Enumerating coloured partitions in 2 and 3 dimensions

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(*Received* 21 *January* 2019)

Abstract

We study generating functions of ordinary and plane partitions coloured by the action of a finite subgroup of the corresponding special linear group. After reviewing known results for the case of ordinary partitions, we formulate a conjecture concerning a basic factorisation property of the generating function of coloured plane partitions that can be thought of as an orbifold analogue of a conjecture of Maulik et al., now a theorem, in three-dimensional Donaldson–Thomas theory. We study natural quantisations of the generating functions arising from geometry, discuss a quantised version of our conjecture, and prove a positivity result for the quantised coloured plane partition function under a geometric assumption.

2010 Mathematics Subject Classification: 05A17

1. *Introduction*

Let $\mathcal{P}$ and $\mathcal{Q}$ denote the set of all *partitions*, respectively *plane partitions*. With a dimension shift, these can be defined as follows: a partition $\lambda \in \mathcal{P}$ is a finite subset $\lambda \subset \mathbb{N}^2$ of the non-negative lattice quadrant, stable under forces acting in the negative direction along each of the coordinate axes; a plane partition $\alpha \in \mathcal{Q}$ is a finite subset $\alpha \subset \mathbb{N}^3$ of the non-negative lattice octant with the same property (see Figure 1). In both cases, we will refer to the lattice points contained in a partition or plane partition as *boxes*. There are other well-known equivalent definitions of these sets [**1**], mostly less symmetric than the one given here. Both partitions and plane partitions have a measure of size $|\ |: \mathcal{P} \to \mathbb{N}$, respectively $|\ |: \mathcal{Q} \to \mathbb{N}$, given by "counting the number of boxes". The corresponding generating functions

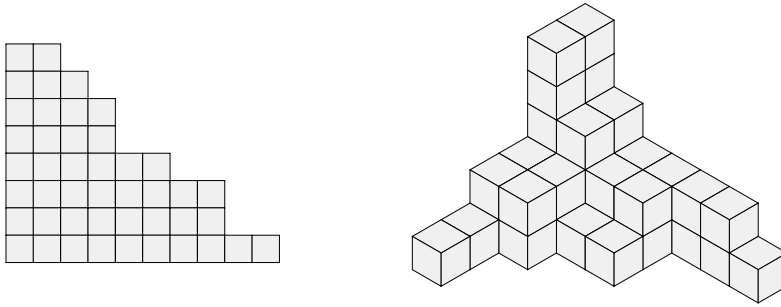$$E(t) = \sum_{\lambda \in \mathcal{P}} t^{|\lambda|}$$

Fig. 1. A partition and a plane partition.

and

$$M(t) = \sum_{\alpha \in \mathcal{Q}} t^{|\alpha|}$$

have product expansions known since the 18th, respectively 19th centuries:

$$E(t) = \prod_{m>0} (1 - t^m)^{-1} \qquad (1\cdot1)$$

and

$$M(t) = \prod_{m>0} (1 - t^m)^{-m}. \qquad (1\cdot2)$$

In this paper, we are interested in "coloured" and "quantised" versions of these generating functions, where the colours are coming from the action of a finite matrix group $G < \mathrm{GL}_d(\mathbb{C})$ for $d = 2, 3$, while the quantisation is coming from geometry. In fact we will only discuss in detail the case when $G < \mathrm{SL}_d(\mathbb{C})$; there is more structure around in this case, and stronger results possible. We start in Section 2 by reviewing the well-known story for $d = 2$ and $G < \mathrm{SL}_2(\mathbb{C})$ abelian; we then explain what can be said in a non-abelian case. Then we turn our attention to $d = 3$, restricting to G abelian. In Section 3, we state as Conjecture 3·1 a factorisation property of the coloured generating function, that can be thought of as an orbifold version of the famous conjecture of Maulik, Nekrasov, Okounkov and Pandharipande [26], now a theorem of Bridgeland and Toda, that states a formally analogous factorisation property of the Donaldson–Thomas partition function attached to a Calabi–Yau threefold (see Remark 4·2). We check one nontrivial consequence of our conjecture, and present some numerical evidence. We recall from the literature some cases in which Conjecture 3·1 is known, and discuss the structural features of the resulting formulae, using the language of the plethystic exponential. In Section 4, we discuss the relationship of the problem to the geometry of the quotient $\mathbb{C}^3/G$ and its minimal resolution(s), leading to a quantised version of the generating function, as well as Conjecture 4·7, a quantised version of Conjecture 3·1. We close by explaining how Hodge theory and the theory of wall-crossing structures gives a framework to attack the quantised conjecture, and proves a related general positivity result, Theorem 4·11, concerning the quantum partition function, under a geometric assumption.

For $d = 3$, geometry dictates that the generating functions should include certain signs. In the combinatorial Section 3, we will ignore these, at the expense of a somewhat unconventional definition of a twisted plethystic exponential PExp^σ. The signed version of the generating function, defined in the geometric Section 4, correctly accounts for all these signs.

In our treatment, the relationship between the results for partitions and those for plane partitions is purely formal; there is a very close similarity between the shapes of the various formulae and their symmetries, but the proofs are different. It is hoped that the reader will find the parallels drawn between the two stories sufficient justification for the joint discussion.

Notation. Given a set of variables $\mathbf{t} = (t_0, \ldots, t_{r-1})$ and $\mathbf{n} = (n_0, \ldots, n_{r-1}) \in \mathbb{N}^r$, denote $\mathbf{t^n} = \prod_i t_i^{n_i}$. Given a set of *non-negative* integers $a_\mathbf{n} \in \mathbb{N}$ for $\mathbf{n} \in \mathbb{N}^r \setminus \{0\}$ and choices of signs $\sigma_\mathbf{n} \in \{\pm 1\}$ for those $\mathbf{n}$ for which $a_\mathbf{n} \neq 0$, we define the twisted plethystic exponential

$$\mathrm{PExp}^\sigma \left(\sum_{\mathbf{n} \in \mathbb{N}^r \setminus \{0\}} \sigma_\mathbf{n} a_\mathbf{n} \mathbf{t^n} \right) = \prod_{\mathbf{n} \in \mathbb{N}^r \setminus \{0\}} (1 - \sigma_\mathbf{n} \mathbf{t^n})^{-\sigma_\mathbf{n} a_\mathbf{n}}.$$

Thus, according to this definition, for a single variable t,

$$\mathrm{PExp}^\sigma (t) = (1 - t)^{-1}$$

but, unusually,

$$\mathrm{PExp}^\sigma (-t) = 1 + t.$$

In fact it is immediate to check that, when expanded around $\mathbf{t} = 0$,

$$\mathrm{PExp}^\sigma \left(\sum_{\mathbf{n} \in \mathbb{N}^r \setminus \{0\}} \sigma_\mathbf{n} a_\mathbf{n} \mathbf{t^n} \right) \in \mathbb{N}[[\mathbf{t}]].$$

This is natural in the context of combinatorial generating functions, but of course destroys all algebraic properties of the ordinary plethystic exponential. With the correct choice of signs, all would be well. Notice in passing that (always expanding around $t = 0$)

$$E(t) = \mathrm{PExp}^\sigma \left(\frac{t}{1 - t} \right)$$

and

$$M(t) = \mathrm{PExp}^\sigma \left(\frac{t}{(1 - t)^2} \right).$$

Some of the statements in the case $d = 3$ are most conveniently expressed in terms of the functions

$$M(s, t) = \prod_{m > 0} (1 - st^m)^{-m} = \mathrm{PExp}^\sigma \left(\frac{st}{(1 - t)^2} \right)$$

and

$$\widetilde{M}(s, t) = M(s, t) \cdot M(s^{-1}, t) = \mathrm{PExp}^\sigma \left(\frac{(s + s^{-1}) t}{(1 - t)^2} \right);$$
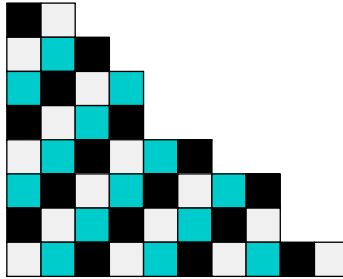
Fig. 2. The partition from Figure 1 coloured by the group $\mu_3(1, -1) < SL_2$.

note

$$\widetilde{M}(-s, t)^{-1} = \mathrm{PExp}^\sigma \left(\frac{-(s + s^{-1})\, t}{(1 - t)^2} \right).$$

Given a finite group G with r conjugacy classes, let $\mathrm{Irrep}(G) = \{\rho_0 = \mathrm{triv}, \rho_1, \ldots, \rho_{r-1}\}$ be the set of irreducible representations of G. For G cyclic of order r, we label so that $\rho_k = \rho_1^{\otimes k}$; for other groups, we will explain the labelling explicitly.

2. *Partitions*

2·1. *The abelian case: r-coloured partitions with diagonal colouring*

As is well known, an abelian subgroup $G < SL_2(\mathbb{C})$ is necessarily cyclic, of some order $r \geq 1$. For concreteness, we fix an isomorphism $G \cong \mu_r$ with the group of rth roots of unity in $\mathbb{C}$, diagonally embedded in $SL_2(\mathbb{C})$ in such a way that the action is given by

$$\xi \longmapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}.$$

We label representations of G so that $\mathbb{C}^2 \cong \rho_1 \oplus \rho_{r-1}$ as G-representations. We will call this action $\mu_r(1, r - 1)$ to record the weights. The G-action extends to an action on the coordinate ring $\mathbb{C}[x, y]$ that we can identify with the semigroup ring $\mathbb{C}[\mathbb{N}^2]$. Each monomial $x^i y^j \in \mathbb{C}[\mathbb{N}^2]$ spans a one-dimensional G-eigenspace in the representation $\rho_{i-j \bmod r}$; this gives a diagonal colouring (labelling) of the monomials in $\mathbb{C}[\mathbb{N}^2]$ with characters of G; see Figure 2.

Given a partition $\lambda \subset \mathbb{N}^2$, we can define an r-tuple of non-negative integers

$$\mathbf{w}(\lambda) = (w_0(\lambda), \ldots, w_{r-1}(\lambda))$$

counting the number of boxes (monomials) in λ of colour j for each j. Clearly

$$|\lambda| = \sum_{j=0}^{r-1} w_j(\lambda).$$

One can then form the generating function

$$Z^{\mu_r(1,r-1)}(\mathbf{t}) = \sum_{\lambda \in \mathcal{P}} \mathbf{t}^{\mathbf{w}(\lambda)}$$

for a set of variables $\mathbf{t} = (t_0, \ldots, t_{r-1})$. Specialising to $t_i = t$ recovers $E(t)$. The following reasonably standard result gives a closed formula for this generating function.

THEOREM 2·1. *Setting* $t = \prod_{j=0}^{r-1} t_j$ *to be the product of the variables, there is a factorisation*

$$Z^{\mu_r(1,r-1)}(\mathbf{t}) = E(t)^r \cdot Z_{\mathrm{red}}^{\mu_r(1,r-1)}(\mathbf{t}) \tag{2·1}$$

into a power of the function $E(t)$ *and a* reduced series $Z_{\mathrm{red}}^{\mu_r(1,r-1)} \in \mathbb{N}[[\mathbf{t}]]$. *This latter series is itself a generating function*

$$Z_{\mathrm{red}}^{\mu_r(1,r-1)}(\mathbf{t}) = \sum_{\lambda \in \mathcal{P}^{(r)}} \mathbf{t}^{\mathbf{w}(\lambda)}, \tag{2·2}$$

where $\mathcal{P}^{(r)} \subset \mathcal{P}$ *is the set of* r-*core partitions* [20]. *Moreover, it admits the following expression*:

$$Z_{\mathrm{red}}^{\mu_r(1,r-1)}(\mathbf{t}) = \sum_{\mathbf{m} \in \Lambda_{A_{r-1}}} x^{\frac{\langle \mathbf{m}, \mathbf{m} \rangle}{2}} \prod_{i=1}^{r-1} t_i^{m_i},$$

where $\Lambda_{A_{r-1}} \cong (\mathbb{Z}^{r-1}, \langle \, \rangle)$ *is the type* A_{r-1} *root (Cartan) lattice, the lattice corresponding to the Dynkin diagram of type* A_{r-1}.

Proof. We recall the outline of the argument. The process of taking the r-quotient of a partition [20] gives a combinatorial bijection

$$\mathcal{P} \longleftrightarrow \mathcal{P}^r \times \mathcal{P}^{(r)}$$

between the set of partitions, and r-tuples of partitions together with an r-core. This gives the factorisation formula (2·1), together with the interpretation (2·2). The reduced partition function can then be determined with the help of a further bijection

$$\mathcal{P}^{(r)} \longleftrightarrow \Lambda_{A_{r-1}}.$$

For details, see for example [17].

Example 2·2. For $r = 2$, we are colouring partitions by the checkerboard colouring. The 2-cores are the staircase partitions. We get the generating function of checkerboard-coloured partitions

$$Z^{\mu_2(1,1)}(t_0, t_1) = E(t_0 t_1)^2 \cdot \sum_{m=-\infty}^{\infty} t_0^{m^2} t_1^{m^2+m}.$$

The sum on the right-hand side is essentially the sum side of the Jacobi triple product identity, and so it also admits an infinite product form.

