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Euler and Navier–Stokes equations in a new time-dependent helically symmetric system: derivation of the fundamental system and new conservation laws

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The present contribution is a significant extension of the work by Kelbin *et al.* (*J. Fluid Mech.*, vol. 721, 2013, pp. 340–366) as a new time-dependent helical coordinate system has been introduced. For this, Lie symmetry methods have been employed such that the spatial dependence of the originally three independent variables is reduced by one and the remaining variables are: the cylindrical radius r and the time-dependent helical variable $\xi = (z/\alpha(t)) + b\varphi$, $b = \text{const.}$ and time t. The variables z and φ are the usual cylindrical coordinates and $\alpha(t)$ is an arbitrary function of time t. Assuming $\alpha = \text{const.}$, we retain the classical helically symmetric case. Using this, and imposing helical invariance onto the equation of motion, leads to a helically symmetric system of Euler and Navier–Stokes equations with a time-dependent pitch $\alpha(t)$, which may be varied arbitrarily and which is explicitly contained in all of the latter equations. This has been conducted both for primitive variables as well as for the vorticity formulation. Hence a significantly extended set of helically invariant flows may be considered, which may be altered by an external time-dependent strain along the axis of the helix. Finally, we sought new conservation laws which can be found from the helically invariant Euler and Navier–Stokes equations derived herein. Most of these new conservation laws are considerable extensions of existing conservation laws for helical flows at a constant pitch. Interestingly enough, certain classical conservation laws do not admit extensions in the new time-dependent coordinate system.

Key words: general fluid mechanics, mathematical foundations, Navier–Stokes equations

1. Introduction

Helical vortices appear in various technological devices with swirl, e.g. in the wake of windmills (Vermeer, Sorensen & Crespo 2003), or as wing tip vortices, in particular, on delta wings (Mitchell, Morton & Forsythe 1997). Furthermore, helical vortex structures were observed by Sarpkaya (1971) who performed experiments

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with swirling flows in a cylindrical tube and observed three different types of vortex breakdown, where one of them is of helical type.

Helically symmetric flows are further observed as natural phenomena, e.g. in laboratory plasma applications (kink instabilities in the 'straight tokamak' approximations, e.g. Johnson *et al.* (1958), Schnack, Caramana & Nebel (1985)), and astrophysical phenomena such as astrophysical jets (Bogoyavlenskij 2000).

In a more theoretical context, flows with a helical symmetry in general have been considered in various contributions in the past. For example, Delbende, Rossi & Daube (2012) developed a direct numerical simulation code for the helical invariant Navier–Stokes equations in a generalized vorticity–streamfunction formulation. Dritschel (1991) reduced the three-dimensional Euler equations to a linear equation, assuming that the flow has helical symmetry and consists of a rigidly rotating basic part and a Beltrami disturbance part. Further, he derived exact solutions for flows in a straight pipe of circular cross-section. Helical flows for a Maxwell fluid between two infinite coaxial circular cylinders were considered by Jamil & Fetecau (2010). Using the finite Hankel transform, they obtained exact solutions which satisfy all imposed initial and boundary conditions. A detailed overview on helical flows may be taken from Kelbin, Cheviakov & Oberlack (2013).

In mathematical physics, symmetries and conservation laws (CLs) are considered to be one of the most fundamental objects. For example, in fluid mechanics, CLs describe physical quantities such as the conservation of mass, energy, momentum or angular momentum. In practice, local CLs are of fundamental importance for several reasons. They are essential for numerical simulations with modern numerical methods, where the equations are assumed to be in divergence form, e.g. for discontinuous Galerkin methods (see Zienkiewicz *et al.* 2003). Additionally, they give the possibility to easily find potential variables (see Bluman, Cheviakov & Anco 2010), which in turn leads to a reduction of the dependent variables and new analytical solutions. Further, CLs are used to establish the existence and uniqueness of solutions as well as in the analysis of stability and global behaviour of solutions (Bluman *et al.* 2010).

For three-dimensional time-dependent fluid flows, CLs were studied in very much detail in Cheviakov & Oberlack (2014). Therein, they considered higher-order CL multipliers and obtained an infinite family of vorticity CLs. Further, Rosenhaus & Shankar (2015) considered the correspondence between symmetries and CLs. They introduced subsymmetries to find further infinite sets of CLs of the Euler equations, involving arbitrary functions of the dependent variables.

Most importantly, additional CLs appear to exist in reduced dimensions such as in plane or axisymmetric flows. Recently new CLs for Euler and Navier–Stokes equations were found for helically invariant flows (see Kelbin *et al.* 2013). Therein they considered a helical coordinate system, given by the radius r and a helical variable $\xi = az + b\varphi$, arising from a linear combination of the cylindrical coordinates z and φ. The parameters a and b, involved in this coordinate, were assumed to be constant. Further, the authors expressed the three-dimensional, incompressible Euler and Navier–Stokes equations in a helical symmetric setting and finally obtained new CLs for primitive variables as well as for the vorticity formulation. Interestingly enough, they also derived new CLs for plane and axisymmetric flows. Due to many citations the publication of Kelbin, Cheviakov & Oberlack (2013) is subsequently denoted as KCO.

In a fluid dynamical context, divergence-type local CLs usually have the form

$$\partial_t \Theta + \nabla \cdot \boldsymbol{\Phi} = 0, \tag{1.1}$$

where $\nabla\cdot\boldsymbol{\Phi}=\partial_i\Phi^i=\partial_1\Phi^1+\partial_2\Phi^2+\cdots+\partial_{n-1}\Phi^{n-1}$ denotes the spatial divergence. The quantity Θ is called density, whereas Φ^i are the spatial fluxes of the CL.

In order to compute a globally conserved quantity one may integrate (1.1) over a fluid domain Ω and apply Gauss's theorem

$$\int_\Omega \partial_t\Theta\,\mathrm{d}^n x+\int_{\partial\Omega}\boldsymbol{\Phi}\cdot\boldsymbol{n}\,\mathrm{d}a=0, \tag{1.2}$$

where $\mathrm{d}^n x$ defines the volume element while $\mathrm{d}a$ corresponds to a surface element on $\partial\Omega$. Assuming that the fluxes Φ^i vanish on the boundary $\partial\Omega$ or if periodicity is assumed and the domain Ω is time-independent ($\Omega\neq\Omega(t)$), one obtains the global conserved quantity given by

$$\frac{\partial}{\partial t}\int_\Omega \Theta\,\mathrm{d}x=0 \quad\Leftrightarrow\quad J=\int_\Omega \Theta\,\mathrm{d}x=\text{const.} \tag{1.3a,b}$$

In practice, one is interested in finding non-trivial CLs (1.1) since trivial CLs usually do not carry a physical or mathematical meaning. To distinguish between trivial and non-trivial CLs, we first explicate the meaning of trivial CLs.

Following Bluman *et al.* (2010), a trivial CL of the first type arises when each of its fluxes Φ vanish identically on the solutions of a given system of partial differential equations. A trivial CL of the second type is a CL that vanishes identically as a differential identity, e.g. $(\mathrm{div}(\mathrm{curl}(\cdot))\equiv 0)$.

The definition of trivial CLs leads to the definition of equivalence and linear dependence of CLs: two CLs $\partial_i\Phi^i=0$ and $\partial_i\Psi^i=0$ are equivalent if $\partial_i(\Phi^i-\Psi^i)=0$ is a trivial CL. An equivalence class of CLs consists of all CLs that can be reduced to a class of non-trivial CLs.

A set of l CLs $\{\partial_i\Phi^i_j=0\}_{j=1}^l$ is linearly dependent if there exists a set of constants $\{a^{(j)}\}_{j=1}^l$ which are not all zero such that the linear combination

$$\partial_i\left(a^{(j)}\Phi^i_{(j)}\right)=0 \tag{1.4}$$

is a trivial CL (Bluman *et al.* 2010). The direct method (see e.g. Anco & Bluman 2002*a*), described and employed in the following, seeks non-trivial sets of local CLs of a given PDE system in non-conservative form.

The direct method is based on two key ideas. The first idea can be explained as follows: consider an arbitrary and non-conservative PDE system given by

$$R^\sigma\left(\boldsymbol{x},\boldsymbol{u},\partial_i\boldsymbol{u}\right)=0,\quad \sigma=1,\ldots,N,\ \ i=1,\ldots,k. \tag{1.5}$$

It is proven in Anco & Bluman (2002*b*) that a PDE system only admits non-trivial CLs arising from linear combinations of these equations with multipliers of kth order given by

$$\{\Lambda_\sigma\left(\boldsymbol{x},\boldsymbol{U},\partial_1\boldsymbol{U},\ldots,\partial_k\boldsymbol{U}\right)\}_{\sigma=1}^N. \tag{1.6}$$

If the multipliers (1.6) are known, they yield divergence expressions of the form

$$\Lambda_\sigma R^\sigma\equiv D_i\Gamma^i \tag{1.7}$$

for arbitrary functions $\boldsymbol{U}(\boldsymbol{x})$ and $D_i=\partial/\partial x_i$. The multipliers Λ_σ are of the form (1.6) and can be chosen to depend on all dependent and independent variables $\boldsymbol{x}$ and $\boldsymbol{U}$

as well as on derivatives $\partial_k \boldsymbol{U}$ up to a certain order k. *A priori* it is not known up to which order one may choose the multipliers. On solutions $\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x})$ of the PDE system (1.5) one obtains the CL

$$D_i \Gamma^i = 0. \tag{1.8}$$

To calculate the multipliers Λ_σ, the second key idea is to apply the Euler operator E_{U^j} with respect to U^j to (1.7). The Euler operator has the property to annihilate an expression if, and only if, it is a divergence expression $D_i \Gamma^i$, and is given by

$$\mathrm{E}_{U^j} = \frac{\partial}{\partial U^j} - D_i \frac{\partial}{\partial U^j_i} + \cdots + (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial U^j_{i_1 \cdots i_s}} + \cdots \tag{1.9}$$

for each $j = 1 \ldots m$, while D_i is defined above.

