have

$$\Gamma = \Gamma_1 \cup \Gamma_2,\ \#(\Gamma) = d^{N_0} \text{ and } \#(\Gamma_2) = m_{i_0}. \tag{2·22}$$

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Now we count the distance between $g_c^{n-N_0}(c)$ and the points in Γ.

For every $\xi \in \Gamma_1$, from our choice of ϵ, we have

$$\left|\xi - g_c^{n-N_0}(c)\right| > 3\epsilon - 2\epsilon = \epsilon. \tag{2·23}$$

For every $\xi \in \Gamma_2$, by Corollary 2·10 with $L := 4L_g$ and $N := N_0$, there is an integer $D > 0$, independent of c, such that

$$|a/b - \xi| < \frac{1}{(2bD)^3} \tag{2·24}$$

has at most $W(d^{N_0}, 1/10)$ rational solutions a/b with $b \geqslant \max\{4^{20}, (|u_d|B_0D)^{d^{N_0}}\}$.

Put

$$N_1 := \lceil \log_d 120 \rceil + N_0 \quad \text{and} \quad N_2 := \lceil 3 \log_d log_2(u_d^2 D + 1) \rceil + 2N_0.$$

Then for every $n \geqslant N_1$, by Lemma 2·5, we have

$$\ln B_{n+n'-N_0} \geqslant \frac{d^{n-N_0}}{3} \ln \widehat{B_{n'}} \geqslant \frac{d^{n-N_0}}{3} \ln 2 \geqslant 20 \ln 4.$$

If $\widehat{B}_0 = 1$, by Lemma 2·5, (2·10) and the choice of N_2, for every $n \geqslant N_2$ we have

$$\ln B_{n+n'-N_0} \geqslant \frac{d^{n-N_0}}{3} \ln \widehat{B_{n'}} \geqslant \frac{d^{n-N_0}}{3} \ln 2 \geqslant d^{N_0} \ln(u_d^2 D) \geqslant d^{N_0} \ln(|u_d|B_0 D).$$

If $\widehat{B}_0 \geqslant 2$, by Lemma 2·5 and (2·10) again, for every $n \geqslant N_2$ we have

$$\ln B_{n-N_0} \geqslant \frac{d^{n-N_0}}{3} \ln \widehat{B}_0 \geqslant \left(\frac{d^{n-N_0}}{3} - d^{N_0}\right) \ln 2 + d^{N_0} \ln \widehat{B}_0$$
$$\geqslant d^{N_0} \ln(u_d^2 \widehat{B}_0 D) \geqslant d^{N_0} \ln(|u_d|B_0 D).$$

Therefore, there are most $W(d^{N_0}, 1/10)$ many integers $n \geqslant \max\{N_1 + n', N_2 + n'\}$ such that $g_c^{n-N_0}(c)$ is a rational solution to (2·24). Combined with

$$|g_c^n(c)| = |g_c^{N_0}\left(g_c^{n-N_0}(c)\right)| = \prod_{\xi\in\Gamma} |g_c^{n-N_0}(c) - \xi| = \prod_{\xi\in\Gamma_1} |g_c^{n-N_0}(c) - \xi| \cdot \prod_{\xi\in\Gamma_2} |g_c^{n-N_0}(c) - \xi|,$$

this implies that for all but at most $\#\Gamma_2 \cdot W(d^{N_0}, 1/10)$ many $n \geqslant \max\{N_1 + n', N_2 + n'\}$ we have

$$\ln |g_c^n(c)| \geqslant \#\Gamma_1 \cdot \ln \epsilon - 3 \cdot \#\Gamma_2 \ln\left(2DB_{n-N_0}\right). \tag{2·25}$$

Combining (2·22) and our assumption $0 < \epsilon < 1$, we have

$$\#\Gamma_1 \cdot \ln \epsilon - 3 \cdot \#\Gamma_2 \ln\left(2DB_{n-N_0}\right) \geqslant d^{N_0} \ln\left(\frac{\epsilon}{8D^3}\right) - 3\#\Gamma_2 \ln B_{n-N_0}.$$

