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ON SEPARABLE $\mathbb{A}^2$ AND $\mathbb{A}^3$-FORMS

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Abstract. In this paper, we will prove that any $\mathbb{A}^3$-form over a field k of characteristic zero is trivial provided it has a locally nilpotent derivation satisfying certain properties. We will also show that the result of Kambayashi on the triviality of separable $\mathbb{A}^2$-forms over a field k extends to $\mathbb{A}^2$-forms over any one-dimensional Noetherian domain containing $\mathbb{Q}$.

§1. Introduction

For any commutative ring R, we will use the notation $A = R^{[n]}$ to mean that A is a polynomial ring in n variables over R. Now let k be a field with algebraic closure $\bar{k}$ and A be a k-algebra. We say that A is an $\mathbb{A}^n$-form over k if $A \otimes_k \bar{k} = \bar{k}^{[n]}$. It is well known that separable $\mathbb{A}^1$-forms are trivial (i.e., $k^{[1]}$) and that there exist nontrivial purely inseparable $\mathbb{A}^1$-forms over fields of positive characteristic. An extensive study of such algebras was made by Asanuma in [3]. Kambayashi established [18] that separable $\mathbb{A}^2$-forms over a field k are trivial. Over any field of positive characteristic, the nontrivial purely inseparable $\mathbb{A}^1$-forms can be used to give examples of nontrivial $\mathbb{A}^n$-forms for any integer $n > 1$. However, the problem of the existence of nontrivial separable $\mathbb{A}^3$-forms over a field is still open in general. A few recent partial results on the triviality of separable $\mathbb{A}^3$-forms are mentioned in Remark 3.3.

Now let R be a ring containing a field k. An R-algebra A is said to be an $\mathbb{A}^n$-form over R with respect to k if $A \otimes_k \bar{k} = (R \otimes_k \bar{k})^{[n]}$, where $\bar{k}$ denotes the algebraic closure of k. In [10], Dutta investigated separable $\mathbb{A}^1$-forms over any ring R containing a field k and obtained Theorem 2.9. He also observed Theorem 2.10 for $\mathbb{A}^2$-forms over any Principal Ideal Domain (PID) containing $\mathbb{Q}$.

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In this paper, we prove a partial result on separable $\mathbb{A}^3$-forms over a field k (Theorem 3.2) and extend the results on $\mathbb{A}^2$-forms (Theorems 2.8 and 2.10) to any one-dimensional Noetherian $\mathbb{Q}$-algebra (Theorem 3.7) and to any $\mathbb{Q}$-algebra having a fixed point free locally nilpotent derivation (Theorem 3.8). After receiving a preprint of our paper, Prof. M. Miyanishi informed us that a part of Theorem 3.2 has also been obtained recently in [13] by a different approach (see Remark 3.3 (4) for a precise statement).

§2. Preliminaries

In this section we recall a few definitions and well-known results. All rings will be assumed to be commutative containing unity.

DEFINITION 2.1. An R-algebra A is said to be an $\mathbb{A}^r$-*fibration over* R if the following hold:

(i) A is finitely generated over R.
(ii) A is flat over R.
(iii) $A \otimes_R k(p) = k(p)^{[r]}$ for every prime ideal p of R.

DEFINITION 2.2. Let k be a field of characteristic p ($\geqslant 0$) with algebraic closure $\bar{k}$ and R a k-algebra. An R-algebra A is said to be an $\mathbb{A}^n$-*form over* R (with respect to k) if $A \otimes_k \bar{k} = (R \otimes_k \bar{k})^{[n]}$.

DEFINITION 2.3. Let $A = R^{[n]}$ and $F \in A$. F is said to be a *residual coordinate* in A if, for every prime ideal p of R, $A \otimes_R k(p) = k(p)[\bar{F}]^{[n-1]}$, where $\bar{F}$ denotes the image of F in $A \otimes_R k(p)$.

DEFINITION 2.4. A derivation D on a ring A is said to be a *locally nilpotent derivation* if, for each $a \in A$, there exists an integer $n \geqslant 0$ (depending on a), such that $D^n(a) = 0$.

DEFINITION 2.5. We say that a locally nilpotent derivation D on a ring A admits a *slice* if there exists s in A for which $D(s) = 1$.

DEFINITION 2.6. A locally nilpotent derivation D on a ring A is said to be *fixed point free* if $(DA) = A$, where (DA) is the ideal of A generated by $D(A)$.

