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KERNELS OF MINIMAL CHARACTERS OF SOLVABLE GROUPS

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Abstract Let G be a finite solvable group. We prove that if $\chi \in \mathrm{Irr}(G)$ has odd degree and $\chi(1)$ is the minimal degree of the nonlinear irreducible characters of G, then $G/\operatorname{Ker}\chi$ is nilpotent-by-abelian.

Keywords: character kernel; minimal character; solvable group

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1. Introduction

A classical theorem of Broline and Garrison implies that if an irreducible character χ of a finite group G has maximal degree then $\operatorname{Ker}\chi$ is nilpotent (Corollary 12.20 of [1]). This result was extended by Isaacs, who considered characters of nth maximal degree in [2], and proved that if $\chi \in \mathrm{Irr}(G)$ has nth maximal degree, then the Fitting height of the solvable radical of $\operatorname{Ker}\chi$ is at most n.

Our goal in this note is to consider irreducible characters at the other extreme. Of course, if $\chi \in \mathrm{Irr}(G)$ is linear, then $G' \leq \operatorname{Ker}\chi$ and $G/\operatorname{Ker}\chi$ is abelian. But can we restrict the structure of $G/\operatorname{Ker}\chi$ if $\chi(1)$ is "small"? This is the content of our main result. We write $m(G) = \min\{\chi(1) \mid \chi \in \mathrm{Irr}(G), \chi(1) > 1\}$. We say that $\chi \in \mathrm{Irr}(G)$ is a **minimal character** if $\chi(1) = m(G)$.

Theorem A. *Let G be a solvable finite group. Suppose that $m(G)$ is odd. If $\chi \in \mathrm{Irr}(G)$ is a minimal character, then $G/\operatorname{Ker}\chi$ is nilpotent-by-abelian.*

As $\mathrm{GL}_2(3)$ shows, some hypothesis on $m(G)$ is definitely necessary. The Frobenius group of order 20 acting faithfully on an extraspecial 2-group of order 2^5 is an example with faithful minimal characters of degree 4. We do not know whether it is enough to assume that $m(G)$ is not a power of 2. On the other hand, some solvability hypothesis is definitely necessary: consider any non-abelian simple group with odd degree minimal characters (for instance, A_5). Theorem A follows from applying the next result to $G/\operatorname{Ker}\chi$.

Theorem B. *Let G be a finite solvable group. Suppose that $\chi \in \operatorname{Irr}(G)$ is a faithful minimal character. If $\chi(1)$ is odd, then G is nilpotent-by-abelian.*

Note that the structure of groups with *all* minimal characters faithful was described in detail by Robinson in [6, 7]. In particular, as shown in Lemma 2.1 of [6], solvable groups with all minimal characters faithful are nilpotent-by-abelian. The examples mentioned above show that this is not the case if we just assume that G has a minimal faithful character. Our proof of Theorem B relies on some of the ideas developed by Robinson.

2. Proofs

We argue as in Lemma 2 of [7] to prove our first lemma.

Lemma 2.1. *Let G be a finite group. Suppose that $\chi \in \operatorname{Irr}(G)$ is a primitive faithful minimal character of G. If $N \trianglelefteq G$ is non-central, then $\chi_N \in \operatorname{Irr}(N)$.*

Proof. Suppose that $\chi_N \notin \operatorname{Irr}(N)$. Then there exists a central extension G^* of G and $\alpha, \beta \in \operatorname{Irr}(G^*)$ such that $\chi = \alpha\beta$, where α, β are primitive nonlinear irreducible characters of G^*. Without loss of generality, we may assume that $\alpha(1) \leq \chi(1)^{1/2}$. Since 1_{G^*} is an irreducible constituent of $\alpha\overline{\alpha}$, the minimality of $\chi(1)$ implies that $\alpha\overline{\alpha}$ is a sum of linear characters. Hence $(G^*)' \leq \operatorname{Ker}(\alpha\overline{\alpha})$. In particular, $(G^*)' \leq \mathbf{Z}(\alpha)$. By Lemma 2.27 of [1], $\mathbf{Z}(\alpha)/\operatorname{Ker}\alpha \leq \mathbf{Z}(G^*/\operatorname{Ker}\alpha)$. If follows that $G^*/\operatorname{Ker}\alpha$ is nilpotent (of class at most 2). But α is nonlinear and primitive. This contradicts Theorem 6.22 of [1]. □

Next, we handle the primitive case of Theorem B. We refer the reader to [3] for the definition and basic properties of Gajendragadkar's p-special characters.

Lemma 2.2. *Let G be a solvable group. Suppose that $\chi \in \operatorname{Irr}(G)$ is a primitive faithful minimal character. Then $\chi(1)$ is a power of a prime p. Furthermore, if $p > 2$, then G is nilpotent-by-abelian.*

Proof. We may assume that $\chi(1) > 1$. By Theorem 2.17 of [3], χ factors as a product of p-special characters, where p runs over the set of prime divisors of $\chi(1)$. Since $\chi(1) = m(G)$, it follows that $\chi(1) = p^n$ is a power of a prime p. This proves the first part of the lemma.