From our assumption that $g(x) \neq u_d x^d$ or $\alpha \neq 0$, we have

$$\#\Gamma_2 = m_{i_0} \leqslant (d-1)^{N_0}.$$

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Therefore, the previous statement implies that for all but at most $(d-1)^{N_0}W(d^{N_0}, 1/10)$ many $n \geqslant \max\{N_1 + n', N_2 + n'\}$ we have

$$\ln|g_c^n(c)| \geqslant d^{N_0} \ln\left(\frac{\epsilon}{8D^3}\right) - 3(d-1)^{N_0} \ln B_{n-N_0}. \tag{2·26}$$

By (2·10) and (2·21), we have

$$3(d-1)^{N_0} \ln B_{n-N_0} \leqslant \frac{d^{N_0-1}(d-1)}{3} \ln B_{n-N_0} \leqslant \frac{d^{N_0-1}(d-1)}{3} \ln(|u_d|\widehat{B_{n-N_0}}).$$

By Lemma 2·5, we obtain

$$\frac{d^{N_0-1}(d-1)}{3} \ln(|u_d|\widehat{B_{n-N_0}}) \leqslant \frac{d^{N_0-1}(d-1)}{3} \ln|u_d| + \frac{d-1}{d} \ln B_n$$
$$\leqslant d^{N_0} \ln|u_d| + \frac{d-1}{d} \ln B_n.$$

The two inequalities above imply that

$$\text{right-hand side of (2·26)} \geqslant d^{N_0} \ln\left(\frac{\epsilon}{8|u_d|D^3}\right) + (-1 + 1/d) \ln B_n,$$

which completes the proof.

3. *Proof of Proposition 2·2*

The basic idea of proving Proposition 2·2 is to show that for each x^2-divisible $g(x) \in \mathbb{Z}[x]$ there exists a finite cover of $\mathbb{S}_g$ as follows:

(1) $(-\infty, -4L_g] \cup [4L_g, \infty)$;
(2) $(\alpha - \delta_{g,\alpha}, \alpha + \delta_{g,\alpha}) \cap \mathbb{U}_g$ for finitely many α in $[-4L_g, 4L_g]$ with $0 < \delta_{g,\alpha} < L_g$;
(3) the finite set $\mathbb{U}_g^0$.

such that g has rapidly increasing numerators on every set in this cover.

Recall that for every $n \in \mathbb{N}$ we denote by $\omega(n)$ the number of its distinct prime divisors. For convenience, we put $s_d(n) := \sum_{p|n} d^{\frac{n}{p}}$. Then we have the following estimation.

LEMMA 3·1. *For every $d \geqslant 2$ and every $n \geqslant 30$, we have $s_d(n) \leqslant d^{\frac{3n}{5}}$.*

Proof. For every integer $n \geqslant 2$ we have $n \geqslant 2^{\omega(n)}$ and hence

$$\omega(n) \leqslant \log_2 n. \tag{3·1}$$

Since for every prime divisor p of n we have $n/p \leqslant n/2$. Combined with (3·1), we have

$$s_d(n) \leqslant d^{\frac{n}{2}} \omega(n) \leqslant d^{\frac{n}{2}} \log_2 n. \tag{3·2}$$

On the other hand, for every $n \geqslant 30$ we have

$$\log_2 n < 5 < 2^3 \leqslant d^{\frac{n}{10}}.$$

Together with (3·2), this finishes the proof.