In order to derive the multipliers Λ_σ, we apply the Euler operator (1.9) to (1.7). Based on the above, the right-hand side vanishes identically and one obtains

$$\mathrm{E}_{U^j} \left(\Lambda_\sigma R^\sigma \right) = 0, \quad j = 1 \ldots m, \tag{1.10}$$

which holds for arbitrary functions $\boldsymbol{U}(\boldsymbol{x})$. Expanding all derivatives in (1.10), a set of linear determining equations for all multipliers Λ_σ arises, where the quantities $\boldsymbol{x}$, $\boldsymbol{U}$, $\partial_1 \boldsymbol{U}$, ... , $\partial_k \boldsymbol{U}$ have to be treated as independent variables. Once the CL multipliers are derived, one may compute the density and fluxes using (1.8). For details, see e.g. Bluman *et al.* (2010).

The main goal of the current contribution is twofold. First, we develop a new helical coordinate system with a time-dependent pitch and, thereof, derived a reduced system of helically invariant Euler and Navier–Stokes equations in primitive variables and vorticity formulation. Second, for both formulations, we derive new local CLs applying the direct construction method. The new results will be compared to the classical case with constant pitch considered in KCO, where various new CLs in primitive variables and in vorticity formulation were derived.

2. Helically invariant Navier–Stokes equations in a time-dependent helical coordinate system

In a cylindrical coordinate system the three-dimensional time-dependent Navier–Stokes equations for a viscous and incompressible fluid without external forces are given by

$$\frac{1}{r} u^r + \frac{\partial u^r}{\partial r} + \frac{1}{r} \frac{\partial u^\varphi}{\partial \varphi} + \frac{\partial u^z}{\partial z} = 0, \tag{2.1a}$$

$$\frac{\partial u^r}{\partial t} + u^r \frac{\partial u^r}{\partial r} + \frac{1}{r} \left(u^\varphi \frac{\partial u^r}{\partial \varphi} - (u^\varphi)^2 \right) + u^z \frac{\partial u^r}{\partial z} = -\frac{\partial p}{\partial r} + \nu \left[\Delta u^r - \frac{1}{r^2} \left(u^r + 2 \frac{\partial u^\varphi}{\partial \varphi} \right) \right], \tag{2.1b}$$

$$\frac{\partial u^\varphi}{\partial t} + u^r \frac{\partial u^\varphi}{\partial r} + \frac{1}{r} \left(u^\varphi \frac{\partial u^\varphi}{\partial \varphi} + u^r u^\varphi \right) + u^z \frac{\partial u^\varphi}{\partial z} = -\frac{1}{r} \frac{\partial p}{\partial \varphi} + \nu \left[\Delta u^\varphi - \frac{1}{r^2} \left(u^\varphi - 2 \frac{\partial u^r}{\partial \varphi} \right) \right], \tag{2.1c}$$

$$\frac{\partial u^z}{\partial t} + u^r \frac{\partial u^z}{\partial r} + \frac{1}{r} u^\varphi \frac{\partial u^z}{\partial \varphi} + u^z \frac{\partial u^z}{\partial z} = -\frac{\partial p}{\partial z} + \nu \Delta u^z. \tag{2.1d}$$

In the inviscid case, i.e. $\nu = 0$, the system (2.1) reduces to the Euler equations.

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The Navier–Stokes equations (2.1) are invariant under the helical symmetry, which relies on the combination of two common Lie symmetry groups: (i) generalized Galilean invariance, which comprises axial translation and classical Galilean boost, and (ii) a rotation group about the same axis. Without restricting generality we choose the coordinate system as such that the common symmetry axis lies along the z-axis.

The global form of the two parameter Lie symmetry group is given by

$$\tilde{\varphi} = \varphi + c, \tag{2.2a}$$

$$\tilde{z} = z + \alpha(t), \tag{2.2b}$$

$$\tilde{u}^z = u^z + \dot{\alpha}, \tag{2.2c}$$

$$\tilde{p} = p - z\ddot{\alpha}. \tag{2.2d}$$

The dot denotes the time derivative and the upper index describes the components of the corresponding velocity field, respectively. For $\alpha(t) = \text{const.}$ we obtain translational invariance in the z-direction, while $\alpha(t) = at$ corresponds to the classical Galilean group in the same direction.

Using the symmetry group (2.2a)–(2.2d), we derived a set of independent variables (i.e. the time-dependent helical coordinates) and dependent variables (for details see appendix A), which are given by

$$\eta = b\varphi, \tag{2.3a}$$

$$\xi = b\varphi + \frac{z}{\alpha}, \tag{2.3b}$$

$$\tilde{r} = r, \tag{2.3c}$$

$$\tau = t, \tag{2.3d}$$

$$u^\xi = \left(\frac{b}{r}u^\varphi + \frac{1}{\alpha}(u^z + \dot{\alpha}b\varphi)\right) \cdot B(r, t), \tag{2.3e}$$

$$u^\eta = \left(\frac{1}{\alpha}u^\varphi - \frac{b}{r}(u^z + \dot{\alpha}b\varphi)\right) \cdot B(r, t), \tag{2.3f}$$

$$\tilde{u}^r = u^r, \tag{2.3g}$$

$$\tilde{p} = p + \frac{1}{2}\frac{\ddot{\alpha}}{\alpha}z^2. \tag{2.3h}$$

The geometric function $B(r, t)$ in (2.3e) and (2.3f) is given by $B(r, t) = r\alpha/\sqrt{r^2 + b^2\alpha^2}$. By assuming $\alpha(t) = \text{const.} = 1/a$ one easily retains the classical helical velocity components, pressure, similarity variable and form function $B(r)$ given in KCO.

In the limiting case $b = 0$, the helical symmetry reduces to an axial symmetry and the similarity variable (2.3b) becomes $\xi = z/\alpha$, although it is still time-dependent owing to the scaling of the z-coordinate by the parameter function $\alpha(t)$. In the opposite case $1/\alpha = 0$, in which helical symmetry reduces to a planar symmetry, the time dependence of the coordinate system vanishes and the classical planar case as discussed in KCO is retained.

Inverting the equations (2.3e)–(2.3h) and replacing the cylindrical coordinates (r, φ, z) by the helical coordinates (r, ξ, η), one obtains the relations given by

$$u^\varphi = B(r, t) \cdot \left(\frac{b}{r}u^\xi + \frac{1}{\alpha}u^\eta\right), \tag{2.4a}$$

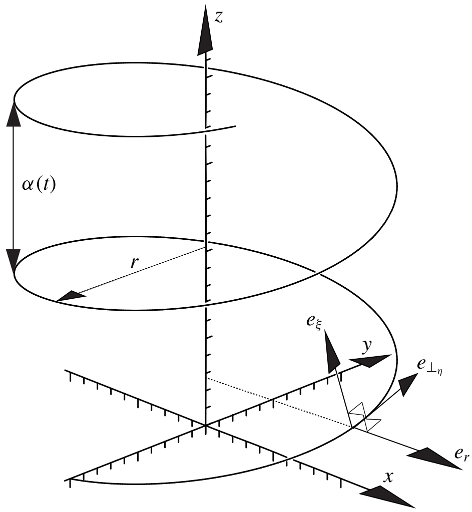

FIGURE 1. An illustration of the helix $\xi = \text{const.}$ with parameter function $\alpha(t)$ (cf. KCO).

$$u^z = B(r, t) \cdot \left(\frac{1}{\alpha} u^\xi - \frac{b}{r} u^\eta \right) - \dot{\alpha}\eta, \tag{2.4b}$$

$$u^r = \tilde{u}^r, \tag{2.4c}$$

$$p = \tilde{p} - \tfrac{1}{2}\ddot{\alpha}\alpha \, (\xi - \eta)^2 \,. \tag{2.4d}$$

The additional term $-\dot{\alpha}\eta$ in the z-component of the velocity (2.4b) describes a relative movement between the time-dependent and the time-independent coordinate system (see figure 1).

2.1. *The Navier–Stokes equations in primitive variables*

In order to obtain the reduced system of helically invariant Navier–Stokes equations we introduce the new variables (2.3e)–(2.3h) into the system (2.1) and impose helical invariance, i.e. $\partial/\partial\eta \equiv 0$, which eliminates η from the system. From this, we obtain the helical invariant continuity equation and the three components of the momentum equations in the r-, η- and ξ-direction

$$\frac{u^r}{r} + u^r_r + \frac{1}{B} u^\xi_\xi = 0, \tag{2.5a}$$

$$\begin{aligned} -\frac{\dot{\alpha}}{\alpha} u^r_\xi \xi + u^r_\tau + u^r u^r_r + \frac{1}{B} u^\xi u^r_\xi - \frac{B^2}{r} \left(\frac{b}{r} u^\xi + \frac{1}{\alpha} u^\eta \right)^2 \\ = -\tilde{p}_r + \nu \left[\frac{1}{r} \left(r u^r_r \right)_r + \frac{1}{B^2} u^r_{\xi\xi} - \frac{1}{r^2} u^r - \frac{2bB}{r^2} \left(\frac{1}{\alpha} u^\eta_\xi + \frac{b}{r} u^\xi_\xi \right) \right], \end{aligned} \tag{2.5b}$$

$$\begin{aligned} \dot{\alpha} \frac{b}{r} u^\xi + \dot{\alpha} \frac{b^2 B^2}{r^2 \alpha} u^\eta - \frac{\dot{\alpha}}{\alpha} \xi u^\eta_\xi + u^\eta_\tau + u^r u^\eta_r + \frac{B^2}{r\alpha^2} u^r u^\eta + \frac{1}{B} u^\xi u^\eta_\xi \\ = -B\ddot{\alpha} \frac{b}{r} \xi + \nu \left[\frac{1}{r} \left(r u^\eta_r \right)_r + \frac{1}{B^2} u^\eta_{\xi\xi} + \left(\frac{\frac{B^2}{\alpha^2} \left(\frac{B^2}{\alpha^2} - 2 \right)}{r^2} \right) u^\eta + \frac{2bB}{r^2 \alpha} \left(u^r_\xi - \left(B u^\xi \right)_r \right) \right], \end{aligned} \tag{2.5c}$$

$$
-2\dot{\alpha}\frac{bB^2}{r\alpha^2}u^\eta - \dot{\alpha}\frac{b^2B^2}{r^2\alpha}u^\xi - \frac{\dot{\alpha}}{\alpha}\xi u^\xi_\xi + u^\xi_\tau + u^r u^\xi_r + \frac{b^2B^2}{r^3}u^r u^\xi + \frac{1}{B}u^\xi u^\xi_\xi + \frac{2bB^2}{r^2\alpha}u^r u^\eta
$$

$$
= -\frac{1}{B}\tilde{p}_\xi + B\frac{\ddot{\alpha}}{\alpha}\xi + \nu\left[\frac{1}{r}\left(ru^\xi_r\right)_r + \frac{1}{B^2}u^\xi_{\xi\xi} + \frac{\frac{B^4}{\alpha^4}-1}{r^2}u^\xi + \frac{2bB}{r}\left(\frac{b}{r^2}u^r_\xi + \left(\frac{B}{\alpha r}u^\eta\right)_r\right)\right]. \tag{2.5d}
$$

It is worth noting that in the case $\alpha = \text{const.} = 1/a$ terms involving the first or second time derivative of α vanish and one retains exactly the same equations derived for classical helical flows, e.g. in KCO.

