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A sharp ultimate bound of the solutions to some second-order evolution equations with nonlinear damping

Faouzia Aloui
UPMC Paris 6, UMR 7598, Laboratoire Jacques Louis Lions,
4 place Jussieu, 75252 Paris Cedex 05, France (aloui@ann.jussieu.fr)

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We establish a sharp ultimate bound of the solution to a second-order dissipative nonlinear equation with bounded forcing term.

Keywords: second-order equation; ultimate boundedness; bounded forcing term

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1. Introduction

Let Ω be a finite measure space, let $H = L^2(\Omega)$ be endowed with the norm denoted by $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|_2$ and let V be a real Hilbert space endowed with the norm denoted by $\|\cdot\|$ such that

$$V \subset H \subset V'$$

with continuous and dense embeddings, where V' is the topological dual of V.

We denote by $(\cdot,\cdot)$ the inner product in H and by $\langle\cdot,\cdot\rangle$ the duality product between V and V'·

In this paper, we look for a sharp ultimate bound of the solution to a dissipative second-order nonlinear evolution equation. More precisely, let us consider the nonlinear evolution equation

$$\left.\begin{aligned} u_{tt}(t,x) + |u_t(t,x)|^\alpha u_t(t,x) + Au(t,x) = f(t,x) \quad \text{in } \mathbb{R}^+ \times \Omega,\\ u(0,x) = u_0, \quad u_t(0,x) = u_1 \quad \text{on } \Omega, \end{aligned}\right\} \tag{1.1}$$

where $\alpha \geqslant 0$ and $f \in L^\infty(\mathbb{R}^+, L^2(\Omega))$.

$A\colon V \to V'$ is the duality map with domain denoted by

$$D(A) = \{v \in V\colon Av \in H\}.$$

We observe that A is characterized by the property

$$\forall (u,v) \in V \times V, \quad \langle Au, v\rangle_{V',V} = (u,v)_V.$$

We set

$$\lambda_1(A) = \lambda_1 = \inf\{\|u\|^2;\ u \in D(A),\ \|u\|_2^2 = 1\}. \tag{1.2}$$

Throughout the paper, we shall denote the norm in $L^p(\Omega)$ by

$$\|z\|_p = \left(\int_\Omega |z|^p \,\mathrm{d}x\right)^{1/p} \quad \text{for any } p \in [1, +\infty[.$$

Assuming $V \subset L^{\alpha+2}(\Omega)$ with continuous embedding, we define

$$c_1(\alpha) = \sup\{\|u\|_{\alpha+2};\ u \in V,\ \|u\| = 1\}. \tag{1.3}$$

Under this condition, we observe that $v \mapsto |v|^\alpha v$ sends V to V' continuously.

We set

$$\|f\|_{\infty,2} = \operatorname*{ess\,sup}_{t \geqslant 0} \|f(t,\cdot)\|_2.$$

Writing (1.1) as a system by introducing $u' = v$, we obtain

$$U'(t) + \mathcal{A}U(t) = F(t), \quad 0 \leqslant t \leqslant T, \tag{1.4}$$

where $U = (u, u')$, $F = (0, f)$ and the operator $\mathcal{A}$ is defined on the Hilbert space $\mathcal{H} = V \times H$ by

$$D(\mathcal{A}) = \{(u, v) \in V \times V;\ Au + |v|^\alpha v \in H\}$$

and

$$\mathcal{A}(u, v) = (-v, Au + |v|^\alpha v) \quad \forall (u, v) \in D(\mathcal{A}).$$

As in [1, 2, 4], it is not difficult to establish that the operator $\mathcal{A}$ is a maximal monotone operator in $V \times H$. Since $\mathcal{A}$ is a maximal monotone, for each $T > 0$, $U_0 \in D(\mathcal{A})$ and $F \in W^{1,1}(0, T; \mathcal{H})$ there is a unique solution $U \in W^{1,1}(0, T; \mathcal{H})$ with $U(t) \in D(\mathcal{A})$ for almost all $t \in (0, T)$, $U(0) = U_0$, satisfying (1.4) for all $t \in (0, T)$.

As a consequence, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+, H)$ and for each $(u_0, u_1) \in V \times H$ there is a unique weak solution $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$ of (1.1) defined by density on (f, u_0, u_1) such that $u(0) = u_0 \in V$, $u'(0) = u_1 \in H$.

Our goal is to obtain an estimate for the ultimate bound

$$\varlimsup_{t \to \infty} [\max(\|u(t)\|, \|u'(t)\|_2)].$$

In order to do this, we use a method introduced by Prouse in 1965 [10]. For $t \geqslant 0$, the main idea is to distinguish two possible cases concerning the energy $E(t)$ of a given solution u:

(a) $E(t + l) \leqslant E(t)$,

(b) $E(t + l) > E(t)$,

where $l = 8\lambda_1^{-1/2}$.