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LEMMA 3·2. *Given any x^2-divisible $g(x) \in \mathbb{Z}[x]$ and $c \in \mathbb{Q}$, for every $n \geqslant 30$ we have*

$$\sum_{p|n} \ln \left|A_{n/p}\right| \leqslant d^{\frac{3n}{5}} \ln \left(2|u_d|^2 \widehat{B_0} \max\left\{|c|, 4L_g\right\}\right).$$

Proof. By Lemma 2·3, for every $n \geqslant 0$ we have

$$\sum_{p|n} \ln \left|A_{n/p}\right| \leqslant \sum_{p|n} d^{n/p} \ln \left(2|u_d| B_0 \max\left\{|c|, 4L_g\right\}\right).$$

Together with Lemma 3·1 and (2·10), this completes the proof.

PROPOSITION 3·3. *Every x^2-divisible $g(x) \in \mathbb{Z}[x]$ of degree $d \geqslant 3$ has rapidly increasing numerators on $(-\infty, -4L_g] \cup [4L_g, \infty)$.*

Proof. Let c be an arbitrary rational number in $(-\infty, -4L_g] \cup [4L_g, \infty)$.

By Lemma 2·5, Corollary 2·7 with $n' = 0$ and $|c| \geqslant 4L_g \geqslant 4$, for every $n \geqslant 0$ we have

$$\ln|A_n| \geqslant d^n \ln|c/2| + \frac{d^n}{3} \ln \widehat{B_0} \geqslant \frac{d^n}{3} \ln|c\widehat{B_0}|. \tag{3·3}$$

Combined with Lemma 3·2 and $|c| \geqslant 4L_g$, this implies that for every $n \geqslant 30$ we have

$$\ln|A_n| - \sum_{p|n} \ln\left|A_{n/p}\right| \geqslant -d^{\frac{3n}{5}} \ln(2|u_d|^2) + \left(d^n/3 - d^{\frac{3n}{5}}\right) \ln\left|c\widehat{B_0}\right|.$$

Therefore, there exists an integer $N > \max\left\{\frac{5}{2}\log_d 3, 30\right\}$, which only depends on g, such that for every $n \geqslant N$ and every rational number $c \in (-\infty, -4L_g] \cup [4L_g, \infty)$ we have

$$\ln|A_n| > \sum_{p|n} \ln\left|A_{n/p}\right|.$$

Thus we prove this proposition.

We next prove the following.

PROPOSITION 3·4. *Every polynomial $g(x) = u_d x^d \in \mathbb{Z}[x]$ of degree $d \geqslant 3$ has rapidly increasing numerators on $(-1/|4u_d|, 1/|4u_d|) \cap \mathbb{S}_g$.*

Since $0 \notin \mathbb{S}_g$, it is sufficient to show the following two lemmas.

LEMMA 3·5. *Every polynomial $g(x) = u_d x^d \in \mathbb{Z}[x]$ of degree $d \geqslant 3$ has rapidly increasing numerators on $(0, 1/4|u_d|) \cap \mathbb{S}_g$.*

Proof. For every $c \in (0, 1/4|u_d|) \cap \mathbb{S}_g$ and $n \geqslant 0$ we have

$$|c| \leqslant |g_c^n(c)| \leqslant (|u_d|+1)^{\frac{d^n-1}{d-1}}|c| \leqslant (|u_d|+1)^{d^n}|c| \tag{3·4}$$

and $\widehat{B_0} \geqslant 2$.

Combining Lemmas 2·3(1), 2·5 for $n' = 0$ with (3·4), for every $n \geqslant 0$ we have

$$\ln|A_n| - \sum_{p|n} \ln\left|A_{n/p}\right| \geqslant (1 - \omega(n)) \ln|c| + \frac{d^n}{3} \ln \widehat{B_0} - s_d(n) \ln\left((|u_d|+1)B_0\right).$$

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Together with (2·10) and Lemma 3·1, this implies that for every $n \geqslant 30$ we have

$$\ln|A_n| - \sum_{p|n} \ln\left|A_{n/p}\right| \geqslant -d^{\frac{3n}{5}} \ln\left((|u_d|+1)|u_d|\right) + \left(\frac{d^n}{3} - d^{\frac{3n}{5}}\right) \ln \widehat{B_0}.$$

From $\widehat{B_0} \geqslant 2$, there exists an integer $N \geqslant 30$ such that for every $n \geqslant N$ we have

$$\ln|A_n| - \sum_{p|n} \ln\left|A_{n/p}\right| > 0,$$

which completes the proof.

