Canad. Math. Bull. Vol. **20** (3), 1977

DILATIONS OF POSITIVE CONTRACTIONS ON L_P SPACES*

BY
M. A. AKCOGLU AND L. SUCHESTON

1. Introduction. Throughout this article p denotes a fixed number such that $1 \le p < \infty$. The definition of a real L_p space associated with a measure space is well known. These spaces are Banach Spaces and, with the usual partial ordering of (equivalence classes of) functions, also Banach Lattices. A (linear) operator between them is called positive if it preserves the order, or, equivalently, if it maps non-negative functions into non-negative functions. A contraction is an operator whose norm is not more than one. Finally, a projection P is an idempotent contraction. Our purpose in this article is to prove the following theorem.

(1.1) THEOREM. *Let $T:L \to L$ be a positive contraction on an L_p Space L. Then there exists another L_p Space B and a positive invertible isometry $Q:B \to B$ so that $DT^n = PQ^nD$ for all $n = 0, 1, 2, \ldots$, where $D:L \to B$ is a positive isometric imbedding of L into B and $P:B \to B$ is a positive projection.*

In the next section we will prove this theorem in the case where L is a finite dimensional L_p space. Then we will show, following an observation of W. B. Johnson [7], that this special case implies the general proof.

The proof of the finite dimensional case follows from the more general results obtained in [5]. We will, however, give here a simpler proof that applies only in the finite dimensional case. This proof is similar to the one given in [1] for a more special case. We will describe the constructions of B, Q, D, P in detail, which is somewhat different from the construction in [1], but we will leave the verification of $DT^n = PQ^nD$ to the reader, which can be done along the same lines as in [1].

The proof in the general case, as observed by W. B. Johnson, is a direct consequence of some general techniques in Banach Spaces, mainly developed by D. Dacunha-Castelle and J. L. Krivine in [6]. We need, however, only very few definitions and results from this theory and we will give a self-contained account of them. We note that the original definitions in [6] use ultrafilters;

Received by the editors June 3, 1977.
* Research supported by NRC Grant A3974.

here we will use the Stone-Cech compactification instead, as discussed e.g. in Royden [9], which may be a more common background for the readers in Analysis.

Finally, the proof of Theorem (1.1) in the general case is a non-constructive proof, as should be obvious from its dependence on the Stone-Cech compactification. A constructive proof is given in [3] for the special case where L is a separable L_p Space. Other constructive proofs for more specialized cases were given in [2] and [4].

The authors are very grateful to Professor W. B. Johnson for his remarks on this problem.

2. Finite dimensional case. We start with a few general remarks. Let $(X, \mathcal{F}, \mu)$ be a measure space and $L_p = L_p(X, \mathcal{F}, \mu)$. We let L_p^+ be the class of non negative functions in L_p and we identify the adjoint of L_p with $L_q = L_q(X, \mathcal{F}, \mu)$ in the usual manner, where $q = p(p-1)^{-1}$ if $p > 1$ and $q = \infty$ if $p = 1$; hence $g \in L_q$ represent the functional that maps $f \in L_p$ into $(f, g) = \int fg \, d\mu$. We need the Hölder's Inequality. If $f \in L_p^+$ and $g \in L_q^+$ then $\int fg \leq \|f\|_p \|g\|_q$ with equality if and only if g is a multiple of f^{p-1}, assuming $\|f\|_p > 0$ and $p > 1$. Also note that if $f \in L_p^+$ then $f^{p-1} \in L_q^+$ and $\|f^{p-1}\|_q = \|f\|_p^{p/q} = \|f\|_p^{p-1}$.

We now consider a positive contraction $T: L_p \to L_p$ and also its adjoint $T^*: L_q \to L_q$. In terms of these operators we define a non-linear operator $M: L_p^+ \to L_q^+$ as $Mf = T^* (Tf)^{p-1}$, $f \in L_p^+$, which will play a central role in this section. Note that $(f, Mf) = (Tf, (Tf)^{p-1}) = \|Tf\|_p^p$.