Prouse [10] applied this method with $l = 1$ to prove the boundedness of solutions on $\mathbb{R}^+$ of the nonlinear dissipative wave equation

$$\begin{aligned} u_{tt} - \Delta u + \beta(u_t) &= f(t, x) && \text{in } \mathbb{R}^+ \times \Omega, \\ u(t, x) &= 0 && \text{on } \mathbb{R}^+ \times \partial\Omega, \end{aligned}$$

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where Ω is an open subset of $\mathbb{R}^N$ with regular boundary, β is coercive with polynomial growth at infinity and $f \in L^\infty(\mathbb{R}^+, L^2(\Omega))$.

Nakao [8, 9] used the approach of Prouse [10] with $l = 1$ to prove the energy decay of the solutions of the following two equations.

The first is a nonlinear evolution equation:

$$u''(t) + B(t)u'(t) + A(t)u(t) = f(t), \quad t \in \mathbb{R}^+.$$

Let W and Z be two Banach spaces such that $W \subset Z$ with dense injection and let W^* and Z^* be the topological duals of W and Z.

$A(t)\colon W \to W^*$ and $B(t)\colon Z \to Z^*$ are nonlinear bounded time-dependent operators. $f \in L^{(r+2)/(r+1)}_{\mathrm{loc}}(\mathbb{R}^+, Z^*)$, $r \geqslant 0$, such that

$$\biggl(\int_t^{t+1} \|f(s)\|_{Z^*}^{(r+2)/(r+1)}\,\mathrm{d}s\biggr)^{(r+1)/(r+2)}$$

tends to 0 fast enough as $t \to \infty$.

The second is a nonlinear dissipative wave equation:

$$\begin{gathered} u_{tt} - \Delta u + \beta(u_t) + \lambda^2(x)u = f(t,x) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0,x) = u_0, \quad u_t(0,x) = u_1 \quad \text{on } \mathbb{R}^N, \end{gathered}$$

where β is a C^1 function defined on $\mathbb{R}$, $\lambda(x) \geqslant 0$ is a locally bounded measurable function defined on $\mathbb{R}^N$, and

$$f \in L^{(r+2)/(r+1)}_{\mathrm{loc}}(\mathbb{R}^+, L^{(r+2)/(r+1)}(\mathbb{R}^N)), \quad r \geqslant 0,$$

such that

$$\biggl(\int_t^{t+1} \|f(s)\|_{(r+2)/(r+1)}^{(r+2)/(r+1)}\,\mathrm{d}s\biggr)^{(r+1)/(r+2)} \to 0$$

fast enough as $t \to \infty$.

In [3], Haraux and Biroli studied the following equation:

$$\left.\begin{aligned} u_{tt} - \Delta u + \beta(u_t) - \lambda u_t = f(t,x) \quad & \text{in } \mathbb{R}^+ \times \Omega, \\ u(t,x) = 0 \quad & \text{on } \mathbb{R}^+ \times \partial\Omega, \end{aligned}\right\} \tag{1.5}$$

where $\lambda > 0$, β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$ and

$$f \in S^2(\mathbb{R}^+, L^2(\Omega)) = \biggl\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}^+, L^2(\Omega));\ \sup_{t \geqslant 0} \int_t^{t+1} \|f(s)\|_2^2\,\mathrm{d}s < +\infty \biggr\}.$$

By using the method of Prouse [10], they chose $l = 4\lambda_0^{-1/2}$ with

$$\lambda_0 = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}$$

to establish the boundedness of solutions of (1.5) in $H_0^1(\Omega) \times L^2(\Omega)$ for $t \geqslant 0$.

In 1982, Haraux [6] established the boundedness properties for global solutions of the abstract evolution equation

$$u'' + g(t, u'(t)) + \Phi'(u) = 0, \quad t \in \mathbb{R}^+,$$

where $\Phi \in C^1(V,\mathbb{R})$ and $g\colon \mathbb{R}^+ \times V \to V'$ is a possibly multi-valued operator satisfying relevant coercivity and growth conditions. Here, the method of Prouse was applied with $l = 4\lambda_0^{-1/2}\alpha^{-1/2}\beta^{-1}$, where α and β appear in the assumptions on g and Φ and

$$\lambda_0 = \inf_{u\in V, u\neq 0} \frac{\|u\|^2}{\|u\|_2^2}.$$

Moreover, he established that there exists $R > 0$ for which

$$\forall t \geqslant 0, \quad \overline{\lim_{t\to\infty}}[\max(\|u(t)\|, \|u'(t)\|_2)] \leqslant R,$$

where R depends only on the assumptions of g and Φ. R is called an ultimate bound of the solutions $U(t) = (u(t), u'(t))$.