2·2. *Geometry and Lie theory*

Given a finite subgroup $G < \mathrm{SL}_2(\mathbb{C})$, the corresponding action on $\mathbb{C}^2$ preserves the two-form $dx \wedge dy$. The quotient space $X = \mathbb{C}^2/G$ has a rational (du Val, simple) surface singularity at the origin, and is smooth elsewhere. It admits a unique Calabi–Yau resolution $\pi : Y \to X$. The exceptional curves $E_i \subset Y$ of π are all rational, and form an intersection

configuration that is described by a simply-laced, connected Dynkin diagram, necessarily of ADE type. For a cyclic $G < \mathrm{SL}_2(\mathbb{C})$ of size r, we get the type A_{r-1} configuration, giving a superficial reason for the appearance of the corresponding root lattice above.

The 2-dimensional McKay correspondence [**31**] gives a different appearance of the Dynkin diagram. As we recall in the next section, McKay defined, using the embedding $G < \mathrm{SL}_2(\mathbb{C})$, a graph structure on the set $\mathrm{Irrep}(G) = \{\rho_0 = \mathrm{triv}, \rho_1, \ldots, \rho_{r-1}\}$. This turns out to give an *affine* Dynkin diagram, of the same type, with distinguished node ρ_0 whose removal returns the finite Dynkin diagram.

The appearance of the finite and affine Dynkin diagrams indicates a possible role for Lie algebras in this picture. The subjects are indeed intimately connected; this is a very large area, so we only mention the issue relevant for the present discussion. For type A, the set of all r-coloured partitions gives one combinatorial model of the so-called *crystal basis* [**21**] of the basic representation of the affine Lie algebra $\widehat{\mathfrak{gl}}_r$, an infinite-dimensional Lie algebra closely related to the affine root system $\hat{A}_{r-1}$. From a representation-theoretic point of view, the formulas (2·1)-(2·2) compute the character of the basic representation of this Lie algebra, compatibly with the Frenkel–Kac construction. For details from the present point of view, see [**18**, section 4 and appendix A]. For the special case $r = 1$, $\widehat{\mathfrak{gl}}_1$ is just the infinite dimensional Heisenberg algebra, acting on its standard (fermionic) Fock space representation.

2·3. *Partitions and Euler characteristics of Hilbert schemes*

In this subsection we again focus on the cyclic/type A case, with $G \cong \mu_r$ acting with weights $(1, -1)$ as above. Denote by

$$\mathrm{Hilb}(\mathbb{C}^2) = \sqcup_n \mathrm{Hilb}^n(\mathbb{C}^2)$$

the Hilbert scheme of points of $\mathbb{C}^2$, whose $\mathbb{C}$-points consist of finite-codimension ideals of $H^0(\mathcal{O}_{\mathbb{C}^2}) \cong \mathbb{C}[x, y]$. The G-fixed locus $\mathrm{Hilb}(\mathbb{C}^2)^G \subset \mathrm{Hilb}(\mathbb{C}^2)$ can also be thought of as the Hilbert scheme $\mathrm{Hilb}([\mathbb{C}^2/G])$ attached to the orbifold $[\mathbb{C}^2/G]$, as any invariant ideal is G-equivariant in a canonical way. This fixed locus naturally decomposes as

$$\mathrm{Hilb}([\mathbb{C}^2/G]) = \bigsqcup_{\rho \in \mathrm{Rep(G)}} \mathrm{Hilb}^\rho([\mathbb{C}^2/G]),$$

where for a finite-dimensional representation $\rho \in \mathrm{Rep(G)}$, we denote

$$\mathrm{Hilb}^\rho([\mathbb{C}^2/G]) = \{I \in \mathrm{Hilb}(\mathbb{C}^2)^G : (\mathbb{C}[x, y]/I) \simeq_G \rho\}$$

the ρ-Hilbert scheme of the orbifold $[\mathbb{C}^2/G]$, with $\simeq_G$ denoting G-equivariant isomorphism. Each of the spaces in the decomposition is smooth; this is a consequence of the smoothness of $\mathrm{Hilb}(\mathbb{C}^2)$.

Given a point in $\mathrm{Hilb}([\mathbb{C}^2/G])$, the associated $\mathbb{C}[x, y]$-module $\mathcal{F} = (\mathbb{C}[x, y]/I)$ decomposes canonically into a direct sum of weight spaces $\mathcal{F} = \oplus_{0 \leq i \leq r-1} \mathcal{F}_i$, where $\mathcal{F}_i$ is a direct sum of copies of the irreducible representation ρ_i. The action of x on $\mathcal{F}$ cyclically permutes these summands, sending $\mathcal{F}_i$ to $\mathcal{F}_{i+1}$ (where i is taken modulo r) and the action of y cyclically permutes these summands in the opposite direction. A G-equivariant $\mathbb{C}[x, y]$-representation, then, corresponds to a representation of a quiver with vertices indexed by the irreducible G-representations, and arrows in both directions between i and $i + 1$ (taken modulo r). We label the arrows starting at i corresponding to the action of x and y with

labels x_i and y_i respectively. The resulting oriented graph is the *McKay quiver* Q for G, and is the double of the McKay graph mentioned above. This quiver can be defined outside of type A, and for arbitrary G-representations: the number of arrows from i to j is defined to be the number of copies of ρ_j inside $V \otimes \rho_i$.

Coming back to our example of $G < \mathrm{SL}_2(\mathbb{C})$, a representation of this quiver corresponds to a G-equivariant representation of $\mathbb{C}[x, y]$ precisely if the commutativity relations $y_{i+1}x_i = x_{i-1}y_i$ hold. The quotient of the free path algebra of Q by these relations is known as the *preprojective algebra*.

Let T be the subgroup of diagonal matrices in $\mathrm{GL}_2(\mathbb{C})$. Then T acts on $\mathrm{Hilb}(\mathbb{C}^2)$ and this action moreover commutes with the G-action, and hence induces an action on each component $\mathrm{Hilb}^\rho([\mathbb{C}^2/G])$; this is where we are making essential use of the assumption that we are in type A. The fixed points of this action correspond to finite-codimension $\mathbb{Z}^2$-graded ideals $\mathcal{I} \subset \mathbb{C}[x, y]$. The monomial basis of $\mathbb{C}[x, y]/\mathcal{I}$ then defines a coloured partition as in Section 2·1. Summing up, for $G = \mu_r(1, r-1)$, there is an equality

$$Z^G(t_0, \ldots, t_{r-1}) = \sum_{\mathbf{d} \in \mathbb{N}^r} \chi(\mathrm{Hilb}^{\rho_{\mathbf{d}}}([\mathbb{C}^2/G]))\mathbf{t}^{\mathbf{d}},$$

where $\rho_{\mathbf{d}} = \bigoplus_{0 \leq i \leq r-1} \rho_i^{\oplus d_i}$, connecting the geometry of Hilbert schemes with coloured partition counting.

This point of view also allows a deformation (or quantisation) of this formula: we can replace Euler characteristics here by the Poincaré polynomials, arriving at the generating functions

$$Z^G(t_0, \ldots, t_{r-1}; q^{1/2}) = \sum_{\mathbf{d} \in \mathbb{N}^r} P(\mathrm{Hilb}^{\rho_{\mathbf{d}}}([\mathbb{C}^2/G]), q^{1/2})\mathbf{t}^{\mathbf{d}},$$

where for a smooth variety X, we denote $P(X, q^{1/2}) = \sum_j \dim_{\mathbb{Q}} H^j(X, \mathbb{Q})q^{j/2}$ its (topological) Poincaré polynomial. In the abelian case, this function is computed by [15] to be

$$Z^{\mu_r(1,r-1)}(\mathbf{t}; q^{1/2}) = \left(\prod_{m>0} (1 - t^m q^{m-1})^{-1}(1 - t^m q^m)^{-(r-1)} \right) \cdot Z^{\mu_r(1,r-1)}_{\mathrm{red}}(\mathbf{t}). \tag{2·3}$$

We will expand upon the relationship between combinatorics, quiver representations and quantised formulae further when we come to plane partitions.

2·4. *A nonabelian family of examples*

Armed with a full understanding of the cyclic case, we can ask whether there are similar stories to be told for *nonabelian* subgroups $G < \mathrm{SL}_2(\mathbb{C})$. Apart from cyclic groups, there is one other infinite set of such subgroups, as well as three sporadic ones. For the infinite series of binary dihedral groups, there is a precise analogue of the diagonally coloured partition story. While from the present point of view this is just an amusing extension of well-known ideas, we could not resist telling it. For geometric applications of the theory in this section, see [18].

The binary dihedral group $G = BD_r < \mathrm{SL}_2(\mathbb{C})$ is generated by the matrices $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, with ξ a $(2r-4)$th root of unity. Another notation for this group would be

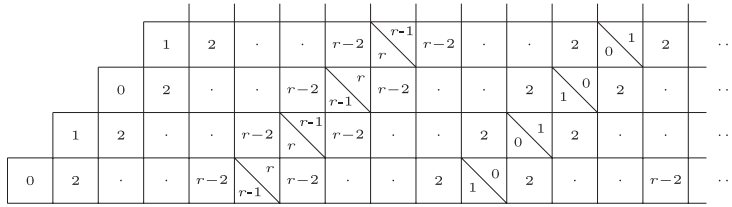| 1 | 2 | . | . | $r-2$ | $r\text{-}1 / r$ | $r-2$ | . | . | 2 | $1/0$ | 2 | $\cdots$ |
0	2	.	.	$r-2$	$r / r\text{-}1$	$r-2$	.	.	2	$0/1$	2	.
1	2	.	.	$r-2$	$r\text{-}1 / r$	$r-2$	.	.	2	$1/0$	2	.
0	2	.	.	$r-2$	$r / r\text{-}1$	$r-2$	.	.	2	$0/1$	2	.	.	$r-2$	$\cdots$

Fig. 3. The D_r-coloured positive octant; single dots in boxes denote a run of numbers between 2 and $r-2$.

BD_{4r-8}, but we will stick with this more economical notation. This group has order $4r-8$, and $r+1$ conjugary classes. In the correspondences recalled above, it gives the (finite and affine) Dynkin diagrams of type D_r. It is natural to label the representations of BD_r so that $\rho_0 = \mathrm{triv}$, ρ_1 is the sign representation, whose kernel is the abelian subgroup generated by the first matrix above, ρ_2 is the irreducible representation coming from the embedding $BD_r < \mathrm{SL}_2(\mathbb{C})$, $\rho_3, \ldots, \rho_{r-2}$ are further 2-dimensional representations (for $r>4$) labelled so that $\rho_j \otimes \rho_2 \cong \rho_{j-1} \oplus \rho_{j+1}$ for $j < r-2$, and finally ρ_{r-1} and ρ_r are again one-dimensional, with $\rho_{r-2} \otimes \rho_2 \cong \rho_{r-3} \oplus \rho_{r-1} \oplus \rho_r$.

A search for a corresponding partition colouring story begins by recalling that the connection between groups and colourings in the abelian case was given by the semigroup ring $\mathbb{C}[\mathbb{N}^2]$ and its decomposition into irreducible μ_r-spaces spanned by monomials. This does not quite work in the dihedral case, but it almost does. It can be checked that for most pairs (i, j), the two-dimensional vector space generated by $x^i y^j$ and $x^j y^i$ forms an irreducible representation of BD_r; for special pairs (i, j), it splits up into two eigenspaces (not generated by monomials). We get Figure 3, first exploited in this language in [18]. The diagram encodes a full description of $\mathbb{C}[\mathbb{N}^2]$ and its decomposition into irreducible BD_r-spaces. There are $r+1$ different colours (labels), some labelling boxes and some half-boxes. Since we wish to study configurations that are equivariant with respect to the dihedral group, it is indeed enough to look at the octant of the plane represented on the figure, and then the full configuration is determined by reflection. Let us call this combinatorial arrangement the D_r-coloured positive octant.

The correct generalisation of partitions can be derived from both the geometric and the representation theoretic points of view. One is led to a combinatorial set $\mathcal{P}_{D_r}$ of D_r-partitions, finite subsets of the D_r-coloured positive octant. The precise definition [22, 18] is as follows: a finite subset β of the D_r-coloured positive octant will be called a D_r-partition, if the following rules are satisfied:

(i)   if a full box is not present in β, then all full or half boxes to the right of it are also not in β, and at least one (full or half) box immediately above it is not in β;
(ii) if a full or half-box is missing from β, then all the boxes on the diagonal in the same position, above and to the right of this box, are also missing;
(iii) if a half box is not in β, then the full box to the right of it is also not in β;
(iv) if both half-boxes sharing the same box position are missing from β, then the full box immediately above this position is also not in β.