LEMMA 3·6. *Every polynomial $g(x) = u_d x^d \in \mathbb{Z}[x]$ of degree $d \geqslant 3$ has rapidly increasing numerators on $(-1/4|u_d|, 0) \cap \mathbb{S}_g$.*

Proof. Note that when d is odd, we may replace c with $-c$ and the forward orbit of 0 will be unchanged, modulo sign. By Lemma 3·5, we immediately prove this case.

Therefore, it is sufficient to study the case that d is even. We first show that for every $c \in (-1/4|u_d|, 0)$ and every $n \geqslant 0$ we have

$$|c|(1 - |u_d||c|^{d-1}) \leqslant |g_c^n(c)| \leqslant |c|. \tag{3·5}$$

For $n = 0$, we have $|g_c^0(c)| = |c|$. Assume that (3·5) holds for some $n \geqslant 0$. Since $g_c(x)$ is negative and decreasing on $(-1/4|u_d|, 0)$, we have

$$|c|(1 - |u_d||c|^{d-1}) = |g_c(c)| \leqslant |g_c(g_c^n(c))| \leqslant |f(0)| = |c|,$$

which proves (3·5) by induction.

Combining Lemmas 2·3(1), 2·5 with (3·5), we have

$$\begin{aligned}\ln|A_n| - \sum_{p|n} \ln\left|A_{n/p}\right| &\geqslant \ln\left(|c| \cdot (1 - |u_d||c|^{d-1})\right) + \frac{d^n}{3} \ln \widehat{B_0} - s_d(n) \ln B_0 - \omega(n) \ln |c| \\ &\geqslant \ln(1 - |u_d||c|^{d-1}) + \frac{d^n}{3} \ln \widehat{B_0} - s_d(n) \ln B_0.\end{aligned}$$

Together with (2·10), Lemma 3·1 and $|c| < 1/4|u_d|$, this implies that for every $n \geqslant 30$ we have

$$\ln|A_n| - \sum_{p|n} \ln\left|A_{n/p}\right| \geqslant \ln(3/4) + d^{\frac{3n}{5}} \ln|u_d| + \left(\frac{d^n}{3} - d^{\frac{3n}{5}}\right) \ln \widehat{B_0}. \tag{3·6}$$

On the other hand, for every $c \in (-1/4|u_d|, 0)$, we have $\widehat{B_0} \geqslant 2$. Combined with (3·6), this proves that there exists an integer $N \geqslant 30$ such that for every $n \geqslant N$ we have

$$\ln|A_n| - \sum_{p|n} \ln\left|A_{n/p}\right| > 0,$$

which completes the proof.

PROPOSITION 3·7. *Given any x^2-divisible $g(x) \in \mathbb{Z}[x]$ of degree $d \geqslant 3$ and $\alpha \in [-4L_g, 4L_g]$ such that $g(x) \neq u_d x^d$ or $\alpha \neq 0$, there is an $0 < \delta < L_g$ such that g has rapidly increasing numerators on $c \in (\alpha - \delta, \alpha + \delta) \cap \mathbb{U}_g$.*

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Proof. Note that for every $c \in \mathbb{U}_g$ we have $\widehat{B_0} \geqslant 2$. By Proposition 2·11 with $n' = 0$, there is a 3-tuple $0 < \delta < L_g$, $C > 0$ and $N_1 > 0$ such that for every $c \in (\alpha - \delta, \alpha + \delta) \cap \mathbb{U}_g$ there is a finite set $S_c \subset \mathbb{N}$ of bounded cardinality N_1 such that for every $n \notin S_c$ we have

$$\ln |g_c^n(c)| \geqslant \min \{(-1 + 1/d) \ln B_n + \ln C, \ \ln \delta\},$$

and therefore

$$\ln |A_n| \geqslant \min \left\{ \frac{1}{d} \ln B_n + \ln C, \ \ln B_n + \ln \delta \right\}. \tag{3·7}$$

