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Blow-up for the wave equation with nonlinear source and boundary damping terms

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The paper deals with blow-up for the solutions of an evolution problem consisting in a semilinear wave equation posed in a bounded $C^{1,1}$ open subset of $\mathbb{R}^n$, supplied with a Neumann boundary condition involving a nonlinear dissipation. The typical problem studied is

$$\begin{aligned} u_{tt} - \Delta u &= |u|^{p-2}u && \text{in } [0,\infty)\times\Omega,\\ u &= 0 && \text{on } (0,\infty)\times\Gamma_0,\\ \partial_\nu u &= -\alpha(x)(|u_t|^{m-2}u_t + \beta|u_t|^{\mu-2}u_t) && \text{on } (0,\infty)\times\Gamma_1,\\ u(0,x) = u_0(x), &\quad u_t(0,x) = u_1(x) && \text{in } \Omega, \end{aligned}$$

where $\partial\Omega = \Gamma_0\cup\Gamma_1$, $\Gamma_0\cap\Gamma_1=\emptyset$, $\sigma(\Gamma_0)>0$, $2<p\leqslant 2(n-1)/(n-2)$ (when $n\geqslant 3$), $m>1$, $\alpha\in L^\infty(\Gamma_1)$, $\alpha\geqslant 0$ and $\beta\geqslant 0$. The initial data are posed in the energy space. The aim of the paper is to improve previous blow-up results concerning the problem.

Keywords: wave equation; boundary damping; blow-up; source

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Secondary 35A01

1. Introduction

We deal with the evolution problem consisting on a semilinear wave equation posed in a bounded subset of $\mathbb{R}^n$, supplied with a Neumann boundary condition involving a nonlinear dissipation. More precisely, we consider the initial–boundary-value problem

$$\left.\begin{aligned} u_{tt} - \Delta u &= f(x,u) && \text{in } (0,\infty)\times\Omega,\\ u &= 0 && \text{on } (0,\infty)\times\Gamma_0,\\ \partial_\nu u &= -Q(x,u_t) && \text{on } (0,\infty)\times\Gamma_1,\\ u(0,x) = u_0(x), &\quad u_t(0,x) = u_1(x) && \text{in } \Omega, \end{aligned}\right\} \tag{1.1}$$

where $u = u(t, x)$, $t \geqslant 0$, $x \in \Omega$, and Δ denotes the Laplacian operator with respect to the x variable. We assume that Ω is a bounded and $C^{1,1}$-open subset of $\mathbb{R}^n$ ($n \geqslant 1$), $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ with Γ_0 and Γ_1 being measurable with respect to the natural (Lebesgue) measure on the manifold $\Gamma = \partial\Omega$, henceforth denoted by σ, and $\sigma(\Gamma_0) > 0$. These properties of Ω, Γ_0 and Γ_1 are adopted, without further comments, throughout the paper. The initial data are in the energy space, i.e. $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$, with the compatibility condition $u_0|_{\Gamma_0} = 0$ (in the trace sense).

Moreover, Q represents a nonlinear boundary damping and, roughly, $Q(x, v) \simeq \alpha(x)(|v|^{m-2}v + \beta|v|^{\mu-2}v)$, $1 < \mu \leqslant m$, $\beta \geqslant 0$, $\alpha \in L^\infty(\Gamma_1)$, $\alpha \geqslant 0$. When $\beta > 0$ and $\mu = 2$ the term Q describes a realistic dissipation rate, linear for small v and superlinear for large v (see, for example, [35]), possibly depending on the space variable, while when $\beta = 0$ and $\alpha = 1$ it is a pure power model nonlinearity. Finally, f is a nonlinear source and roughly $f(x, u) \simeq |u|^{p-2}u$, $2 < p \leqslant 2^*$, where as usual 2^* denotes the Sobolev critical exponent $2n/(n-2)$ when $n \geqslant 3$ and $2^* = \infty$ when $n = 1, 2$.

The presence of the boundary damping in (1.1) plays a critical role in the context of boundary control (see, for example, [12–15, 28, 29, 31, 34, 58]). For this reason, and for their clear physical meaning, problems such as (1.1) are the subject of a wide literature. In addition to the papers already quoted, see also [9–11, 16, 17, 21, 24, 32, 33, 44, 47, 56].

The analysis of problems like (1.1) is related to the treatment of quasi-linear wave equations with Neumann boundary conditions involving source terms (see [4–6, 30, 43, 55]).

