*Bull. Aust. Math. Soc.* **104** (2021), 238–248
doi:10.1017/S0004972720001513

# DIVISIBILITY OF CERTAIN SINGULAR OVERPARTITIONS BY POWERS OF 2 AND 3

AJIT SINGH and RUPAM BARMAN

(Received 14 October 2020; accepted 30 November 2020; first published online 14 January 2021)

**Abstract**

Andrews introduced the partition function $\overline{C}_{k,i}(n)$, called the singular overpartition function, which counts the number of overpartitions of $n$ in which no part is divisible by $k$ and only parts $\equiv \pm i \pmod{k}$ may be overlined. We prove that $\overline{C}_{6,2}(n)$ is almost always divisible by $2^k$ for any positive integer $k$. We also prove that $\overline{C}_{6,2}(n)$ and $\overline{C}_{12,4}(n)$ are almost always divisible by $3^k$. Using a result of Ono and Taguchi on nilpotency of Hecke operators, we find infinite families of congruences modulo arbitrary powers of 2 satisfied by $\overline{C}_{6,2}(n)$.

2020 *Mathematics subject classification*: primary 11P83; secondary 05A17.

*Keywords and phrases*: arithmetic density, eta-quotients, modular forms, singular overpartitions.

## 1. Introduction and statement of results

In [5], Corteel and Lovejoy introduced overpartitions. An overpartition of $n$ is a nonincreasing sequence of natural numbers whose sum is $n$ in which the first occurrence of a number may be overlined. In order to give overpartition analogues of Rogers–Ramanujan type theorems for the ordinary partition function, Andrews [1] defined the so-called singular overpartitions. Andrews' singular overpartition function $\overline{C}_{k,i}(n)$ counts the number of overpartitions of $n$ in which no part is divisible by $k$ and only parts $\equiv \pm i \pmod{k}$ may be overlined. For example, $\overline{C}_{3,1}(4) = 10$ with the relevant partitions being $4, \overline{4}, 2+2, \overline{2}+2, 2+1+1, \overline{2}+1+1, 2+\overline{1}+1, \overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1$. For $k \geq 3$ and $1 \leq i \leq \lfloor k/2 \rfloor$, the generating function for $\overline{C}_{k,i}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n) q^n = \frac{(q^k;q^k)_\infty(-q^i;q^k)_\infty(-q^{k-i};q^k)_\infty}{(q;q)_\infty}, \tag{1.1}$$

where $(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j)$. Andrews proved the following Ramanujan-type congruences satisfied by $\overline{C}_{3,1}(n)$:

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3} \quad \text{for } n \geq 0.$$

© 2021 Australian Mathematical Publishing Association Inc.



https://doi.org/10.1017/S0004972720001513 Published online by Cambridge University Press

Chen *et al*. [4] later showed that Andrews' congruences modulo 3 are two examples of an infinite family of congruences modulo 3 which hold for the function $\overline{C}_{3,1}(n)$. More precisely, they showed that for prime $p \equiv 3 \pmod 4$,

$$\overline{C}_{3,1}(p^{2k+1}m) \equiv 0 \pmod 3 \quad \text{for all } k, m \geq 0 \text{ with } p \nmid m.$$

In [4], Chen *et al*. also studied the parity of $\overline{C}_{k,i}(n)$. They showed that $\overline{C}_{3,1}(n)$ is always even and that $\overline{C}_{6,2}(n)$ is even (or odd) if and only if $n$ is not (or is) a pentagonal number. Recently, Aricheta [2] studied the parity of $\overline{C}_{3k,k}(n)$. Represent any positive integer $k$ as $k = 2^a m$ where the integer $a \geq 0$ and $m$ is odd. Assume further that $2^a \geq m$. Then Aricheta proved that $\overline{C}_{3k,k}(n)$ is almost always even, that is,

$$\lim_{X\to\infty} \frac{\#\{0 < n \leq X : \overline{C}_{3k,k}(n) \equiv 0 \pmod 2\}}{X} = 1.$$

Aricheta also showed that for any pair $(k, i)$, $\overline{C}_{k,i}(n)$ satisfies infinitely many Ramanujan-type congruences modulo any power of a prime coprime to $6k$.