DEFINITION 2.7. Let R be a ring and D a locally nilpotent R-derivation on the polynomial ring $A = R^{[n]}$. Then the *rank of the derivation* D, denoted by rk (D), is defined to be the least integer i such that there exist $X_1, \ldots, X_{n-i} \in \ker D$ satisfying $A = R[X_1, \ldots, X_{n-i}]^{[i]}$.

We first state Kambayashi's theorem [18, Theorem 3] on the triviality of separable $\mathbb{A}^2$-forms over fields.

THEOREM 2.8. *Let k and L be fields such that L is separably algebraic over k. Suppose A is a k-algebra such that $A \otimes_k L = L^{[2]}$. Then $A = k^{[2]}$.*

We now state a theorem on separable $\mathbb{A}^1$-forms over rings and a theorem on $\mathbb{A}^2$-forms over a PID due to Dutta [10, Theorem 7 and Remark 8].

THEOREM 2.9. *Let k be a field, L a separable field extension of k, R a k-algebra and A an R-algebra such that $A \otimes_k L$ is isomorphic to the symmetric algebra of a finitely generated rank one projective module over $R \otimes_k L$. Then A is isomorphic to the symmetric algebra of a finitely generated rank one projective module over R.*

THEOREM 2.10. *Let k be a field of characteristic zero, R a PID containing k and A an R-algebra such that A is an $\mathbb{A}^2$-form over R with respect to k. Then $A = R^{[2]}$.*

Next we quote a result on $\mathbb{A}^2$-fibrations due to Asanuma and Bhatwadekar [2, Theorem 3.8 and Remark 3.13].

THEOREM 2.11. *Let R be a one-dimensional Noetherian $\mathbb{Q}$-algebra. Let A be an $\mathbb{A}^2$-fibration over R. Then there exists $H \in A$ such that A is an $\mathbb{A}^1$-fibration over $R[H]$.*

The following result on residual coordinates was proved by Bhatwadekar and Dutta for Noetherian rings containing $\mathbb{Q}$ [5, Theorem 3.2] and later generalized by van den Essen and van Rossum for general $\mathbb{Q}$-algebras [11, Theorem 3.4].

THEOREM 2.12. *Let R be a $\mathbb{Q}$-algebra, $A = R^{[2]}$ and $F \in A$. If F is a residual coordinate in A then $A = R[F]^{[1]}$.*

Next we state a theorem which follows from a fundamental result in the theory of locally nilpotent derivations [12, Corollary 1.26].

THEOREM 2.13. *Let k be a field of characteristic zero, A a k-algebra, D a locally nilpotent derivation on A and $B := \ker D$. Then the following are equivalent:*

(1) *D admits a slice s.*
(2) *$A = B[s] = B^{[1]}$ and $D = d/ds$ on A.*
(3) *$D(A) = A$.*

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The following rigidity theorem is due to Daigle [8, Theorem 2.5].

THEOREM 2.14. *Let k be a field of characteristic zero and D be a locally nilpotent derivation on the polynomial ring $A = k^{[3]}$ with $\operatorname{rk}(D) = 2$. Let $X, W \in \ker D$ be such that $A = k[X]^{[2]} = k[W]^{[2]}$. Then $k[X] = k[W]$.*

The following result on fixed point free locally nilpotent derivations was obtained by Bhatwadekar and Dutta [6, Theorem 4.7] for any Noetherian $\mathbb{Q}$-algebra and later generalized to any $\mathbb{Q}$-algebra by Berson *et al.* [4, Theorem 3.5]; [12, Theorem 4.15].

THEOREM 2.15. *Let R be a $\mathbb{Q}$-algebra, $A = R[X, Y] = R^{[2]}$, D a fixed point free locally nilpotent R-derivation of A and $B = \ker D$. Then D admits a slice, $B = R^{[1]}$ and $A = B^{[1]}$.*

REMARK 2.16. A fixed point free locally nilpotent derivation on $k[X, Y, Z]$ has a slice [17]. But a fixed point free locally nilpotent R-derivation on $R[X, Y, Z]$ need not have a slice even if R is a PID [7, Example 5.6].

§3. Main results

In this section we will prove our main results. Note that if k is a field of characteristic zero, A a k-algebra and L a field extension of k, then any k-linear locally nilpotent derivation D on A can be extended to a locally nilpotent derivation $D \otimes 1_L$ on $A \otimes_k L$ such that $(D \otimes 1_L)(a \otimes \lambda) = D(a) \otimes \lambda$ for all $a \in A$ and $\lambda \in L$. We will first establish our main theorem on $\mathbb{A}^3$-forms over k (Theorem 3.2). We begin with a special case of this result which holds for $\mathbb{A}^3$-forms over a PID R with respect to k.