(2.1) LEMMA. *If* $\lambda = \sup \{Tf \mid f \in L_p^+, \|f\|_p = 1\}$ *then* $\lambda = \|T\|$.

Proof. It is clear that $\lambda \leq \|T\|$. Also, if $f \in L_p^+$ then $\|Tf\|_p = \|Tf^+ - Tf^-\|_p \leq \|Tf^+ + Tf^-\|_p \leq \lambda \|f^+ + f^-\|_p = \lambda \|f\|_p$. Hence $\|T\| \leq \lambda$.

(2.2) LEMMA. *Let* $p > 1$ *and let* $f \in L_p^+$ *satisfy* $\|Tf\|_p = \|T\| \, \|f\|_p > 0$. *Then* $Mf = \|T\|^p f^{p-1}$.

Proof. First note that $\|Mf\|_q \leq \|T^*\| \, \|(Tf)^{p-1}\|_q = \|T\| \, \|Tf\|_p^{p-1} = \|T\|^p \|f\|_p^{p-1}$. Also, $\|T\|^p \|f\|_p^p = \|Tf\|_p^p = (Tf, (Tf)^{p-1}) = (f, Mf)$ which shows that

$$(*) \qquad \|Mf\|_q = \|T\|^p \, \|f\|_p^{p-1}$$

and also that we have the equality case in Hölder's inequality. Therefore $Mf = kf^{p-1}$ and (*) implies that $k = \|T\|^p$. This completes the proof.

If $E \in \mathcal{F}$ is a measurable set, let $L_p(E)$ be the class of L_p functions with support in E.

(2.3) LEMMA. *Let* $p > 1$ *and let* $\lambda_E = \sup\{\|Tf\| \mid f \in L_p^+(E), \|f\|_p = 1\}$. *If* $u \in L_p^+(E)$ *with* $\|Tu\|_p = \lambda_E \|u\|_p > 0$ *then* $\chi_E Mu = \lambda_E^p u^{p-1}$, *where* χ_E *is the characteristic function of* E.

Proof. Let $T_E : L_p \to L_p$ be defined as $T_E f = T\chi_E f$, $f \in L_p$. Then $(T_E)^* g = \chi_E T^* g$, $g \in L_q$, from the definitions. Also, $\|T_E\| = \lambda_E$ by Lemma (2.1). Hence the proof follows from Lemma (2.2) applied to T_E, by noticing that $Tu = T_E u$.

(2.4) Lemma. *Let f, $g \in L_p^+$, $f \cdot g = 0$ and $Mf \le f^{p-1}$. Then $fMg = 0$ and $M(f+g) = Mf + Mg$.*

Proof. To see that $fMg = 0$, we note that $(f, Mg) = (Tf, (Tg)^{p-1}) = 0$. In fact, this is equivalent to the fact that Tf and $(Tg)^{p-1}$ have disjoint supports, or that $(Tf)^{p-1}$ and Tg have disjoint supports. But this is true, since $0 \le (Tg, (Tf)^{p-1}) = (g, Mf) \le (g, f^{p-1}) = 0$. Therefore $fMg = 0$. Now $[T(f+g)]^{p-1} = (Tf + Tg)^{p-1} = (Tf)^{p-1} + (Tg)^{p-1}$, since Tf and Tg have disjoint supports. Hence $M(f+g) = Mf + Mg$.

We now restrict ourselves to the finite dimensional case. Hence we assume that $X = \{1, \ldots, n\}$ consists of n points with masses $m_i > 0$. We denote functions on X as n-dimensional vectors $r = (r_i)$ and represent $T : L_p \to L_p$ by an $n \times n$ matrix $T = (t_{ij})$ so that $(Tr)_j = \Sigma_i T_{ij} r_i$. Note that $(T^* s)_i = \Sigma_j m_j m_i^{-1}\, T_{ij} s_j$.