Thus the rules that define these objects are "partition-like" at the full boxes; indeed, (i) and (ii) (which in the case of ordinary partitions is superfluous) give the standard definition
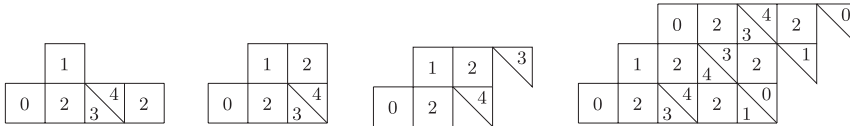
Fig. 4. Some D_4-partitions.

of ordinary partitions. However, the half-boxes have their own, different rules, leading to interesting complications in the theory. Figure 4 shows some D_r-partitions for $r = 4$.

Be that as it may, to a D_r-partition $\beta \in \mathcal{P}_{D_r}$ we can associate a set of $r + 1$ nonnegative integers

$$\mathbf{w}(\beta) = (w_0(\beta), \ldots, w_r(\beta))$$

counting the numbers of (full or half) boxes in β of different colours (labels). For a set of variables $\mathbf{t} = (t_0, \ldots, t_r)$, we are lead to the generating function

$$Z^{BD_r}(\mathbf{t}) = \sum_{\beta \in \mathcal{P}_{D_r}} \mathbf{t}^{\mathbf{w}(\beta)}.$$

The following result is the precise analogue of Theorem 2·1 above.

THEOREM 2·3. *Set* $t = t_0 t_1 t_{r-1} t_r \prod_{j=2}^{r-2} t_j^2$. *Then there is a factorisation*

$$Z^{BD_r}(\mathbf{t}) = E(t)^{r+1} \cdot Z_{\text{red}}^{BD_r}(\mathbf{t}) \tag{2·4}$$

into a power of the function $E(t)$ *and a reduced series* $Z_{\text{red}}^{BD_r} \in \mathbb{N}[[\mathbf{t}]]$. *This latter series is itself a generating function*

$$Z_{\text{red}}^{BD_r}(\mathbf{t}) = \sum_{\lambda \in \mathcal{P}_{D_r}^{(r)}} \mathbf{t}^{\mathbf{w}(\lambda)}, \tag{2·5}$$

where $\mathcal{P}_{D_r}^{(r)} \subset \mathcal{P}$ *is a certain set of* core D_r-*partitions. Moreover, this function admits the following expression:*

$$Z_{\text{red}}^{BD_r}(\mathbf{t}) = \sum_{\mathbf{m} \in \Lambda_{D_r}} t^{\frac{\langle \mathbf{m}, \mathbf{m} \rangle}{2}} \prod_{i=1}^{r} t_i^{m_i},$$

where $\Lambda_{D_r} \cong (\mathbb{Z}^r, \langle \ \rangle)$ *is the type* D_r *root (Cartan) lattice, the lattice corresponding to the Dynkin diagram of type* D_r.

Proof. The proof works along the same lines as the proof of Theorem 2·1; details are spelled out in [**18**].

We are not aware of a proof of the obvious generalisation of formula (2·3) for Poincaré polynomials in the type D case.

Remark 2·4. As already hinted above, the set of all D_r-partitions gives one combinatorial model of the crystal basis of the basic representation of an infinite-dimensional Lie algebra closely related to the affine root system $\widehat{D}_r$; see [**22**].
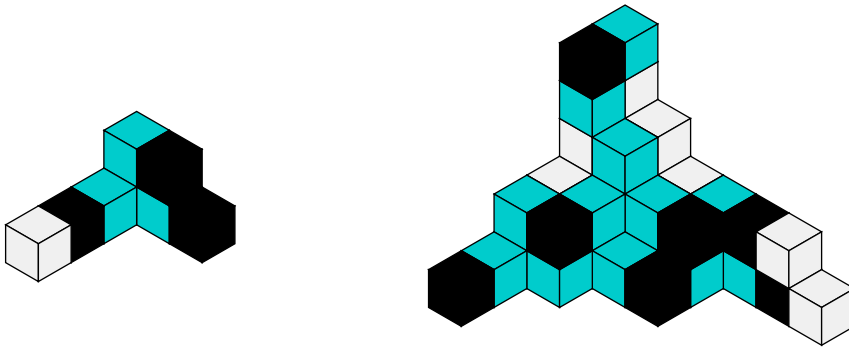
Fig. 5. Plane partitions α_1, α_2 coloured by $\mu_3(1, 1, 1) < \mathrm{SL}_3$. The weights are $\mathbf{w}(\alpha_1) = (2, 3, 3)$, respectively $\mathbf{w}(\alpha_2) = (13, 14, 14)$.

For subgroups of $\mathrm{SL}_2(\mathbb{C})$ of types E_6, E_7, E_8, in other words the symmetry groups of the tetrahedron, cube and dodecahedron, we are not aware of such a construction of the crystal basis. Perhaps the corresponding groups are too "three-dimensional".

2·5. *Groups outside* SL

For any abelian group $G < \mathrm{GL}_2(\mathbb{C})$, the colouring of partitions by periodic colours of course makes sense. However, we are not aware of any results computing the resulting partition function in full if G contains elements of determinant different from 1. See [3] for some nontrivial results.

3. *Plane partitions: combinatorics*

3·1. *The setup and the main conjecture*

Now let G be a finite abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$ of size r, that we will assume again to be diagonally embedded. As before, let $\mathrm{Irrep}(G) = \{\rho_0 = \mathrm{triv}, \dots, \rho_{r-1}\}$ denote the irreducible representations of G. The embedding $G < \mathrm{SL}_3(\mathbb{C})$ corresponds to a three-dimensional representation of G that can be decomposed as $\mathbb{C}^3 \cong \rho_a \oplus \rho_b \oplus \rho_c$; if needed, we will denote this group $G(a, b, c) < \mathrm{SL}_3$.

As before, we can colour (label) each box of a plane partition $\alpha \in \mathcal{Q}$ by one of r labels as follows. Each box of coordinate (i, j, k) in α corresponds to a monomial $x^i y^j z^k \in \mathbb{C}[\mathbb{N}^3]$, and thus corresponds to a one-dimensional G-eigenspace with character $\rho_{\chi(i,j,k)} \in \mathrm{Irrep}(G)$ for some $\chi(i, j, k) \in \{0, \dots, r-1\}$; give this box colour $\chi(i, j, k)$. Let $w_i(\alpha)$ to be the number of boxes in α of colour i and set $\mathbf{w}(\alpha) = (w_0(\alpha), \dots, w_{r-1}(\alpha))$. See Figure 5 for some plane partitions coloured by $\mu_3(1, 1, 1) < \mathrm{SL}_3$.

Let

$$Z^{G(a,b,c)}(\mathbf{t}) = \sum_{\alpha \in \mathcal{Q}} \mathbf{t}^{\mathbf{w}(\alpha)} \tag{3·1}$$

be the generating function of $G(a, b, c)$-coloured plane partitions. Once again, setting $t_i = t$ for all i recovers the generating function $M(t)$ of all plane partitions.

Motivated on the one hand by the formal analogy with coloured partitions, and on the other by some geometric ideas to be discussed in the last part of the paper, we make the following conjecture about this generating function.

CONJECTURE 3·1. *Setting $t = \prod_{j=0}^{r-1} t_j$ to be the product of the variables, there exists a factorisation*

$$Z^{G(a,b,c)}(\mathbf{t}) = M(t)^r \cdot Z_{\text{red}}^{G(a,b,c)}(\mathbf{t}), \qquad (3\cdot2)$$

where the reduced series $Z_{\text{red}}^{G(a,b,c)}(\mathbf{t})$ has non-negative coefficients only.

The series $M(t)^r$ begins $1 + rt_0t_1 \cdots t_{r-1} + O(t^2)$. Thus our putative factorisation with positive quotient can only work if the coefficient of $t_0t_1 \cdots t_{r-1}$ in $Z^{G(a,b,c)}(\mathbf{t})$ is at least r. In fact, the following stronger statement is implicit in the literature.

PROPOSITION 3·2. *The coefficient of the $t_0t_1 \cdots t_{r-1}$ term in $Z^{G(a,b,c)}(\mathbf{t})$ is exactly r. In other words, independently of the choice of the group $G < \mathrm{SL}_3$, there are exactly r plane partitions with one box of colour i for each $i = 0, \ldots, r-1$.*

Proof. This is stated in [**30**, lemma 4·1], but, it seems to us, without proof. In the next section, we discuss a geometric argument for this statement. We are not aware of a direct combinatorial proof.

It is natural to speculate whether the reduced series $Z_{\text{red}}^{G(a,b,c)}(\mathbf{t})$ might be the generating function of some natural subset of the set $\mathcal{Q}$ of plane partitions, analogous to the second part of Theorems 2·1, 2·3. Unfortunately, as we discuss in one of the examples below in Section 3·4, this cannot be true, at least without breaking the natural symmetries of the problem.

In the following sections we discuss some instances in which Conjecture 3·1 is known to hold, and present some numerical evidence for it in other cases. Section 4 of the paper puts the conjecture in a geometric context, discusses a quantisation of it, and outlines a possible line of attack.

3·2. *The action $\mu_r(1, r-1, 0)$ and $\mathfrak{sl}_r$ symmetry*

Consider the case when $G < \mathrm{SL}_2(\mathbb{C}) < \mathrm{SL}_3(\mathbb{C})$ is abelian; thus $G \cong \mu_r$ acts with weights $(1, r-1, 0)$. The generating function $Z^{\mu_r(1,r-1,0)}(\mathbf{t})$ admits an explicit infinite product representation.

THEOREM 3·3 (Young [**32**]). *We have*

$$Z^{\mu_r(1,r-1,0)}(t_0 \ldots, t_{r-1}) = M(t)^r \prod_{0<a<b<r} \widetilde{M}\left(\prod_{a\le j\le b} t_j, t\right). \qquad (3\cdot3)$$

Proof. As discussed in [**32**], some of the standard proofs of (1·2) also prove this result. The key is to "slice" a plane partition into partition slices that are monochromatic in this colouring.

In particular, Conjecture 3·1 holds in this case. To bring out symmetries of this formula more explicitly, we can re-write it using the plethystic exponential as follows.

COROLLARY 3·4. *Let $\Sigma \subset \Lambda_{A_{r-1}} \cong \mathbb{Z}^{r-1} \subset \mathbb{Z}^r$ denote the root system of the Lie algebra of type A_{r-1}. Then*

$$Z^{\mu_r(1,r-1,0)}(\mathbf{t}) = \mathrm{PExp}^\sigma \left(\frac{t}{(1-t)^2} \left(r + \sum_{\alpha \in \Sigma} \mathbf{t}^\alpha \right) \right),$$

where $t = \prod_{j=0}^{r-1} t_j$ still, and in the last bracket, only the variables $t_1, \ldots, t_{r-1}$ appear.

This formula nicely confirms expectations of the theoretical physics literature, where this geometry is expected to lead to "enhanced A_{r-1} symmetry" [23]. The $\mathfrak{sl}_r$-symmetry of the last formula is apparent: it is essentially the (exponential of the) character of the adjoint representation.

3·3. *The case $G \cong \mu_2 \times \mu_2$ and $\mathfrak{sl}_2^3$-symmetry*

The main result of [32] treats the case $G = \mu_2 \times \mu_2 < \mathrm{SL}_3(\mathbb{C})$, embedded as the group of all diagonal matrices with determinant 1 that square to the identity. The group G has four one-dimensional irreducible representations $\mathrm{Irrep}(G) = \{\rho_0 = \mathrm{triv}, \rho_1, \rho_2, \rho_3\}$. The G-coloured partition function again admits an explicit infinite product representation.

THEOREM 3·5 (Young [32]). *With $t = \prod_{i=0}^3 t_i$ as before,*

$$Z^{\mu_2 \times \mu_2}(t_0, t_1, t_2, t_3) = M(t)^4 \frac{\widetilde{M}(t_1 t_2, t)\widetilde{M}(t_1 t_2, t)\widetilde{M}(t_1 t_2, t)}{\widetilde{M}(-t_1, t)\widetilde{M}(-t_2, t)\widetilde{M}(-t_3, t)\widetilde{M}(-t_1 t_2 t_3, t)}.$$

Proof. This is much harder; the proof in [32] is indirect, and uses combinatorial wall-crossing to arrive at a different problem, solved earlier, involving pyramid partitions.

This result confirms Conjecture 3·1 in this case. The right-hand side can again be written in a more symmetric form using the plethystic exponential. Indeed, a short calculation shows:

COROLLARY 3·6. *We have*

$$Z^{\mu_2 \times \mu_2}(\mathbf{t}) = \mathrm{PExp}^\sigma \left(\frac{t}{(1-t)^2} \left(2 + (t_1^{\frac{1}{2}} - t_1^{-\frac{1}{2}})(t_2^{\frac{1}{2}} - t_2^{-\frac{1}{2}})(t_3^{\frac{1}{2}} - t_3^{-\frac{1}{2}})((t_1 t_2 t_3)^{-\frac{1}{2}} - (t_1 t_2 t_3)^{\frac{1}{2}}) \right) \right).$$

This formula is compatible with expectations in the physics that expects $\mathfrak{sl}_2^3$-symmetry in the partition functions connected to this geometry [6]. Indeed, the first three terms in the last bracket give the character of the fundamental representation of $\mathfrak{sl}_2^3$. Introducing a new variable t_4 with $t_1 t_2 t_3 t_4 = 1$, we can re-write the last formula as

$$Z^{\mu_2 \times \mu_2}(\mathbf{t}) = \mathrm{PExp}^\sigma \left(\frac{t}{(1-t)^2} \left(2 + \prod_{j=1}^4 (t_j^{\frac{1}{2}} - t_j^{-\frac{1}{2}}) \right) \right),$$

suggesting that there is more symmetry around. It is not clear to us whether there is any significance to this remark.