2.2. *Navier–Stokes equations in vorticity formulation*

The generic Navier–Stokes equations in vorticity formulation are given by

$$
\nabla \cdot \boldsymbol{\omega} = 0, \tag{2.6a}
$$

$$
\boldsymbol{\omega} = \nabla \times \boldsymbol{u}, \tag{2.6b}
$$

$$
\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) - \nu\Delta\boldsymbol{\omega} = 0, \tag{2.6c}
$$

which consist of the continuity equation, the definition of vorticity, and the vorticity transport equation.

Following KCO, we introduce a local orthogonal basis system to compute the helical vorticity components $\omega^r, \omega^\xi, \omega^\eta$. The unit vectors are given by

$$
\boldsymbol{e}_r = \frac{\nabla r}{|\nabla r|}, \quad \boldsymbol{e}_\xi = \frac{\nabla \xi}{|\nabla \xi|}, \quad \boldsymbol{e}_{\perp\eta} = \boldsymbol{e}_\xi \times \boldsymbol{e}_r. \tag{2.7a-c}
$$

The vorticity vector in the helical basis is given by

$$
\boldsymbol{\omega} = \omega^r\boldsymbol{e}_r + \omega^\varphi\boldsymbol{e}_\varphi + \omega^z\boldsymbol{e}_z = \omega^r\boldsymbol{e}_r + \omega^\eta\boldsymbol{e}_{\perp\eta} + \omega^\xi\boldsymbol{e}_\xi. \tag{2.8}
$$

The helical vorticity components are related to the cylindrical vorticity components given by

$$
\omega^\eta = \boldsymbol{\omega}\cdot\boldsymbol{e}_{\perp\eta} = B\left(\frac{1}{\alpha}\omega^\varphi - \frac{b}{r}\omega^z\right), \quad \omega^\xi = B\left(\frac{b}{r}\omega^\varphi + \frac{1}{\alpha}\omega^z\right). \tag{2.9a,b}
$$

The definition of vorticity (2.6*b*) in cylindrical coordinates is given by

$$
\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \left(\frac{1}{r}u^z_\varphi - u^\varphi_z\right)\boldsymbol{e}_r + \left(u^r_z - u^z_r\right)\boldsymbol{e}_\varphi + \left(\frac{1}{r}u^\varphi + u^\varphi_r - \frac{1}{r}u^r_\varphi\right)\boldsymbol{e}_z. \tag{2.10}
$$

Replacing the expressions for the velocity components (2.4*a*)–(2.4*c*) and their derivatives and, further, assuming helical invariance ($\partial/\partial\eta \equiv 0$) one obtains the respective components of $\boldsymbol{\omega}$ given by

$$
\omega^r = -\frac{1}{B}u^\eta_\xi - \frac{b}{r}\dot{\alpha}, \tag{2.11a}
$$

$$\omega^\eta = \frac{1}{B}u^r_\xi - \frac{1}{r}\left(ru^\xi\right)_r - \frac{2bB^2}{\alpha r^2}u^\eta + \frac{B^2}{\alpha^2 r}u^\xi, \tag{2.11b}$$

$$\omega^\xi = u^\eta_r + \frac{B^2}{\alpha^2 r}u^\eta. \tag{2.11c}$$

Employing a transformation similarly to the Navier–Stokes equations in primitive variables, one obtains the helically invariant Navier–Stokes equations in vorticity formulation given by

$$\frac{\omega^r}{r} + \omega^r_r + \frac{1}{B}\omega^\xi_\xi = 0, \tag{2.12a}$$

$$\begin{aligned} &-\frac{\dot\alpha}{\alpha}\omega^r_\xi\xi + \omega^r_\tau + u_r\omega^r_r + \frac{1}{B}u^\xi\omega^r_\xi \\ &= \omega^r u^r_r + \frac{1}{B}\omega^\xi u^r_\xi + \nu\left[\frac{1}{r}\left(r\omega^r_r\right)_r + \frac{1}{B^2}\omega^r_{\xi\xi} - \frac{1}{r^2}\omega^r - \frac{2bB}{r^2}\left(\frac{1}{\alpha}\omega^\eta_\xi + \frac{b}{r}\omega^\xi_\xi\right)\right], \end{aligned} \tag{2.12b}$$

$$\begin{aligned} &\frac{b}{r}\dot\alpha\left(-\frac{b^2}{r^2} + \frac{1}{\alpha^2}\right)B^2\omega^\xi - \frac{\dot\alpha}{\alpha}\omega^\eta_\xi\xi - \frac{b^2\dot\alpha}{\alpha r^2}B^2\omega^\eta + \omega^\eta_\tau + u^r\omega^\eta_r + \frac{1}{B}u^\xi\omega^\eta_\xi \\ &\quad -\frac{B^2}{r\alpha^2}(u^r\omega^\eta - u^\eta\omega^r) + \frac{2bB^2}{\alpha r^2}(u^\xi\omega^r - u^r\omega^\xi) \\ &= \omega^r u^\eta_r + \frac{1}{B}\omega^\xi u^\eta_\xi + \nu\left[\frac{1}{r}\left(r\omega^\eta_r\right)_r + \frac{1}{B^2}\omega^\eta_{\xi\xi} + \frac{B^2\left(\dfrac{1}{\alpha^2}B^2 - 2\right)}{\alpha^2 r^2}\omega^\eta + \frac{2bB}{\alpha r^2}\left(\omega^r_\xi - (B\omega^\xi)_r\right)\right], \end{aligned} \tag{2.12c}$$

$$\begin{aligned} &-\frac{\dot\alpha}{\alpha}\omega^\xi_\xi\xi + \frac{b^2\dot\alpha}{r^2\alpha}B^2\omega^\xi + \omega^\xi_\tau + u^r(\omega^\xi)_r + \frac{1}{B}u^\xi\omega^\xi_\xi + \frac{1 - \dfrac{B^2}{\alpha^2}}{r}(u^\xi\omega^r - u^r\omega^\xi) \\ &= \omega^r u^\xi_r + \frac{1}{B}\omega^\xi u^\xi_\xi + \nu\left[\frac{1}{r}\left(r\omega^\xi_r\right)_r + \frac{1}{B^2}\omega^\xi_{\xi\xi} + \frac{\dfrac{1}{\alpha^4}B^4 - 1}{r^2}\omega^\xi \right.\\ &\quad \left. + \frac{2bB}{r}\left(\frac{b}{r^2}\omega^r_\xi + \left(\frac{B}{\alpha r}\omega^\eta\right)_r\right)\right]. \end{aligned} \tag{2.12d}$$

Likewise in KCO, the first two terms on the right-hand side of each equation in (2.12*b*)–(2.12*d*) correspond to vortex stretching.

3. Conservation laws of the helically invariant Euler system in time-dependent helical coordinates

In order to seek local CLs, we subsequently apply the direct construction method (Bluman *et al.* 2010) to the Euler system (2.5*a*)–(2.5*d*) with $\nu = 0$ in primitive variables as well as in vorticity formulation (2.12*a*)–(2.12*d*) with $\nu = 0$. In both cases, the CL multipliers Λ_σ are chosen to be of zeroth order and, thus, only depend on the dependent and independent variables

$$\Lambda_\sigma = \Lambda_\sigma \left(t, r, \xi, u^r, u^\eta, u^\xi, \tilde{p}\right). \tag{3.1}$$

Computations with CL multipliers of first order, i.e. Λ_σ in (3.1) containing first derivatives of $\boldsymbol{u}$ and $\tilde{p}$, have also been done. However, no additional multipliers exist and, hence, also no extended set of CLs.

All computations to derive local CLs were conducted with the aid of the symbolic software package GEM for MAPLE (Cheviakov 2007) and the MAPLE RIFSIMP tool. Further, with these tools we also obtained the corresponding fluxes of the general form Γ^i in equation (1.8). The relation between the fluxes Γ^i and the density Θ and the spatial fluxes Φ^i in equation (1.1) can be derived as follows.