(2.5) Theorem. *There exists a vector $u = (u_i) \in L_p^+$ with strictly positive coordinates so that $Mu \le u^{p-1}$.*

Proof. If $p = 1$ then we may let $u_i = 1$ for all $i = 1, \ldots, n$. If $p > 1$ then the theorem follows from a finite number of applications of the following lemma, starting, for example, with the vector $\alpha = 0$.

(2.6) Lemma. *Let $\alpha \in L_p^+$ satisfy $M\alpha \le \alpha^{p-1}$ and assume that some coordinates of α are zero. Then there exists an $\tilde{\alpha} \in L_p^+$, whose support is strictly larger than the support of α, so that $M\tilde{\alpha} \le \tilde{\alpha}^{p-1}$.*

Proof. Let $E = \{i \mid i \in X, \alpha_i = 0\}$ and $B = \{r \mid r \in L_p^+(E), \|r\|_p = 1\}$. Since we are in a finite dimensional space, B is a compact set. Hence if $\lambda_E = \sup\{\|Tr\|_p \mid r \in B\}$, as also defined in the statement of Lemma (2.3), then there exists a $\beta \in B$ so that $\|T\beta\|_p = \lambda_E\, \|\beta\|_p = \lambda_E$. Therefore, by Lemma (2.3), $\chi_E M\beta = \lambda_E^p \beta^{p-1} \le \beta^{p-1}$. But, applying Lemma (2.4) with $f = \alpha$ and $g = \beta$, we first see that $\alpha M\beta = 0$, i.e. that $\chi_E M\beta = M\beta$, and then also that

$$M(\alpha+\beta) = M\alpha + M\beta \le \alpha^{p-1} + \beta^{p-1} = (\alpha+\beta)^{p-1},$$

where the last equality follows from the fact that $\alpha\beta = 0$. Hence $\tilde{\alpha} = \alpha + \beta$ gives the required vector.

We will now prove Theorem (1.1) in the finite dimensional case. Hence we assume that $L = L_p(X, \mathscr{F}, \mu)$, where, as we have already defined, $X = \{1, \ldots, n\}$ consists of n points. We will construct B, Q, D and P explicitly and then show that they have the properties stated in Theorem (1.1). We fix a vector $u \in L^+$ as obtained in Theorem (2.5) and let $v = Tu$. the construction we are about to give is similar to the one given in [1] and reduces exactly to that construction if v has also strictly positive coordinates and if $Mu = u^{p-1}$.

We first construct a measure space $(Z, \mathcal{G}, \nu)$ and then define B as $B = L_p(Z, \mathcal{G}, \nu)$. The set Z will be a subset of the two dimensional cartesian plane Oxy, the σ-algebra $\mathcal{G}$ and the measure ν will be the restriction of the ordinary two dimensional Lebesgue measure to Z. We denote the one and two dimensional Lebesgue measures as ℓ and ℓ^2, with the corresponding differentials dx and $dx\,dy$, respectively.

Let I_i's be n disjoint intervals on the x-axis with $\ell(I_i) = m_i$ and J_i's n disjoint intervals on the y-axis with $\ell(J_i) = 1$. We let $E_i = I_i \times J_i$, $Z_0 = \bigcup_{i=1}^n E_i$ and complete this set Z_0 to a doubly infinite disjoint sequence of sets Z_k, $k = 0, \pm 1, \pm 2, \ldots$, by choosing the other Z_k's arbitrarily with $\ell^2(Z_k) > 0$. We then let $Z = \bigcup_{-\infty<k<\infty} Z_k$.

This defines $(Z, \mathcal{G}, \nu)$ and also B. To define $Q: B \to B$ we will first define a transformation $\tau: Z \to Z$ as follows.