3·4. *The action $\mu_3(1, 1, 1)$ and other cyclic cases*

The first case not covered by the results above is when $G \cong \mu_3$, acting with weights $(1, 1, 1)$ on $\mathbb{C}^3$. The generating function $Z^{\mu_3(1,1,1)}(\mathbf{t})$ does not appear to admit an infinite product expansion. We checked Conjecture 3·1 with the help of computer enumeration up to

degree 24; we include the resulting reduced series $Z_{\text{red}}^{\mu_3(1,1,1)}(\mathbf{t})$ in the Appendix. The support of this series, the rational cone in $\mathbf{N}^3$ spanned by (a_0, a_1, a_2) with the corresponding monomial $t_0^{a_0} t_1^{a_1} t_2^{a_2}$ having nonzero coefficient, appears to be the simplicial cone generated by $(1, 0, 0)$, $(1, 3, 0)$ and $(1, 3, 6)$. These monomials come with coefficient 1, and correspond to the first, second and third power of the maximal ideal of the origin respectively.

As we mentioned before, it would have been natural to hope that the reduced series $Z_{\text{red}}^{G(a,b,c)}(\mathbf{t})$ is the coloured generating function of some natural subset of the set $\mathcal{Q}$ of plane partitions. However, in the case $G = \mu_3(1, 1, 1)$, we show that there is no such subset chosen in a way that fully respects the symmetries of the problem. Let S_3 be the natural group of (rotation and reflection) symmetries of the positive octant $\mathbb{N}^3$. Note that the colouring employed in this case is S_3-invariant.

PROPOSITION 3·7. *For the action of the group μ_3 with weights $(1, 1, 1)$ on $\mathbb{C}^3$, there does not exist an S_3-invariant rule that selects a subset of $\mathcal{Q}$ whose coloured generating function agrees with the reduced series $Z_{\text{red}}^{\mu_3(1,1,1)}(\mathbf{t})$.*

Proof. It can be checked by computer enumeration that the number of plane partitions of multi-weight $(3, 3, 3)$ is 108. Furthermore, this set is partitioned into orbits of size 3 or 6 under the natural action of S_3: there are no plane partitions of this multi-weight with full S_3 or A_3-symmetry. However, the corresponding coefficient of the reduced series $Z_{\text{red}}^{\mu_3(1,1,1)}(\mathbf{t})$ is 44. This number is not divisible by 3, so there is no symmetric rule that would select 44 partitions among the 108 with this multi-weight.

We further checked Conjecture 3·1 to degree 24 by computer enumeration also for the cases $\mu_4(1, 1, 2)$, $\mu_5(1, 1, 3)$ and $\mu_6(1, 2, 3)$.

3·5. *The case $G \cong \mu_3 \times \mu_3$ and the elusive $\mathfrak{e}_6$-symmetry*

We discuss one last case; consider $G \cong \mu_3 \times \mu_3$ embedded in $\text{SL}_3(\mathbb{C})$. Numerical evidence is of limited value for this example, since Proposition 3·2 means that the non-trivial checks for Conjecture 3·1 happen in total degree 18 and above, and it is increasingly difficult to calculate values in this range; but the conjecture does pass the tests we were able to set it (up to degree 20). On the other hand, physics not only predicts the existence of a $\mathfrak{sl}_3^3$-symmetry in this example, in analogy with Section 3·3, but an enhancement of this symmetry to $\mathfrak{e}_6$; see [**6**, section 4·2]. Indeed, one of the resolutions of the quotient $\mathbb{C}^3/G$ is the total space of the anticanonical bundle on a specialised cubic surface, and thus symmetry under $\mathfrak{e}_6$ is expected. It would be interesting to see some remnant of this symmetry in the reduced orbifold series $Z^{\mu_3 \times \mu_3}(\mathbf{t})$, but we have not been able to do so.

4. *Plane partitions: geometry*

4·1. *The geometry of the quotient and its resolutions*

The action of $G < \text{SL}_3(\mathbb{C})$ on $\mathbb{C}^3$ preserves the three-form $dx \wedge dy \wedge dz$, and hence the quotient $X = \mathbb{C}^3/G$ has Gorenstein (Calabi–Yau) singularities; the singular set of X includes the origin, and can also contain curves of singularities. In the three-dimensional case, there is still a Calabi–Yau resolution, but it is not unique. There turns out to be a distinguished resolution Y_G, Nakamura's G-Hilbert scheme [**30**]; the resolution map is given by the Hilbert–Chow morphism $\pi_G \colon Y_G \to X = \mathbb{C}^3/G$. Other resolutions $\pi_i \colon Y_i \to X$ are

obtained by flops from Y_G. One key feature [2, theorem 1·10] of the McKay correspondence is the numerical equality of the topological Euler characteristic of one (hence all) Calabi–Yau resolution(s) of X, and the "orbifold Euler characteristic" of the equivariant geometry, which in the case of $G < \mathrm{SL}_3(\mathbb{C})$ acting on $\mathbb{C}^3$ is just the number of conjugacy classes of G, or for G abelian, simply its order r.

In simple cases, all fibres of the resolution $\pi_G \colon Y_G \to X$ are at most one-dimensional; the examples discussed in Sections 3·2-3·3 were of this type. In general however, the map π_G has a two-dimensional fibre over the (image of the) origin $0 \in X$, leading to increased complexity in the problem, as shown by the examples in Sections 3·4-3·5. Indeed, for the example $G \cong \mu_3$ acting with weights $(1, 1, 1)$ on $\mathbb{C}^3$, the quotient $X = \mathbb{C}^3/G$ admits the (unique) Calabi–Yau resolution $Y = Y_G = \mathcal{O}_{\mathbb{P}^2}(-3)$ with $\pi \colon Y \to X$ contracting the zero section. Thus the resolution of singularities π of X has the two dimensional fibre $\mathbb{P}^2$ over the origin (smooth and irreducible in this case).

The tautological bundle of $\mathbb{C}[x, y, z] \rtimes G$-modules on Y_G, considered as a G-equivariant sheaf on $Y_G \times \mathbb{C}[x, y, z]$, provides the Fourier–Mukai kernel of an equivalence of categories [8]

$$\Phi \colon D^b(\mathrm{Coh}(Y_G)) \longrightarrow D^b((\mathbb{C}[x, y, z] \rtimes G)\text{-mod}) \tag{4·1}$$

with an inverse that we will denote by Ψ.

4·2. *Plane partitions and Euler characteristics*

In complete analogy with the discussion of Section 2·3, the generating function $Z^{G(a,b,c)}(\mathbf{t})$ can be thought of as the generating function of topological Euler characteristics of spaces of G-invariant subschemes of $\mathbb{C}^3$, or equivalently sheaves on the orbifold $[\mathbb{C}^3/G]$. The equivariant Hilbert schemes $\mathrm{Hilb}^\rho([\mathbb{C}^3/G])$ for $\rho \in \mathrm{Rep}(G)$ are in general singular, but still have finitely many fixed points under the action of the group T of diagonal matrices in $\mathrm{GL}_3(\mathbb{C})$, parametrised by plane partitions (we are once again relying on the fact here that G is abelian so can be assumed to be diagonal). We thus get an equality

$$Z^{G(a,b,c)}(\mathbf{t}) = \sum_{\mathbf{d} \in \mathbb{N}^r} \chi\left(\mathrm{Hilb}^{\rho_\mathbf{d}}([\mathbb{C}^3/G])\right)\mathbf{t}^\mathbf{d}, \tag{4·2}$$

with $\rho_\mathbf{d} = \bigoplus_{0 \le i \le r-1} \rho_i^{\oplus d_i}$.

Nakamura's G-Hilbert scheme is by definition $Y_G = \mathrm{Hilb}^{\rho_{\mathrm{reg}}}([\mathbb{C}^3/G])$, where $\rho_{\mathrm{reg}} \in \mathrm{Rep}(G)$ is the regular representation of G. This particular Hilbert scheme is known to be nonsingular [8, 30] and indeed is a crepant resolution of the quotient $X = \mathbb{C}^3/G$, as noted already above. This fact then provides a proof of Proposition 3·2.

Proof of Proposition 3·2. By the previous discussion, remembering that G is abelian, the coefficient of $t = \prod_{i=0}^{r-1} t_i$ in $Z^{G(a,b,c)}(\mathbf{t})$ is equal to the Euler characteristic $\chi(Y_G)$ of Nakamura's G-Hilbert scheme. Again as commented above, this number is known to be equal to the "orbifold Euler characteristic" of $[\mathbb{C}^3/G]$ which is simply r since G is abelian [2].

In the three-dimensional case, alongside the series $Z^{G(a,b,c)}(\mathbf{t})$ defined in (3·1) and identified with a partition function of Euler characteristics in (4·2), it is also natural to consider a *signed* series

$$Z_{\text{signed}}^{G(a,b,c)}(\mathbf{t}) = \sum_{\alpha \in \mathcal{Q}} (-1)^{d_0 + (\mathbf{d}, \mathbf{d})} \mathbf{t}^{\mathbf{w}(\alpha)} \tag{4·3}$$

where the bilinear form in the exponent is defined by

$$(\mathbf{d}, \mathbf{e}) = \sum_{i \text{ a vertex of } Q} d_i e_i - \sum_{a: i \to j \text{ an arrow of } Q} d_i e_j$$

and the sign change will arise from taking a virtual Euler characteristic, ubiquitous in three-dimensional sheaf counting problems. In the abelian case, this is still a sum over torus-fixed points [5], so the generating function gives a signed count of planar partitions [32, A·2]. We will return to these signs below.

4·3. *Representations of the McKay quiver in dimension* 3

We can also approach coloured plane partitions via moduli spaces of representations of a McKay quiver, just as in dimension 2. We will denote by $Q = Q(a, b, c)$ the McKay quiver for $G(a, b, c)$, constructed as follows. The vertex set Q_0 of $Q = Q(a, b, c)$ is still identified with the set of irreducible representations $\rho_0, \ldots, \rho_{r-1}$ of G. For each $i \leq r - 1$ there is an arrow x_i going from ρ_i to ρ_{i+a}, where addition is modulo r, and similarly, arrows y_i, z_i from ρ_i to ρ_{i+b} and ρ_{i+c} respectively. As before, G-equivariant sheaves on $\mathbb{C}[x, y, z]$ are naturally isomorphic to representations of this quiver, satisfying the obvious quadratic (commutation) relations. A new feature in the 3-dimensional setting is that these relations can be packaged as the noncommutative derivatives [16] of a potential $W = W(a, b, c)$:

$$W(a, b, c) = \sum_{0 \leq i \leq r-1} (z_{i+a+b} y_{i+a} x_i - y_{i+a+c} z_{i+a} x_i). \tag{4·4}$$

The variables should be considered as (non-commuting) quiver arrows here, and subscripts should still be taken modulo r. Since $a + b + c = 0 \mod r$, the above expression is a sum of cyclic paths in Q. We write $\mathbb{C}(Q, W)$ for the quotient of the free path algebra by the noncommutative derivatives of W. Then there is an isomorphism

$$\mathbb{C}[x, y, z] \rtimes G \cong \mathbb{C}(Q, W)$$

which at the level of underlying vector spaces of representations, takes a $\mathbb{C}[x, y, z] \rtimes G$-module to its decomposition according to the characters of G.

It was observed by Ito and Nakajima [29, section 3], that Nakamura's G-Hilbert scheme Y_G makes an appearance as a moduli space of stable representations of the McKay quiver with potential (Q, W). Recall that a stability parameter [24] for a quiver Q is an element

$$\zeta \in \mathbb{Q}^{Q_0}.$$

For concreteness, for the rest of this paper we fix the stability parameter

$$\zeta_0 = (r - 1, -1, \ldots, -1) \in \mathbb{Q}^{Q_0} \tag{4·5}$$

on the McKay quiver Q. The slope of a Q-representation ρ with respect to a stability parameter ζ is defined by

$$\mu(\rho) = \frac{\dim(\rho) \cdot \zeta}{\dim(\rho)},$$

where $\underline{\dim}(\rho) \in \mathbb{N}^{Q_0}$ is the dimension vector of the representation, and $\dim(\rho) = \sum \underline{\dim}(\rho)_i$ is the ordinary dimension. A representation M is defined to be ζ-(semi)stable if all proper submodules have slope less than (or equal to) that of M. Then we have

PROPOSITION 4·1 (Ito and Nakajima). *Fix the dimension vector* $\mathbf{d}_0 = (1, \ldots, 1)$. *There exists an open neighbourhood* U *of stability parameters of* $\zeta_0 \in \mathbb{Q}^{Q_0}$, *such that the space* Y_G *can be identified with the fine moduli space of* ζ-*stable* $\mathbf{d}_0$-*dimensional* $\mathbb{C}(Q, W)$-*modules for any* $\zeta \in U$.