In order to clearly describe the specific subject of this paper we consider problem (1.1) when f and Q are exactly the model nonlinearities, i.e. when problem (1.1) reduces to

$$\left.\begin{aligned} u_{tt} - \Delta u &= |u|^{p-2}u && \text{in } (0,\infty)\times\Omega,\\ u &= 0 && \text{on } (0,\infty)\times\Gamma_0,\\ \partial_\nu u &= -\alpha(x)(|u_t|^{m-2}u_t + \beta|u_t|^{\mu-2}u_t) && \text{on } (0,\infty)\times\Gamma_1,\\ u(0,x) = u_0(x),&\quad u_t(0,x) = u_1(x) && \text{in } \Omega \end{aligned}\right\} \tag{1.2}$$

with $1 < \mu \leqslant m$, $\beta \geqslant 0$, $\alpha \in L^\infty(\Gamma_1)$, $\alpha \geqslant 0$ and $2 < p \leqslant 2^*$.

The local existence and uniqueness for weak solutions of problem (1.2) when $2 < p \leqslant 1 + 2^*/2$ was first proved in [56, theorem 4] (see theorem 2.6 herein). In the literature this parameter range is often referred to as the subcritical/critical one, since the Nemytskii operator $u \mapsto |u|^{p-2}u$ is locally Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$. In this case the nonlinear semigroup theory is directly applicable.

Theorem 4 in [56] was subsequently extended to more general nonlinearities Q and f, of non-algebraic type, in [9, 11]. Moreover, at least when α is constant, Hadamard well-posedness for problem (1.2) follows from the results in [5], dealing with more general versions of problem (1.1), possibly involving internal nonlinear damping and boundary source terms. It is worth observing that, when no internal damping is present in the equation, the well-posedness result in [5] only applies to the subcritical/critical range $2 < p \leqslant 1+2^*/2$, due to [5, assumption 1.1]. Moreover, when u_0 and u_1 are small (in the energy space) the solutions of (1.2) are global in time.

On the other hand, blow-up results for problem (1.2) are much less common in the literature. In the particular case $\Gamma_1 = \emptyset$ (the same arguments work also when $\alpha \equiv 0$) it is well known that, for a particular choice of data, local solutions of problem (1.2), when they exist, blow up in finite time (see, for example, [2, 22, 25–27, 36, 37, 48]; we also refer the reader to the related papers [38, 39], which deal with boundary source terms). Payne and Sattinger [45] introduced the so-called 'potential-well theory' for the semilinear wave equation with Dirichlet boundary condition, and, in particular, blow-up for positive initial energy was proved. We also mention [20], which deals with the equation $u_{tt} - \Delta u + |u_t|^{m-2}u_t = |u|^{p-2}u$ in $[0, \infty) \times \Omega$ with homogeneous Dirichlet boundary conditions when $2 < p \leqslant 1 + 2^*/2$ and $m > 1$; it was the first paper to show the competition between nonlinear damping and source terms. In particular, it was proved in [20] that solutions may blow up in finite time (depending on initial data) if and only if $m < p$. The result was subsequently generalized to positive initial energy and abstract evolution equations in several papers (see, for example, [40, 46, 52]).

The problem of global non-existence for solutions of (1.2) when $\Gamma_1 \neq \emptyset$ and $m = 2$ was studied in [54] using the classical concavity method of Levine, which is no longer available for nonlinear damping terms. The first blow-up result for problem (1.2) in the general case $m > 1$ (and $2 < p \leqslant 1 + 2^*/2$) is contained in [56]. In order to relate it, we need to introduce some basic notation. We denote by $\|\cdot\|_p$ the norm in $L^p(\Omega)$ and the norm in $[L^p(\Omega)]^n$. We also introduce the Hilbert space

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) \colon u|_{\Gamma_0} = 0\}$$

(where $u|_{\Gamma_0}$ is intended in the trace sense), equipped with the norm $\|\nabla u\|_2$, which is equivalent, by a Poincaré-type inequality (see [57]), to the standard one. We also introduce the functionals

$$J(u) = \frac{\|\nabla u\|_2^2}{2} - \frac{\|u\|_p^p}{p} \quad \text{and} \quad K(u) = \|\nabla u\|_2^2 - \|u\|_p^p \tag{1.3}$$

for $u \in H^1_{\Gamma_0}(\Omega)$. The energy associated with initial data $u_0 \in H^1_{\Gamma_0}(\Omega)$ and $u_1 \in L^2(\Omega)$ is denoted by $E(u_0, u_1) := \frac{1}{2}\|u_1\|_2^2 + J(u_0)$. Moreover, we set

$$d = \inf_{u \in H^1_{\Gamma_0}(\Omega) \setminus \{0\}} \sup_{\lambda > 0} J(\lambda u). \tag{1.4}$$