Let $k$ be a fixed positive integer. Recently, Barman and Ray [3] proved that for any positive integer $k$, $\overline{C}_{3,1}(n)$ is almost always divisible by $2^k$ and $3^k$. In this paper we study divisibility of $\overline{C}_{6,2}(n)$ and $\overline{C}_{12,4}(n)$ by $2^k$ and $3^k$. More precisely, we prove that $\overline{C}_{6,2}(n)$ is divisible by arbitrary powers of 2 for almost all $n$. We also prove that $\overline{C}_{6,2}(n)$ and $\overline{C}_{12,4}(n)$ are divisible by arbitrary powers of 3 for almost all $n$.

THEOREM 1.1. *Let k be a fixed positive integer. Then the set*

$$\{n \in \mathbb{N} : \overline{C}_{6,2}(n) \equiv 0 \pmod{2^k}\}$$

*has arithmetic density* 1.

Serre observed and Tate proved that the action of Hecke algebras on spaces of modular forms of level 1 modulo 2 is locally nilpotent (see [10–12]). Ono and Taguchi [9] showed that this phenomenon generalises to higher levels. Using this, we prove the following congruence for $\overline{C}_{6,2}(n)$ modulo arbitrary powers of 2.

THEOREM 1.2. *Let n be a nonnegative integer. Then there is an integer $s \geq 0$ such that for every $t \geq 1$ and distinct primes $p_1, \ldots, p_{s+t}$ coprime to* 6,

$$\overline{C}_{6,2}\Big(\frac{p_1 \cdots p_{s+t} \cdot n - 1}{24}\Big) \equiv 0 \pmod{2^t}$$

*whenever n is coprime to $p_1, \ldots, p_{s+t}$.*

We further prove that the partition functions $\overline{C}_{6,2}(n)$ and $\overline{C}_{12,4}(n)$ are divisible by $3^k$ for almost all $n$.

THEOREM 1.3. *Let k be a fixed positive integer. Then the set*

$$\{n \in \mathbb{N} : \overline{C}_{3\ell,\ell}(n) \equiv \pmod{3^k}\}$$

*has arithmetic density 1 for $\ell = 2, 4$.*

https://doi.org/10.1017/S0004972720001513 Published online by Cambridge University Press

## 2. Preliminaries

In this section we recall some definitions and basic facts on modular forms and eta-quotients. For more details, see [6, 8].

**2.1. Spaces of modular forms.** We first define the matrix groups

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$
$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \ (\mathrm{mod}\ N) \right\},$$
$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \ (\mathrm{mod}\ N) \right\},$$
$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \ (\mathrm{mod}\ N) \text{ and } b \equiv c \equiv 0 \ (\mathrm{mod}\ N) \right\},$$

where $N$ is a positive integer. A subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subseteq \Gamma$ for some $N$. The smallest $N$ such that $\Gamma(N) \subseteq \Gamma$ is called the level of $\Gamma$. For example, $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level $N$.

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half of the complex plane. The group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on $\mathbb{H}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d}.$$

We identify $\infty$ with $\frac{1}{0}$ and define

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{r}{s} = \frac{ar + bs}{cr + ds}, \quad \text{where } \frac{r}{s} \in \mathbb{Q} \cup \{\infty\}.$$

This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half-plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that $\Gamma$ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A cusp of $\Gamma$ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of $\Gamma$.

The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$. If $f(z)$ is a meromorphic function on $\mathbb{H}$ and $\ell$ is an integer, we define the slash operator $|_\ell$ by

$$(f|_\ell\gamma)(z) := (\det \gamma)^{\ell/2}(cz + d)^{-\ell} f(\gamma z), \quad \text{where } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R}).$$

DEFINITION 2.1. Let $\Gamma$ be a congruence subgroup of level $N$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with integer weight $\ell$ on $\Gamma$ if the following statements hold.

https://doi.org/10.1017/S0004972720001513 Published online by Cambridge University Press

(1) We have
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\ell} f(z) \quad \text{for all } z \in \mathbb{H} \text{ and all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma.$$

(2) If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_{\ell}\gamma)(z)$ has a Fourier expansion of the form
$$(f|_{\ell}\gamma)(z) = \sum_{n \geq 0} a_{\gamma}(n) q_N^n,$$
where $q_N := e^{2\pi i z/N}$.

For a positive integer $\ell$, the complex vector space of modular forms of weight $\ell$ with respect to a congruence subgroup $\Gamma$ is denoted by $M_{\ell}(\Gamma)$.