The remainder of the paper is organized as follows: in § 2 we estimate the ultimate bound of solutions of (1.1) by using the method of Prouse with $l = 8\lambda_1^{-1/2}$. Section 3 is devoted to some applications.

2. Main theorem

THEOREM 2.1. *Let $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, L^2(\Omega))$ be a weak solution of (1.1). Then $u \in L^\infty(\mathbb{R}^+, V)$, $u' \in L^\infty(\mathbb{R}^+, L^2(\Omega))$ and we have*

$$\overline{\lim_{t\to\infty}}[\max(\|u(t)\|, \|u'(t)\|_2)] \leqslant \sqrt{2K},$$

where

$$K = \left(\frac{1}{\lambda_1} + 4|\Omega|^{\alpha/(\alpha+2)}(c_1(\alpha))^2 + \frac{16}{\sqrt{\lambda_1}}|\Omega|^{\alpha/2(\alpha+1)}\right)\|f\|_{\infty,2}^2 + \left(3|\Omega|^{\alpha/(\alpha+1)} + \frac{16}{\sqrt{\lambda_1}}|\Omega|^{\alpha/2(\alpha+1)}\right)\|f\|_{\infty,2}^{2/(\alpha+1)}.$$

REMARK 2.2. The result of the above theorem is sharp in the sense that both terms in K are necessary in general and cannot be replaced by any other power of the norm. Indeed, let us consider the following special cases.

(1) If $f(t,\cdot) = f_0(\cdot)$, we consider the following equation, whose solutions can be considered as special solutions of (1.1):

$$Au = f_0.$$

If $f_0 = \lambda\varphi_1$, then

$$u = A^{-1}(\lambda\varphi_1) = \frac{\lambda}{\lambda_1}\varphi_1,$$

where λ is a positive constant. Hence,

$$\|u\|^2 = \left(\frac{\lambda}{\lambda_1}\right)^2\|\varphi_1\|^2 = C\lambda^2$$

and

$$\|f_0\|_2^{2/(\alpha+1)} = (\lambda\|\varphi_1\|_2)^{2/(\alpha+1)} = C'\lambda^{2/(\alpha+1)}.$$

If $\lambda \to \infty$, then the term $\|f\|_2^{2/(\alpha+1)}$ cannot control the ultimate bound. More precisely, the term $\|f\|_2^2$ is necessary for all $\alpha \geqslant 0$.

(2) For any solution v of

$$v'' - \Delta v = 0, \qquad v|_{\partial\Omega} = 0,$$

we can write

$$v'' - \Delta v + |v'|^\alpha v' = |v'|^\alpha v', \qquad v|_{\partial\Omega} = 0.$$

Let us consider the special solution $v(t,x) = \varepsilon \sin(\sqrt{\lambda_1} t)\varphi_1(x)$.

Then

$$\|v'(t,\cdot)\|_2 = \varepsilon\sqrt{\lambda_1}|\cos(\sqrt{\lambda_1}t)|\|\varphi_1\|_2$$

and

$$\overline{\lim_{t\to\infty}} \|v'(t,\cdot)\|_2 = \sup_{t\geqslant 0} \|v'(t,\cdot)\|_2 = \varepsilon\sqrt{\lambda_1}\|\varphi_1\|_2.$$

Setting $f = |v'|^\alpha v'$, we have

$$\int_\Omega |f|^2 \,\mathrm{d}x = \int_\Omega |v'|^{2\alpha+2} \,\mathrm{d}x \leqslant \int_\Omega \varepsilon^{2\alpha+2}\lambda_1^{\alpha+1}|\varphi_1|^{2\alpha+2} \,\mathrm{d}x \leqslant A\varepsilon^{2\alpha+2}$$

for some positive constant A.

The term $\|f\|_2^2$ cannot control the ultimate bound for ε small. To be more precise, the term $\|f\|_2^{2/(\alpha+1)}$ is also necessary for all $\alpha \geqslant 0$.