On the other hand, by Lemma 2·5, we have

$$\ln(B_n) + \ln \delta - d^{\frac{3n}{5}} \ln \left|2cu_d^2 \widehat{B_0}\right| \geqslant \ln \delta + \left(\frac{d^n}{3} - d^{\frac{3n}{5}} \right) \ln \widehat{B_0} - d^{\frac{3n}{5}} \ln |2cu_d^2|,$$

$$\frac{1}{d} \ln(B_n) + \ln C - d^{\frac{3n}{5}} \ln \left|2cu_d^2 \widehat{B_0}\right| \geqslant \ln C + \left(\frac{d^{n-1}}{3} - d^{\frac{3n}{5}} \right) \ln \widehat{B_0} - d^{\frac{3n}{5}} \ln |2cu_d^2|.$$

Combined with Lemma 3·2, (3·7) and $|c| \leqslant 5L_g$, this implies that there exists an integer $N_2 \geqslant 30$ such that for every rational number $c \in (\alpha - \delta, \alpha + \delta)$ and $n \in \{N_2, N_2 + 1, \ldots, \} \backslash S_c$ we have

$$\ln |A_n| > \sum_{p|n} \ln \left|A_{n/p}\right|.$$

Taking $N := N_1 + N_2$, we prove this proposition.

Now we turn our attention to the finite set $\mathbb{U}_g^0$.

PROPOSITION 3·8. *Every x^2-divisible $g(x) \in \mathbb{Z}[x]$ of degree $d \geqslant 3$ has rapidly increasing numerators on the finite set $\mathbb{U}_g^0$.*

Proof. It is sufficient to show that for each individual rational number in $\mathbb{U}_g^0$ there are finite many $n \in \mathbb{N}$ satisfying (2·1).

Let c be an arbitrary rational number in $\mathbb{U}_g^0$. We first show that there must exist an integer n' such that either $|g_c^{n'}(c)| > 4L_g$ or $\widehat{B_{n'}} \geqslant 2$. Suppose that for every $n \geqslant 0$ we have $\widehat{B_n} = 1$, i.e., $B_n | u_d$. Since $c \in \mathbb{S}_g$ and there are only finitely many integers in $[-4L_g, 4L_g]$ with denominator dividing u_d, we know that there must exist an $n' \geqslant 0$ such that $|g_c^{n'}(c)| > 4L_g$.

(1) When $|c| \leqslant 4L_g$, $\widehat{B_0} = 1$ and there exists an integer $n' \geqslant 1$ such that $|g_c^{n'}(c)| > 4L_g$. Combining these conditions with Corollary 2·7 and Lemma 3·2, for every $n \geqslant \max\{30, n'\}$ we have

$$\ln |A_n| - \sum_{p|n} \ln \left|A_{n/p}\right| \geqslant -d^{\frac{3n}{5}} \ln(8L_g|u_d|^2) + d^{n-n'} \left|g_c^{n'}(c)/2\right|.$$

Clearly, there exists an integer $N > \max\{30, n'\}$ such that for every $n \geqslant N$ we have

$$\ln |A_n| > \sum_{p|n} \ln \left|A_{n/p}\right|.$$

(2) When $|c| \leqslant 4L_g$, $\widehat{B_0} = 1$ and there is an integer $n' \geqslant 1$ such that $\widehat{B_{n'}} \geqslant 2$. Similar to Proposition 3·7, we combine Lemma 2·5 with Proposition 2·11, and obtain a finite set $S_c \subset \mathbb{N}$, an integer N, $\delta_c > 0$ and $C_c > 0$ such that for every $n \geqslant n'$ if $n \notin S_c$, then

$$\ln|g_c^n(c)| \geqslant \min\{(-1+1/d)\ln B_n + \ln C_c,\ \ln\delta_c\},$$

and therefore

$$\ln|A_n| \geqslant \min\left\{\frac{1}{d}\ln B_n + \ln C_c,\ \ln B_n + \ln\delta_c\right\}. \tag{3·8}$$