Let $X_0 = \{j \mid j \in X,\ v_j > 0\}$, where $v = Tu$, and let $P = X \times X_0$. For each $(i, j) \in P$ we let $\xi_{ij} = T_{ij}\, u_i / v_j$, $n_{ij} = T_{ij}\,(v_j^{p-1} / u_i^{p-1})\, m_j / m_i$ and note that for each $j \in X_0$ we have $\sum_i \xi_{ij} = 1$, since $v = Tu$, and also that for each $i \in X$ we have $\sum_{j \in X_0} \eta_{ij} \leq 1$, because of $Mu \leq u^{p-1}$. Hence we can divide each I_j, $j \in X_0$, into n disjoint subintervals I_{ij} with $\ell(I_{ij}) = \xi_{ij} m_j$ and for each $i \in X$ we can find subintervals J_{ij}, $j \in X_o$, in J_i so that $\ell(j_{ij}) = \eta_{ij}$. We then let $S_{ij} = I_{ij} \times J_i$, $R_{ij} = I_i \times J_{ij}$, $(i, j) \in P$, and $S = \cup\, S_{ij}$, $R = UR_{ij}$, where both unions are taken over $(i, j) \in P$.

For each $(i, j) \in P$, R_{ij} and S_{ij} are two rectangles with non-zero ℓ^2-measures. Hence one can find an affine transformation $\tau_{ij}: R_{ij} \to S_{ij}$ of the form

$$\tau_{ij}(x, y) = (a_{ij}x + b_{ij},\ c_{ij}y + d_{ij}),$$

with constants a_{ij}, b_{ij}, c_{ij}, d_{ij}, so that $\tau_{ij} R_{ij} = S_{ij}$, up to ℓ^2-null sets. We then define τ on R as τ_{ij} on each R_{ij}. Hence τ transforms R onto S. If $\ell^2(Z_0 - R) = 0$ then we define τ as the identify transformation on $\bigcup_{k=1}^\infty Z_k$. If $\ell^2(Z_0 - R) > 0$ we define τ to map $Z_0 - R$ onto Z_1 and to map Z_k onto Z_{k+1}, $k \geq 1$. Similarly, if $\ell^2(Z_0 - S) = 0$ then we define τ as the identity on $\bigcup_{k=1}^\infty Z_{-k}$. If $\ell^2(Z_0 - S) > 0$ we then define τ to map Z_{-k} onto Z_{-k+1}, $k \geq 2$, and to map Z_{-1} onto $Z_0 - S$. Hence $\tau: Z \to Z$ is defined and it is clear that we can make τ invertible and measurable and non-singular in both directions.

Let τ transport the measure ν to σ, defined as $\sigma(G) = \nu(\tau^{-1}G)$, $G \in \mathcal{G}$. Let $\rho = d\sigma/d\nu$ and define $Q: B \to B$ as $(Qf)(x, y) = (\rho(x, y))^{1/p} f(\tau^{-1}(x, y))$, $(x, y) \in Z$, $f \in B$. It is then well known (and very easy to verify) that Q is a positive invertible isometry of B.

The definition of $D: L \to B$ is simple. If χ_{E_i} is the characteristic function of $E_i = I_i \times J_i$ and $r = (r_i) \in L$ then $Dr = \sum_{i=1}^n r_i \chi_{E_i}$. Finally, $P: B \to B$ is defined as $Pf = E(\chi_{z_0} f)$ where E is the conditional expectation operator with respect to the partition $\{E_1, \ldots, E_n\}$ of Z_0. More explicitly, $Pf = \sum_{i=1}^{n} \chi_{E_i} 1/m_i \int_{E_i} f\, d\nu$. A routine generalization of the arguments given in (2.10), (2.12), (2.13) of [1] shows that $DT^n = PQ^nD$ for all $n = 0,1,\ldots$. As already mentioned, this will be left to the reader.