Proof of theorem 2.1. Let $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+, V) \cap W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^+, L^2(\Omega))$ be a solution of (1.1) such that $u(0) = u'(0) = 0$. The energy of u is defined by

$$E(t) = \tfrac{1}{2}(\|u(t)\|^2 + \|u'(t)\|_2^2) \quad \text{for } t \geqslant 0. \tag{2.1}$$

Multiplying (1.1) by u' and integrating over Ω, we have, for $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = (f(t,\cdot), u'(t,\cdot)) - \|u'(t,\cdot)\|_{\alpha+2}^{\alpha+2}. \tag{2.2}$$

Let us introduce

$$l = 8\lambda_1^{-1/2}.$$

We first estimate $\sup_{t\geqslant 0} E(t)$ and to do that we distinguish the following two cases:

$$E(t+l) \leqslant E(t), \tag{2.3}$$

$$E(t+l) > E(t). \tag{2.4}$$

By using Young's inequality in (2.2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqslant \frac{\alpha+1}{\alpha+2}\|f(t,\cdot)\|_{(\alpha+2)/(\alpha+1)}^{(\alpha+2)/(\alpha+1)} + \frac{1}{\alpha+2}\|u'(t,\cdot)\|_{\alpha+2}^{\alpha+2} - \|u'(t,\cdot)\|_{\alpha+2}^{\alpha+2}.$$

Writing $\beta = (\alpha+2)/(\alpha+1)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leqslant \|f(t,\cdot)\|_\beta^\beta. \tag{2.5}$$

Let $s \in [t, t+l]$. Integrating (2.5) from t to s, we deduce

$$E(s) \leqslant E(t) + l\|f\|_{\infty,\beta}^\beta.$$

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As a consequence of (2.4), we obtain

$$\sup_{s\in[t,t+l]} E(s) \leqslant E(t+l) + l\|f\|_{\infty,\beta}^{\beta}. \tag{2.6}$$

By integrating (2.5) from $s \in [t, t+l]$ to $t+l$, we obtain

$$E(t+l) \leqslant E(s) + l\|f\|_{\infty,\beta}^{\beta}.$$

Integrating the last inequality from t to $t+l$ and dividing through by l, we find

$$E(t+l) \leqslant \frac{1}{l}\int_t^{t+l} E(s)\,\mathrm{d}s + l\|f\|_{\infty,\beta}^{\beta}. \tag{2.7}$$

Then, we find

$$\sup_{s\in[t,t+l]} E(s) \leqslant \frac{1}{l}\int_t^{t+l} E(s)\,\mathrm{d}s + 2l\|f\|_{\infty,\beta}^{\beta}. \tag{2.8}$$

Now multiplying (1.1) by u', integrating from t to $t+l$ and using (2.4), we obtain

$$\int_t^{t+l} \|u'(s,\cdot)\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}s \leqslant \int_t^{t+l} (f(s,\cdot), u'(s,\cdot))\,\mathrm{d}s.$$

By Hölder's inequality, we get

$$\int_t^{t+l} \|u'(s,\cdot)\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}s \leqslant \int_t^{t+l} \|f(s,\cdot)\|_{\beta}\|u'(s,\cdot)\|_{\alpha+2}\,\mathrm{d}s,$$

and by using Young's inequality we obtain

$$\int_t^{t+l} \|u'(s,\cdot)\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}s \leqslant l\|f\|_{\infty,\beta}^{\beta}. \tag{2.9}$$

Multiplying (1.1) by u and integrating from t to $t+l$, we have

$$\begin{aligned}\int_t^{t+l} \|u(s,\cdot)\|^2\,\mathrm{d}s &= \int_t^{t+l} (f(s,\cdot), u(s,\cdot))\,\mathrm{d}s + \int_t^{t+l} \|u'(s,\cdot)\|_2^2\,\mathrm{d}s \\ &\quad - [\langle u(s,\cdot), u'(s,\cdot)\rangle]_t^{t+l} - \int_t^{t+l} \langle |u'(s,\cdot)|^{\alpha} u'(s,\cdot), u(s,\cdot)\rangle\,\mathrm{d}s.\end{aligned} \tag{2.10}$$

Applying Young's inequality to the first term and using (1.2), we have

$$\int_t^{t+l} (f(s,\cdot), u(s,\cdot))\,\mathrm{d}s \leqslant \frac{l}{2\lambda_1}\|f\|_{\infty,2}^2 + \frac{1}{2}\int_t^{t+l} \|u(s,\cdot)\|^2\,\mathrm{d}s. \tag{2.11}$$

By using Hölder's inequality in the second term, we obtain

$$\int_t^{t+l} \|u'(s,\cdot)\|_2^2\,\mathrm{d}s \leqslant (l|\Omega|)^{\alpha/(\alpha+2)}\biggl(\int_t^{t+l} \|u'(s,\cdot)\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}s\biggr)^{2/(\alpha+2)}. \tag{2.12}$$

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From (2.9), it follows that

$$\int_t^{t+l} \|u'(s,\cdot)\|_2^2 \,\mathrm{d}s \leqslant l|\Omega|^{\alpha/(\alpha+2)}\|f\|_{\infty,\beta}^{2/(\alpha+1)}. \tag{2.13}$$