DEFINITION 2.2 [8, Definition 1.15]. If $\chi$ is a Dirichlet character modulo $N$, then we say that a modular form $f \in M_{\ell}(\Gamma_1(N))$ has Nebentypus character $\chi$ if
$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^{\ell} f(z) \quad \text{for all } z \in \mathbb{H} \text{ and all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N).$$
The space of such modular forms is denoted by $M_{\ell}(\Gamma_0(N), \chi)$.

**2.2. Modularity of eta-quotients.** In this paper the relevant modular forms are those that arise from eta-quotients. The Dedekind eta-function $\eta(z)$ is defined by
$$\eta(z) := q^{1/24}(q;q)_{\infty} = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$
where $q := e^{2\pi i z}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form
$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_{\delta}},$$
where $N$ is a positive integer and the $r_{\delta}$ are integers.

We now recall two theorems from [8, page 18] on modularity of eta-quotients. We will use these two results to verify modularity of certain eta-quotients appearing in the proofs of our main results.

THEOREM 2.3 [8, Theorem 1.64]. *If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_{\delta}}$ is an eta-quotient such that $\ell = \frac{1}{2} \sum_{\delta|N} r_{\delta} \in \mathbb{Z}$, $\sum_{\delta|N} \delta r_{\delta} \equiv 0 \pmod{24}$ and*
$$\sum_{\delta|N} \frac{N}{\delta} r_{\delta} \equiv 0 \pmod{24},$$
*then $f(z)$ satisfies*
$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^{\ell} f(z) \quad \textit{for every } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N).$$
*Here the character $\chi$ is defined by $\chi(d) := \left(\frac{(-1)^{\ell} s}{d}\right)$, where $s := \prod_{\delta|N} \delta^{r_{\delta}}$ and $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol.*

https://doi.org/10.1017/S0004972720001513 Published online by Cambridge University Press

Suppose that $f$ is an eta-quotient satisfying the conditions of Theorem 2.3 and that the associated weight $\ell$ is a positive integer. If $f(z)$ is holomorphic at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_\ell(\Gamma_0(N), \chi)$. The following theorem gives the necessary criterion for determining orders of an eta-quotient at cusps.

THEOREM 2.4 [8, Theorem 1.65]. *Let $c, d$ and $N$ be positive integers with $d \mid N$ and $\gcd(c, d) = 1$. If $f$ is an eta-quotient satisfying the conditions of Theorem 2.3 for $N$, then the order of vanishing of $f(z)$ at the cusp $(c/d)$ is*

$$\frac{N}{24}\sum_{\delta \mid N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d, N/d)d\delta}.$$

Finally, we recall the definition of Hecke operators. Let $m$ be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$. Then the action of the Hecke operator $T_m$ on $f(z)$ is defined by

$$f(z)|T_m := \sum_{n=0}^{\infty}\Big(\sum_{d \mid \gcd(n,m)} \chi(d)d^{\ell-1}a\Big(\frac{nm}{d^2}\Big)\Big)q^n.$$

In particular, if $m = p$ is prime, then

$$f(z)|T_p := \sum_{n=0}^{\infty}\Big(a(pn) + \chi(p)p^{\ell-1}a\Big(\frac{n}{p}\Big)\Big)q^n.$$

We adopt the convention that $a(n) = 0$ when $n$ is a nonnegative integer.

## 3. Proof of Theorem 1.1

Using (1.1), we find that the generating function of $\overline{C}_{6,2}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{C}_{6,2}(n)q^n = \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty^2}{(q;q)_\infty(q^2;q^2)_\infty(q^{12};q^{12})_\infty}. \tag{3.1}$$

Given a prime $p$, let

$$A_p(z) = \prod_{n=1}^{\infty}\frac{(1-q^{48n})^p}{(1-q^{48pn})} = \frac{\eta^p(48z)}{\eta(48pz)}.$$

By the binomial theorem,

$$A_p^{p^k}(z) = \frac{\eta^{p^{k+1}}(48z)}{\eta^{p^k}(48pz)} \equiv 1 \pmod{p^{k+1}}.$$

Define $B_{p,k}(z)$ by

$$B_{p,k}(z) = \Big(\frac{\eta(96z)\eta(144z)^2}{\eta(24z)\eta(48z)\eta(288z)}\Big)A_p^{p^k}(z). \tag{3.2}$$

https://doi.org/10.1017/S0004972720001513 Published online by Cambridge University Press

TABLE 1. Calculation of values of $L$ for Theorem 1.1.

