Elegant special cases of Van Aubel's theorem

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Van Aubel's theorem for quadrilaterals concerns any quadrilateral on whose sides squares are constructed*. The theorem states that the segments connecting the centres of the squares lying on opposing sides are equal and perpendicular. In other words, *PR* is equal and perpendicular to *SQ*, as shown in Figure 1. The theorem also holds for non-convex quadrilaterals, as shown in Figure 2, and for a self-intersecting quadrilateral, as shown in Figure 3. The squares may be constructed outwards (Figure 1) or inwards (Figure 4) on the quadrilateral. The theorem holds even when the quadrilateral degenerates into a triangle, a straight segment, and even two connected segments.

Van Aubel's theorem for quadrilaterals has different proofs, including some using elementary geometry [2, 3, 4, 5].

In Figures 5-17 we give examples of different arrangements of 3 or 4 squares, where in each arrangement *PR* is equal and perpendicular to *SQ*. The example in Figure 18 demonstrates the fact that if the centres of the squares constructed on one pair of opposing sides of the quadrilateral (points *Q* and *S*) coincide, then so do the centres of the squares constructed on the other pair (points *P* and *R*). The examples of Van Aubel's degenerate quadrilateral in Figures 5, 6, 9 and 10 are found in [6, p. 179]. The rest of the examples are original, and their goal is to enrich a subject that has fascinated many geometers for many years. We are certain that the reader will find some additional elegant cases of the theorem and so emphasise the beauty geometry.

For each example, it is quite easy to find mathematical proofs that segments *PR* and *SQ* are equal in length and perpendicular using different methods (Euclidean geometry, trigonometry, analytic geometry, vector analysis, complex analysis). It is interesting and pleasing to note that each example represents a particular case of Van Aubel's theorem, and the reader can be easily convinced of this. Quadrilateral *ABCD* is marked in bold black. In Figures 5-9 the quadrilateral degenerates into a segment, in Figures 10-11 it degenerates into a triangle, in Figures 12-15 it degenerates into a triangle with a segment, and in Figure 17 it degenerates into two connected segments. If one knows the theorem, in order to prove the correctness in each example, one can cite the theorem without having to give a proof requiring the use of mathematical tools and other theorems.

In conclusion we suggest two nice quickies (the answers are obtained immediately by the use of Van Aubel's theorem). Given an isosceles trapezium with the parallel sides $AB = a$, $CD = b$ and with height h.

* Van Aubel is also well-known for a theorem about ratios and Cevians. Both are mentioned in [1].

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1. Find the distance QS between the centres of the squares lying on the legs of the trapezium – see Figure 19 (the answer is: $\frac{1}{2}(a + b) + h$).
2. Find the distance QS between the centres of the squares lying on the diagonals of the trapezium – see Figure 20 (the answer is: $\frac{1}{2}(a - b) + h$).