By using Young's inequality in the third term, we have

$$|[\langle u(s,\cdot),u'(s,\cdot)\rangle]_t^{t+l}| \leqslant \frac{1}{\sqrt{\lambda_1}}[\|u'(s,\cdot)\|_2\|u(s,\cdot)\|]_t^{t+l} \leqslant 2\lambda_1^{-1/2}\sup_{s\in[t,t+l]} E(s). \tag{2.14}$$

Using Hölder's inequality in the last term, we have

$$\left|\int_t^{t+l}\langle |u'(s,\cdot)|^\alpha u'(s,\cdot),u(s,\cdot)\rangle \,\mathrm{d}s\right| \leqslant \int_t^{t+l}\|u'(s,\cdot)\|_{\alpha+2}^{\alpha+1}\|u(s,\cdot)\|_{\alpha+2}\,\mathrm{d}s.$$

By using (1.3), we get

$$\|u\|_{\alpha+2} \leqslant c_1(\alpha)\|u\|.$$

Therefore, we find

$$\left|\int_t^{t+l}\langle |u'(s,\cdot)|^\alpha u'(s,\cdot),u(s,\cdot)\rangle \,\mathrm{d}s\right| \leqslant c_1(\alpha)\int_t^{t+l}\|u'(s,\cdot)\|_{\alpha+2}^{\alpha+1}\|u(s,\cdot)\|.$$

Then, by Hölder's inequality, it follows that

$$\begin{aligned}\left|\int_t^{t+l}\langle |u'(s,\cdot)|^\alpha u'(s,\cdot),u(s,\cdot)\rangle \,\mathrm{d}s\right|&\\ \leqslant l^{1/(\alpha+2)}c_1(\alpha)&\left(\int_t^{t+l}\|u'(s,\cdot)\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}s\right)^{(\alpha+1)/(\alpha+2)}\sup_{s\in[t,t+l]}\|u(s,\cdot)\|.\end{aligned}$$

Using (2.1), we obtain

$$\begin{aligned}\left|\int_t^{t+l}\langle |u'(s,\cdot)|^\alpha u'(s,\cdot),u(s,\cdot)\rangle \,\mathrm{d}s\right|&\\ \leqslant \sqrt{2}c_1(\alpha)l^{1/(\alpha+2)}&\left(\int_t^{t+l}\|u'(s,\cdot)\|_{\alpha+2}^{\alpha+2}\,\mathrm{d}s\right)^{(\alpha+1)/(\alpha+2)}\sqrt{\sup_{s\in[t,t+l]}E(s)}.\end{aligned}$$

Finally, by (2.9) we obtain

$$\left|\int_t^{t+l}\langle |u'(s,\cdot)|^\alpha u'(s,\cdot),u(s,\cdot)\rangle \,\mathrm{d}s\right| \leqslant \sqrt{2}c_1(\alpha)l\|f\|_{\infty,\beta}\sqrt{\sup_{s\in[t,t+l]}E(s)}. \tag{2.15}$$

Then, by (2.11) and (2.13)–(2.15), we obtain

$$\begin{aligned}\frac{1}{2}\int_t^{t+l}\|u(s,\cdot)\|^2\,\mathrm{d}s \leqslant{}& \frac{l}{2\lambda_1}\|f\|_{\infty,2}^2 + l|\Omega|^{\alpha/(\alpha+2)}\|f\|_{\infty,\beta}^{2/(\alpha+1)}\\ &+\frac{2}{\sqrt{\lambda_1}}\sup_{s\in[t,t+l]}E(s)+\sqrt{2}c_1(\alpha)l\|f\|_{\infty,\beta}\sqrt{\sup_{s\in[t,t+l]}E(s)}.\end{aligned} \tag{2.16}$$

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Hence, by (2.13) and (2.16),

$$\frac{1}{l}\int_t^{t+l} E(s)\,\mathrm{d}s \leqslant \frac{1}{2\lambda_1}\|f\|_{\infty,2}^2 + \tfrac{3}{2}|\Omega|^{\alpha/(\alpha+2)}\|f\|_{\infty,\beta}^{2/(\alpha+1)} + \frac{2}{l\sqrt{\lambda_1}}\sup_{s\in[t,t+l]} E(s) + \sqrt{2}c_1(\alpha)\|f\|_{\infty,\beta}\sqrt{\sup_{s\in[t,t+l]} E(s)}.$$

Using Young's inequality in the last term of the above inequality, it follows that

$$\frac{1}{l}\int_t^{t+l} E(s)\,\mathrm{d}s \leqslant \frac{1}{2\lambda_1}\|f\|_{\infty,2}^2 + \tfrac{3}{2}|\Omega|^{\alpha/(\alpha+2)}\|f\|_{\infty,\beta}^{2/(\alpha+1)} + \frac{1}{2}\sup_{s\in[t,t+l]} E(s) + 2(c_1(\alpha))^2\|f\|_{\infty,\beta}^2. \tag{2.17}$$