| $d \mid 576$ | $\frac{\gcd(d,48)^2}{\gcd(d,288)^2}$ | $\frac{\gcd(d,96)^2}{\gcd(d,288)^2}$ | $\frac{\gcd(d,144)^2}{\gcd(d,288)^2}$ | $\frac{\gcd(d,24)^2}{\gcd(d,288)^2}$ | $L$ |
|---|---|---|---|---|---|
| 1, 2, 3, 4, 6, 8, 12, 24 | 1 | 1 | 1 | 1 | $9 \cdot 2^k - 12$ |
| 16, 48 | 1 | 1 | 1 | 0.2500 | $9 \cdot 2^k - 3$ |
| 32, 64, 96, 192 | 0.2500 | 1 | 0.2500 | 0.0625 | 0.7500 |
| 9, 18, 36, 72 | 0.1111 | 0.1111 | 1 | 0.1111 | $2^k + 1.33$ |
| 144 | 0.1111 | 0.1111 | 1 | 0.0278 | $2^k + 2.33$ |
| 288, 576 | 0.0278 | 0.1111 | 0.2500 | 0.0069 | 0.0833 |

Modulo $p^{k+1}$,

$$B_{p,k}(z) \equiv \frac{\eta(96z)\eta(144z)^2}{\eta(24z)\eta(48z)\eta(288z)} = q\bigg(\frac{(q^{96};q^{96})_\infty (q^{144};q^{144})_\infty^2}{(q^{24};q^{24})_\infty (q^{48};q^{48})_\infty (q^{288};q^{288})_\infty}\bigg). \tag{3.3}$$

Combining (3.1) and (3.3),

$$B_{p,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{6,2}(n) q^{24n+1} \pmod{p^{k+1}}. \tag{3.4}$$

PROOF OF THEOREM 1.1. We put $p = 2$ in (3.2) to obtain

$$B_{2,k}(z) = \bigg(\frac{\eta(96z)\eta(144z)^2}{\eta(24z)\eta(48z)\eta(288z)}\bigg) A_2^{2^k}(z) = \frac{\eta(48z)^{2^{k+1}-1}\,\eta(144z)^2}{\eta(96z)^{2^k-1}\,\eta(24z)\eta(288z)}.$$

Now, $B_{2,k}$ is an eta-quotient with $N = 576$. The cusps of $\Gamma_0(576)$ are represented by fractions $c/d$ where $d \mid 576$ and $\gcd(c, d) = 1$ (see [7, page 5]). By Theorem 2.4, we find that $B_{2,k}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$\frac{\gcd(d,48)^2}{48}(2^{k+1}-1) + \frac{\gcd(d,96)^2}{96}(1-2^k) + \frac{\gcd(d,144)^2}{72} - \frac{\gcd(d,24)^2}{24} - \frac{\gcd(d,288)^2}{288} \geq 0,$$

that is, if and only if

$$L := 6\frac{\gcd(d,48)^2}{\gcd(d,288)^2}(2^{k+1}-1) + 3\frac{\gcd(d,96)^2}{\gcd(d,288)^2}(1-2^k) + 4\frac{\gcd(d,144)^2}{\gcd(d,288)^2} - 12\frac{\gcd(d,24)^2}{\gcd(d,288)^2} - 1 \geq 0.$$

Table 1 shows the possible values of $L$. The table was prepared using MATLAB.

Since $L \geq 0$ for all $d \mid 576$, we see that $B_{2,k}(z)$ is holomorphic at every cusp $c/d$. From Theorem 2.3, the weight of $B_{2,k}(z)$ is $\ell = 2^{k-1}$ and the associated character for

https://doi.org/10.1017/S0004972720001513 Published online by Cambridge University Press

$B_{2,k}(z)$ is

$$\chi_1 = \left(\frac{2^{3\cdot 2^k+1}3^{2^k+1}}{\bullet}\right).$$

Finally, it follows from Theorem 2.3 that $B_{2,k}(z) \in M_{2^{k-1}}(\Gamma_0(576), \chi_1)$ for $k \geq 2$. For given any positive integer $m$, by a deep theorem of Serre [8, page 43], if $f(z) \in M_\ell(\Gamma_0(N), \chi)$ has Fourier expansion $f(z) = \sum_{n=0}^{\infty} c(n)q^n \in \mathbb{Z}[[q]]$, then there is a constant $\alpha > 0$ such that

$$\#\{n \leq X : c(n) \not\equiv 0 \pmod{m}\} = O\left(\frac{X}{(\log X)^\alpha}\right).$$