More generally, there is an isomorphism of schemes between $\text{Hilb}^{\rho_\mathbf{d}}([\mathbb{C}/G])$ and the moduli space $\text{ncHilb}^{\mathbf{d}}(Q, W)$ of pairs of a $\mathbf{d}$-dimensional $\mathbb{C}(Q, W)$-representation M with a generating vector $v \in M^G$, which in turn is realised as a subscheme of the smooth moduli scheme of all stable framed representations of the free path algebra $\mathbb{C}Q$ as the critical locus of the function $\text{Tr}(W)$. To study such representations, we will write Q' for the quiver obtained from the McKay quiver Q by adding one extra vertex, which we label ∞, and one arrow, from ∞ to ρ_0.

Let $T_0 \cong (\mathbb{C}^*)^2 \subset T \cong (\mathbb{C}^*)^3$ be the sub-torus of the torus T acting on $\mathbb{C}^3$ and fixing the three-form $dx \wedge dy \wedge dz$. The function $\text{Tr}(W)$ is T_0-invariant, and its fixed locus has isolated fixed points equal to the fixed points of the T-action on $\text{ncHilb}^{\mathbf{d}}(Q, W)$; the latter are precisely parametrised by coloured plane partitions. The results of [5] then tell us that the correct weighted Euler characteristic to consider is given by summing over these fixed points, with each fixed point p contributing $(-1)^{T_p Y}$, the parity of the Zariski tangent space at p, which is equal to $(-1)^{d_0 + (\mathbf{d}, \mathbf{d})}$ by [28, theorem 7·1]. In conclusion, there is an equality

$$Z_{\text{signed}}^{G(a,b,c)}(\mathbf{t}) = \sum_{\mathbf{d} \in \mathbb{N}^r} \chi_{\text{vir}}(\text{ncHilb}^{\mathbf{d}}(Q, W)) \mathbf{t}^{\mathbf{d}} \tag{4·6}$$

between the signed version of the coloured plane partition generating function, and the so-called Donaldson–Thomas (DT) generating function attached to the collection of spaces $\{\text{ncHilb}^{\mathbf{d}}(Q, W) : \mathbf{d} \in \mathbb{N}^r\}$.

Remark 4·2. In [26], Maulik et al. attach an analogous generating function to ordinary (non-orbifold) Calabi–Yau geometries Y with topological Euler characteristic $e(Y)$, and conjecture a factorisation into a term $M(-t)^{e(Y)}$, corresponding to point sheaves on Y, and a reduced series. In fact the conjecture in [26] is more general, but we only need the Calabi–Yau case. One way to think about Conjecture 3·1 is as an orbifold analogue of the conjecture of [26], where $r = e(Y) = e_{\text{orb}}(\mathbb{C}^3, G)$ is the (orbifold) Euler characteristic.

4·4. *Refinements from cohomological DT theory*

The link between DT theory and coloured plane partitions can be enriched by considering mixed Hodge structures on the cohomology of the moduli spaces $\text{ncHilb}^{\mathbf{d}}(Q, W)$ instead of their weighted Euler characteristics. The technical tools for refined Donaldson–Thomas theory come from Saito's theory of mixed Hodge modules.

Given a function f on a smooth variety Z, we write

$$\varphi_f := \varphi_f \underline{\mathbb{Q}}_Z \otimes \mathbb{L}^{-\dim(Z)/2}$$

for the normalised mixed Hodge module of vanishing cycles, where $\mathbb{L} = H_c(\mathbb{A}^1, \mathbb{Q})$ is a pure weight 2 Hodge structure in cohomological degree 2. Strictly speaking, to make sense

of this expression when the exponent is not an integer, we have to formally add a tensor square root $\mathbb{L}^{1/2}$ to the category of mixed Hodge modules — see [25] for details. In the general framework of cohomological Donaldson–Thomas theory, the cohomology of the above mixed Hodge module is supposed to be considered as a *monodromic* mixed Hodge structure; however in our case, since the quiver with potential admits a cut, in the sense of [19], the monodromy is trivial — see the appendix of [10] for details and proofs.

We package the mixed Hodge structures corresponding to all $\mathbf{d} \in \mathbb{N}^r$ together into a $\mathbb{N}^r$-graded mixed Hodge structure

$$\mathcal{H}(a, b, c) := \bigoplus_{\mathbf{d} \in \mathbb{N}^r} \mathrm{H}(\mathrm{ncHilb}^{\mathbf{d}}(Q), \varphi_{\mathrm{Tr}(W)}).$$

On the right-hand side we have taken the cohomology of the smooth ambient moduli scheme $\mathrm{ncHilb}^{\mathbf{d}}(Q)$; since the mixed Hodge module $\varphi_{\mathrm{Tr}(W)}$ is supported on $\mathrm{ncHilb}^{\mathbf{d}}(Q, W)$, the critical locus of $\mathrm{Tr}(W)$, this makes no difference. In terms of stability conditions, $\mathrm{ncHilb}^{\mathbf{d}}(Q)$ can be realised as the (fine) moduli space of ζ'-stable $(\mathbf{d}, 1)$-dimensional Q'-representations, for $\zeta' = (-1, \ldots, -1, |\mathbf{d}|)$.

For later use, we generalise this construction as in [14, section 3] to take account of nontrivial stability conditions on Q. Let $\zeta \in \mathbb{N}_{Q_0}$ be such a stability condition. Fixing a dimension vector $\mathbf{d}$ and $M \gg 0$ a large integer, we may assume that $\zeta \cdot \mathbf{d} = -1/M$ by adding a constant dimension vector to ζ, leaving the notion of ζ-(semi)stability unchanged. Then we set $\zeta' = (M\zeta, 1) \in \mathbb{Q}^{Q_0}$. It is easy to check that a $(\mathbf{d}, 1)$-dimensional Q'-representation is stable if it is semistable, and it is semistable precisely if the underlying Q-representation ρ is ζ-semistable and there is no proper subrepresentation $\rho' \subset \rho$ containing the image of the one-dimensional vector space at the vertex ∞, and such that $\mu(\rho') = \mu(\rho)$. We denote the resulting fine moduli space $\mathrm{ncHilb}^{\zeta, \mathbf{d}}(Q)$. For a stability parameter ζ, slope function μ and fixed slope $\theta \in (-\infty, \infty)$ we define,

$$\mathcal{H}(a, b, c)_\theta := \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0} | \mu(\mathbf{d}) = \theta} \mathrm{H}(\mathrm{ncHilb}^{\zeta, \mathbf{d}}(Q), \varphi_{\mathrm{Tr}(W)}).$$

Definition 4·3. Let $\mathcal{L}$ be a $\mathbb{N}^{Q_0}$-graded, cohomologically graded mixed Hodge structure, such that for each $\mathbf{d} \in \mathbb{N}^r$ there is an equality $\mathrm{Gr}_n^W \mathrm{H}(\mathcal{L}_{\mathbf{d}}) = 0$ for $n \gg 0$, and moreover for every n, $\mathrm{Gr}_n^W \mathrm{H}^i(\mathcal{L}_{\mathbf{d}}) = 0$ for all but finitely many i. We define the characteristic function of $\mathcal{L}$:

$$\chi_{Q,\mathrm{wt}}(\mathcal{L}, q^{1/2}, \mathbf{t}) = \sum_{n,i \in \mathbb{Z}} \sum_{\mathbf{d} \in \mathbb{N}^{Q_0}} (-1)^i \dim(\mathrm{Gr}_n^W \mathrm{H}^i(\mathcal{L}_{\mathbf{d}}))(q^{1/2})^n \mathbf{t}^{\mathbf{d}}, \qquad (4\cdot7)$$

as an element of the *quantum torus*

$$\mathcal{A}_Q = \mathbb{Z}((q^{-1/2}))[[t_i|_{i \in Q_0}]]$$

endowed with the usual additive structure, and the $\mathbb{Z}((q^{-1/2}))$-linear multiplication defined on t-variables by $\mathbf{t}^{\mathbf{d}} \cdot \mathbf{t}^{\mathbf{d}'} = q^{(\mathbf{d}, \mathbf{d}')/2} \mathbf{t}^{\mathbf{d}+\mathbf{d}'}$.

If the total cohomology of each of the graded pieces $\mathcal{L}_{\mathbf{d}}$ is finite-dimensional, then we define

$$\chi_Q(\mathcal{L}, \mathbf{t}) = \chi_{Q,\mathrm{wt}}(\mathcal{L}, q^{1/2} = 1, \mathbf{t}) \in \mathbb{Z}[[t_i|_{i \in Q_0}]].$$

The mixed Hodge structure $\mathcal{H}(a, b, c)$ satisfies the stronger finite-dimensionality condition of Definition 4·3, and by [4] we have that the Euler characteristic of its **d**th graded piece $\chi(\mathcal{H}(a, b, c)_{\mathbf{d}})$ is the virtual Euler characteristic of ncHilb$^{\mathbf{d}}(Q, W)$. Combining this with (4·6) we deduce that there is an equality

$$\chi_Q(\mathcal{H}(a, b, c), \mathbf{t}) = \sum_{\alpha \in \mathcal{Q}} (-1)^{d_0 + (\mathbf{d}, \mathbf{d})} \mathbf{t}^{\mathbf{w}(\alpha)} = Z_{\text{signed}}^{G(a,b,c)}(\mathbf{t}) \tag{4·8}$$

giving an interpretation of the signed plane partition function in terms of the representation theory of the McKay quiver with potential.

We also consider the mixed Hodge structure on the vanishing cycle cohomology

$$\mathcal{H}^{\text{stack}}(a, b, c) = \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathrm{H}_c\left(\mathfrak{M}_{\mathbf{d}}(Q), \varphi_{\text{Tr}(W)}\right)$$

of the stack $\mathfrak{M}_{\mathbf{d}}(Q)$ of *all* finite-dimensional $\mathbb{C}Q$-modules. The cohomological degree is unbounded below for every nonzero $\mathbb{N}^{Q_0}$-degree, and so only the q-refined characteristic function of Definition 4·3 is defined for this mixed Hodge structure. For a stability parameter $\zeta \in \mathbb{Q}^{Q_0}$ and slope $\theta \in (-\infty, \infty)$ we can also define

$$\mathcal{H}^{\text{stack}}(a, b, c)_{\theta} = \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0} | \mu(\mathbf{d}) = \theta} \mathrm{H}_c\left(\mathfrak{M}_{\mathbf{d}}^{\zeta - \text{sst}}(Q), \varphi_{\text{Tr}(W)}\right)$$

where $\mathfrak{M}_{\mathbf{d}}^{\zeta - \text{sst}}(Q)$ is the stack of ζ-semistable modules with dimension vector **d**, and μ is the slope function defined with respect to ζ.

All of these mixed Hodge structures are related to each other in an elegant way by wall crossing isomorphisms which are special cases of [13, theorem B]. The isomorphisms we will need are

$$\mathcal{H}^{\text{stack}}(a, b, c) \cong \overset{\text{tw}}{\underset{-\infty \overset{\theta}{\rightarrow} \infty}{\bigotimes}} \mathcal{H}^{\text{stack}}(a, b, c)_{\theta} \tag{4·9}$$

$$\mathcal{H}(a, b, c) \otimes^{\text{tw}} \mathcal{H}^{\text{stack}}(a, b, c) \cong \mathcal{H}^{\text{stack}}(a, b, c) \otimes^{\text{tw}} \mathbb{Q}_{1_{\infty}} \tag{4·10}$$

and

$$\mathcal{H}(a, b, c)_{\theta} \otimes^{\text{tw}} \mathcal{H}^{\text{stack}}(a, b, c)_{\theta} \cong \mathcal{H}^{\text{stack}}(a, b, c)_{\theta} \otimes^{\text{tw}} \mathbb{Q}_{1_{\infty}}. \tag{4·11}$$

The isomorphism (4·9) takes place in the category of $\mathbb{N}^r$-graded mixed Hodge structures, while (4·10) and (4·11) take place in the category of $\mathbb{N}^{r+1}$-graded mixed Hodge structures, i.e. the categories of mixed Hodge structures graded by dimension vectors for Q and Q' respectively. The category of $\mathbb{N}^r$-graded mixed Hodge structures is equipped with a twisted tensor product via the rule $\mathcal{N} \otimes^{\text{tw}} \mathcal{P} = \mathcal{N} \otimes \mathcal{P} \otimes \mathbb{L}^{\frac{(\mathbf{e}, \mathbf{d}) - (\mathbf{d}, \mathbf{e})}{2}}$ for $\mathcal{N}$ and $\mathcal{P}$ homogeneous of degree **d** and **e** respectively. We extend this to a tensor product for $\mathbb{N}^{r+1}$-graded mixed Hodge structures by using the Euler pairing for Q' instead of Q. Finally, the object $\mathbb{Q}_{1_{\infty}}$ above is the constant pure weight zero Hodge structure $\mathbb{Q}$, situated in $\mathbb{N}^{r+1}$-degree $(0, \ldots, 0, 1)$, so

$$\chi_{Q', \text{wt}}(\mathbb{Q}_{1_{\infty}}, q^{1/2}, \mathbf{t}) = t_{\infty}.$$

This twist is introduced so that the equation $\chi_Q(\mathcal{L} \otimes^{\mathrm{tw}} \mathcal{B}) = \chi_Q(\mathcal{L})\chi_Q(\mathcal{B})$ holds, for $\mathcal{L}, \mathcal{B}$ two $\mathbb{N}^r$-graded mixed Hodge structures, and the same statement holds for $\mathbb{N}^{r+1}$-graded mixed Hodge structures with χ_Q replaced by $\chi_{Q'}$.