By (2.17) and (2.8), we find

$$\sup_{s\in[t,t+l]} E(s) \leqslant \frac{1}{\lambda_1}\|f\|_{\infty,2}^2 + 3|\Omega|^{\alpha/(\alpha+2)}\|f\|_{\infty,\beta}^{2/(\alpha+1)} + 4(c_1(\alpha))^2\|f\|_{\infty,\beta}^2 + \frac{32}{\sqrt{\lambda_1}}\|f\|_{\infty,\beta}^{\beta}. \tag{2.18}$$

Since Ω is bounded, we have

$$\|f\|_\beta \leqslant |\Omega|^{\alpha/2(\alpha+2)}\|f\|_2. \tag{2.19}$$

Then

$$\sup_{s\in[t,t+l]} E(s) \leqslant \frac{1}{\lambda_1}\|f\|_{\infty,2}^2 + 3|\Omega|^{\alpha/(\alpha+1)}\|f\|_{\infty,2}^{2/(\alpha+1)} + 4|\Omega|^{\alpha/(\alpha+2)}(c_1(\alpha))^2\|f\|_{\infty,2}^2 + \frac{32}{\sqrt{\lambda_1}}|\Omega|^{\alpha/2(\alpha+1)}\|f\|_{\infty,2}^{\beta}.$$

Applying Young's inequality, we have

$$\|f\|_2^\beta = \|f\|_2^{(\alpha+2)/(\alpha+1)} = \|f\|_2\|f\|_2^{1/(\alpha+1)} \leqslant \tfrac{1}{2}\|f\|_2^2 + \tfrac{1}{2}\|f\|_2^{2/(\alpha+1)}.$$

Hence, we get

$$\sup_{s\in[t,t+l]} E(s) \leqslant \left(\frac{1}{\lambda_1} + 4|\Omega|^{\alpha/(\alpha+2)}(c_1(\alpha))^2 + \frac{16}{\sqrt{\lambda_1}}|\Omega|^{\alpha/2(\alpha+1)}\right)\|f\|_{\infty,2}^2 + \left(3|\Omega|^{\alpha/(\alpha+1)} + \frac{16}{\sqrt{\lambda_1}}|\Omega|^{\alpha/2(\alpha+1)}\right)\|f\|_{\infty,2}^{2/(\alpha+1)}.$$

We set

$$K = \left(\frac{1}{\lambda_1} + 4|\Omega|^{\alpha/(\alpha+2)}(c_1(\alpha))^2 + \frac{16}{\sqrt{\lambda_1}}|\Omega|^{\alpha/2(\alpha+1)}\right)\|f\|_{\infty,2}^2 + \left(3|\Omega|^{\alpha/(\alpha+1)} + \frac{16}{\sqrt{\lambda_1}}|\Omega|^{\alpha/2(\alpha+1)}\right)\|f\|_{\infty,2}^{2/(\alpha+1)}.$$

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Hence,

$$\sup_{s\in[t,t+l]} E(s) \leqslant K. \tag{2.20}$$

Then, from (2.3) and (2.20), we conclude that in all cases,

$$\forall t \geqslant 0, \quad E(t+l) \leqslant \sup\{E(t), K\}.$$

Therefore, we deduce

$$\forall t \geqslant 0, \quad E(t) \leqslant \max\Big(K, \sup_{0\leqslant s\leqslant l} E(s)\Big).$$

Thus,

$$\sup_{t\geqslant 0} E(t) \leqslant \max\Big(K, \sup_{0\leqslant s\leqslant l} E(s)\Big).$$

For $s \in [0, l]$, by using (2.5), we have

$$\begin{aligned} E(s) &= E(0) + \int_0^s \frac{\mathrm{d}}{\mathrm{d}\tau} E(\tau)\,\mathrm{d}\tau \\ &\leqslant E(0) + \int_0^s \|f(\tau,\cdot)\|_\beta^\beta \,\mathrm{d}\tau \\ &\leqslant E(0) + \|f\|_{\infty,\beta}^\beta. \end{aligned}$$

Then, since $u(0) = u'(0) = 0$, we have

$$\forall s \in [0, l], \quad E(s) \leqslant \|f\|_{\infty,\beta}^\beta.$$

By using (2.19), we have

$$\|f\|_{\infty,\beta}^\beta \leqslant K.$$

Then we get

$$\sup_{0\leqslant s\leqslant l} E(s) \leqslant K.$$

Finally, we obtain

$$\sup_{t\geqslant 0} E(t) \leqslant K.$$

Once this estimate has been established for a strong solution, we can extend it to the weak solution $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$ of (1.1), corresponding to the initial data $(u_0, u_1) = (0, 0)$, by a density argument.