This yields

$$\lim_{X\to\infty} \frac{\#\{n \leq X : c(n) \equiv 0 \pmod{m}\}}{X} = 1.$$

Since $B_{2,k}(z) \in M_{2^{k-1}}(\Gamma_0(576), \chi_1)$, the Fourier coefficients of $B_{2,k}(z)$ are almost always divisible by $m = 2^k$. Now using (3.4) completes the proof of the theorem. □

## 4. Proof of Theorem 1.2

We prove Theorem 1.2 using nilpotency of Hecke operators. We use the following result which is implied by a much more general result of Ono and Taguchi [9, Theorem 1.3]. This result was also used by Aricheta (see, for example, [2, Theorem 4.5]).

THEOREM 4.1. *Let $n$ be a nonnegative integer and $k$ be a positive integer. Let $\chi$ be a quadratic Dirichlet character of conductor $9 \cdot 2^n$. There is an integer $c \geq 0$ such that for every $f(z) \in M_k(\Gamma_0(9 \cdot 2^a), \chi) \cap \mathbb{Z}[q]$ and every $t \geq 1$,*

$$f(z)|T_{p_1}|T_{p_2}|\cdots|T_{p_{c+t}} \equiv 0 \pmod{2^t}$$

*whenever the primes $p_1, \ldots, p_{c+t}$ are coprime to 6.*

PROOF OF THEOREM 1.2. Taking $p = 2$ in (3.4),

$$B_{2,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{6,2}(n)q^{24n+1} \pmod{2^{k+1}}.$$

This yields

$$B_{2,k}(z) := \sum_{n=0}^{\infty} A(n)q^n \equiv \sum_{n=0}^{\infty} \overline{C}_{6,2}\left(\frac{n-1}{24}\right)q^n \pmod{2^{k+1}}. \tag{4.1}$$

Note that $B_{2,k}(z) \in M_{2^{k-1}}(\Gamma_0(N), \chi_1)$, where the level $N = 576 = 9 \cdot 2^6$. Using Theorem 4.1, we find that there is an integer $s \geq 0$ such that for any $t \geq 1$,

$$B_{2,k}(z)|T_{p_1}|T_{p_2}|\cdots|T_{p_{s+t}} \equiv 0 \pmod{2^t}$$

whenever $p_1, \ldots, p_{s+t}$ are coprime to 6. It follows from the definition of Hecke operators that if $p_1, \ldots, p_{s+t}$ are distinct primes and if $n$ is coprime to $p_1 \cdots p_{s+t}$

https://doi.org/10.1017/S0004972720001513 Published online by Cambridge University Press

then

$$A(p_1 \cdots p_{s+t} \cdot n) \equiv 0 \pmod{2^t}. \tag{4.2}$$

Combining (4.1) and (4.2) completes the proof of the theorem. □

## 5. Proof of Theorem 1.3

The generating function of $\overline{C}_{12,4}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{12,4}(n)q^n = \frac{(q^8;q^8)_\infty (q^{12};q^{12})_\infty^2}{(q;q)_\infty (q^4;q^4)_\infty (q^{24};q^{24})_\infty}. \tag{5.1}$$

Given a prime $p$, let

$$E_p(z) = \prod_{n=1}^{\infty} \frac{(1-q^{32n})^p}{(1-q^{32pn})} = \frac{\eta^p(32z)}{\eta(32pz)}.$$

From the binomial theorem,

$$E_p^{p^k}(z) = \frac{\eta^{p^{k+1}}(32z)}{\eta^{p^k}(32pz)} \equiv 1 \pmod{p^{k+1}}.$$

Define $R_{p,k}(z)$ by

$$R_{p,k}(z) = \left(\frac{\eta(64z)\eta(96z)^2}{\eta(8z)\eta(32z)\eta(192z)}\right) E_p^{p^k}(z). \tag{5.2}$$

Modulo $p^{k+1}$,

$$R_{p,k}(z) \equiv \frac{\eta(64z)\eta(96z)^2}{\eta(8z)\eta(32z)\eta(192z)} = q\left(\frac{(q^{64};q^{64})_\infty (q^{96};q^{96})_\infty^2}{(q^8;q^8)_\infty (q^{32};q^{32})_\infty (q^{192};q^{192})_\infty}\right). \tag{5.3}$$