Putting together (4·10) and (4·8) we deduce that

$$\chi_Q(\mathcal{H}(a, b, c), \mathbf{t}) = \left[\mathrm{Ad}_{\chi_{Q,\mathrm{wt}}(\mathcal{H}^{\mathrm{stack}}(a,b,c), q^{1/2}, \mathbf{t})}(t_\infty)\right]_{q^{1/2}=1=t_\infty} = Z_{\mathrm{signed}}^{G(a,b,c)}(\mathbf{t}),$$

where, for the left-hand side, we have given $\mathcal{H}(a, b, c)$ the $\mathbb{N}^r$ grading given by only remembering the dimension vector restricted to Q. Motivated by the above discussion, we make the following:

Definition 4·4. The q-deformed coloured plane partition function is

$$Z^{G(a,b,c)}(\mathbf{t}, q^{1/2}) = \chi_{Q,\mathrm{wt}}(\mathcal{H}(a, b, c), q^{1/2}, \mathbf{t}) \in \mathcal{A}_Q.$$

Setting $q^{1/2} = 1$ in $Z^{G(a,b,c)}(\mathbf{t}, q^{1/2})$ recovers the signed partition function $Z_{\mathrm{signed}}^{G(a,b,c)}(\mathbf{t})$.

The connection between these ideas and our Conjecture 3·1 comes from the following construction. Recall the dimension vector $\mathbf{d}_0 = (1, \ldots, 1) \in \mathbb{N}^{Q_0}$, stability parameter $\zeta_0 = (r - 1, -1, \ldots, -1) \in \mathbb{Q}^{Q_0}$, and the open neighbourhood $\zeta_0 \in U \subset \mathbb{Q}^{Q_0}$ from Proposition 4·1. Fixing a bound $N \in \mathbb{N}$, it is clearly possible to choose a small rational perturbation $\zeta_N \in U \subset \mathbb{Q}^{Q_0}$ such that a dimension vector $\mathbf{d}$ with all components smaller than N has fixed slope

$$\theta_0 = \mathbf{d}_0 \cdot \zeta_0 = 0$$

if and only if $\mathbf{d}$ is a rational multiple of $\mathbf{d}_0$. The mixed Hodge structure $\mathcal{H}(a, b, c)_{\theta_0}$ stabilises in each degree as we take the limit in the sequence of stability conditions $\zeta_0, \zeta_1, \ldots$ so the entire graded mixed Hodge structures limits to a "generic" mixed Hodge structure which we denote $\mathcal{P}(a, b, c)$. Likewise, we denote by $\mathcal{P}^{st}(a, b, c)$ the limit of the mixed Hodge structures $\mathcal{H}^{\mathrm{stack}}(a, b, c)_{\theta_0}$ as we vary the stability condition.

There is a natural inclusion of quantum tori $\mathcal{A}_Q \subset \mathcal{A}_{Q'}$. Note that setting as usual $t = \prod_{i=0}^{r-1} t_i = \mathbf{t}^1$, the ring $\mathbb{Z}((q^{-1/2}))[[t]]$ lies in the centre of $\mathcal{A}_Q$, but not the centre of $\mathcal{A}_{Q'}$. Indeed, if $\alpha \in t_\infty \cdot \mathcal{A}_Q$ then applying $\chi_{Q',\mathrm{wt}}$ to (4·11), with $\theta = \mu(\mathbf{1}) = \mathbf{0}$, we deduce

$$\mathrm{Ad}_{\chi_{Q,\mathrm{wt}}(\mathcal{P}^{st}(a,b,c))}(\alpha) = \alpha * \chi_{Q',\mathrm{wt}}(\mathcal{P}(a, b, c)) \tag{4·12}$$

and we have

PROPOSITION 4·5. *With $t = \prod_{i=0}^{r-1} t_i$ as before, there is an equality*

$$\chi_Q(\mathcal{P}(a, b, c), \mathbf{t}) = M(-t)^{\chi(Y_G)}. \tag{4·13}$$

Proof. Set $\theta = 0$. For the stability condition ζ_N constructed above, the geometric Proposition 4·13 to be discussed below implies that $\mathcal{H}(a, b, c)_\theta$ is the vanishing cycle cohomology of the Hilbert scheme of points of Y_G up to degree $(N, \ldots, N)$. Taking $N \to \infty$ and using the results of [5], the stated equality follows.

This proposition, along with (4·12), says that the operator $\mathrm{Ad}_{\chi_{Q,\mathrm{wt}}(\mathcal{P}^{st}(a,b,c))}$ is a q-refinement of multiplication by the right-hand side of (4·13), the term we factor out in the definition of the reduced partition function in (3·2). On the other hand, by (4·10) we can

write

$$Z_{\text{signed}}^{G(a,b,c)}(\mathbf{t}, q^{1/2}) = \left[\text{Ad}_{\chi_{Q,\text{wt}}(\mathcal{H}^{\text{stack}}(a,b,c),q^{1/2},\mathbf{t})}(t_\infty)\right]_{t_\infty=1}$$

and by centrality of $\chi_{Q,\text{wt}}(\mathcal{P}^{st}(a, b, c))$ in $\mathcal{A}_Q$ we can factorise

$$\text{Ad}_{\chi_{Q,\text{wt}}(\mathcal{H}^{\text{stack}}(a,b,c),q^{1/2},\mathbf{t})} = \text{Ad}_{\chi_{Q,\text{wt}}(\mathcal{P}^{st}(a,b,c),q^{1/2},\mathbf{t})} \circ \text{Ad}_{\chi_{Q,\text{wt}}(\mathcal{H}^{\text{stack}}(a,b,c)',q^{1/2},\mathbf{t})},$$

where

$$\mathcal{H}^{\text{stack}}(a, b, c)' = \overset{\text{tw}}{\underset{-\infty \overset{\theta \neq 0}{\longrightarrow} \infty}{\bigotimes}} \mathcal{H}^{\text{stack}}(a, b, c)_\theta.$$

The discussion above motivates the following definition.

Definition 4·6. We define the q-deformed reduced plane partition function

$$Z_{\text{red}}^{G(a,b,c)}(\mathbf{t}, q^{1/2}) = \chi_{Q,\text{wt}}(\mathcal{H}(a, b, c), q^{1/2}, \mathbf{t})/\chi_{Q,\text{wt}}(\mathcal{P}(a, b, c), q^{1/2}, \mathbf{t}). \qquad (4\cdot14)$$

We formulate the following, quantised version of Conjecture 3·1:

CONJECTURE 4·7. *The q-deformed reduced partition function* (4·14) *can be written*

$$Z_{\text{red}}^{G(a,b,c)}(\mathbf{t}, q^{1/2}) = \sum_{\mathbf{d} \in \mathbb{N}^r} h_{\mathbf{d}}(q)(-q^{1/2})^{d_0+(\mathbf{d},\mathbf{d})}\mathbf{t}^{\mathbf{d}},$$

where $h_{\mathbf{d}}(q) \in \mathbb{N}[q^{\pm 1}]$.

Setting $q^{1/2} = 1$ in Conjecture 4·7 recovers Conjecture 3·1.

Example 4·8. For the case $G = \mu_r(1, r - 1, 0)$ studied in Section 3·2, Conjecture 4·7 is also known to hold. In fact, [**27**, theorem 1·2] gives a somewhat involved but explicit formula for the quantised generating function, a quantum version of (3·3), from which Conjecture 4·7 can be derived easily.

4·5. *Purity and positivity*

As noted in Proposition 3·7, in general there is little hope of proving positivity of the reduced partition function by showing that the $\mathbf{d}$th coefficient counts a certain subset of weight $\mathbf{d}$ coloured partitions. Rather than proving positivity combinatorially, i.e. by showing that the coefficients count elements in a set, we may try to prove positivity via categorification, i.e. by showing that these numbers are the dimensions of vector spaces. In this section we establish the base camp for such an attempt, and prove Theorem 4·11, the positivity of the *unreduced q-refined partition function* for certain abelian $G(a, b, c) < \text{SL}_3(\mathbb{C})$, by exactly this kind of approach.

Our general approach to proving a conjecture like Conjecture 4·7 regarding a partition function $\mathcal{Z}(q^{1/2}, \mathbf{t}) \in \mathbb{Z}((q^{-1/2}))[[\mathbf{t}]]$ is as follows:

(i) show that $\mathcal{Z}(q^{1/2}, \mathbf{t}) = \chi_{\text{wt}}(\mathcal{H}, q^{1/2}, \mathbf{t})$ for $\mathcal{H}$ some $\mathbb{N}^r$-graded, cohomologically graded mixed Hodge structure (for example, arising in Donaldson–Thomas theory);

(ii) show that $\mathcal{H}$ is pure, in the sense that the ith cohomologically graded piece of $\mathcal{H}$ is pure of weight i;

(iii) show that each $\mathcal{H}_\mathbf{d}$ is concentrated entirely in even cohomological degree, or entirely in odd cohomological degree, of the predicted parity.

The point is that for pure mixed Hodge structures, the only terms that contribute to the sum in (4·7) have $n = i$.

We will use an extra geometric assumption to prove a purity statement. We do not expect it to be essential.

THEOREM 4·9. *Let $G < SL_3$ be abelian, and such that the exceptional locus of $\pi_G\colon Y_G \to \mathbb{C}^3/G$ is mapped to the origin. Then the mixed Hodge structure $\mathcal{H}_G^{stack}$ is pure, and $\mathcal{H}_G^{stack}$ is concentrated entirely in even or odd degree, depending on whether $(\mathbf{d}, \mathbf{d})$ is even or odd, respectively.*

We relegate the technical proof of this statement to the last section of our paper.

COROLLARY 4·10. *Under the same assumption as Theorem 4·9, the mixed Hodge structure on $\mathcal{H}(a, b, c)$ is pure, and $\mathcal{H}(a, b, c)_\mathbf{d}$ is concentrated entirely in even or odd degree, depending on whether $(\mathbf{d}, \mathbf{d}) + d_0$ is even or odd, respectively.*

Proof. From (4·10) there is an inclusion of $\mathbb{N}^{r+1}$-graded mixed Hodge structures

$$\mathcal{H}(a, b, c)_\mathbf{d} \subset \mathcal{H}^{stack}(a, b, c) \otimes^{tw} \mathbb{Q}_{1_\infty}.$$

The result follows from Theorem 4·9 and the definition of the twisted tensor product.

We deduce

THEOREM 4·11. *Let $G < SL_3$ be abelian, and assume that the exceptional locus of $\pi_G\colon Y_G \to \mathbb{C}^3/G$ is mapped to the origin. Then*

$$Z^{G(a,b,c)}(q^{1/2}, \mathbf{t}) = \sum_{\mathbf{d}\in\mathbb{N}^r} g_\mathbf{d}(q)(-q^{1/2})^{d_0+(\mathbf{d},\mathbf{d})}\mathbf{t}^\mathbf{d},$$

where $g_\mathbf{d} \in \mathbb{N}[q^{\pm 1}]$.

Remark 4·12. If $G = \mu_r(a, b, c) < SL_3$ is cyclic, then it is easy to see that the condition that the exceptional locus of $\pi_G\colon Y_G \to \mathbb{C}^3/G$ be mapped to the origin is equivalent to $(a, r) = (b, r) = (c, r) = 1$.

4·6. *Proof of Theorem 4·9*

In this section, the most technical in our paper, we give the proof of the purity result Theorem 4·9. We begin with a geometric statement, which we used in Section 4·4; the main idea of the proof is exactly the same as the proof of Nakamura's conjecture in [8], as kindly explained to us by Tom Bridgeland.

As before, let (Q, W) be the McKay quiver of $G(a, b, c) < SL(3, \mathbb{C})$ with its potential (4·4). Define $n\mathbf{d}_0 = (n, \ldots, n) \in \mathbb{N}^{Q_0}$, a dimension vector on Q, $\zeta_0 = (r - 1, -1, \ldots, -1) \in \mathbb{Q}^{Q_0}$ our fixed stability parameter on Q and $U \subset \mathbb{Q}^{Q_0}$ the small open neighbourhood around it in the space of stability parameters from Proposition 4·1.