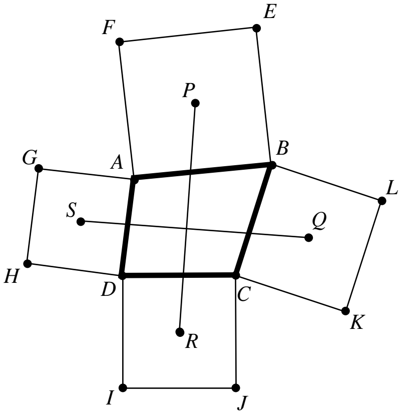

FIGURE 1: Convex case

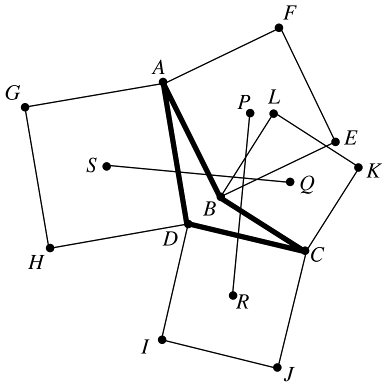

FIGURE 2: Non-convex case

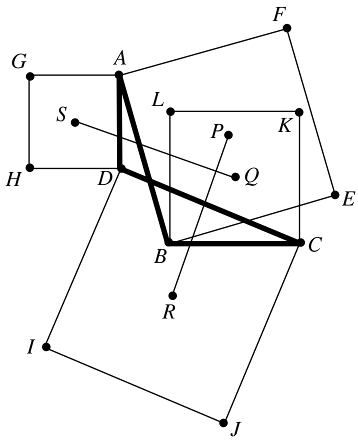

FIGURE 3: Self-intersecting case

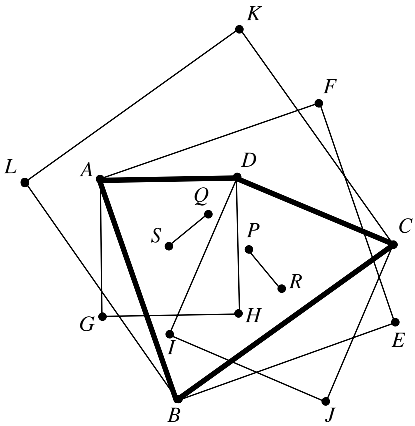

FIGURE 4: Squares internally

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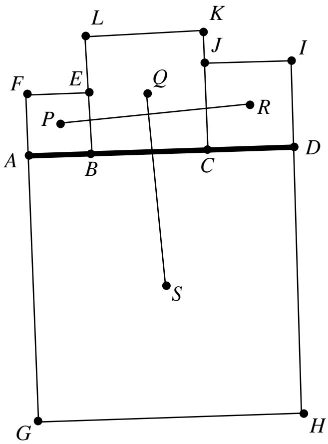

FIGURE 5: Collinear in order
A, *B*, *C*, *D*

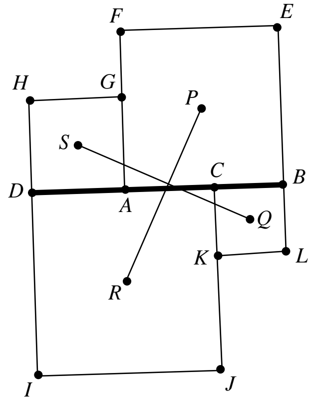

FIGURE 6 : Collinear in order
B, *C*, *A*, *D*

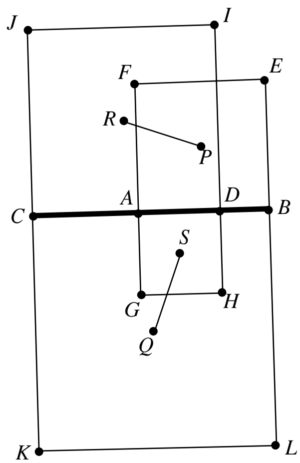

FIGURE 7 : Collinear in order
C, *A*, *D*, *B*

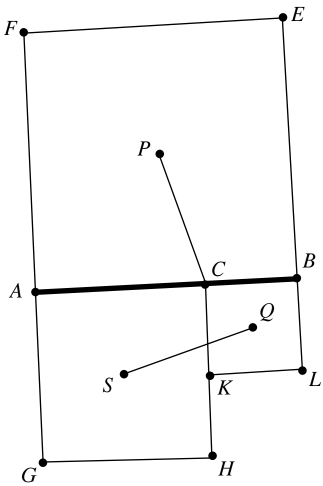

FIGURE 8 : Collinear with
$C = D = R$

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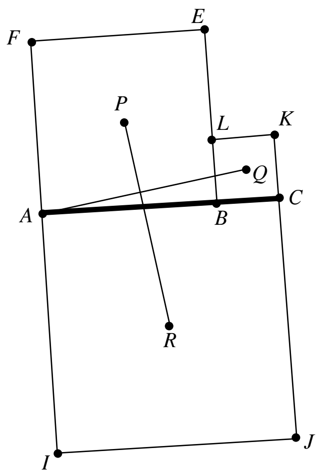