Combining (5.1) and (5.3),

$$R_{p,k}(z) \equiv \sum_{n=0}^{\infty} \overline{C}_{12,4}(n) q^{8n+1} \pmod{p^{k+1}}. \tag{5.4}$$

Proof of Theorem 1.3. We put $p = 3$ in (5.2) to obtain

$$R_{3,k}(z) = \left(\frac{\eta(64z)\eta(96z)^2}{\eta(8z)\eta(32z)\eta(192z)}\right) E_3^{3^k}(z) = \frac{\eta(32z)^{3^{k+1}-1}\,\eta(64z)}{\eta(96z)^{3^k-2}\,\eta(8z)\eta(192z)}. \tag{5.5}$$

Now, $R_{3,k}$ is an eta-quotient with $N = 192$. The cusps of $\Gamma_0(192)$ are represented by fractions $c/d$ where $d \mid 192$ and $\gcd(c,d) = 1$. Hence, by Theorem 2.4, $R_{3,k}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$\frac{\gcd(d,32)^2}{32}(3^{k+1}-1) + \frac{\gcd(d,64)^2}{64} - \frac{\gcd(d,8)^2}{8} - \frac{\gcd(d,96)^2}{96}(3^k-2) - \frac{\gcd(d,192)^2}{192} \ge 0,$$

https://doi.org/10.1017/S0004972720001513 Published online by Cambridge University Press

TABLE 2. Calculation of $S$ for Theorem 1.3.

| $d \mid 192$ | $\frac{\gcd(d,96)^2}{\gcd(d,192)^2}$ | $\frac{\gcd(d,64)^2}{\gcd(d,192)^2}$ | $\frac{\gcd(d,8)^2}{\gcd(d,192)^2}$ | $\frac{\gcd(d,32)^2}{\gcd(d,192)^2}$ | $S$ |
|---|---|---|---|---|---|
| 1, 2, 4, 8 | 1 | 1 | 1 | 1 | $16 \cdot 3^k - 24$ |
| 3, 6, 12, 24 | 1 | 0.1111 | 0.1111 | 0.1111 | 0 |
| 16 | 1 | 1 | 0.2500 | 1 | $16 \cdot 3^k - 6$ |
| 32 | 1 | 1 | 0.0625 | 1 | $16 \cdot 3^k - 1.5$ |
| 48 | 1 | 0.1111 | 0.0278 | 0.1111 | 2 |
| 64 | 0.2500 | 1 | 0.0156 | 0.2500 | $4.3^k + 1.12$ |
| 96 | 1 | 0.1111 | 0.0069 | 0.1111 | 2.5 |
| 192 | 0.2500 | 0.1111 | 0.0017 | 0.0278 | 0.1250 |

that is, if and only if

$$S := 6\frac{\gcd(d,32)^2}{\gcd(d,192)^2}(3^{k+1}-1) + 3\frac{\gcd(d,64)^2}{\gcd(d,192)^2} - 24\frac{\gcd(d,8)^2}{\gcd(d,192)^2} - 2\frac{\gcd(d,96)^2}{\gcd(d,192)^2}(3^k-2) - 1 \geq 0.$$

Table 2 shows all the possible values of $S$. The table was prepared using MATLAB.

Since $S \geq 0$ for all $d \mid 192$ and $k \geq 1$, it follows that $R_{3,k}(z)$ is holomorphic at every cusp $c/d$. Using Theorem 2.3, we find that the weight of $R_{3,k}(z)$ is $\ell = 3^k$. Also, the associated character for $R_{3,k}(z)$ is given by

$$\chi_3 = \left(\frac{-2^{10\cdot 3^k+2}3^{3^{-k}+1}}{\bullet}\right).$$

Finally, from Theorem 2.3, $R_{3,k}(z) \in M_{3^k}(\Gamma_0(192), \chi_3)$ for $k \geq 1$ and, by Serre's density result, the Fourier coefficients of $R_{3,k}(z)$ are almost always divisible by $3^k$. This proves that $\overline{C}_{12,4}(n)$ is divisible by $3^k$ for almost all $n$ because of (5.4).