In the helical setting, any divergence form reads

$$\frac{\partial \Gamma^t}{\partial t} + \frac{\partial \Gamma^r}{\partial r} + \frac{\partial \Gamma^\xi}{\partial \xi} = 0, \tag{3.2a}$$

while a divergence expression of an evolution process in helically symmetric form reads

$$\frac{\partial \Theta}{\partial t} + \nabla \cdot \boldsymbol{\Phi} = \frac{\partial \Theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Phi^r) + \frac{1}{B}\frac{\partial \Phi^\xi}{\partial \xi} = 0, \tag{3.2b}$$

which is apparently not identical to (3.2a) due to metric terms. However, multiplying (3.2b) by r and using (3.2a) leads to

$$\frac{\partial \Gamma^t}{\partial t} + \frac{\partial \Gamma^r}{\partial r} + \frac{\partial \Gamma^\xi}{\partial \xi} = \frac{\partial}{\partial t}(r\Theta) + \frac{\partial}{\partial r}(r\Phi^r) + \frac{\partial}{\partial \xi}\left(\frac{r}{B}\Phi^\xi\right). \tag{3.3}$$

Comparing the left- and right-hand side of (3.3) one obtains

$$\Gamma^t = r\Theta, \quad \Gamma^r = r\Phi^r, \quad \Gamma^\xi = \frac{r}{B}\Phi^\xi. \tag{3.4a–c}$$

Subsequently, we always list the density Θ and the spatial fluxes $\boldsymbol{\Phi}$ for each CL. The obvious CLs, such as the continuity equations $\nabla \cdot \boldsymbol{u} = 0$, $\nabla \cdot \boldsymbol{\omega} = 0$ and the scaling of these equations with a time-dependent function leading to $\nabla \cdot (G(t)\boldsymbol{u}) = 0$, $\nabla \cdot (G(t)\boldsymbol{\omega}) = 0$, will not be listed explicitly.

In the next section we seek local CLs that arise from the helically invariant Euler equations in primitive variables, given by (2.5) with $\nu = 0$. These CLs will be denoted by the prefix 'EP'. The CLs obtained from the helically invariant Euler system in vorticity formulation will be denoted by the prefix 'EV'.

3.1. *Primitive variables*

EP1. Extension of the conservation of the z-projection of momentum

The first CL is given by

$$\Theta = \alpha^2 B\left(\frac{1}{\alpha}u^\xi - \frac{b}{r}u^\eta\right), \tag{3.5a}$$

$$\Phi^r = \alpha^2\left(u^r B\left(\frac{1}{\alpha}u^\xi - \frac{b}{r}u^\eta\right) - \dot{\alpha}\xi u^r - \frac{1}{2}\ddot{\alpha}r\xi\right), \tag{3.5b}$$

$$\Phi^{\xi} = \alpha^2 \left(u^{\xi} B \left(\frac{1}{\alpha} u^{\xi} - \frac{b}{r} u^{\eta} \right) + \frac{B}{\alpha} p \right) - \alpha^2 \dot{\alpha} \xi u^{\xi} - \alpha \dot{\alpha} \xi B^2 \left(\frac{1}{\alpha} u^{\xi} - \frac{b}{r} u^{\eta} \right). \quad (3.5c)$$

Compared to the classical conservation of the z-projection of momentum, one determines that the density and the spatial fluxes are multiplied by the square of the arbitrary function $\alpha(t)$. It may further be noted that Θ in (3.5*a*) is linked to u^z in cylindrical coordinates by $\Theta = \alpha^2(u^z + \dot{\alpha}\eta)$. Moreover, the spatial fluxes are extended by terms involving first and second time derivatives of the arbitrary function. ($\alpha =$ const. corresponds to EP2 in KCO).

EP2. Extension of the conservation of the z-projection of angular momentum

The second CL is given by

$$\Theta = \alpha r B \left(\frac{1}{\alpha} u^{\eta} + \frac{b}{r} u^{\xi} \right) = \alpha r u^{\varphi}, \quad (3.6a)$$

$$\Phi^r = \alpha r u^r u^{\varphi}, \quad (3.6b)$$

$$\Phi^{\xi} = \alpha \left(r u^{\xi} u^{\varphi} + bBp \right) - \dot{\alpha} r \xi B u^{\varphi}. \quad (3.6c)$$

In contrast to the velocity component in the z-direction, the component in the angular direction has no relative movement. Therefore, we are able to express the density and the fluxes in terms of cylindrical velocities ($\alpha =$ const. corresponds to EP3 in KCO).

EP3. Extension of the conservation of the generalized momenta/angular momenta

Here we obtain an infinite family of CLs

$$\Theta = \alpha F \left(\frac{\alpha r}{B} u^{\eta} + b \alpha \dot{\alpha} \xi \right) - b \alpha^2 \dot{\alpha} \xi, \quad (3.7a)$$

$$\Phi^r = \alpha F \left(\frac{\alpha r}{B} u^{\eta} + b \alpha \dot{\alpha} \xi \right) u^r + \frac{1}{2} \alpha^2 \ddot{\alpha} b r \xi, \quad (3.7b)$$

$$\Phi^{\xi} = \alpha F \left(\frac{\alpha r}{B} u^{\eta} + b \alpha \dot{\alpha} \xi \right) u^{\xi} - B \dot{\alpha} \xi F \left(\frac{\alpha r}{B} u^{\eta} + b \alpha \dot{\alpha} \xi \right) + B b \alpha \dot{\alpha}^2 \xi^2, \quad (3.7c)$$

where $F(\cdot)$ is a once differentiable arbitrary function.

The density and the spatial fluxes contain the first and second time derivative of α. Referring to KCO, the quantity

$$\zeta = \frac{\alpha r}{B} u^{\eta} + b \alpha \dot{\alpha} \xi \quad (3.8)$$

can be physically interpreted as a 'blend' of momentum and angular momentum density in the η-direction since with the use of (2.3*e*) and $z = \alpha(\xi - \eta)$ the quantity reads

$$\zeta = \frac{\alpha r}{B} u^{\eta} + b \alpha \dot{\alpha} \xi = r u^{\varphi} - \alpha b u^z + b \dot{\alpha} z \quad (3.9)$$

$$= \alpha \left(\frac{r}{\alpha} u^{\varphi} - b u^z + \frac{b}{\alpha} \dot{\alpha} z \right), \quad (3.10)$$

where the term $(b/\alpha)\dot{\alpha}z$ describes the relative movement. For the special cases $b = 0$ the quantity is proportional to the angular momentum density in the z-direction

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$\tilde{\zeta} \sim (r/\alpha)u^{\varphi}$ in (3.10) while for $1/\alpha = 0$ it is proportional to the linear momentum density in the z-direction $\tilde{\zeta} \sim u^z$ in (3.10) ($\alpha =$ const. corresponds to EP4 in KCO).

Comparing the above results to the classical cases one observes that the only quantity which is not conserved is kinetic energy. In the next section we will see that also helicity is not conserved. In order to ensure that this proposition is appropriate in any helical, time-dependent coordinate system, we will prove the non-existence of conservation of energy and helicity in appendices B and C.

3.2. *The vorticity formulation*

The following CLs are derived from the helically invariant Euler equations in vorticity formulation, consisting of the continuity equation for velocity as well as for vorticity, equations (2.5*a*) and (2.12*a*), the equations defining the vorticity components (2.11) and the vorticity dynamics equations (2.12*b*)–(2.12*d*) with $\nu = 0$. Similar to the multipliers (3.1) for the primitive variables we assume the present multipliers to be of the form

$$\varLambda_\sigma = \varLambda_\sigma \left(t, r, \xi, u^r, u^\eta, u^\xi, \tilde{p}, \omega^r, \omega^\eta, \omega^\xi\right), \tag{3.11}$$

i.e. we limit ourselves again to zeroth-order multipliers.

EV1. Extended family of conservation laws involving ω^φ

The family of CLs is given by

$$\varTheta = \frac{q(t)}{r}\omega^\varphi, \tag{3.12a}$$

$$\varPhi^r = \frac{1}{r}\left(q(t)\left[u^r\omega^\varphi - \omega^r u^\varphi\right] + \dot{q}(t)B\left(\frac{1}{\alpha}u^\xi - \frac{b}{r}u^\eta\right) - \dot{\alpha}q(t)B\left(\frac{1}{\alpha^2}u^\xi - \frac{b}{r\alpha}u^\eta\right)\right), \tag{3.12b}$$

$$\varPhi^\xi = -\frac{B}{r\alpha}\left(q(t)\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right) + \dot{q}(t)u^r + q(t)\dot{\alpha}\xi\omega^\varphi - q(t)\frac{\dot{\alpha}}{\alpha}u^r\right), \tag{3.12c}$$

where $q(t)$ and $\dot{q}(t)$ is an arbitrary function and its time derivative, respectively ($\alpha =$ const. corresponds to EV3 in KCO).

EV2. Vorticity conservation law

The CL is given by the density and fluxes

$$\varTheta = -\alpha^4 rB\left(\frac{1}{\alpha^3}\omega^\eta - \frac{b^3}{r^3}\omega^\xi\right), \tag{3.13a}$$

$$\begin{aligned}\varPhi^r &= \alpha^4\left(-\frac{B}{r^2}\left(\frac{r^3}{\alpha^3}\left(u^r\omega^\eta - u^\eta\omega^r\right) - b^3\left(u^r\omega^\xi - u^\xi\omega^r\right)\right) - \frac{2B}{\alpha^2}u^r\left(-\frac{b}{r}u^\eta + \frac{1}{\alpha}u^\xi\right)\right)\\ &\quad + \frac{B}{r^2}\dot{\alpha}\left(-r^3u^\xi - 2\alpha^3b^3u^\eta - \alpha r^2bu^\eta + \frac{2}{B}\alpha^2r^2\xi u^r\right),\end{aligned} \tag{3.13b}$$

$$\begin{aligned}\varPhi^\xi &= \alpha^4\left(\frac{B}{\alpha^3}\left((u^r)^2 + (u^\eta)^2 - \left(u^\xi\right)^2 + r\left(u^\eta\omega^\xi - u^\xi\omega^\eta\right)\right) + \frac{2bB}{\alpha^2 r}u^\eta u^\xi\right)\\ &\quad - \frac{B^2}{r^2}\dot{\alpha}\left(b^3\alpha^3\xi\omega^\xi - 2b^2\alpha^2\xi u^\xi - r^3\xi\omega^\eta - 2r^2\xi u^\xi - \frac{r^3}{B}u^r\right).\end{aligned} \tag{3.13c}$$

The assumption $b=0$ leads to $\Theta=-\alpha^2 r\omega^\varphi$. Corresponding to KCO problems, where the velocity vanishes on the boundary of the time-independent flow domain $\Omega\neq\Omega(t)$, the quantity $\alpha^2 r\omega^\varphi$ corresponds to the conservation of linear momentum in the z-direction since

$$\frac{1}{2}\alpha^2\int\int_\Omega r\omega^\varphi\,\mathrm{d}A=\alpha^2\int\int_\Omega u^z\,\mathrm{d}A, \tag{3.14}$$

where $\alpha(t)$ is the parameter function ($\alpha=$ const. corresponds to EV4 in KCO).