PROPOSITION 4·13. *Let $\zeta \in U$ be generic. The equivalences of categories Φ of (4·1) and its inverse Ψ induce for every positive integer n an isomorphism of stacks*

$$f_n : \mathfrak{Tors}^n(Y_G) \cong \mathfrak{M}(Q, W)^{\zeta-\mathrm{sst}}_{n\mathbf{d}_0}$$

between the moduli stack of ζ-semistable representations of the QP (Q, W) of dimension vector $n\mathbf{d}_0$ and the stack $\mathfrak{Tors}^n(Y_G)$ of torsion sheaves on Y_G of length n. If, further, the exceptional locus of $\pi_G : Y_G \to \mathbb{C}^3/G$ maps to the origin, then f_n restricts to an isomorphism of groupoids

$$f_{n,\mathrm{nilp}} : (\mathfrak{Tors}^n(Y_G)(\mathbb{C}))_{\mathrm{ex}} \cong (\mathfrak{M}(Q, W)^{\zeta-\mathrm{sst}}_{n\mathbf{d}_0}(\mathbb{C}))_{\mathrm{nilp}},$$

where on the right we restrict to nilpotent $\mathbb{C}Q$-modules, and on the left we restrict to sheaves set-theoretically supported on the exceptional locus of $\pi_G : Y_G \to \mathbb{C}^3/G$.

Proof. Since the equivalence Φ is derived from a Fourier–Mukai kernel $\mathcal{Q}$, there is a canonical way to try to upgrade it to a morphism of stacks — we take the Fourier–Mukai kernel $\mathcal{Q} \boxtimes \mathcal{O}_S$ to define a functor from S-flat families of coherent sheaves on $S \times Y_G$ to complexes of $\mathcal{O}_S \otimes \mathbb{C}(Q, W)$-modules on S. We need to show that the complexes we obtain this way are supported in cohomological degree zero and are moreover S-flat families of ζ-semistable modules. By [7, lemma 4·3] this statement follows from the claim that this construction gives an isomorphism of groupoids of K-points for any $K \supset \mathbb{C}$, which furthermore proves that the morphism of stacks is an isomorphism.

For simplicity we assume that $K = \mathbb{C}$, as for a general field the proofs are unchanged. For $n = 1$, we have $\mathfrak{Tors}^n(Y_G) \cong Y_G$, so the result follows from Proposition 4·1. We prove the general case by induction on n. First take $\mathcal{F} \in \mathfrak{Tors}^n(Y_G)$. Any such sheaf is a finite iterated extension of point sheaves $\mathcal{O}_{y_i}$ for (not necessarily distinct) points $y_i \in Y$, and in particular for some $y \in Y_G$ there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_y \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

The sheaf $\mathcal{F}'$ is then a torsion sheaf of length $n - 1$, and we get a distinguished triangle

$$\Phi(\mathcal{O}_y) \longrightarrow \Phi(\mathcal{F}) \longrightarrow \Phi(\mathcal{F}') \xrightarrow{[1]}$$

in $\mathcal{D}(\mathbb{C}(Q, W)\text{-mod})$. By induction, $\Phi(\mathcal{F}')$ and $\Phi(\mathcal{O}_y)$ are ζ-semistable representations of (Q, W), of dimension vectors $\mathbf{n} - \mathbf{1}$ and $\mathbf{1}$ respectively. Hence $\Phi(\mathcal{F})$ is also an actual representation, and also ζ-semistable. This gives a morphism

$$g_n : \mathfrak{Tors}^n(Y)(\mathbb{C}) \longrightarrow \mathfrak{M}(Q, W)^{\zeta-\mathrm{sst}}_{n\mathbf{d}_0}(\mathbb{C})$$

that is injective on isomorphism classes and an isomorphism on morphism spaces since Φ is an equivalence of categories.

To show that g_n is surjective, we follow [8, section 8]. Suppose that M is a ζ-stable $\mathbb{C}(Q, W)$-representation of dimension vector $n\mathbf{d}_0$, for $n > 1$. Then for all ζ-(semi)stable representations N of dimension vector $(1, 1, 1)$, we have $\mathrm{Hom}(M, N) = \mathrm{Hom}(N, M) = 0$, so $\mathrm{Ext}^i(M, N)$ lives in degrees 1 and 2 only. Thus $\Psi(M)$ is a 2-term complex of vector bundles $[L^{-2} \xrightarrow{d} L^{-1}]$ on Y_G. The cohomology of this complex is compactly supported, and so d is injective. So $\Psi(M)$ is a coherent sheaf on Y living in degree (-1). This sheaf cannot have a summand supported away from the exceptional locus E, since the corresponding summand

of M would have negative dimension vector. So $\Psi(M)[1]$ is set-theoretically supported on E, and thus has a filtration by coherent sheaves on E. The Chern characters of these sheaves sum to $-n$ times the chern Character of the skyscraper sheaf of a point, which is a contradiction. Hence any semistable representation of dimension vector $n\mathbf{d}_0$ is strictly semistable and in particular an iterated extension of stable representations of dimension vector $\mathbf{d}_0$. The converse of the inductive argument above then shows that its image under Ψ is a torsion sheaf on Y. Thus g_n is an isomorphism of groupoids as required.

Finally, suppose that $\mathcal{F}$ in the above argument is supported along the exceptional locus. By our assumption on π, $\mathcal{F}$ is annihilated by all monomials $x^r y^s z^t$, considered as functions on Y_G, for $r + s + t \gg 0$. As such, $\Phi(\mathcal{F})$ is annihilated by multiplication by the central elements

$$\left(\sum_{0 \leq i \leq r-1} x_i\right)^r \left(\sum_{0 \leq i \leq r-1} y_i\right)^s \left(\sum_{0 \leq i \leq r-1} z_i\right)^t$$

in $\mathbb{C}(Q, W)$, for $r + s + t \gg 0$ i.e. it is nilpotent. The converse works in the same way.

Proof of Theorem 4·9. We introduce an auxiliary mixed Hodge structure

$$\mathcal{H}^{\text{stack}}(a, b, c)^{\text{z-nilp}} = \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} H_c\left(\mathfrak{M}_{\mathbf{d}}(Q), \varphi_{\text{Tr}(W)_{\mathbf{d}}}|_{\text{z-nilp}}\right)$$

by restricting the vanishing cycle mixed Hodge module to the reduced substack, the $\mathbb{C}$-points of which are defined by the condition that the action of $z \in \mathbb{C}[x, y, z] \rtimes G$ is nilpotent. We will prove that the purity and parity statements of the theorem are true of both $\mathcal{H}^{\text{stack}}(a, b, c)$ and $\mathcal{H}^{\text{stack}}(a, b, c)^{\text{z-nilp}}$. By [13, theorem B] we have the following variant of (4·9):

$$\mathcal{H}^{\text{stack}}(a, b, c)^{\text{z-nilp}} \cong \bigotimes_{-\infty \xrightarrow{\theta} \infty}^{\text{tw}} H_c\left(\mathfrak{M}_{\theta}^{\text{sst}}(Q), \varphi_{\text{Tr}(W)}|_{\text{z-nilp}}\right). \tag{4·15}$$

The parity and purity statements for the left-hand side of (4·9) and (4·15) are equivalent to the same parity and purity statements for the components of the tensor product decompositions. This is because purity is preserved by tensor product, $\mathbb{L}^{1/2}$ is pure, and the degree shift introduced by the twisted monoidal product is equal to the part of the following equation that we have put in square brackets:

$$(\mathbf{d}, \mathbf{d}) + (\mathbf{e}, \mathbf{e}) + [(\mathbf{e}, \mathbf{d}) - (\mathbf{d}, \mathbf{e})] = (\mathbf{d} + \mathbf{e}, \mathbf{d} + \mathbf{e}) \mod 2.$$

Under the derived equivalence Φ, complexes of $\mathbb{C}(Q, W)$-modules with finite-dimensional cohomology correspond to complexes of coherent sheaves with compactly supported cohomology sheaves, and nilpotency for the cohomology of $\mathbb{C}(Q, W)$-modules corresponds to cohomology sheaves being set-theoretically supported on the exceptional locus of $\pi_G : Y_G \to \mathbb{C}^3/G$.

Without affecting the definition of ζ-stability we may adjust our stability condition ζ so that the dimension vector $\mathbf{d}_0 = (1, \ldots, 1)$ has slope zero, by adding some scalar multiple of $(1, \ldots, 1)$ to ζ.

Let $\mathbf{d}$ be a dimension vector of nonzero slope with respect to ζ. If M is ζ-semistable and $\mathbf{d}$-dimensional, the sheaf $\Psi(M)$ must be supported set-theoretically on the exceptional locus of π_G, for if $\mathcal{G} \subset \Psi(M)$ is the summand supported away from the exceptional locus, then

$\dim(\mathrm{Supp}(\Phi(\mathcal{G}))) = 0$ and so $\Phi(\mathcal{G})$ is a direct summand of ρ with slope zero. It follows that the support of $\varphi_{\mathrm{Tr}(W)_{\mathbf{d}}}$, restricted to the semistable locus, only contains nilpotent modules, and in particular, modules for which the action of z is nilpotent. So for $\theta \neq 0$ there is an equality

$$\mathrm{H}_c\left(\mathfrak{M}^{\mathrm{sst}}_\theta(Q), \varphi_{\mathrm{Tr}(W)}\right) = \mathrm{H}_c\left(\mathfrak{M}^{\mathrm{sst}}_\theta(Q), \varphi_{\mathrm{Tr}(W)}|_{\text{z-nilp}}\right). \tag{4.16}$$

If we let $\mathbb{C}^*$ act on the edges of Q by scaling the edges $x_1, \ldots, x_r$ and leaving the other edges invariant, the function $\mathrm{Tr}(W)$ is a weight one eigenfunction for $\mathbb{C}^*$, and so as a special case of the dimensional reduction isomorphism (i.e. we "reduce" along the x direction, from a categorically three-dimensional moduli problem to a two-dimensional one) we have

$$\mathcal{H}^{\mathrm{stack}}(a, b, c)^{\text{z-nilp}}_{\mathbf{d}} \cong \mathrm{H}_c(\mathfrak{R}_{\mathbf{d}}, \mathbb{Q}) \otimes \mathbb{L}^{(\mathbf{d}, \mathbf{d})/2 + r},$$

where $r = \sum_{0 \leq i \leq r-1} d_i d_{i+c}$ and

$$\mathfrak{R}_{\mathbf{d}} = \mathrm{Rep}^{\text{z-nilp}}_{\mathbf{d}}(\mathbb{C}[y, z] \rtimes G),$$

is the stack of $\mathbb{C}[y, z] \rtimes G$-modules for which z acts nilpotently, and the underlying G-representation is given by

$$\bigoplus_{0 \leq i \leq r-1} \rho_i^{\oplus d_i}.$$

Because of the constraint on z, for a given $\mathbf{d}$ there are finitely many possibilities for the isomorphism class of the underlying $\mathbb{C}[z] \rtimes G$-representation. We can partition the stack $\mathfrak{R}_{\mathbf{d}}$ according to the isomorphism type of this representation, and it is straightforward to check as in [11] that each piece of this partition has pure compactly supported cohomology, supported entirely in even degree. From the resulting long exact sequences in compactly supported cohomology, it follows that $\mathfrak{R}_{\mathbf{d}}$ has pure compactly supported cohomology, supported in even degree. It follows that $\mathcal{H}^{\mathrm{stack}}(a, b, c)^{\text{z-nilp}}_G$ is pure, supported in degrees of the required parity, and so the same is true of the terms in (4.9), for $\theta \neq 0$, as well as the right-hand side of (4.15) for all θ.

Let $\theta = 0$, so that by Proposition 4.13 the only semistable $\mathbb{C}(Q, W)$-modules of slope θ have dimension vector $n\mathbf{d}_0$ for some n and correspond under the derived equivalence Φ to coherent sheaves with zero-dimensional support. As in [12] we can write

$$\mathrm{H}_c\left(\mathfrak{M}^{\mathrm{sst}}_\theta(Q), \varphi_{\mathrm{Tr}(W)}\right) \cong \mathrm{Sym}\left(\bigoplus_{n \geq 1} \mathrm{H}_c(Y_G, \mathcal{L}_n) \otimes \mathrm{H}_c(\mathrm{pt}/\mathbb{C}^*) \otimes \mathbb{L}^{1/2}\right) \tag{4.17}$$

$$\mathrm{H}_c\left(\mathfrak{M}^{\mathrm{sst}}_\theta(Q), \varphi_{\mathrm{Tr}(W)}|_{\mathrm{nilp}}\right) \cong \mathrm{Sym}\left(\bigoplus_{n \geq 1} \mathrm{H}_c(Y_G, \mathcal{L}_n|_{\mathrm{nilp}}) \otimes \mathrm{H}_c(\mathrm{pt}/\mathbb{C}^*) \otimes \mathbb{L}^{1/2}\right), \tag{4.18}$$

where $\mathcal{L}_n$ is a pure rank one variation of mixed Hodge structure on Y_G, and $\mathcal{L}_n|_{\mathrm{nilp}}$ is its restriction to the locus where z acts nilpotently. Since $(n\mathbf{d}_0, n\mathbf{d}_0) = 0$, we need to show that the left-hand side of (4.17) is pure, of even degree, which is equivalent to showing that each $\mathrm{H}_c(Y_G, \mathcal{L}_n)$ is pure of odd degree, because of the extra twist by $\mathbb{L}^{1/2}$ on the right hand side.