PROPOSITION 3.1. *Let k be a field of characteristic zero with algebraic closure $\bar{k}$, R a PID containing k and A be an $\mathbb{A}^3$-form over R with respect to k. Suppose that there exists an R-linear locally nilpotent derivation D on A such that $\operatorname{rk}(D \otimes 1_{\bar{k}}) = 1$. Then $A = R^{[3]}$.*

Proof. Since A is an $\mathbb{A}^3$-form over R with respect to k, there exists a finite extension L over k such that $A \otimes_k L = (R \otimes_k L)^{[3]}$ and $\operatorname{rk}(D \otimes_k 1_L) = 1$. Let $B = \ker D$. Set $\bar{R} := R \otimes_k L$, $\bar{A} := A \otimes_k L$, $\bar{B} := B \otimes_k L$ and $\bar{D} := D \otimes 1_L$. Then $\bar{A} = \bar{R}^{[3]}$ and $\ker \bar{D} = \bar{B}$. Since $\operatorname{rk}(\bar{D}) = 1$, we have $\bar{A} = \bar{B}^{[1]}$ and $\bar{B} = \bar{R}^{[2]}$. Hence, $B = R^{[2]}$ by Theorem 2.10. As $\operatorname{Pic}(B)$ is trivial, $A = B^{[1]}$ by Theorem 2.9. Thus, $A = R^{[3]}$. ∎

We now prove our main result on $\mathbb{A}^3$-forms.

THEOREM 3.2. *Let k be a field of characteristic zero with algebraic closure $\bar{k}$ and A be an $\mathbb{A}^3$-form over k. Suppose that there exists a k-linear locally nilpotent derivation D on A such that* $\operatorname{rk}(D \otimes 1_{\bar{k}}) \leqslant 2$. *Then $A = k^{[3]}$.*

Proof. Since A is an $\mathbb{A}^3$-form over k, there exists a finite Galois extension L over k with Galois group G such that $A \otimes_k L = L^{[3]}$ and $\operatorname{rk}(D \otimes_k 1_L) \leqslant 2$. Let $B = \ker D$. Set $\bar{A} := A \otimes_k L$, $\bar{B} := B \otimes_k L$ and $\bar{D} := D \otimes 1_L$. Then $\bar{A} = L^{[3]}$ and $\ker \bar{D} = \bar{B}$.

If $\operatorname{rk}(\bar{D}) = 1$, then $A = k^{[3]}$ by Proposition 3.1 (taking $R = k$).

We now consider the case $\operatorname{rk}(\bar{D}) = 2$. We then have $X \in \bar{B}$ such that $\bar{A} = L[X]^{[2]}$. We show that there exists $W \in L[X] \cap A$ such that $L[X] = L[W]$.

We identify A with its image in $\bar{A}$ under the map $a \to a \otimes 1$. Any $\sigma \in G$ can be extended to an A-automorphism of $\bar{A}$ by defining $\sigma(a \otimes l) = a \otimes \sigma(l)$, for all $a \in A$ and $l \in L$. Let

$$X = 1 \otimes l_0 + e_1 \otimes l_1 + \cdots + e_r \otimes l_r$$

where $1, e_1, \ldots, e_r$ form a part of a k-basis of A and l_i's are in L. Since the bilinear map $L \times L \longrightarrow k$ given by $(x, y) \mapsto \operatorname{Tr}(xy)$ is nondegenerate (where $\operatorname{Tr}(a) := \operatorname{Trace}(a)$ for all a in L), replacing X by lX (for some $l \in L$) if necessary we can assume that $\operatorname{Tr}(l_i) \neq 0$ for some $i \geqslant 1$. Thus

$$W := \sum_{\sigma \in G} \sigma(X) = 1 \otimes \operatorname{Tr}(l_0) + e_1 \otimes \operatorname{Tr}(l_1) + \cdots + e_r \otimes \operatorname{Tr}(l_r)$$

is an element of $A \setminus k$. Note that $\sigma \bar{D} = \bar{D}\sigma$ and hence $\sigma(X) \in \bar{B}$. Since σ is an automorphism of $\bar{A}$, by Theorem 2.14, $L[X] = L[\sigma(X)]$. Hence $\sigma(X)$ is linear in X for each σ and hence $\deg_X W \leqslant 1$. But as $B \cap L = k$, $W \notin L$, so that $\deg_X W = 1$ which implies $L[X] = L[W]$.