On the other hand, by Lemma 2·5, we have

$$\ln B_n + \ln\delta_c - d^{\frac{3n}{5}}\ln\left|2cu_d^2\widehat{B}_0\right| \geqslant \ln\delta_c + \frac{d^{n-n'}}{3}\ln\widehat{B_{n'}} - d^{\frac{3n}{5}}\ln|2cu_d^2|,$$
$$\frac{1}{d}\ln B_n + \ln C_c - d^{\frac{3n}{5}}\ln\left|2cu_d^2\widehat{B}_0\right| \geqslant \ln C_c + \frac{d^{n-n'-1}}{3}\ln\widehat{B_{n'}} - d^{\frac{3n}{5}}\ln|2cu_d^2|.$$

Combined with Lemma 3·2, (3·8) and $|c| \leqslant 4L_g$, this implies that there exists an integer $N_1 > \max\{n', 30\}$ such that for every $n \in \{N_1, N_1+1, \ldots\}\backslash S_c$ we have

$$\ln|A_n| > \sum_{p|n}\ln\left|A_{n/p}\right|.$$

Put $N := \#S_c + N_1$. Then we complete the proof.

Proof of Proposition 2·2. By Proposition 3·7, for every $g(x) \neq u_d x^d$ or $\alpha \neq 0$ and every real number $\alpha \in [-4L_g, 4L_g]$, there is an $0 < \delta_{g,\alpha} < L_g$ such that g has rapidly increasing numerators on $c \in (\alpha - \delta_{g,\alpha}, \alpha + \delta_{g,\alpha}) \cap \mathbb{U}_g$.

For $g(x) = u_d x^d$ and $\alpha = 0$, if we put $\delta_{g,\alpha} := 1/4|u_d|$, then we proved in Proposition 3·4 that g has rapidly increasing numerators on $c \in (-\delta_{g,\alpha}, \delta_{g,\alpha}) \cap \mathbb{U}_g$.

Now for every x^2-divisible $g(x) \in \mathbb{Z}[x]$ we obtain a cover of $\mathbb{S}_g$ as

$$(-\infty, -4L_g] \cup [4L_g, \infty) \cup \mathbb{U}_g^0 \cup \bigcup_{\alpha\in[-4L_g,4L_g]}\Big((\alpha - \delta_{g,\alpha}, \alpha + \delta_{g,\alpha}) \cap \mathbb{U}_g\Big). \tag{3·9}$$

Note that

$$\bigcup_{\alpha\in[-4L_g,4L_g]}(\alpha - \delta_{g,\alpha}, \alpha + \delta_{g,\alpha})$$

is an open cover of the closed interval $[-4L_g, 4L_g]$, which has a finite cover. We use the center α to represent the interval $(\alpha - \delta_{g,\alpha}, \alpha + \delta_{g,\alpha})$ in this finite cover, and put T to be the index set of α.

Therefore, we obtain a finite cover of $\mathbb{S}_g$ as follows:

$$(-\infty, -4L_g] \cup [4L_g, \infty) \cup \mathbb{U}_g^0 \cup \bigcup_{\alpha\in T}\Big((\alpha - \delta_{g,\alpha}, \alpha + \delta_{g,\alpha}) \cap \mathbb{U}_g\Big).$$

By Propositions 3·3, 3·4, 3·7 and 3·8, we know that g has rapidly increasing numerators on each set in this cover. It implies that g also has rapidly increasing numerators on $\mathbb{S}_g$, which completes the proof.

4. *Theorem* 1·4 *implies Theorem* 1·2

At the end of this section, we will prove that Theorem 1·4 implies the following proposition which leads to Theorem 1·2.