It is well known that $d > 0$ (see lemma 4.1, which clarifies this property, and also remark 4.2, where a variational characterization of d is recalled). Finally, we introduce the 'bad part of the potential well' (this terminology was coined in [7])

$$W_u := \{(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \colon K(u_0) \leqslant 0 \text{ and } E(u_0, u_1) < d\}. \tag{1.5}$$

Trivially, if $E(u_0, u_1) < 0$, then $(u_0, u_1) \in W_u$, since $p > 2$. The situation is described clearly by figure 1.

In particular [56, theorem 7] asserts that solutions blow up in finite time if $(u_0, u_1) \in W_u$ and the further condition

$$m < m_0(p) := \frac{2(n+1)p - 4(n-1)}{n(p-2) + 4} \tag{1.6}$$

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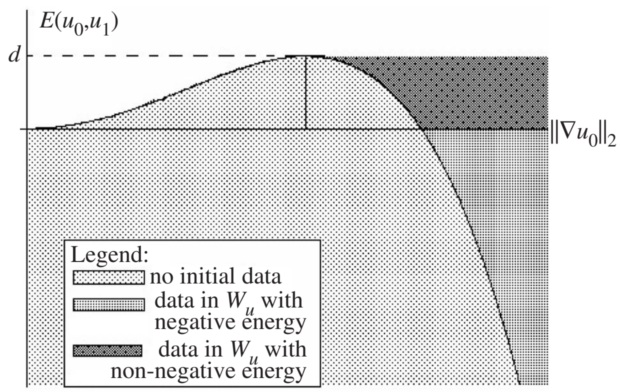

Figure 1. The sets of initial data considered by [3], having negative initial energy, and those considered only in the present paper.

holds. Note that $m_0(p) > 2$ when $p > 2$, so the case when $1 < m \leqslant 2$ is fully covered; but, when $m > 2$, condition (1.6) is rather restrictive (see figure 2).

In [6, 11] the blow-up problem is also considered. These papers deal with a modified version of (1.2), where internal damping and boundary source terms are also present. These papers do not include the assumption (1.6), since the combination of internal and boundary sources is more effective in producing blow-up.

As for problem (1.2) without boundary sources, Gerbi and Said-Houari [21] prove exponential growth, but not blow-up, for solutions of (1.2) when $m < p$. A generalized version of assumption (1.6) also appears in the recent paper [1], dealing with much more general Kirchhoff systems and a larger class of initial data.

Assumption (1.6) was first skipped in [3], where blow-up for a modified version of problem (1.2) is proved when $m < 1 + p/2$ and $E(u_0, u_1) < 0$. Even if the blow-up result in [3] is stated in the presence of an internal damping, one easily sees that the arguments in the proof also apply to problem (1.2). Clearly, assumption $m < 1 + p/2$ is more general than (1.6), since $m_0(p) < 1 + p/2$ for $p > 2$ (figure 2). The improvement in the assumption was obtained by using an interpolation estimate in the full scale of Besov spaces instead of in the Hilbert scale used in [56].

Assumption (1.6) was also skipped in the more recent papers [18, 41], which deal with the one-dimensional case $n = 1$, when $\beta = 0$ and $\alpha \equiv 1$. Blow-up for problem (1.2) is proved there when $E(u_0, u_1) < d$ and either

(i) $m < 1 + p/2$ or

(ii) $m \geqslant 1 + p/2$ and $|\Omega|$ is sufficiently large.

The arguments used by Feng *et al*. [18] and Liu *et al*. [41] in the two cases are different. Consequently, in dimension 1 the line $m = p$ is not the threshold between global existence and blow-up for suitable data. A natural conjecture is then that the same phenomenon occurs in a higher spatial dimension, n, even if the one-dimensional case is sometimes different from the higher-dimensional one (see, for example, [49,50], where a similar situation occurs for well-posedness, and the related paper [51]). Unfortunately, the arguments used to handle the case $m \geqslant 1 + p/2$ cannot be adapted to $n \geqslant 2$.