So $\bar{A} = L[W]^{[2]} = (k[W] \otimes_k L)^{[2]}$. By Theorem 2.10, we get $A = k[W]^{[2]}$. ∎

REMARK 3.3. Let k be a field of characteristic zero with algebraic closure $\bar{k}$ and A an $\mathbb{A}^3$-form over k. We record below a few other results on the triviality of A.

(1) Daigle and Kaliman have proved [9, Corollary 3.3] that if A admits a fixed point free locally nilpotent derivation D, then $A = k^{[3]}$.

(2) Daigle and Kaliman have also proved [9, Proposition 4.9] that if A contains an element f which is a coordinate of $A \otimes_k \bar{k}$, then $A = k^{[3]}$ and f is a coordinate of A.

(3) Koras and Russell have proved [19, Theorem C] that if A admits an effective action of a reductive algebraic k-group of positive dimension, then $A = k^{[3]}$.

(4) Recently, Gurjar, Masuda and Miyanishi have shown [13] that $A = k^{[3]}$ if A admits either a fixed point free locally nilpotent derivation or a nonconfluent action of a unipotent group of dimension two. Their results give an alternative approach to Theorem 3.2 for the case rk $(\bar{D}) = 1$.

We now extend Theorems 2.8 and 2.10 to more general rings. For convenience, we first record a few easy lemmas.

LEMMA 3.4. *Let R be a ring containing $\mathbb{Q}$ and $A = R^{[2]}$. If $H \in A$ is such that A is an $\mathbb{A}^1$-fibration over $R[H]$, then $A = R[H]^{[1]}$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R and let $\bar{H}$ denote the image of H in $A \otimes_R k(\mathfrak{p})$. Then $A \otimes_R k(\mathfrak{p})$ is an $\mathbb{A}^1$-fibration over the PID $k(\mathfrak{p})[\bar{H}]$ and hence $A \otimes_R k(\mathfrak{p}) = k(\mathfrak{p})[\bar{H}]^{[1]}$. Thus, H is a residual coordinate of A. Hence, by Theorem 2.12, $A = R[H]^{[1]}$. ∎

We now observe that Theorem 2.8 extends to separable $\mathbb{A}^2$-forms over a field K with respect to a subfield k.

LEMMA 3.5. *Let k be a field and K a field extension of k. Let A be a K-algebra such that $A \otimes_k L = (K \otimes_k L)^{[2]}$, for some finite separable field extension L of k. Then $A = K^{[2]}$.*

Proof. By hypothesis, we have $A \otimes_K (K \otimes_k L) = (K \otimes_k L)^{[2]}$. Since L over k is a finite separable extension, $K \otimes_k L$ is a finite direct product of separable extensions L_i over K. Hence, we have $A \otimes_K L_i = {L_i}^{[2]}$ (for each i), which implies $A = K^{[2]}$ by Theorem 2.8. ∎

We now show that $\mathbb{A}^2$-forms are $\mathbb{A}^2$-fibrations.

LEMMA 3.6. *Let k be a field of characteristic zero, R be a k-algebra and A be an R-algebra. Let A be an $\mathbb{A}^2$-form over R with respect to k. Then A is an $\mathbb{A}^2$-fibration over R.*

Proof. Let $A \otimes_k \bar{k} = (R \otimes_k \bar{k})[X, Y]$, where $\bar{k}$ is an algebraic closure of k. Let $X = \sum_{i=0}^{n} a_i \otimes \lambda_i$ and $Y = \sum_{i=0}^{m} b_i \otimes \mu_i$, where $a_i, b_i \in A$ and $\lambda_i, \mu_i \in \bar{k}$. Then $R[a_1, \ldots, a_n, b_1, \ldots, b_m] \subseteq A$ and the induced map $R[a_1, \ldots, a_n, b_1, \ldots, b_m] \otimes_k \bar{k} \longrightarrow A \otimes_k \bar{k}$ is an isomorphism. Hence $\bar{k}$ being faithfully flat over k, we have $A = R[a_1, \ldots, a_n, b_1, \ldots, b_m]$. Thus A is a finitely generated R-algebra. Again, as $A \otimes_k \bar{k}$ is faithfully flat over

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$R \otimes_k \bar{k}$ and $\bar{k}$ is faithfully flat over k, A is flat over R. Now it suffices to show $A \otimes_R k(p) = k(p)^{[2]}$, for each prime ideal p of R.

Let p be an arbitrary prime ideal of R. By hypothesis there exists a finite separable extension L of k such that $A \otimes_k L = (R \otimes_k L)^{[2]}$. Hence, $k(p) \otimes_R (A \otimes_k L) = k(p) \otimes_R (R \otimes_k L)^{[2]} = (k(p) \otimes_k L)^{[2]}$. Hence by Lemma 3.5, $A \otimes_R k(p) = k(p)^{[2]}$.