Indeed, from [1], if $f \in L^\infty(\mathbb{R}^+, L^2(\Omega))$, then there exists a sequence $f_n \in C^\infty(\mathbb{R}^+, L^2(\Omega))$ with $f_n - f \in L^1(\mathbb{R}^+, L^2(\Omega))$ for all $n \in \mathbb{N}$ and

$$\|f_n - f\|_{L^1(\mathbb{R}^+, L^2(\Omega))} \leqslant \varepsilon_n$$

with $\varepsilon_n \to 0$ as $n \to \infty$.

Let $\tilde{f}_n = P_{B_M} f_n \in W^{1,\infty}(\mathbb{R}^+, L^2(\Omega))$ with $B_M = \{u \in L^2(\Omega);\ \|u\| \leqslant M\}$ and $M = \|f\|_{\infty,2}$.

Then

$$\|\tilde{f}_n\|_{L^\infty(\mathbb{R}^+, L^2(\Omega))} \leqslant M$$

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and

$$\begin{aligned}\|\tilde{f}_n - f\|_{L^1(\mathbb{R}^+,L^2(\Omega))} &= \|P_{B_M} f_n - f\|_{L^1(\mathbb{R}^+,L^2(\Omega))}\\ &= \|P_{B_M} f_n - P_{B_M} f\|_{L^1(\mathbb{R}^+,L^2(\Omega))}\\ &\leqslant \|f_n - f\|_{L^1(\mathbb{R}^+,L^2(\Omega))}\\ &\leqslant \varepsilon_n.\end{aligned}$$

Let $u_n(0) = u_n'(0) = 0$. We consider the strong solution $u_n \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^+, V) \cap W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^+, L^2(\Omega))$ of

$$u_n''(t) + |u_n'(t)|^\alpha u_n'(t) + Au_n(t) = \tilde{f}_n(t), \quad t \in \mathbb{R}^+.$$

Then, as a consequence of [1], we have the following inequality:

$$\|u_n - u\|_{L^\infty(\mathbb{R}^+,V)\cap W^{1,\infty}(\mathbb{R}^+,L^2(\Omega))} \leqslant \|\tilde{f}_n - f\|_{L^1(\mathbb{R}^+,L^2(\Omega))} \leqslant \varepsilon_n.$$

On the other hand, u_n converges to u in $L^\infty(\mathbb{R}^+, V) \cap W^{1,\infty}(\mathbb{R}^+, L^2(\Omega))$ as $n \to \infty$.

Therefore,

$$\begin{aligned}E_n(t) &= \tfrac{1}{2}(\|u_n(t)\|^2 + \|u_n'(t)\|_2^2)\\ &\leqslant \left(\frac{1}{\lambda_1} + 4|\Omega|^{\alpha/(\alpha+2)}(c_1(\alpha))^2 + \frac{16}{\sqrt{\lambda_1}}|\Omega|^{\alpha/2(\alpha+2)}\right)\|\tilde{f}_n\|^2_{\infty,2}\\ &\quad + \left(3|\Omega|^{\alpha/(\alpha+1)} + \frac{16}{\sqrt{\lambda_1}}|\Omega|^{\alpha/2(\alpha+2)}\right)\|\tilde{f}_n\|^{2/(\alpha+1)}_{\infty,2}.\end{aligned} \tag{2.21}$$

So, by passing to the limit in (2.21), we find

$$\overline{\lim_{t\to\infty}}\, E(t) \leqslant \sup_{t\geqslant 0} E(t) \leqslant K.$$

This is valid for the special solution with $(u(0), u'(0)) = (0, 0)$ and then as an immediate consequence of theorem 3.1 of [7] for any solution with initial data $(u_0, u_1) \in V \times H$, since $\limsup_{t\to\infty} E(t)$ is independent of the initial data. □

3. Some applications

Let Ω be a C^2-bounded open domain of $\mathbb{R}^N$ and $\alpha \geqslant 0$.

We apply theorem 2.1 to the following examples.

Example 3.1. We consider the equation

$$\left.\begin{aligned} u_{tt} - \Delta u + |u_t|^\alpha u_t = f(t, x) &\quad \text{in } \mathbb{R}^+ \times \Omega,\\ u = 0 &\quad \text{on } \mathbb{R}^+ \times \partial\Omega.\end{aligned}\right\} \tag{3.1}$$

Here $V = H^1_0(\Omega)$ is endowed with the norm $\|u\| = \|\nabla u\|$, $H = L^2(\Omega)$ and the duality map $A = -\Delta$.