EV3. Vorticity conservation law

The CL is given by

$$\begin{aligned}\Theta &= -\alpha^4\frac{B}{r^2}\left(\frac{r^2b^2}{B^2}\omega^\xi+\frac{r^4}{\alpha^3}\left(-\frac{b}{r}\omega^\eta+\frac{1}{\alpha}\omega^\xi\right)\right)\\ &= -\alpha^4\frac{B}{r^2}\left(\frac{r^2b^2}{B^2}\omega^\xi+\frac{r^4}{\alpha^3 B}\omega^z\right),\end{aligned} \tag{3.15a}$$

$$\begin{aligned}\Phi^r &= \alpha^4\frac{rB}{\alpha^3}\left(2u^r\left(\frac{1}{\alpha}u^\eta+\frac{b}{r}u^\xi\right)+b\left(u^r\omega^\eta-u^\eta\omega^r\right)\right.\\ &\quad\left.-\frac{r^4+r^2\alpha^2b^2+b^4\alpha^4}{r^3\alpha}\left(u^r\omega^\xi-u^\xi\omega^r\right)+\dot\alpha\left(\frac{b}{\alpha}u^\xi+2\alpha^2\frac{b^4}{r^3}u^\eta+\frac{b^2}{r}u^\eta\right)\right),\end{aligned} \tag{3.15b}$$

$$\begin{aligned}\Phi^\xi &= \alpha^4\left(-\frac{1}{\alpha^3}bB\right)\left((u^r)^2+(u^\eta)^2-\left(u^\xi\right)^2+r\left(u^\eta\omega^\xi-u^\xi\omega^\eta\right)\right.\\ &\quad\left.-2\frac{r}{\alpha b}u^\eta u^\xi+\dot\alpha\left(B\frac{r}{\alpha}\xi\omega^\eta+\frac{r}{\alpha}u^r-\frac{r^4+r^2\alpha^2b^2+b^4\alpha^4}{r^2\alpha^2 b}B\xi\omega^\xi\right)\right).\end{aligned} \tag{3.15c}$$

As already seen, the vorticity component ω^z has no relative movement. Hence, the helical vorticity components ω^ξ, ω^η were replaced by ω^z in the density (3.15*a*). For rotationally symmetric flows, $b=0$, the density reduces to $-\alpha r^2\omega^z/2$. In a similar way to (3.14), for problems with vanishing flow velocities on the boundary and $\Omega\neq\Omega(t)$, it corresponds to the conservation of angular momentum in the z-direction. As before, the time-dependent parameter function $\alpha(t)$ is still involved ($\alpha=$ const. corresponds to EV5 in KCO).

EV4. Vorticity conservation law

We further obtain a family of CLs given by the density and fluxes

$$\Theta=0, \tag{3.16a}$$

$$\Phi^r=N\omega^r-\frac{1}{B}N_\xi u^\eta+\frac{b}{r}\dot\alpha N, \tag{3.16b}$$

$$\Phi^\xi=N\omega^\xi+N_r u^\eta, \tag{3.16c}$$

where $N=N(r,\xi,t)$. This CL is a linear combination of the continuity equation for the vorticity (2.12*a*) and the equations defining the vorticity components (2.11). For $\alpha=$ const. we obtain an additional CL of the classical case that has not been listed in KCO. In KCO only a dimensionally reduced form of (3.16*b*)/(3.16*c*) was found, where the function only depended on ξ and t, i.e. $N=N(\xi,t)$ (EV6 in KCO).

EV5. Vorticity conservation law

In this case the CL multipliers again depend on an arbitrary function $M = M(\xi, r, t)$ and one obtains the following family of CLs, given by the density and fluxes

$$\Theta = M_r \omega^r + \frac{M_\xi}{B} \omega^\xi, \tag{3.17a}$$

$$\Phi^r = \frac{M_\xi}{B} \left(u^r \omega^\xi - u^\xi \omega^r \right) - M_t \omega^r + \frac{\dot{\alpha}}{\alpha} \omega^r \left(M + \xi M_\xi \right), \tag{3.17b}$$

$$\Phi^\xi = M_r \left(u^\xi \omega^r - u^r \omega^\xi \right) - M_t \omega^\xi + \frac{\dot{\alpha}}{\alpha} \left(M \omega^\xi - B \xi M_r \omega^r \right). \tag{3.17c}$$

This CL is a subset of the infinite family of vorticity CLs for helical flows presented in Cheviakov & Oberlack (2014) for arbitrary time-dependent three-dimensional flows and is closely linked to Ertel's theorem.

4. Conservation laws of the helically invariant Navier–Stokes system in time-dependent coordinates

Presently, the application of the direct method to the reduced Navier–Stokes system (2.5*a*)–(2.5*d*) in primitive variables leads to two new CLs. For the related vorticity formulation (2.12*a*)–(2.12*d*) five new CLs will be derived. For the present calculations all multipliers (3.1) have been chosen to be of zeroth order. Similarly to the classical case, all new present CLs are subsets of those admitted by the helically symmetric Euler equations in the previous section. Whereas the densities are the same, the fluxes are extended by additional viscous terms.

4.1. *Primitive variables*

NSP1. Extension of the conservation of the z-projection of momentum

The CL is respectively defined by the density and fluxes

$$\Theta = \alpha^2 B \left(\frac{1}{\alpha} u^\xi - \frac{b}{r} u^\eta \right), \tag{4.1a}$$

$$\Phi^r = \alpha^2 \left(u^r B \left(\frac{1}{\alpha} u^\xi - \frac{b}{r} u^\eta \right) - \dot{\alpha} \xi u^r - \frac{1}{2} \ddot{\alpha} r \xi \right) - \alpha^2 \nu \left(B \left(\frac{1}{\alpha} u^\xi - \frac{b}{r} u^\eta \right) \right)_r, \tag{4.1b}$$

$$\begin{aligned} \Phi^\xi = {} & \alpha^2 \left(u^\xi B \left(\frac{1}{\alpha} u^\xi - \frac{b}{r} u^\eta \right) + \frac{B}{\alpha} p \right) \\ & - \alpha^2 \dot{\alpha} \xi u^\xi - \alpha \dot{\alpha} \xi B^2 \left(\frac{1}{\alpha} u^\xi - \frac{b}{r} u^\eta \right) - \nu \frac{\alpha^2}{B} \left(B \left(\frac{1}{\alpha} u^\xi - \frac{b}{r} u^\eta \right) \right)_\xi . \end{aligned} \tag{4.1c}$$

Equation (4.1) is a viscous extension of the CL (3.5). In order to clearly see the momentum conservation, we may write the density Θ in terms of a cylindrical coordinate system to obtain

$$\Theta = \alpha^2 \left(u^z + \dot{\alpha} \eta \right) \tag{4.2}$$

($\alpha =$ const. corresponds to NSP1 in KCO).

NSP2. Extension of the conservation of generalized momentum

For the present viscous case, the family of CLs for the helical Euler system (3.7) reduces to one single CL, given by the density and fluxes

$$\Theta = \alpha^2 \frac{r}{B} u^\eta, \tag{4.3a}$$

$$\Phi^r = \alpha^2 \left(\frac{r}{B} u^\eta u^r + \dot{\alpha}\xi b u^r + \frac{1}{2}\ddot{\alpha} b r \xi \right) - \alpha^2 \nu \left[-2\frac{B}{\alpha}\left(\frac{1}{\alpha}u^\eta + \frac{b}{r}u^\xi \right) + \left(\frac{r}{B}u^\eta \right)_r \right] \tag{4.3b}$$

$$= \alpha^2 \left(\frac{r}{B} u^\eta u^r + \dot{\alpha}\xi b u^r + \frac{1}{2}\ddot{\alpha} b r \xi \right) - \alpha^2 \nu \left[-\frac{2}{\alpha}u^\varphi + \left(\frac{r}{B}u^\eta \right)_r \right], \tag{4.3c}$$

$$\Phi^\xi = \alpha^2 \left(\frac{r}{B} u^\eta u^\xi + \alpha^2 \dot{\alpha}\xi b u^\xi \right) - \alpha\dot{\alpha} r \xi u^\eta - \alpha^2 \nu \frac{1}{B}\left[\frac{2bB^2}{\alpha r}u^r + \left(\frac{r}{B}u^\eta \right)_\xi \right]. \tag{4.3d}$$

Presently, the analysis and interpretation below (3.7) also holds true, namely that (4.3*a*)–(4.3*d*) refers to a blend of momentum and angular momentum ($\alpha =$ const. corresponds to NSP2 in KCO).

4.2. *The vorticity formulation*

In the case of CLs in vorticity formulation we obtained five distinct cases all of which are one-to-one extensions of the five cases in KCO.

NSV1. Extension of an infinite family of vorticity conservation laws

The family of CLs (3.12) is extended by the viscous terms and the density and fluxes are given by

$$\Theta = \frac{q(t)}{r} B \left(\frac{1}{\alpha}\omega^\eta + \frac{b}{r}\omega^\xi \right) = \frac{q(t)}{r}\omega^\varphi, \tag{4.4a}$$

$$\Phi^r = \frac{1}{r}\left(q(t)\left[u^r\omega^\varphi - \omega^r u^\varphi \right] + \dot{q}(t) B \left(\frac{1}{\alpha}u^\xi - \frac{b}{r}u^\eta \right) - \dot{\alpha} q(t) B \left(\frac{1}{\alpha^2}u^\xi - \frac{b}{r\alpha}u^\eta \right) \right)$$
$$- \frac{q(t)}{r}\nu \left[\frac{B}{r\alpha}\omega^\eta + \frac{b^2}{r\left(\dfrac{r^2}{\alpha^2} + b^2 \right)}\omega^\varphi + B\left(\frac{1}{\alpha}\omega^\eta_r + \frac{b}{r}\omega^\xi_r \right) \right], \tag{4.4b}$$

$$\Phi^\xi = -\frac{B}{r\alpha}\left(q(t)\left(u^\eta\omega^\xi - u^\xi\omega^\eta \right) + \dot{q}(t) u^r + q(t)\dot{\alpha}\xi\omega^\varphi - q(t)\frac{\dot{\alpha}}{\alpha}u^r \right)$$
$$- \frac{q(t)}{r^4} B\nu \left[\frac{r^3}{B}\left(\frac{1}{\alpha}\omega^\eta_\xi + \frac{b}{r}\omega^\xi_\xi \right) + 2br\omega^r \right], \tag{4.4c}$$

where $q(t)$ is an arbitrary function of time ($\alpha =$ const. corresponds to NSV1 in KCO).