FIGURE 9 : Collinear with
$A = D = S$

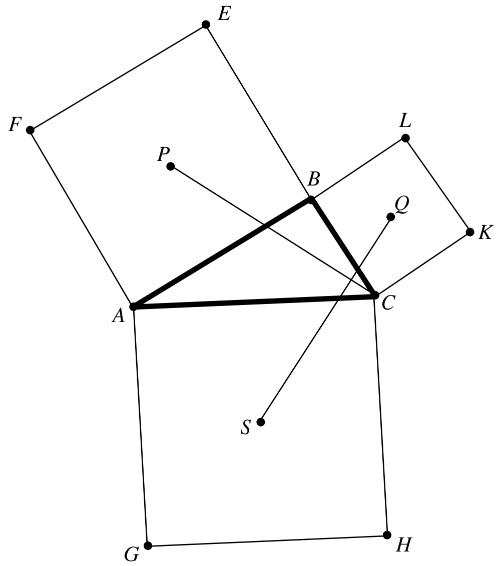

FIGURE 10 : Triangle with
$C = D = R$

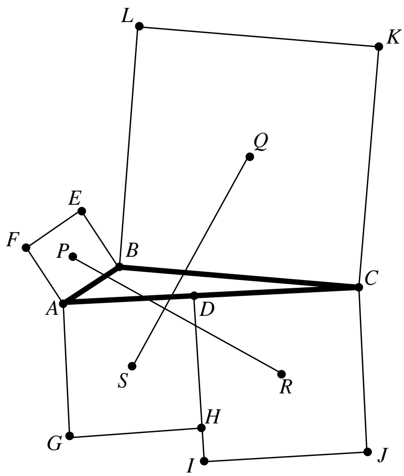

FIGURE 11 : Triangle with D on AC internally

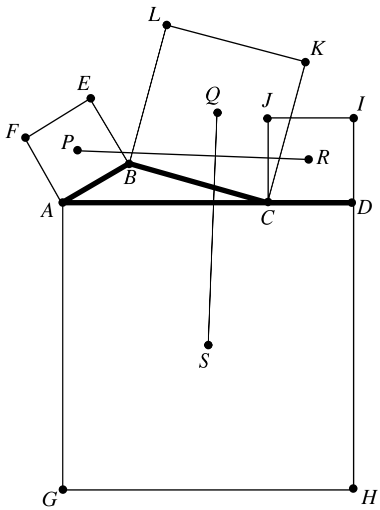

FIGURE 12 : Triangle with D on AC externally (C on AD)

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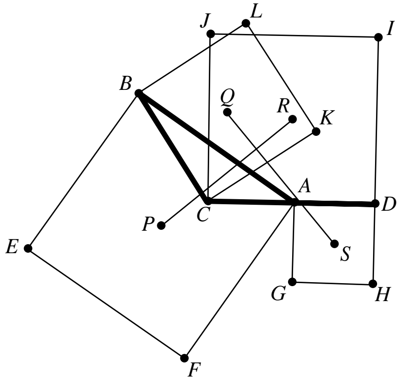

FIGURE 13 : Triangle with
D on *AC* externally (*A* on *CD*)