Suppose now that $p > 2$. Let $q \neq p$ be a prime. Then $\chi_{\mathbf{O}_q(G)}$ is not irreducible. It follows from Lemma 2.1 that $\mathbf{O}_q(G)$ is central in G. Hence $\mathbf{F}(G) = E\mathbf{Z}(G)$, where $E = \mathbf{O}_p(G)$. Furthermore, using again Lemma 2.1, every normal abelian subgroup of G is central. Since G has a faithful irreducible character, Theorem 2.32 of [1] implies that $\mathbf{Z}(G)$ is cyclic. Now, Corollary 1.10 of [5] implies that E is extraspecial of exponent p. Since $\chi_E \in \operatorname{Irr}(E)$, we necessarily have that $|E| = p^{2n+1}$.

Note that $\mathbf{C}_G(E) = \mathbf{C}_G(\mathbf{F}(G)) = \mathbf{Z}(G)$, so $G/\mathbf{Z}(G)$ is isomorphic to a subgroup of $\operatorname{Aut}_{\mathbf{Z}(E)}(E)$. By [8], using again that $p > 2$, we deduce that $G/\mathbf{F}(G)$ is isomorphic to

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a subgroup of $\mathrm{Sp}(2n, p)$. By [4], $\mathrm{Sp}(2n, p)$ has a faithful irreducible representation of dimension $(p^n - 1)/2$. Hence $G/\mathbf{F}(G)$ has a faithful character of degree $\leq (p^n - 1)/2$. Since $m(G) = p^n$, we conclude that $G/\mathbf{F}(G)$ has a faithful character that is a sum of linear characters. We conclude that $G/\mathbf{F}(G)$ is abelian, as we wanted to prove. □

Now, we complete the proof of a slightly strengthened version of Theorem B.

Theorem 2.3. *Let G be a solvable group. Suppose that $\chi \in \mathrm{Irr}(G)$ is a faithful minimal character. If χ is induced from an odd degree character, then G is nilpotent-by-abelian.*

Proof. Let $H \leq G$ and $\beta \in \mathrm{Irr}(H)$ primitive such that $\beta^G = \chi$. Suppose first that $\beta(1) = 1$, so that $\chi(1) = |G : H|$. Since 1_G is an irreducible constituent of $(1_H)^G$ and $m(G) = |G : H| = (1_H)^G(1)$, we deduce that $(1_H)^G$ is a sum of linear characters. Hence $G' \leq \mathrm{Ker}(1_H)^G \leq H$. Thus $H \trianglelefteq G$ and by Clifford's theorem (Theorem 6.2 of [1]), χ_H is a sum of conjugates of β. In particular, χ_H is a sum of linear characters, so $H' \leq \mathrm{Ker}\,\chi = 1$. Hence G is metabelian and the result follows.

Now, we may assume that $\beta(1) > 1$ is odd. First, we will see that $H \trianglelefteq G$ and $G' = H'$. Note that $\chi(1) = |G : H|\beta(1) > |G : H|$. Hence $(1_H)^G$ is a sum of linear characters and $G' \leq H$, as before. In particular, $H \trianglelefteq G$. Thus H' is also normal in G and all the irreducible characters of G/H' have degree divisible by $|G : H| < \chi(1) = m(G)$. Hence, $\mathrm{Irr}(G/H')$ is a set of linear characters, and we conclude that $G' = H'$, as desired.

Now, we claim that $\beta(1) = m(H)$. Let $\mu \in \mathrm{Irr}(H)$ be nonlinear. Hence, there exists $\nu \in \mathrm{Irr}(G)$ nonlinear such that $[\mu^G, \nu] \neq 0$. Thus

$$|G : H|\beta(1) = \chi(1) = m(G) \leq \nu(1) \leq \mu^G(1) = |G : H|\mu(1).$$

We conclude that $\beta(1) \leq \mu(1)$. The claim follows.

Thus β is a primitive faithful minimal character of $H/\mathrm{Ker}\,\beta$. By Lemma 2.2, we have that $\beta(1)$ is a power of a prime p. Let $\beta = \beta_1, \ldots, \beta_t$ be the G-conjugates of β. Let $K_i = \mathrm{Ker}\,\beta_i$. Since $\chi = \beta^G$ is faithful, Lemma 5.11 of [1] implies that $\bigcap_{i=1}^t K_i = 1$. Since $p > 2$, by the second part of Lemma 2.2, we have that H/K_i is nilpotent-by-abelian. Write $F_i/K_i = \mathbf{F}(H/K_i)$, so that $G' = H' \leq F_i$ for every i. By Proposition 9.5 of [5], $\bigcap_{i=1}^t F_i = \mathbf{F}(H)$. Therefore, $G' \leq \mathbf{F}(H)$ and we conclude that G is nilpotent-by-abelian, as wanted. □

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