NSV2. Extension of the vorticity conservation law

This CL is given by

$$\Theta = -\alpha^4 r B \left(\frac{1}{\alpha^3}\omega^\eta - \frac{b^3}{r^3}\omega^\xi \right), \tag{4.5a}$$

$$
\begin{aligned}
\Phi^r =\ & \alpha^4\left(-\frac{B}{r^2}\left(\frac{r^3}{\alpha^3}\left(u^r\omega^\eta-u^\eta\omega^r\right)-b^3\left(u^r\omega^\xi-u^\xi\omega^r\right)\right)-\frac{2B}{\alpha^2}u^r\left(-\frac{b}{r}u^\eta+\frac{1}{\alpha}u^\xi\right)\right)\\
&+\frac{B}{r^2}\dot{\alpha}\left(-r^3u^\xi-2\alpha^3b^3u^\eta-\alpha r^2bu^\eta+\frac{2}{B}\alpha^2r^2\xi u^r\right)\\
&-\alpha^4\frac{B}{r^2}\nu\left[\frac{r^2}{B^2}\left(\frac{1}{\alpha}\omega^\eta+\frac{b}{r}\omega^\xi\right)-r^3\left(\frac{1}{\alpha^3}\omega^\eta_r-\frac{b^3}{r^3}\omega^\xi_r\right)+\frac{b}{\alpha}B^2r\left(\frac{b^3}{r^3}\omega^\eta+\frac{1}{\alpha^3}\omega^\xi\right)\right],
\end{aligned}
\tag{4.5b}
$$

$$
\begin{aligned}
\Phi^\xi =\ & \alpha^4\left(\frac{B}{\alpha^3}\left((u^r)^2+(u^\eta)^2-\left(u^\xi\right)^2+r\left(u^\eta\omega^\xi-u^\xi\omega^\eta\right)\right)+\frac{2bB}{\alpha^2r}u^\eta u^\xi\right)\\
&-\frac{B^2}{r^2}\dot{\alpha}\left(b^3\alpha^3\xi\omega^\xi-2b\alpha^2\xi u^\xi-r^3\xi\omega^\eta-2r^2\xi u^\xi-\frac{r^3}{B}u^r\right)\\
&+\alpha^4\frac{2bB}{\alpha^2r}\nu\left[\left(1-\frac{b^2\alpha^2}{r^2}\right)\omega^r+\frac{r^2\alpha^2}{2bB}\left(\frac{1}{\alpha^3}\omega^\eta_\xi-\frac{b^3}{r^3}\omega^\xi_\xi\right)\right],
\end{aligned}
\tag{4.5c}
$$

which is a viscous extension of the CL (3.13) ($\alpha=$ const. corresponds to NSV2 in KCO).

NSV3. Extension of the vorticity conservation law

This CL is given by

$$
\begin{aligned}
\Theta &= -\alpha^4\frac{B}{r^2}\left(\frac{r^2b^2}{B^2}\omega^\xi+\frac{r^4}{\alpha^3}\left(-\frac{b}{r}\omega^\eta+\frac{1}{\alpha}\omega^\xi\right)\right)\\
&= -\alpha^4\frac{B}{r^2}\left(\frac{r^2b^2}{B^2}\omega^\xi+\frac{r^4}{\alpha^3B}\omega^z\right),
\end{aligned}
\tag{4.6a}
$$

$$
\begin{aligned}
\Phi^r =\ & \alpha^4\frac{rB}{\alpha^3}\Bigg(2u^r\left(\frac{1}{\alpha}u^\eta+\frac{b}{r}u^\xi\right)+b\left(u^r\omega^\eta-u^\eta\omega^r\right)\\
&-\frac{r^4+r^2\alpha^2b^2+b^4\alpha^4}{r^3\alpha}\left(u^r\omega^\xi-u^\xi\omega^r\right)+\dot{\alpha}\left(\frac{b}{\alpha}u^\xi+2\alpha^2\frac{b^4}{r^3}u^\eta+\frac{b^2}{r}u^\eta\right)\Bigg)\\
&+\alpha^4\nu\Bigg[4\frac{B}{\alpha^3}\left(\frac{1}{\alpha}u^\eta+\frac{b}{r}u^\xi\right)-\frac{br}{\alpha^3}B\omega^\eta_r+\frac{B}{r^3}\left(b^4-\frac{r^4}{\alpha^4}-\frac{r^6}{\alpha^4r^2+\alpha^6b^2}\right)\omega^\xi\\
&+\frac{B}{\alpha^4r^2}\left(r^4+\alpha^2r^2b^2+\alpha^4b^4\right)\omega^\xi_r+\frac{b}{\alpha B}\left(2+\frac{r^4}{\left(r^2+\alpha^2b^2\right)^2}\right)\omega^\eta\Bigg],
\end{aligned}
\tag{4.6b}
$$

$$
\begin{aligned}
\Phi^\xi =\ & \alpha^4\left(-\frac{1}{\alpha^3}bB\right)\Bigg(\left(u^r\right)^2+\left(u^\eta\right)^2-\left(u^\xi\right)^2+r\left(u^\eta\omega^\xi-u^\xi\omega^\eta\right)\\
&-2\frac{r}{\alpha b}u^\eta u^\xi+\dot{\alpha}\left(B\frac{r}{\alpha}\xi\omega^\eta+\frac{r}{\alpha}u^r-\frac{r^4+r^2\alpha^2b^2+b^4\alpha^4}{r^2\alpha^2b}B\xi\omega^\xi\right)\Bigg)\\
&+\alpha^4\nu\left[\frac{1}{\alpha^4r^2}\left(r^4+\alpha^2r^2b^2+\alpha^4b^4\right)\omega^\xi_\xi-\frac{br}{\alpha^3}\omega^\eta_\xi-\frac{4bB}{\alpha^3r}u^r+\frac{2B^4b}{r^3}\omega^r\right],
\end{aligned}
\tag{4.6c}
$$

which is a viscous extension of (3.15) ($\alpha=$ const. corresponds to NSV3 in KCO).

NSV4. Extension of an infinite family of vorticity conservation laws

The family of CLs (3.16), which holds for the helically invariant Euler equations, is carried over to the viscous case without change, as the viscosity does not appear explicitly ($\alpha = \text{const.}$ corresponds to NSV4 in KCO).

NSV5. Extension of an infinite family of vorticity conservation laws

The infinite set of CLs (3.17) is extended by a viscous term. Its density and fluxes read

$$\Theta = M_r\omega^r + \frac{M_\xi}{B}\omega^\xi, \tag{4.7a}$$

$$\begin{aligned}\Phi^r = {} & \frac{M_\xi}{B}\left(u^r\omega^\xi - u^\xi\omega^r\right) - M_t\omega^r + \frac{\dot{\alpha}}{\alpha}\omega^r\left(M + \xi M_\xi\right) \\ & + \nu\left[-\frac{M_{\xi\xi}}{B^2}\omega^r - \frac{M_r}{r}\left(r\omega^r\right)_r - \frac{2bBM_\xi}{\alpha r^2}\omega^\eta + \left(\frac{BM_\xi}{\alpha^2 r} + \frac{M_{r\xi}}{B}\right)\omega^\xi - \frac{M_\xi}{rB}\left(r\omega^\xi\right)_r\right],\end{aligned} \tag{4.7b}$$

$$\begin{aligned}\Phi^\xi = {} & M_r\left(u^\xi\omega^r - u^r\omega^\xi\right) - M_t\omega^\xi + \frac{\dot{\alpha}}{\alpha}\left(M\omega^\xi - B\xi M_r\omega^r\right) + \nu\left[\left(\frac{2B}{\alpha^2 r} - \frac{2}{rB}\right)M_\xi\omega^r\right. \\ & \left. + \frac{M_{r\xi}}{B}\omega^r - \frac{M_r}{B}\omega^r_\xi + \frac{2bB^2M_r}{\alpha r^2}\omega^\eta - \left(\frac{2B^2M_r}{\alpha^2 r} + r\left(\frac{M_r}{r}\right)_r\right)\omega^\xi - \frac{M_\xi}{B^2}\omega^\xi_\xi\right],\end{aligned} \tag{4.7c}$$

while $M = M(\xi, r, t)$. As for (3.17) the previous CLs are a special case of the infinite family for viscous helical flows in Cheviakov & Oberlack (2014).

5. Summary and conclusions

In the current contribution the classical case of a helical coordinate system is extended to a new time-dependent helical coordinate system by applying the method of group invariant reduction. Based on these new coordinates, a reduced system of time-dependent helically invariant Euler and Navier–Stokes equations was derived, where the spatial dependence of all dependent variables has been reduced by one. For the development of the new helical coordinate system, we considered the helical symmetry of the Euler and Navier–Stokes equations, which relies on the combination of two Lie symmetry groups: (i) generalized Galilean invariance, which comprises classical Galilean group and axial translation and (ii) the rotation about the same axis. Compared to the coordinate system that is used for classical helical flows (see e.g. KCO), where all coordinates are time-independent, the present helical coordinate ξ is time-dependent. Nevertheless, particularly for the helical coordinate, there is a good analogy between the classical and the extended case. The only difference is that the parameter a in the classical helical setting is replaced by the time-dependent parameter $1/\alpha(t)$. Moreover, the helically invariant Euler and Navier–Stokes equations are extended by terms involving the time derivative of the parameter function $\alpha(t)$.