3. Ultraproducts of Banach Spaces. Let A be a directed set and let ζ be the class of all bounded real valued functions $z: A \to$. These functions are called bounded nets and also identified by the collection of their values as $z = \{z_\alpha\}$ or as $\{z_\alpha\}_{\alpha \in A}$. Note that ζ is an algebra of functions with the usual pointwise definitions of linear operations and multiplication. We also let $\lim\inf_\alpha z_\alpha = \sup_{\alpha_1 \in A}[\inf_{\alpha_2 \geq \alpha_1} z_{\alpha_2}]$ and $\lim\sup_\alpha z_\alpha = -\lim\inf_\alpha(-z_\alpha)$. Finally note that if $u: \mathbb{R} \to \mathbb{R}$ is continuous and if $z \in \zeta$ then $(u \circ z)_\alpha = u(z_\alpha)$ defines a bounded net.

(3.1) LEMMA. *There exists a homomorphism (i.e. a linear and multiplicative function)* $\mathrm{LIM}: \zeta \to \mathbb{R}$ *so that if* $z \in \zeta$ *then* $\lim\inf_\alpha Z_\alpha \leq \mathrm{LIM}\ z \leq \lim\sup_\alpha z_\alpha$ *and if* $u: \mathbb{R} \to \mathbb{R}$ *is continuous then* $\mathrm{LIM}(u \circ z) = u(\mathrm{LIM}\ z)$.

Proof. Consider A as a topological space with its discrete topology (i.e. each subset of A is open). Then A is a locally compact Hausdorff space. Let A^* be the Stone-Cech compactification of A. Then, by definition, A^* is a compact Hausdorff space and A is imbedded homeomorphically as a dense open subset of A^* so that any bounded (automatically continuous) function $z: A \to \mathbb{R}$ has a (necessarily unique) extension to a continuous function $z^*: A^* \to \mathbb{R}$. For each $\alpha \in A$, let C_α be the closure of $\{\beta \mid \beta \in A, \beta \geq \alpha\}$ in A^*. Since A is a directed set, the family $\{C_\alpha\}_{\alpha \in A}$ has the finite intersection property. Therefore $\bigcap_{\alpha \in A} C_\alpha$ contains a point α^*. We then let $\mathrm{LIM}\ z = z^*(\alpha^*)$. It is easy to see that this satisfies the requirements of the lemma.

A function as obtained in this lemma will be called a limit functional. For the rest of this paper we are going to choose and fix a limit functional. We denote its value also as $\mathrm{LIM}_\alpha z_\alpha$. If $\{z_\alpha\}$ is a convergent net then $\mathrm{LIM}_\alpha z_\alpha = \lim_\alpha z_\alpha$. Note that if $z, z' \in \zeta$ for which there is an α_0 so that $z_\alpha = z'_\alpha$ for all $\alpha \geq \alpha_0$, then $\mathrm{LIM}_\alpha z_\alpha = \mathrm{LIM}_\alpha z'_\alpha$.

After these preliminaries we define the ultraproducts of Banach Spaces as follows. Let A be a directed set and let W_α be a Banach space for each $\alpha \in A$. From this collection $\{W_\alpha\}$ of Banach Spaces we will define a new Banach Space W which will be called the ultraproduct of W_α's. Points in W are collections of the form $w = \{w_\alpha\}$, indexed by $\alpha \in A$, so that $w_\alpha \in W_\alpha$ for each $\alpha \in A$ and so that $\{\|w_\alpha\|\}$ is a bounded net. Linear combinations in W are defined as $av + bw = \{av_\alpha + bw_\alpha\}$ and the norm as $\|w\| = \mathrm{LIM}_\alpha \|w_\alpha\|$. Here, $v, w \in W$ and $a, b \in \mathbb{R}$. It is clear that this is only a pseudonorm, since $\|w\| = 0$ does not imply that $w = 0$, i.e. that $w_\alpha = 0$ for all $\alpha \in A$. We define an equivalence relation in W as $w \sim w'$ if and only if $\|w - w'\| = 0$. To obtain a norm, W must be replaced, as usual, by the set of equivalence classes. It will be more convenient, however, to work directly with the elements of W and distinguish between the equalities and equivalences in B.