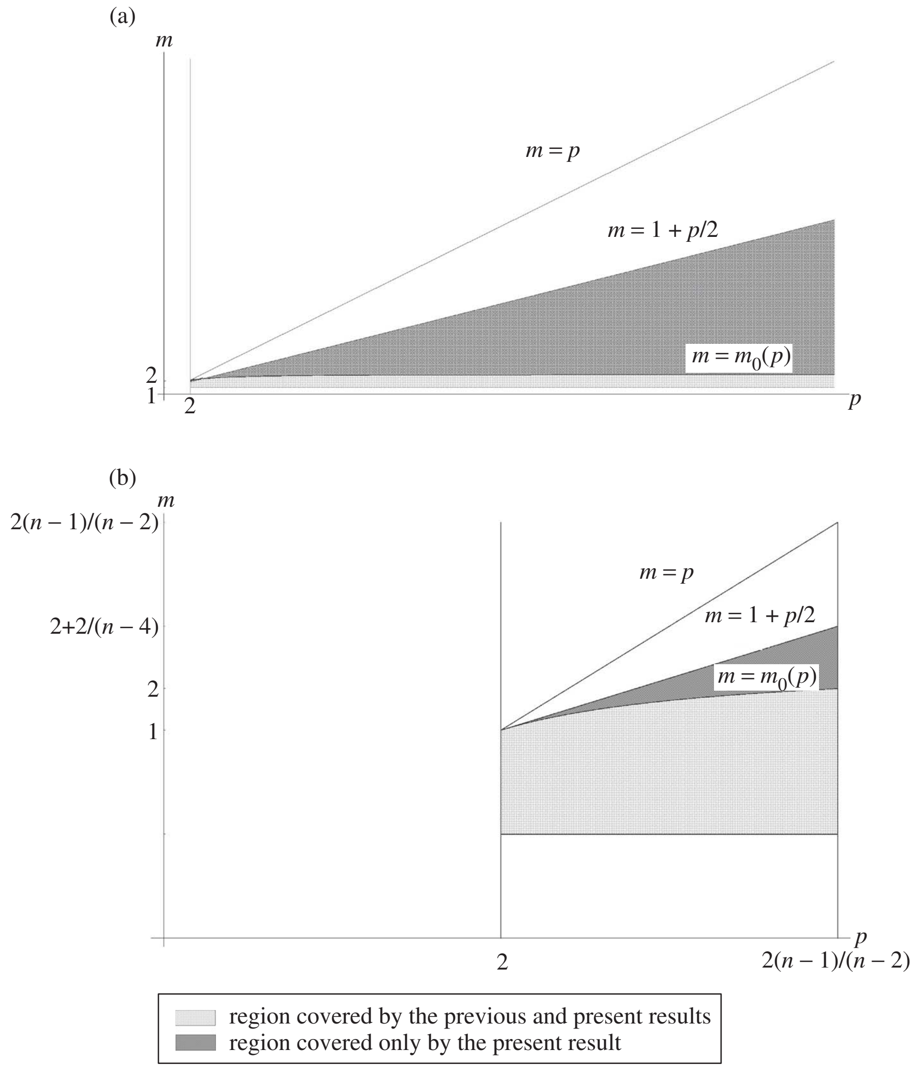

Figure 2. The sets of the (p, m) considered in [56] and in the present paper, in the two cases (a) $n = 1, 2$ and (b) $n \geqslant 3$. The two cases are shown with different scales due to the unboundedness of the sets considered in the first case.

The aim of this paper is to show that the technique in [56] can be adapted to cover at least the case $m < 1 + p/2$. In this way we extend the blow-up result from [3] to positive initial energy while extending the result from [41] to $n \geqslant 1$. Instead of using interpolation theory, we adapt a more elementary estimate, used in [18, 41] when $n = 1$, to the case when $n \geqslant 1$.

Our main result concerning problem (1.2) is as follows.

Theorem 1.1. *Let* $\alpha \in L^\infty(\Gamma_1)$, $\alpha \geqslant 0$, $\beta \geqslant 0$,

$$2 < p \leqslant 1 + 2^*/2, \qquad 1 < m < 1 + p/2$$

and $(u_0, u_1) \in W_u$. Then the weak solution u of problem (1.2) blows up in finite time, i.e. there exists $T_{\max} < \infty$ such that $\|u(t)\|_p \to \infty$ (and so also $\|u(t)\|_\infty \to \infty$ and $\|\nabla u(t)\|_2 \to \infty$) as $t \to T_{\max}^-$.