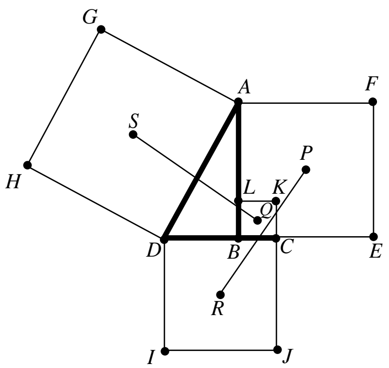

FIGURE 14: Two right triangles
ABC and *ABD*

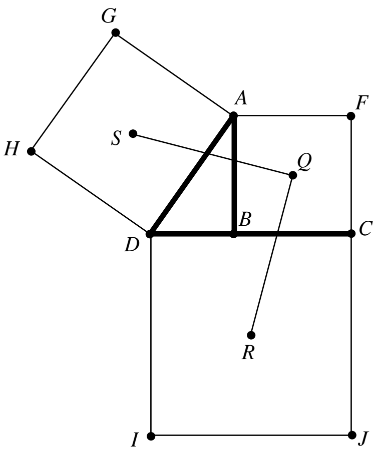

FIGURE 15: Two right triangles
ABC and *ABD* ($Q = P$)

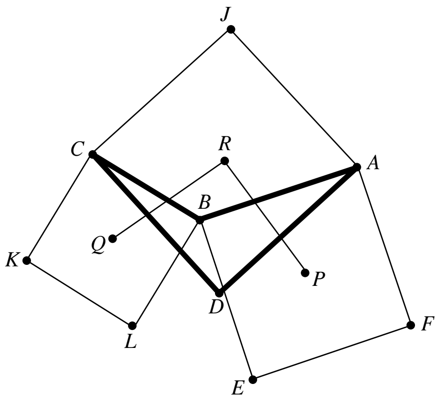

FIGURE 16: Non-concave case with *D*
a right-angle and $CD = AD$

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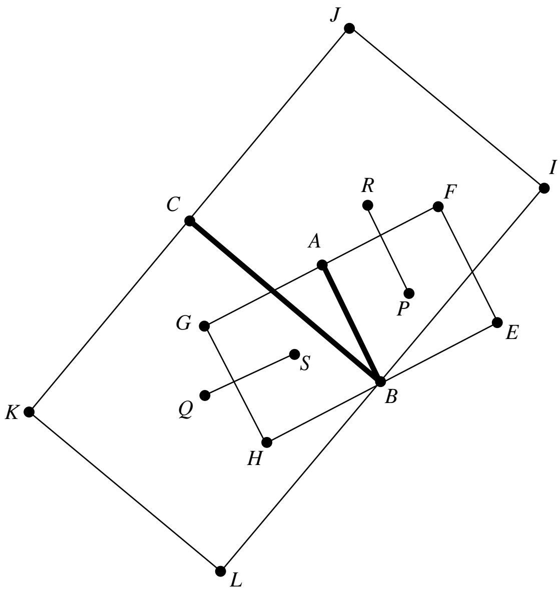

FIGURE 17: AB and CB segments with $D = B$

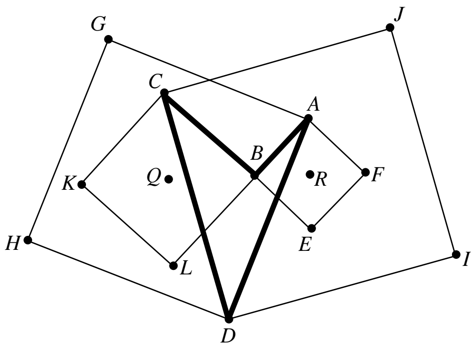

FIGURE 18: Degenerate case when $Q = S$ and $R = P$

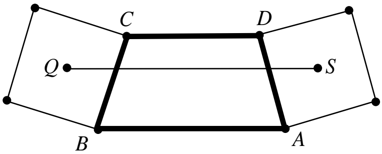

FIGURE 19: $QS = ?$

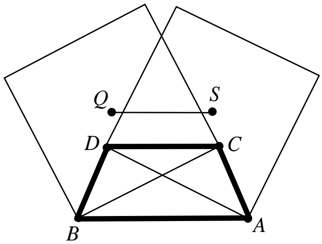

FIGURE 20: $QS = ?$

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