(3.2) THEOREM. *W is a Banach Space.*

Proof. It is clear that W is a (pseudo) normed vector space. As it is well known, the completeness is equivalent to Lemma 3.4 below. Before then we note a technical fact.

(3.3) LEMMA. *For each $w \in W$ there exists a $v \in V$ so that $v \sim w$ and so that $\|v_\alpha\| \leq \|w\| (= \|v\|)$ for all $\alpha \in A$.*

Proof. If $\|w\| = 0$ then let $v_\alpha = 0$. If $\|w\| > 0$, then define $\lambda_\alpha = (\|w\| / \|w\| \vee \|w_\alpha\|)$ and let $v_\alpha = \lambda_\alpha w_\alpha$ Then $\|v_\alpha\| \leq \|w\|$ and also $v \sim w$, since $\mathrm{LIM}_\alpha \lambda_\alpha = 1$ and, consequently $\|v - w\| = \mathrm{LIM}_\alpha |1 - \lambda_\alpha| \, \|w_\alpha\| = 0$.

(3.4) LEMMA. *Let $\{w^n\}_{n=1}^\infty$ be a sequence in W so that $\sum_{n=1}^\infty \|w^n\| < \infty$. Then there is a $w \in W$ so that $\sum_{n=1}^\infty w^n = w$ in W,* i.e. *that* $\lim_n \| \sum_{i=1}^n w^i - w\| = 0$.

Proof. For each n find a $v^n \sim w^n$ so that $\|v_\alpha^n\| \leq \|w^n\|$, by the previous lemma. Hence $\sum_{n=1}^\infty \|v_\alpha^n\| < \infty$ for each $\alpha \in A$ and since W_α is a Banach space and there is a w_α so that $\sum_{n=1}^\infty v_\alpha^n = w_\alpha$ in W_α. Then $\{w_\alpha\} \in W$, since $\|w_\alpha\| \leq \sum_{n=1}^\infty \|v_\alpha^n\| \leq \sum_{n=1}^\infty \|w^n\|$ for all $\alpha \in A$. Also, $\|\sum_{i=1}^n w^i - w\| = \|\sum_{i=1}^n v^i - w\| = \mathrm{LIM}_\alpha \|\sum_{i=1}^n v_\alpha^i - w_\alpha\| \leq \mathrm{LIM}_\alpha \sum_{i=n+1}^\infty \|v_\alpha^i\| \leq \mathrm{LIM}_\alpha \sum_{i=n+1}^\infty \|w^i\| = \sum_{i=n+1}^\infty \|w^i\|$ converges to zero as $n \to \infty$.

Now we will observe that if each W_α is an L_p space then W is isomorphic to an L_p Space. In fact, introduce a partial order into W as $v \leq w$ being equivalent to $v_\alpha \leq w_\alpha$ for each $\alpha \in A$. The corresponding maximum and minimum operations are$v \vee w = \{v_\alpha \vee w_\alpha\}$, $v \wedge w = \{v_\alpha \wedge w_\alpha\}$, respectively, and the positive cone of W is $w^+ = \{w \mid w \in W, \; w \geq 0\}$. The following lemma shows that these operations can be defined on the equivalence classes of W and that they are continuous with respect to the norm topology. Hence it is easily seen that W becomes a Banach Lattice with these definitions.

(3.5) LEMMA. *If $v \sim v'$ and $w \sim w'$ then $v \vee w \sim v' \vee w'$ and $v \wedge w \sim v' \wedge w'$. If v^n and w^n converge respectively to v and w in W then $v^n \vee w^n$ and $v^n \wedge w^n$ converge, respectively, to $v \vee w$ and $v \wedge w$ in W.*

Proof. We prove only the statements for the maximum operation. They will obviously follow from

$$\|v \vee w - v' v w'\| \leq \|v - v'\| + \|w - w'\|,$$

https://doi.org/10.4153/CMB-1977-044-4 Published online by Cambridge University Press

which is obtained from

$$|v_\alpha \vee w_\alpha - v'_\alpha \vee w'_\alpha| \leq |v_\alpha - v'_\alpha| + |w_\alpha - w'_\alpha|,$$

first using the Minkowski's inequality and then applying the limit functional.