Thus, A is an $\mathbb{A}^2$-fibration over R. ∎

We now extend Theorems 2.8 and 2.10 to any one-dimensional Noetherian ring containing a field of characteristic zero.

THEOREM 3.7. *Let k be a field of characteristic zero and R a one-dimensional Noetherian k-algebra. If A is an $\mathbb{A}^2$-form over R with respect to k, then there exists a finitely generated rank one projective R-module Q such that $A \cong (\mathrm{Sym}_R(Q))^{[1]}$.*

Proof. By Lemma 3.6, A is an $\mathbb{A}^2$-fibration over R and hence by Theorem 2.11, there exists $H \in A$ such that A is an $\mathbb{A}^1$-fibration over $R[H]$. Let $\bar{k}$ be an algebraic closure of k, $\bar{A} := A \otimes \bar{k}$ and $\bar{R} := R \otimes \bar{k}$. Since $\bar{A} = \bar{R}^{[2]}$ and $\bar{A}$ is an $\mathbb{A}^1$-fibration over $\bar{R}[H]$, we have $\bar{A} = \bar{R}[H]^{[1]}$ by Lemma 3.4. Thus by Theorem 2.9, $A \cong \mathrm{Sym}_{R[H]}(Q_1)$, for some finitely generated rank one projective $R[H]$-module Q_1. Set $R_{\mathrm{red}} := R/nil(R)$. Now

$$A/nil(R)A \cong \mathrm{Sym}_{R[H]}(Q_1) \otimes_R R_{\mathrm{red}} = \mathrm{Sym}_{R_{\mathrm{red}}[H]}(Q_1 \otimes_R R_{\mathrm{red}}).$$

Now by [14, Section 2, Lemma 1], there exists a finitely generated rank one projective R_{red}-module Q' such that $Q_1 \otimes_R (R_{\mathrm{red}}) = Q' \otimes_{R_{\mathrm{red}}} R_{\mathrm{red}}[H]$. Thus,

$$A/nil(R)A \cong \mathrm{Sym}_{R_{\mathrm{red}}}(Q') \otimes_{R_{\mathrm{red}}} R_{\mathrm{red}}[H] = (\mathrm{Sym}_{R_{\mathrm{red}}}(Q'))^{[1]}.$$

Now by [15, Proposition 2.3.5], there exists a finitely generated rank one projective R-module Q such that $Q \otimes_R R_{\mathrm{red}} = Q'$ and hence $A = (\mathrm{Sym}_R(Q))^{[1]}$. ∎

The following result shows that under the additional hypothesis that A has a fixed point free locally nilpotent R-derivation, Theorem 3.7 can be extended to any ring containing a field of characteristic zero.

THEOREM 3.8. *Let k be a field of characteristic zero, R a ring containing k and A be an $\mathbb{A}^2$-form over R with respect to k. Suppose A has a fixed point free locally nilpotent R-derivation. Then there exists a finitely generated rank one projective R-module Q such that $A \cong (\mathrm{Sym}_R(Q))^{[1]}$.*

Proof. Let L be a finite extension of k such that $A \otimes_k L = (R \otimes_k L)^{[2]}$. Let D be a fixed point free locally nilpotent R-derivation of A and $B = \ker D$. Set $\bar{R} := R \otimes_k L$, $\bar{A} := A \otimes_k L$, $\bar{B} := B \otimes_k L$ and $\bar{D} := D \otimes 1_L$. Then $\bar{A} = \bar{R}^{[2]}$, $\ker \bar{D} = \bar{B}$ and $\bar{D}$ is a fixed point free locally nilpotent derivation of $\bar{A}$. Hence, by Theorem 2.15, $\bar{D}$ has a slice and $\bar{B} = \bar{R}^{[1]}$. Now, by Theorem 2.13, $D(\bar{A}) = \bar{A}$. Thus, $D(A) \otimes_k L = D(\bar{A}) = \bar{A} = A \otimes_k L$. Hence, by faithful flatness of L over k, $D(A) = A$. So $A = B^{[1]}$ by Theorem 2.13. Since $\bar{B} = \bar{R}^{[1]}$, by Theorem 2.9, $B = \mathrm{Sym}_R(Q)$ and hence $A = (\mathrm{Sym}_R(Q))^{[1]}$, for some finitely generated rank one projective R-module Q. ∎

REMARK 3.9. Kahoui and Ouali have shown [16, Corollary 3.2] that when R is regular the above result holds for any $\mathbb{A}^2$-fibration over R.

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