Remark 1.2. The meaning of weak solutions will be made precise below. Moreover, it will be clear (after the proof) that the parameter range $2 < p \leqslant 1 + 2^*/2$ in theorem 1.1 can be extended to $2 < p \leqslant 2^*$, but when $1 + 2^*/2 < p \leqslant 2^*$ we merely obtain global non-existence of weak solutions, since a local existence theorem is missing.

The paper is organized as follows. In § 2 we recall (from [56]) our main assumptions, local existence and potential-well theories for problem (1.1), with some additional remarks. Section 3 is devoted to stating and proving our main result, i.e. theorem 3.2, on problem (1.1). In § 4 we show that, when applying theorem 3.2 to problem (1.2), we obtain theorem 1.1.

2. Preliminaries

In this section we recall some material from [56], to which we refer for most of the proofs. We start by recalling the assumptions on Q and f needed for local existence.

(Q1) Q is a Carathéodory real function in $\Gamma_1 \times \mathbb{R}$, and there exist $\alpha \in L^1(\Gamma_1)$, $\alpha \geqslant 0$[1], and an exponent $m > 1$ such that, if $m \geqslant 2$,

$$(Q(x, v) - Q(x, w))(v - w) \geqslant \alpha(x)|v - w|^m$$

for all $x \in \Gamma_1$, $v, w \in \mathbb{R}$, while, if $1 < m < 2$,

$$(Q(x, v) - Q(x, w))(v - w) \geqslant \alpha(x)||v|^{m-2}v - |w|^{m-2}w|^{m'}$$

for all $x \in \Gamma_1$, $v, w \in \mathbb{R}$, where $1/m + 1/m' = 1$;

(Q2) there exist $1 < \mu \leqslant m$ and $c_1 > 0$ such that

$$|Q(x, v)| \leqslant c_1\alpha(x)(|v|^{\mu-1} + |v|^{m-1})$$

for all $x \in \Gamma_1$, $v \in \mathbb{R}$.

Remark 2.1. The model nonlinearity

$$Q_0(x, v) = \alpha(x)(|v|^{\mu-2}v + |v|^{m-2}v), \quad 1 < \mu \leqslant m, \quad \alpha \geqslant 0, \quad \alpha \in L^1(\Gamma_1), \tag{2.1}$$

satisfies (Q1) and (Q2). Indeed, while (Q2) is verified trivially, assumption (Q1) holds, when $m \geqslant 2$, up to multiplying α by an inessential positive constant, due to the elementary inequality

$$(|v|^{m-2}v - |w|^{m-2}w)(v - w) \geqslant \text{const.}|v - w|^m, \quad v, w \in \mathbb{R}^2. \tag{2.2}$$

When $1 < m < 2$ we get (Q1) by applying (2.2) to $m' > 2$, $|v|^{m-2}v$ and $|w|^{m-2}w$.

[1]The integrability of α on Γ_1, although not explicitly assumed in [56, theorem 4], was tacitly used there.

[2]This is a consequence of the boundedness of the real function $(|t - 1|^{m-2}(t - 1))/(|t|^{m-2}t - 1)$ when $m \geqslant 2$.

We note, for future use, some consequences of (Q1) and (Q2). First, it follows that

$$Q(x,v)v \geqslant \alpha(x)|v|^m \tag{2.3}$$

for all $x \in \Gamma_1$, $v \in \mathbb{R}$. Moreover, $Q(x,\cdot)$ is increasing for all $x \in \Gamma_1$, and $Q(\cdot,0) \equiv 0$. Then, after setting

$$\Phi(x,u) = \int_0^u Q(x,s)\,\mathrm{d}s, \tag{2.4}$$

we obtain

$$\Phi(x,u) \geqslant \frac{\alpha(x)}{m}|v|^m \quad \text{for all } x \in \Gamma_1,\ v \in \mathbb{R}. \tag{2.5}$$

We now introduce some notation. When $1 < q \leqslant \infty$ we denote by $L^q(\Gamma,\alpha)$ the L^q-space on Γ associated with the measure μ_α defined by $\mu_\alpha(A) = \int_A \alpha(x)\,\mathrm{d}\sigma$ for any measurable subset A of Γ, while $L^q(\Gamma)$ denotes the standard L^q-space, i.e. $L^q(\Gamma) = L^q(\Gamma,1)$. An analogous convention will be adopted on Γ_1 and in $(0,T)\times\Gamma_1$ for $T > 0$ (the measure μ_α being replaced by $\mathrm{d}t \times \mu_\alpha$ in the latter case). Moreover, for simplicity we shall write

$$\|\cdot\|_{q,\Gamma,\alpha} := \|\cdot\|_{L^q(\Gamma,\alpha)}, \qquad \|\cdot\|_{q,\Gamma} := \|\cdot\|_{L^q(\Gamma)},$$
$$\|\cdot\|_{q,\Gamma_1,\alpha} := \|\cdot\|_{L^q(\Gamma_1,\alpha)}, \qquad \|\cdot\|_{q,\Gamma_1} := \|\cdot\|_{L^q(\Gamma_1)}.$$

Our assumption concerning f is as follows.