Furthermore, new local CLs were sought, which were obtained from the new helically invariant system of equations. However, for every CL in KCO, a corresponding CL has been found for time-dependent coordinates, except for the conservation of energy and helicity. Due to their great importance in a physical context, their absence was proven in appendices B and C.

For the helically invariant Euler equations in primitive variables an extension of the conservation of the z-projection of momentum and angular momentum was obtained. Further, an infinite family of CLs holds for helical flows in time-dependent coordinates, which may be considered a blend of momentum and angular momentum.

For the helically invariant Euler equations in vorticity formulation, three extended families of CLs were obtained. The family of CLs EV1 (3.12) involves the vorticity component ω^{φ} and depends on the arbitrary function $q(t)$. The family of CLs EV4 (3.16) is a combination of the continuity equation for vorticity (2.12*a*) and the equations defining the vorticity components (2.11). The family of CLs (EV5) is a subset of the infinite family of CLs for helical flows presented in Cheviakov & Oberlack (2014). Further, two new CLs, which are extensions of the classical case, hold (formulae (3.13) and (3.15)).

For the viscous case in primitive variables, extensions of the conservation of the z-projection of momentum and generalized momentum were obtained (formulae (4.1), (4.3)).

Finally, for the viscous case in vorticity formulation, all the CLs obtained for the inviscid case are carried over and extended by a viscous term. They were listed in the § 4.2.

In summary, with the use of the Lie symmetries, a setting to describe helical flows with a time-dependent moving pitch could be derived in a straightforward way. Moreover, the assumption of helical invariance gives rise to new CLs which exist in addition to the known CL in three dimensions, e.g. conservation of mass, momentum and energy.

One objective for future work is seeking exact solutions of the helical invariant Navier–Stokes equations (2.5). However, it is rather difficult to find an appropriate ansatz for a solution of the equations even for the classical helical case with a constant pitch. The stability of special solutions to the Navier–Stokes equations is a topic of countless publications, although very little is known for helical flows. However, it is known that many swirling flows are unstable to helical disturbances, in fact, very often these are the most unstable modes. In this sense, the present study may be valuable for the understanding of the nonlinear stability of these flows.

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Appendix A. Derivation of the time-dependent helical coordinates

Presently, we derived a new time-dependent helical coordinate system for the Euler and Navier–Stokes equations by using the method of group invariant reduction. Here, group invariance is meant in the sense of reduction of the spatial coordinates such as e.g. a reduction to plane flows.

The derivation is based on the helical symmetry which is a combination of the generalized Galilean invariance and the rotation group, given by (2.2). The corresponding infinitesimal generators for rotation and generalized Galilean invariance of the Euler and Navier–Stokes equations are given by (see e.g. Oberlack 2000)

$$X_R = \frac{\partial}{\partial \varphi}, \tag{A 1a}$$

$$X_G = \alpha(t)\frac{\partial}{\partial z} + \dot{\alpha}(t)\frac{\partial}{\partial u^z} - z\ddot{\alpha}(t)\frac{\partial}{\partial p}. \tag{A 1b}$$

A new extended helical symmetry of the Euler and Navier–Stokes equations consists of a superposition of a rotation group X_R and the generalized Galilean symmetry X_G. To maintain the nomenclature to earlier results on helical flows (see e.g. KCO), we choose the following linear combination of symmetries (A 1*a*) and (A 1*b*) to obtain

$$X = \frac{1}{b}X_R - X_G \tag{A 2}$$

$$= \frac{1}{b}\frac{\partial}{\partial\varphi} - \alpha(t)\frac{\partial}{\partial z} - \dot{\alpha}(t)\frac{\partial}{\partial u^z} + z\ddot{\alpha}(t)\frac{\partial}{\partial p}. \tag{A 3}$$

For the final aim of a helically symmetric coordinate system we need to define $\eta(r, \varphi, z, t)$ to be the variable, which should be eliminated from the system of equations such that the reduced system only contains two spatial variables.

Based on this idea, we derive a new set of variables, i.e. the reduced helical coordinates $(\tilde{r}, \xi)$, the helical velocities $(\tilde{u}^r, u^\eta, u^\xi)$ and the pressure $\tilde{p}$ summarized in the vector

$$\boldsymbol{\sigma} := (\xi, \tilde{r}, \tau, \tilde{u}^r, u^\eta, u^\xi, \tilde{p}). \tag{A 4}$$

For their derivation we employ the method of canonical coordinates (see e.g. Bluman *et al.* 2010), which results in two linear partial differential equations of first order, given by

$$X\eta = 1, \tag{A 5a}$$

$$X\boldsymbol{\sigma} = 0. \tag{A 5b}$$

Their solutions generate the new variables $(\eta, \boldsymbol{\sigma})$ and the symmetry (A 3) transforms into the symmetry

$$X = \frac{\partial}{\partial\eta}, \tag{A 6}$$

which is a translational symmetry in η.

Substituting (A 3) in (A 5*a*) leads to

$$X\eta = \frac{1}{b}\frac{\partial\eta}{\partial\varphi} - \alpha(t)\frac{\partial\eta}{\partial z} - \dot{\alpha}(t)\frac{\partial\eta}{\partial u^z} + z\ddot{\alpha}(t)\frac{\partial\eta}{\partial p} = 1, \tag{A 7}$$

which can be equivalently written in the form of a characteristic system

$$b\,\mathrm{d}\varphi = -\frac{\mathrm{d}z}{\alpha(t)} = -\frac{\mathrm{d}u^z}{\dot{\alpha}(t)} = \frac{\mathrm{d}p}{z\ddot{\alpha}(t)} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}r}{0} = \frac{\mathrm{d}u^\varphi}{0} = \frac{\mathrm{d}u^r}{0} = \frac{\mathrm{d}\eta}{1}. \tag{A 8}$$

Solving the system (A 8), we obtain a general solution of equation (A 7) for the helical variable η given by

$$\eta = b\varphi + F\left(r, t, b\varphi + \frac{z}{\alpha}, p + \frac{1}{2}\frac{\ddot{\alpha}}{\alpha}z^2, u^\varphi, u^r, u^z + \dot{\alpha}b\varphi\right). \tag{A 9}$$

Similarly, substituting (A 3) in (A 5*b*) yields

$$X\boldsymbol{\sigma} = \frac{1}{b}\frac{\partial\boldsymbol{\sigma}}{\partial\varphi} - \alpha(t)\frac{\partial\boldsymbol{\sigma}}{\partial z} - \dot{\alpha}(t)\frac{\partial\boldsymbol{\sigma}}{\partial u^z} + z\ddot{\alpha}(t)\frac{\partial\boldsymbol{\sigma}}{\partial p} = 0, \tag{A 10}$$

and, written in the equivalent form of the characteristic system, we obtain

$$b\,\mathrm{d}\varphi = -\frac{\mathrm{d}z}{\alpha(t)} = -\frac{\mathrm{d}u^z}{\dot{\alpha}(t)} = \frac{\mathrm{d}p}{z\ddot{\alpha}(t)} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}r}{0} = \frac{\mathrm{d}u^\varphi}{0} = \frac{\mathrm{d}u^r}{0} = \frac{\mathrm{d}\boldsymbol{\sigma}}{0}. \tag{A 11}$$

A general solution of (A 11) reads

$$\boldsymbol{\sigma} = \boldsymbol{F}\left(r, t, b\varphi + \frac{z}{\alpha}, p + \frac{1}{2}\frac{\ddot{\alpha}}{\alpha}z^2, u^\varphi, u^r, u^z + \dot{\alpha}b\varphi\right), \tag{A 12}$$

where $\boldsymbol{F} = (f^1, f^2, f^3, f^4, f^5, f^6, f^7)$ is a vector consisting of seven arbitrary functions depending on the arguments given in (A 12).

The specific new helical coordinates may now be manufactured from (A 9) and (A 12), respectively. In order to keep the complexity for the resulting Euler and Navier–Stokes equations in the helical coordinate system as low as possible, we choose the new independent coordinates as given in (2.3*a*)–(2.3*d*).

Analogous to the classical case the helical velocity components and the pressure are given by (2.3*e*)–(2.3*h*). They are a particular choice of the special solutions of (A 12).

Appendix B. Proof of the absence of conservation of kinetic energy in time-dependent helical coordinates

It is well known that for the three-dimensional time-dependent Euler system, the kinetic energy is conserved in every coordinate system. In the present section we will prove that in our spatially reduced time-dependent helically symmetric coordinate system the conservation of kinetic energy does not hold.

The proof can be done by contradiction: we rewrite the kinetic energy in conserved form in the helical time-dependent coordinates, i.e. (r, ξ, η) defined in (2.3*a*)–(2.3*d*). However, after imposing helical invariance, i.e. eliminating any η-dependence from the velocity and pressure, we would expect an energy equation, which is independent of η. However, we obtain an equation involving the helical coordinate η itself. As a result, it is not possible to write an energy conservation equation in a helically symmetric time-dependent coordinate system. The following steps may clarify the proceeding.