We next put $p = 3$ in (3.2) to obtain

$$B_{3,k}(z) = \left(\frac{\eta(96z)\eta(144z)^2}{\eta(24z)\eta(48z)\eta(288z)}\right)A_3^{3^k}(z) = \frac{\eta(48z)^{3^{k+1}-1}\,\eta(96z)}{\eta(144z)^{3^k-2}\,\eta(24z)\eta(288z)}. \tag{5.6}$$

As before, the cusps of $\Gamma_0(576)$ are represented by fractions $c/d$ where $d \mid 576$ and $\gcd(c,d) = 1$. By Theorem 2.4, $B_{3,k}(z)$ is holomorphic at a cusp $c/d$ if and only if

$$\frac{\gcd(d,48)^2}{48}(3^{k+1}-1) + \frac{\gcd(d,144)^2}{144}(2-3^k) + \frac{\gcd(d,96)^2}{96} - \frac{\gcd(d,24)^2}{24} - \frac{\gcd(d,288)^2}{288} \geq 0,$$

https://doi.org/10.1017/S0004972720001513 Published online by Cambridge University Press

that is, if and only if

$$Q := 6\frac{\gcd(d,48)^2}{\gcd(d,288)^2}(3^{k+1}-1) + 2\frac{\gcd(d,144)^2}{\gcd(d,288)^2}(2-3^k) + 3\frac{\gcd(d,96)^2}{\gcd(d,288)^2} - 12\frac{\gcd(d,24)^2}{\gcd(d,288)^2} - 1 \geq 0.$$

From Table 1, we find that $Q \geq 0$ for all $d \mid 576$. As before, using Theorem 2.3, we find that $B_{3,k}(z) \in M_{3^k}(\Gamma_0(576), \chi_2)$, where the character $\chi_2$ is given by

$$\chi_2 = \left(\frac{-2^{8\cdot 3^k+1}3^{3^k+1}}{\bullet}\right).$$

Using the same reasoning and (3.4), we find that $\overline{C}_{6,2}(n)$ is divisible by $3^k$ for almost all $n \geq 0$. This completes the proof of the theorem. □

## Acknowledgement

We thank Professor Ken Ono for many valuable comments on the article.

## References


[1] G. E. Andrews, 'Singular overpartitions', *Int. J. Number Theory* **11** (2015), 1523–1533.

[2] V. M. Aricheta, 'Congruences for Andrews' $(k, i)$-singular overpartitions', *Ramanujan J.* **43** (2017), 535–549.

[3] R. Barman and C. Ray, 'Divisibility of Andrews' singular overpartitions by powers of 2 and 3', *Res. Number Theory* **5** (2019), 22.

[4] S. C. Chen, M. D. Hirschhorn and J. A. Sellers, 'Arithmetic properties of Andrews' singular overpartitions', *Int. J. Number Theory* **11** (2015), 1463–1476.

[5] S. Corteel and J. Lovejoy, 'Overpartitions', *Trans. Amer. Math. Soc.* **356** (2004), 1623–1635.

[6] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms* (Springer-Verlag, New York, 1991).

[7] K. Ono, 'Parity of the partition function in arithmetic progressions', *J. reine angew. Math.* **472** (1996), 1–15.

[8] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series*, CBMS Regional Conference Series in Mathematics, 102 (American Mathematical Society, Providence, RI, 2004).

[9] K. Ono and Y. Taguchi, '2-adic properties of certain modular forms and their applications to arithmetic functions', *Int. J. Number Theory* **1** (2005), 75–101.

[10] J.-P. Serre, 'Divisibilité de certaines fonctions arithmétiques', *Séminaire Delange-Pisot-Poitou. Théorie des Nombres* **16** (1974), 1–28.

[11] J.-P. Serre, 'Valeurs propres des opérateurs de Hecke modulo $\ell$', *Astérisque* **24** (1975), 109–117.

[12] J. Tate, 'Extensions of $\mathbb{Q}$ unramified outside 2', In: *Arithmetic Geometry: Conference on Arithmetic Geometry with an Emphasis on Iwasawa Theory, March 15–18, 1993, Arizona State University*, Contemporary Mathematics, 174 (American Mathematical Society, Providence, RI, 1994).


https://doi.org/10.1017/S0004972720001513 Published online by Cambridge University Press

AJIT SINGH, Department of Mathematics,
Indian Institute of Technology Guwahati, Assam,
India, PIN-781039
e-mail: ajit18@iitg.ac.in

RUPAM BARMAN, Department of Mathematics,
Indian Institute of Technology Guwahati, Assam,
India, PIN-781039
e-mail: rupam@iitg.ac.in

https://doi.org/10.1017/S0004972720001513 Published online by Cambridge University Press