(3.6) LEMMA: *If v, $w \in W^+$ and if $v \wedge w = 0$ then* $\|v+w\|^p = \|v\|^p + \|w\|^p$.

Proof. Integrate

$$|v_\alpha|^p + |w_\alpha|^p \leq |v_\alpha + w_\alpha|^p \leq |v_\alpha + (v_\alpha \wedge w_\alpha)|^p + |w_\alpha + (v_\alpha \wedge w_\alpha)|^p$$

to get the corresponding inequalities in the norm of W_α and apply the limit functional to get

$$\|v\|^p + \|w\|^p \leq \|v+w\|^p \leq \|v + v \wedge w\|^p + \|w + v \wedge w\|^p = \|v\|^p + \|w\|^p.$$

A generalization of a Theorem of Kakutani (see, e.g. p. 112 of [8]) shows that a Banach Lattice with the property stated in Lemma 3.6 is order isomorphic to an L_p Space. Hence if each W_α is an L_p Space then there exists another L_p Space B so that W can be identified with B by means of a positive isometric isomorphism $\Psi: W \to B$.

4. The main proof in the general case. Let L be the L_p Space associated with an arbitrary measure space $(X, \mathcal{F}, \mu)$. By a semi-partition of X we mean a finite disjoint collection of measurable sets with finite measures. Let A be the set of all semi-partitions of X. Introduce a partial order into A as $\alpha \leq \alpha'$ meaning that each set in α is a union of some sets in α'. Then it is clear that A becomes a directed set. For each $\alpha \in A$ let $E_\alpha: L \to L$ be the conditional expectation operator with respect to the semi-partition α, mapping functions to their average values on the sets of α and to zero outside of these sets. Note that for each fixed $f \in L$ the net $\{E_\alpha f\}$ converges to f in the sense that $\lim_\alpha \|f - E_\alpha f\| = 0$. Finally let $L_\alpha = E_\alpha L$ be the range of E_α, which is a finite dimensional L_p Space.

Now let $T: L \to L$ be a positive contraction. We define $T_\alpha: L \to L_\alpha$ as $T_\alpha = E_\alpha T E_\alpha$. A simple argument shows that $\lim_\alpha \|T_\alpha^n f - T^n f\| = 0$ for each $f \in L$, and for integer $n = 1, 2, \ldots$. The operator T_α can also be considered as acting on L_α. Hence we have a positive contraction $T_\alpha: L_\alpha \to L_\alpha$ of a finite dimensional L_p Space. Therefore the dilation theorem for finite dimensional spaces shows that for each $\alpha \in A$ there exists an L_p Space W_α, a positive invertible isometry $Q_\alpha: W_\alpha \to W_\alpha$ a positive projection $P_\alpha: W_\alpha \to W_\alpha$ and a positive isometry $D_\alpha: L_\alpha \to W_\alpha$ so that $D_\alpha T_\alpha^n = P_\alpha Q_\alpha^n D_\alpha$ for each $n = 0, 1, 2, \ldots$. We then construct the ultraproduct W of W_αs and define $Q: W \to W$, $P: W \to W$ and $D: L \to W$ as $Q\{w_\alpha\} = \{Q_\alpha w_\alpha\}$, $P\{w_\alpha\} = \{P_\alpha w_\alpha\}$ and $Df = \{D_\alpha E_\alpha f\}$, where $w = \{w_\alpha\} \in W$ and $f \in L$. Now to see that $PQ^nD = DT^n$, we apply both sides to a function $f \in L$:

$$PQ^nDf = \{P_\alpha Q_\alpha^n D_\alpha E_\alpha f\}, \qquad DT^nf = \{D_\alpha E_\alpha T^n f\}$$

and note that $P_\alpha Q_\alpha^n D_\alpha E_\alpha f = D_\alpha T_\alpha^n E_\alpha f = D_\alpha T_\alpha^n f$ and also that $\|D_\alpha T_\alpha^n f - D_\alpha E_\alpha T^n f\| = \|T_\alpha^n f - ET^n f\|$ since D_α is an isometry. But $\lim_\alpha \|T_\alpha^n f - E_\alpha T^n f\| = 0$, which shows that $\{D_\alpha T_\alpha^n f\} \sim \{D_\alpha E_\alpha T^n f\}$ or that $PQ^n Df \sim DT^n f$. Hence $PQ^n D = DT^n$.