The energy equation can be written in a cylindrical coordinate system

$$\frac{\partial \Gamma^t}{\partial t} + \frac{\partial \Gamma^\varphi}{\partial \varphi} + \frac{\partial \Gamma^r}{\partial r} + \frac{\partial \Gamma^z}{\partial z} = 0, \tag{B 1}$$

where the fluxes read

$$\Gamma^t = rK, \tag{B 2}$$

$$\Gamma^\varphi = u^\varphi\,(K + p)\,, \tag{B 3}$$

$$\Gamma^r = ru^r\,(K + p)\,, \tag{B 4}$$

$$\Gamma^z = ru^z\,(K + p)\,. \tag{B 5}$$

K denotes the kinetic energy density, given by

$$K=\frac{1}{2}|\boldsymbol{u}|^2 = \frac{1}{2}\left((u^r)^2+(u^\varphi)^2+(u^z)^2\right) \tag{B 6a}$$

$$= \frac{1}{2}\left[(u^r)^2+B^2\left(\frac{b}{r}u^\xi+\frac{1}{\alpha}u^\eta\right)^2+\left(B\left(\frac{1}{\alpha}u^\xi-\frac{b}{r}u^\eta\right)-\dot{\alpha}\eta\right)^2\right]. \tag{B 6b}$$

Replacing the cylindrical derivatives by the helical derivatives, the energy equation reads

$$\frac{\partial}{\partial\tau}(rK)+\frac{\partial}{\partial r}\left(ru^r\left(K+p\right)\right)-\frac{\dot{\alpha}}{\alpha}\xi\frac{\partial}{\partial\xi}(rK) + \frac{\partial}{\partial\xi}\left(\frac{\dot{\alpha}}{\alpha}\eta\,(rK)+\left(\frac{r}{\alpha}u^z-\frac{\dot{\alpha}}{\alpha}\eta\right)(K+p)\right)+\frac{\partial}{\partial\eta}\left(bu^\varphi\left(K+p\right)\right)=0. \tag{B 7}$$

Since the helical invariant coordinate η itself only occurs in the velocity component u^z and in the pressure p, these variables will be replaced by their corresponding helical expressions. Assuming helical invariance and simplifying, the energy equation reads

$$\begin{aligned} R\left(\boldsymbol{\sigma},\partial_j\boldsymbol{\sigma}\right) &+ \left[\dot{\alpha}\left(r\frac{\dot{\alpha}}{\alpha}\xi\hat{u}_\xi-r\hat{u}_\tau-bu^\varphi_\xi\hat{u}+bu^\varphi\dot{\alpha}+bu^\varphi\hat{u}_\xi-\frac{r}{\alpha}\hat{u}_\xi\hat{u}\right.\right.\\ &+\left.\left(u^r+ru^r_r\right)\left(-\hat{u}-ru^r\hat{u}_r\right)-2\hat{u}\hat{u}_\xi-\left(u^ru^r_\xi+u^\varphi u^\varphi_\xi\right)-\tilde{p}_\xi\right)\\ &+\left.\ddot{\alpha}\left(-r\hat{u}+r\alpha bu^\varphi_\xi\xi+\left(u^r+ru^r_r\right)\alpha\xi+r\xi\hat{u}_\xi+\alpha\hat{u}\right)\right]\eta\\ &+\left[\dot{\alpha}\left(r\ddot{\alpha}-\frac{\dot{\alpha}}{\alpha}r\hat{u}_\xi+\frac{1}{2}\left(u^r+ru^r_r\right)\dot{\alpha}+\frac{1}{2}u^\varphi_\xi\dot{\alpha}+\frac{1}{2}\frac{r}{\alpha}\dot{\alpha}\hat{u}_\xi+\dot{\alpha}\hat{u}_\xi-\ddot{\alpha}\alpha\right)\right.\\ &+\left.\ddot{\alpha}\left(-\frac{1}{2}r\alpha bu^\varphi_\xi-\frac{1}{2}\left(u^r+ru^r_r\right)\alpha-\frac{1}{2}r\right)\right]\eta^2=0, \end{aligned} \tag{B 8}$$

where R is the collection of all η-independent terms and thus is a function of the helical dependent and independent variables $\boldsymbol{\sigma}\in\mathbb{R}^7$ defined in (A 4) and their first derivatives $\partial_j\boldsymbol{\sigma}\,(j\in\{r,\xi,\tau\})$.

Here, the quantity $\hat{u}$ is the short form of the η-independent part of the u^z-velocity component, given by

$$\hat{u}:=B\left(\frac{1}{\alpha}u^\xi-\frac{b}{r}u^\eta\right). \tag{B 9}$$

Apparently, (B 8) displays the above-mentioned contradiction as velocity and pressure are assumed to be independent of η, while the equation still contains η. For $\alpha=\text{const.}$ all η-dependent terms vanish in equation (B 8) and, hence, the conservation of energy only holds true for the classical helically symmetric case.

Appendix C. Proof of the absence of conservation of helicity in time-dependent helical coordinates

In this section we prove that for the case of Euler's equations the conservation of helicity in helically symmetric flows in a spatially reduced time-dependent coordinate system does not hold. The proof is done in the same manner as for the conservation of kinetic energy. In cylindrical coordinates the density and fluxes of helicity conservation are given by

$$\Gamma^t = rh, \tag{C 1}$$

$$\Gamma^\varphi = \omega^\varphi \left(E - (u^r)^2 - (u^z)^2\right) + u^\varphi \left(h - u^\varphi \omega^\varphi\right), \tag{C 2}$$

$$\Gamma^r = r\omega^r \left(E - (u^\varphi)^2 - (u^z)^2\right) + ru^r \left(h - u^r \omega^r\right), \tag{C 3}$$

$$\Gamma^z = r\omega^z \left(E - (u^r)^2 - (u^\varphi)^2\right) + ru^z \left(h - u^z \omega^z\right), \tag{C 4}$$

where $h = \boldsymbol{u} \cdot \boldsymbol{\omega} = u^r\omega^r + u^\varphi\omega^\varphi + u^z\omega^z$ is helicity and

$$E = E\frac{1}{2}|\boldsymbol{u}|^2 + p = \frac{1}{2}\left((u^r)^2 + (u^\varphi)^2 + (u^z)^2\right) + p \tag{C 5a}$$

$$= \frac{1}{2}\left[(u^r)^2 + B^2\left(\frac{b}{r}u^\xi + \frac{1}{\alpha}u^\eta\right)^2 + \left(B\left(\frac{1}{\alpha}u^\xi - \frac{b}{r}u^\eta\right) - \dot{\alpha}\eta\right)^2\right]$$
$$+ \tilde{p} - \frac{1}{2}\ddot{\alpha}\alpha\,(\xi - \eta)^2 \tag{C 5b}$$

is the total energy density (see e.g. KCO).

Rewriting the helicity conservation equation in the full three-dimensional helical coordinate system (r, ξ, η) we obtain

$$\frac{\partial}{\partial\tau}(rh) + \frac{\partial}{\partial r}\left(r\omega^r\left(K + p - \left(u^\xi\right)^2 - (u^\eta)^2 + 2B\left(\frac{1}{\alpha}u^\xi - \frac{b}{r}u^\eta\right)\dot{\alpha}\eta - \dot{\alpha}^2\eta^2\right)\right)$$
$$- \frac{\dot{\alpha}}{\alpha}\xi\frac{\partial}{\partial\xi}(rh) + \frac{\partial}{\partial\xi}\left(\frac{\dot{\alpha}}{\alpha}\eta\,(rh) + \left(\frac{r}{B}\omega^\xi\right)\left(K + p - (u^r)^2\right) + \left(\frac{r}{B}u^\xi - \frac{\dot{\alpha}}{\alpha}\eta\right)h\right.$$
$$\left.- \frac{r}{B}\omega^\xi\left(\left(u^\xi\right)^2 + (u^\eta)^2 - 2B\left(\frac{1}{\alpha}u^\xi - \frac{b}{r}u^\eta\right)\dot{\alpha}\eta + \dot{\alpha}^2\eta^2\right)\right) + \frac{\partial}{\partial\eta}\left(b\omega^\varphi\left(K + p\right.\right.$$
$$\left.\left.- (u^r)^2\right) - b\omega^\varphi\left(\left(u^\xi\right)^2 + (u^\eta)^2 - 2B\left(\frac{1}{\alpha}u^\xi - \frac{b}{r}u^\eta\right)\dot{\alpha}\eta + \dot{\alpha}^2\eta^2\right) + bu^\varphi h\right) = 0. \tag{C 6}$$

In a second step imposing helical invariance onto equation (C 6), i.e. vorticity, velocity and pressure are η-independent, the helicity equation yields

$$\tilde{R}\left(\boldsymbol{\sigma}, \partial_j\boldsymbol{\sigma}\right) + \left[\dot{\alpha}\left(\omega^z_\xi\frac{r}{\alpha}\hat{u} - r\omega^z_\tau + \frac{\dot{\alpha}}{\alpha}\xi r\omega^z_\xi + b\left(\omega^\varphi_\xi\hat{u} - \dot{\alpha}\omega^\varphi\hat{u}_\xi - u^\varphi_\xi\omega^z - u^\varphi\omega^z_\xi\right)\right.\right.$$
$$\left.\left.+ (r\omega^r)_{,r}\,\hat{u} + r\hat{u}_{,r}\omega^r - (ru^r\omega^z)_{,r}\right) + \ddot{\alpha}\left(\omega^z_\xi r\xi + b\alpha\omega^\varphi_\xi + (r\omega^r)_{,r}\,2\alpha\xi\right)\right]\eta$$
$$+ \left[\dot{\alpha}\frac{1}{2}\left(-r\omega^z_\xi\frac{\dot{\alpha}}{\alpha} + b\omega^\varphi_\xi\dot{\alpha} - (r\omega^r)_{,r}\,\dot{\alpha}\right) + \ddot{\alpha}\left(\frac{1}{2}r\omega^z_\xi - b\omega^\varphi_\xi\alpha - (r\omega^r)_{,r}\,\alpha\right)\right]\eta^2 = 0, \tag{C 7}$$

where $\tilde{R}$ is a collection of all η-independent terms and thus is a function of the reduced helical independent and dependent variables $\tilde{\boldsymbol{\sigma}} \in \mathbb{R}^{10}$, which consist of the two helical coordinates (r, ξ), the time (t), the three velocity components, pressure and the three vorticity components and their first derivatives $\partial_j\tilde{\boldsymbol{\sigma}}$, for $j \in \{r, \xi, \tau\}$. As for the energy equation, the contradiction becomes apparent, as in (C 7) η-independence of all dependent variables was employed, still the equation contains η explicitly.

Likewise for the kinetic energy, for $\alpha = \text{const.}$ all η-dependent terms vanish in (C 7) and the conservation of helicity holds for the classical helically symmetric case.

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