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PROPOSITION 4·1. *For every x^2-divisible polynomial $g(x) \in \mathbb{Q}[x]$ there is a constant $\mathbf{M}_g > 0$, depending only on g, such that*

$$\#\mathcal{Z}(g_c, 0) \leqslant \mathbf{M}_g$$

for every $c \in \mathbb{S}_g$.

For every $n \in \mathbb{Z}\backslash\{0\}$, we denote by $\omega(n)$ the number of distinct prime factors of n.

LEMMA 4·2. *Let $g(x) \in \mathbb{Q}[x]$ be x^2-divisible and $a \in \mathbb{Z}\backslash\{0\}$. Then with $h(x) := g(ax)/a$ we have*

$$|\#\mathcal{Z}(g_c, 0) - \#\mathcal{Z}(h_{c/a}, 0)| \leqslant \omega(a) \quad \textit{for every } c \in \mathbb{Q}.$$

Proof. Consider that

$$g_c^n(ax) = uh_{c/a}^n(x) \quad \text{and} \quad g_c^n(0) = ah_{c/a}^n(0).$$

For every $p \nmid u$ we have p is a primitive prime divisor of $g_c^n(0)$ if and only if it is a primitive prime divisor of $h_{c/a}^n(0)$. Therefore, the difference between $\#\mathcal{Z}(g_c, 0)$ and $\#\mathcal{Z}(h_{c/a}, 0)$ can not exceed the number of prime factors of a, which completes the proof.

LEMMA 4·3. *For any x^2-divisible $g(x) \in \mathbb{Q}[x]$ and $t \in \mathbb{Q}\backslash\{0\}$, Proposition 4·1 holds for g if and only if it holds for $g(tx)/t$.*

Proof. Let $t = a/b$ be an arbitrary rational number. Put

$$h(x) := \frac{1}{t}g(tx) \quad \text{and} \quad h_1(x) := \frac{1}{a}g(ax).$$

Note that $h_1(x) = \frac{1}{b}h(bx)$.

By Lemma 4·2 for every $c \in \mathbb{Q}$ we have

$$|\#\mathcal{Z}(g_c, 0) - \#\mathcal{Z}((h_1)_{c/a}, 0)| \leqslant \omega(a),$$
$$|\#\mathcal{Z}(h_{c/t}, 0) - \#\mathcal{Z}((h_1)_{c/a}, 0)| \leqslant \omega(b),$$

which implies

$$|\#\mathcal{Z}(g_c, 0) - \#\mathcal{Z}(h_{c/t}, 0)| \leqslant \omega(a) + \omega(b).$$

Since $\omega(a)$ and $\omega(b)$ are both independent of c, we complete the proof.

PROPOSITION 4·4. *Theorem 1·4 implies Theorem 1·2.*

Proof. Step I. We first prove that Theorem 1·4 implies Proposition 4·1.

Given any x^2-divisible polynomial $g(x) \in \mathbb{Q}[x]$ of degree $d \geqslant 2$, if $d = 2$, then we can find $t \in \mathbb{Q}$ such that $g(tx)/t = x^2$. Combining Lemma 4·3 with [**6**, theorem 1.1], we prove Proposition 4·1 for this case.

If $d \geqslant 3$, there is $t \in \mathbb{Z}$ such that $g(tx)/t \in \mathbb{Z}[x]$. Combined with Lemma 4·3, this proves Proposition 4·1 for this case.

Step II. Now we prove that Proposition 4·1 implies Theorem 1·2.

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For every polynomial $g(x) \in \mathbb{Q}[x]$ with a critical point $u \in \mathbb{Q}$, if we put $f(x) := g(x+u) - u$, then we know that 0 is a critical point of f and for every $c \in \mathbb{Q}$ we have $f_c^n(0) = g_c^n(u) - u$ and hence

$$\mathcal{Z}(g_c, u) = \mathcal{Z}(f_c, 0).$$

This completes the proof.

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