We claim that in fact $\mathcal{L}_n|_{\text{nilp}} \cong \underline{\mathbb{Q}}_{Y_G} \otimes \mathbb{L}^{-3/2}$. This follows from the (analytic) local triviality of this mixed Hodge module, along with the claim that $\pi_1(Y_G) = 1$. For this claim, note that by [**30**], Y_G is a smooth toric variety, and since it has Euler characteristic r, the associated fan contains a 3-dimensional cone. The claim then follows from [**9**, theorem 12·1·10]. Finally, then, we have to prove that either of the isomorphic mixed Hodge structures

$$\mathrm{H}_c(Y_G, \mathbb{Q}) \otimes \mathbb{L}^{-3/2} \cong \left(\mathrm{H}(Y_G, \mathbb{Q}) \otimes \mathbb{L}^{-3/2}\right)^*$$

are pure, with cohomology in entirely odd degree. The purity, of weight i, of $\mathrm{H}^i(Y_G, \mathbb{Q})$ follows by a standard argument: the mth graded piece of the weight filtration is zero for $m < i$ since Y_G is smooth, on the other hand, Y_G contracts onto its exceptional locus $E = \pi_G^{-1}(0)$, and the mth graded piece of the weight filtration on $\mathrm{H}^i(E, \mathbb{Q})$ is zero for $m > i$ since E is proper.

For the parity statement, from the composition of isomorphisms

$$\mathrm{H}(Y_G, \mathbb{Q}) \cong \mathrm{H}(E, \mathbb{Q}) \cong \mathrm{H}_c(E, \mathbb{Q})$$

we deduce that it is enough to show that $\mathrm{H}_c(E, \mathbb{Q})$ is concentrated in even degree. Let $Z \subset Y_G$ be the locus cut out by the equation $z = 0$. Then the complement $U = Z \setminus E$ admits a Galois cover from $\mathbb{C}^2 \setminus \{0\}$, i.e. the locus inside $\mathbb{C}^3$ for which $z = 0$ and one of x, y are nonzero. In particular, the only impure compactly supported cohomology of U is concentrated in odd cohomological degree, as the same is true of $\mathbb{C}^2 \setminus \{0\}$. Moreover, the compactly supported cohomology of both Z and E are pure – purity for Z follows from purity of the left hand side of (4·18) and our description of $\mathcal{L}_n$. Then since the morphisms in the long exact sequence in cohomology

$$\longrightarrow \mathrm{H}_c^i(U, \mathbb{Q}) \longrightarrow \mathrm{H}_c^i(Z, \mathbb{Q}) \longrightarrow \mathrm{H}_c^i(E, \mathbb{Q}) \longrightarrow$$

are morphisms of mixed Hodge structures it follows that $\mathrm{H}_c(E, \mathbb{Q})$ is concentrated in even degree.

Appendix

Here we record the coefficients of the reduced $\mu_3(1, 1, 1)$ partition function up to total degree 24.

$$Z_{\mathrm{red}}^{\mu_3^{(1,1,1)}}(t_0,t_1,t_2) = 1 + t_0 + 3t_0t_1 + 3t_0t_1^2 + t_0t_1^3 + 9t_0t_1^2t_2 + 6t_0t_1^2t_2 + 9t_0t_1^3t_2 + 9t_0^2t_1^2t_2 + 15t_0t_1^3t_2^2 +$$
$$3t_0t_1^2t_2^3 + 12t_0^2t_1^3t_2^2 + 12t_0^2t_1^2t_2^3 + 20t_0t_1^3t_2^3 + 18t_0^3t_1^2t_2^3 + 46t_0t_1^3t_2^3 + 15t_0t_1^3t_2^4 + 12t_0^4t_1^2t_2^3 +$$
$$44t_0^3t_1^3t_2^3 + 66t_0^2t_1^3t_2^4 + 6t_0t_1^3t_2^5 + 3t_0^5t_1^2t_2^3 + 36t_0^4t_1^3t_2^3 + 15t_0^3t_1^4t_2^3 + 114t_0^3t_1^3t_2^4 + 42t_0^2t_1^3t_2^5 +$$
$$t_0t_1^3t_2^6 + 18t_0^5t_1^3t_2^3 + 36t_0^4t_1^3t_2^3 + 96t_0^4t_1^3t_2^4 + 45t_0^3t_1^4t_2^4 + 126t_0^3t_1^3t_2^5 + 10t_0^2t_1^3t_2^6 + 45t_0^5t_1^4t_2^3 +$$
$$12t_0^4t_1^5t_2^3 + 39t_0^5t_1^3t_2^4 + 153t_0^4t_1^4t_2^4 + 210t_0^4t_1^3t_2^5 + 45t_0^3t_1^4t_2^5 + 45t_0^3t_1^3t_2^6 + 60t_0^5t_1^5t_2^3 +$$
$$6t_0^6t_1^3t_2^4 + 144t_0^5t_1^4t_2^4 + 54t_0^4t_1^5t_2^4 + 210t_0^5t_1^3t_2^5 + 234t_0^4t_1^4t_2^5 + 120t_0^4t_1^3t_2^6 + 15t_0^3t_1^4t_2^6 +$$
$$45t_0^5t_1^6t_2^3 + 36t_0^6t_1^4t_2^4 + 225t_0^5t_1^5t_2^4 + 126t_0^6t_1^3t_2^5 + 486t_0^5t_1^4t_2^5 + 90t_0^4t_1^5t_2^5 + 210t_0^5t_1^3t_2^6 +$$
$$120t_0^4t_1^4t_2^6 + 18t_0^5t_1^7t_2^3 + 90t_0^6t_1^5t_2^4 + 201t_0^5t_1^6t_2^4 + 42t_0^4t_1^3t_2^5 + 504t_0^6t_1^4t_2^5 + 468t_0^5t_1^5t_2^5 +$$
$$252t_0^6t_1^3t_2^6 + 420t_0^5t_1^4t_2^6 + 66t_0^4t_1^5t_2^6 + 3t_0^5t_1^8t_2^3 + 120t_0^6t_1^6t_2^4 + 108t_0^5t_1^7t_2^4 + 6t_0^8t_1^3t_2^5 +$$
$$261t_0^7t_1^4t_2^5 + 354t_0^5t_1^6t_2^5 + 846t_0^6t_1^5t_2^5 + 210t_0^7t_1^3t_2^6 + 840t_0^6t_1^4t_2^6 + 429t_0^5t_1^5t_2^6 + 18t_0^4t_1^5t_2^7 +$$
$$90t_0^6t_1^7t_2^4 + 27t_0^5t_1^8t_2^4 + 54t_0^8t_1^4t_2^5 + 684t_0^7t_1^5t_2^5 + 828t_0^6t_1^5t_2^5 + 270t_0^5t_1^7t_2^5 + 120t_0^8t_1^3t_2^6 +$$
$$1050t_0^7t_1^4t_2^6 + 1296t_0^6t_1^5t_2^6 + 306t_0^5t_1^6t_2^6 + 126t_0^5t_1^5t_2^7 + 36t_0^6t_1^8t_2^4 + 216t_0^8t_1^5t_2^5 + 975t_0^7t_1^6t_2^5 +$$
$$558t_0^6t_1^7t_2^5 + 108t_0^5t_1^8t_2^5 + 45t_0^9t_1^3t_2^6 + 840t_0^8t_1^4t_2^6 + 2319t_0^7t_1^5t_2^6 + 1420t_0^6t_1^6t_2^6 + 360t_0^5t_1^7t_2^6 +$$
$$378t_0^6t_1^5t_2^7 + 129t_0^5t_1^6t_2^7 + 6t_0^6t_1^9t_2^4 + 504t_0^8t_1^5t_2^5 + 810t_0^7t_1^7t_2^5 + 252t_0^6t_1^8t_2^5 + 10t_0^{10}t_1^3t_2^6 +$$
$$420t_0^9t_1^4t_2^6 + 2586t_0^8t_1^5t_2^6 + 3207t_0^7t_1^6t_2^6 + 1284t_0^6t_1^7t_2^6 + 252t_0^5t_1^8t_2^6 + 630t_0^7t_1^5t_2^7 + 852t_0^6t_1^6t_2^7 +$$
$$270t_0^5t_1^7t_2^7 + 21t_0^5t_1^6t_2^8 + 756t_0^8t_1^7t_2^5 + 387t_0^7t_1^8t_2^5 + 54t_0^6t_1^9t_2^5 + t_0^{11}t_1^3t_2^6 + 120t_0^{10}t_1^4t_2^6 + 1755t_0^9t_1^5t_2^6 +$$
$$4668t_0^8t_1^6t_2^6 + 3138t_0^7t_1^7t_2^6 + 795t_0^6t_1^8t_2^6 + 630t_0^8t_1^5t_2^7 + 2355t_0^7t_1^6t_2^7 + 1404t_0^6t_1^7t_2^7 + 378t_0^5t_1^8t_2^7 +$$
$$147t_0^6t_1^5t_2^8 + 108t_0^5t_1^7t_2^8 + 756t_0^8t_1^8t_2^5 + 96t_0^7t_1^9t_2^5 + 15t_0^{11}t_1^4t_2^6 + 660t_0^{10}t_1^5t_2^6 + 4320t_0^9t_1^6t_2^6 +$$
$$5622t_0^8t_1^7t_2^6 + 2022t_0^7t_1^8t_2^6 + 216t_0^6t_1^9t_2^6 + 378t_0^9t_1^5t_2^7 + 3480t_0^8t_1^6t_2^7 + 4140t_0^7t_1^7t_2^7 + 1494t_0^6t_1^8t_2^7 +$$
$$441t_0^7t_1^5t_2^8 + 738t_0^6t_1^7t_2^8 + 378t_0^5t_1^8t_2^8 + 18t_0^5t_1^7t_2^9 + 504t_0^8t_1^9t_2^5 + 9t_0^7t_1^{10}t_2^5 + 105t_0^{11}t_1^5t_2^6 +$$
$$2200t_0^{10}t_1^6t_2^6 + 6930t_0^9t_1^7t_2^6 + 4920t_0^8t_1^8t_2^6 + 693t_0^7t_1^9t_2^6 + 126t_0^{10}t_1^5t_2^7 + 2895t_0^9t_1^6t_2^7 + 8172t_0^8t_1^7t_2^7 +$$
$$4365t_0^7t_1^8t_2^7 + 504t_0^6t_1^9t_2^7 + 735t_0^8t_1^6t_2^8 + 2412t_0^7t_1^7t_2^8 + 1845t_0^6t_1^8t_2^8 + 150t_0^6t_1^7t_2^9 + 252t_0^5t_1^8t_2^9 +$$
$$216t_0^8t_1^{10}t_2^5 + 455t_0^{11}t_1^6t_2^6 + 4950t_0^{10}t_1^7t_2^6 + 7560t_0^9t_1^8t_2^6 + 3270t_0^8t_1^9t_2^6 + 81t_0^7t_1^{10}t_2^6 + 18t_0^{11}t_1^5t_2^7 +$$
$$1284t_0^{10}t_1^6t_2^7 + 9612t_0^9t_1^7t_2^7 + 10800t_0^8t_1^8t_2^7 + 2220t_0^7t_1^9t_2^7 + 735t_0^9t_1^6t_2^8 + 4680t_0^8t_1^7t_2^8 +$$
$$5346t_0^7t_1^8t_2^8 + 756t_0^6t_1^9t_2^8 + 546t_0^7t_1^7t_2^9 + 1536t_0^6t_1^8t_2^9 + 108t_0^5t_1^8t_2^{10} + 54t_0^8t_1^{11}t_2^5 +$$
$$1365t_0^{11}t_1^7t_2^6 + 7920t_0^{10}t_1^8t_2^6 + 5670t_0^9t_1^9t_2^6 + 1596t_0^8t_1^{10}t_2^6 + 237t_0^{11}t_1^6t_2^7 + 5886t_0^{10}t_1^7t_2^7 +$$
$$18045t_0^9t_1^8t_2^7 + 9114t_0^8t_1^9t_2^7 + 324t_0^7t_1^{10}t_2^7 + 441t_0^{10}t_1^6t_2^8 + 5580t_0^9t_1^7t_2^8 + 11934t_0^8t_1^8t_2^8 +$$
$$4143t_0^7t_1^9t_2^8 + 1134t_0^8t_1^7t_2^9 + 4497t_0^7t_1^8t_2^9 + 756t_0^6t_1^9t_2^9 + 837t_0^6t_1^8t_2^{10} + 27t_0^5t_1^8t_2^{11} +$$
$$O((t_0,t_1,t_2)^{25}).$$

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