It is now clear that, in the partial ordering of W given in the previous section, $Q: W \to W$ is a positive invertible isometry, $D: L \to W$ is a positive isometry, and $P: W \to W$ is a positive projection. Although W is obtained as a general Banach Lattice, the theorem of Kakutani mentioned at the end of the previous section shows that there is an L_p Space B and a positive isometric isomorphism $\Psi: W \to B$. This Ψ can be used to transport Q, D, P to similar operators Q', D', P' related to B as $Q' = \Psi Q \Psi^{-1}: B \to B$, $D' = \Psi D: L \to B$, $P' = \Psi P \Psi^{-1}: B \to B$. Then it is clear that all the requirements of Theorem (1.1) are satisfied.

Finally, we will mention the following point. If L and B are two L_p spaces and if $D: L \to B$ is a positive isometry then DL can be characterized as follows. If $B = L_p(Z, \mathcal{G}, \nu)$, then there exists a sub σ-algebra $\mathcal{G}_0 \subset \mathcal{G}$ and a set $Z_0 \in \mathcal{G}_0$ so that $DL = L_p(Z_0, Z_0 \cap \mathcal{G}_0, \nu_0)$, where ν_0 is the restriction of ν to $Z_0 \cap \mathcal{G}_0$. Hence there is a natural positive projection $\Pi: B \to B$ so that $\Pi B = DL$. This is defined as $\Pi f = E(\chi_{z_0} f)$, $f \in B$, where E is the conditional expectation with respect to $\mathcal{G}_0$. Although in Theorem (1.1) we have this positive isometry $D: L \to B$, the positive projection $P: B \to B$ (we omit the primes from D and P to simplify the notation) obtained in the above proof is not the natural projection Π. A more careful analysis of the representation of W as an L_p Space shows that $\Pi Q^n D = PQ^n D$, i.e. that Π can also be used as the positive projection required in Theorem (1.1). We will, however, omit this.

REFERENCES

1. M. A. Akcoglu: *A pointwise ergodic theorem for L_p Spaces*, Can. J. Math. **27,** 1075–1082, (1975).
2. M. A. Akcoglu: *Positive contractions of L_1 Spaces*, Math. Z. **143,** 5–13, (1975).
3. M. A. Akcoglu and P. E. Kopp: *Positive contractions of L_p Spaces.* To appear in 2 Math.
4. M. A. Akcoglu and L. Sucheston: *On convergence of iterates of positive contractions in L_p Spaces*, J. Approx. Theory, **13,** 348–362, (1975).
5. M. A. Akcoglu and L. Sucheston: *On positive dilations to isometries in L_p Spaces.* Lecture Notes in Math. Vol **541,** 389–401, Springer Verlag 1976.
6. D. Dacunha-Castelle and J. L. Krivine: *Applications des ultraproduits a l'ètude des espaces et des algèbres de Banach*, Studia Math. **41,** 313–334, (1972).
7. W. B. Johnson: *Private communication*, January, 1976.
8. J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces.* Lecture Notes in Mathematics, **338,** Springer-Verlag, 1973.
9. H. L. Royden: *Real Analysis.* Macmillan, New York, 1968.
10. G. C. Rota: "*An Alternierende Verfahren*" for general positive operators. Bull. A.M.S. **68,** 95–102, 1962.

DEPT OF MATH
UNIVERSITY OF TORONTO