We have $H^1_0(\Omega) \subset L^{\alpha+2}(\Omega)$ with

$$\alpha \in \begin{cases} [0, 4/(N-2)] & \text{if } N > 2,\\ [0, \infty[& \text{if } N \leqslant 2.\end{cases}$$

Then

$$c_1(\alpha) = \sup\{\|u\|_{\alpha+2};\ u \in H_0^1(\Omega),\ \|\nabla u\|_2 = 1\}$$

and

$$\lambda_1(A) = \inf\{\|\nabla u\|_2^2;\ u \in D(A),\ \|u\|_2^2 = 1\}.$$

EXAMPLE 3.2. We consider the equation

$$\left.\begin{aligned} u_{tt} + \Delta^2 u + |u_t|^\alpha u_t = f(t,x) \quad \text{in } \mathbb{R}^+ \times \Omega, \\ u = \Delta u = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega. \end{aligned}\right\} \tag{3.2}$$

Here $V = H^2(\Omega) \cap H_0^1(\Omega)$ is endowed with the norm $\|u\| = \|\Delta u\|_2$, $H = L^2(\Omega)$ and the duality map $A = \Delta^2$.

By [5] we know that, for all $\alpha \geqslant 1$, $V \subset L^{\alpha+2}$ with

$$\alpha \in \begin{cases} [1,\infty] & \text{if } N < 4, \\ [1,\infty[& \text{if } N = 4, \\ [1, 8/(N-4)] & \text{if } N > 4. \end{cases}$$

Then

$$c_1(\alpha) = \sup\{\|u\|_{\alpha+2};\ u \in H^2(\Omega) \cap H_0^1(\Omega),\ \|\Delta u\|_2 = 1\}$$

and

$$\lambda_1(A) = \inf\{\|\Delta u\|_2^2;\ u \in D(A),\ \|u\|_2^2 = 1\}.$$

EXAMPLE 3.3. We consider the equation

$$\left.\begin{aligned} u_{tt} + \Delta^2 u + |u_t|^\alpha u_t = f(t,x) \quad \text{in } \mathbb{R}^+ \times \Omega, \\ u = |\nabla u| = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega. \end{aligned}\right\} \tag{3.3}$$

Here $V = H_0^2(\Omega)$ is endowed with the norm $\|u\| = \|\Delta u\|_2$, $H = L^2(\Omega)$ and the duality map $A = \Delta^2$.

We have $V \subset L^{\alpha+2}$ for any $\alpha \geqslant 1$ with

$$\alpha \in \begin{cases} [1,\infty] & \text{if } N < 4, \\ [1,\infty[& \text{if } N = 4, \\ [1, 8/(N-4)] & \text{if } N > 4. \end{cases}$$

Then

$$c_1(\alpha) = \sup\{\|u\|_{\alpha+2};\ u \in H_0^2(\Omega),\ \|\Delta u\|_2 = 1\}$$

and

$$\lambda_1(A) = \inf\{\|\Delta u\|_2^2;\ u \in D(A),\ \|u\|_2^2 = 1\}.$$

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References

1 F. Aloui, I. Ben Hassen and A. Haraux. Compactness of trajectories to some nonlinear second order evolution equations and applications. *J. Math. Pures Appl.* **100** (2013), 295–326.

2 P. Benilan and H. Brézis. Solutions faibles d'équations d'évolution dans les espaces de Hilbert. *Annales Inst. Fourier* **22** (1972), 311–329.

3 M. Biroli and A. Haraux. Asymptotic behavior for an almost periodic, strongly dissipative wave equation. *J. Diff. Eqns* **38** (1980), 422–440.

4 H. Brézis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland Mathematics Studies, vol. 5 (North-Holland, Amsterdam, 1972).

5 T. Cazenave. *An introduction to semilinear elliptic equations* (Rio de Janeiro: Editora do IM da UFRJ, 2006).

6 A. Haraux. Dissipativity in the sense of Levinson for a class of second-order nonlinear evolution equations. *Nonlin. Analysis* **6** (1982), 1207–1220.

7 A. Haraux and E. Zuazua. Decay estimates for some semilinear damped hyperbolic problems. *Arch. Ration. Mech. Analysis* **100** (1988), 191–206.

8 M. Nakao. Asymptotic stability for some nonlinear evolution equations of second order with unbounded dissipative terms. *J. Diff. Eqns* **30** (1978), 54–63.

9 M. Nakao. Energy decay of the wave equation with nonlinear dissipative term. *Funkcial. Ekvac.* **26** (1983), 237–250.

10 G. Prouse. Soluzioni quasi-periodiche dell'equazione non omogenea delle onde, con termine dissipativo non lineare. IV. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **38** (1965), 38–39.