(F1) f is a Carathéodory real function in $\Omega \times \mathbb{R}$, $f(x,0) = 0$ and there exist $p > 2$ and $c_2 > 0$ such that

$$|f(x,u) - f(x,v)| \leqslant c_2|u-v|(1 + |u|^{p-2} + |v|^{p-2})$$

for all $x \in \Omega$, $u, v \in \mathbb{R}$.

Remark 2.2. The model nonlinearity

$$f_0(x,u) = a|u|^{q-2}u + b|u|^{p-2}u, \quad 2 \leqslant q < p, \quad a, b \in \mathbb{R}, \tag{2.6}$$

satisfies (F1), due to the elementary inequality

$$||u|^{s-2}u - |v|^{s-2}v| \leqslant \text{const.}|v-w|(1 + |u|^{s-2} + |v|^{s-2}), \quad u, v \in \mathbb{R},$$

which holds for $s \geqslant 2$.

We define precisely the definition of weak solution used (implicitly) in [56].

Definition 2.3. When (Q1), (Q2) and (F1) hold and $2 < p \leqslant 2^*$ we say that u is a weak solution of problem (1.1) in $[0,T]$, $T > 0$, if

(a) $u \in C([0,T]; H^1_{\Gamma_0}(\Omega)) \cap C^1([0,T]; L^2(\Omega))$,

(b) the spatial trace of u on $(0,T)\times\Gamma$ (which exists by the trace theorem) has a distributional time derivative on $(0,T)\times\Gamma_1$, belonging to $L^m((0,T)\times\Gamma_1,\alpha)$,

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(c) for all $\varphi \in C([0,T]; H^1_{\Gamma_0}(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \cap L^m((0,T)\times\Gamma_1,\alpha)$ and for almost all $t \in [0,T]$ the distribution identity

$$\int_\Omega u_t\varphi\big|_0^t = \int_0^t\int_\Omega u_t\varphi_t - \nabla u\nabla\varphi + \int_0^t\int_\Omega f(\cdot,u)\varphi - \int_0^t\int_{\Gamma_1} Q(\cdot,u_t)\varphi \quad (2.7)$$

holds,

(d) $u(0) = u_0$ and $u_t(0) = u_1$.

We say that u is a weak solution of problem (1.1) in $[0,T)$ if u is a weak solution in $[0,T']$ for all $T' \in (0,T)$. Finally, we say that a weak solution u in $[0,T)$ is maximal if u cannot be seen as a restriction of a weak solution in $[0,T')$, $T < T'$.

Remark 2.4. The term $\int_0^t\int_\Omega f(\cdot,u)\varphi$ in (2.7) makes sense by (F1), the continuity of Nemytskii operators and the Sobolev embedding theorem. Recognizing that the last term in the right-hand side of (2.7) makes sense requires some deliberation. First, we note that, by (b), we have $\alpha^{1/m}u_t \in L^m((0,T)\times\Gamma_1)$ and then $\alpha^{1/m'}|u_t|^{m-1} \in L^{m'}((0,T)\times\Gamma_1)$. Since $\varphi \in L^m((0,T)\times\Gamma_1,\alpha)$, we have $\alpha^{1/m}\varphi \in L^m((0,T)\times\Gamma_1)$. Consequently, $\alpha|u_t|^{m-1}\varphi \in L^1((0,T)\times\Gamma_1)$. Now, since $\mu_\alpha(\Gamma_1) < \infty$ and $\mu \leqslant m$, we have $L^m((0,T)\times\Gamma_1,\alpha) \subset L^\mu((0,T)\times\Gamma_1,\alpha)$. Hence, we can repeat previous arguments with μ instead of m to show that $\alpha|u_t|^{\mu-1}\varphi \in L^1((0,T)\times\Gamma_1)$. Consequently, by (Q2) we get $Q(\cdot,u_t)\varphi \in L^1((0,T)\times\Gamma_1)$.

Remark 2.5. For clarity, we state the following facts. Since the equation and boundary conditions in problem (1.1) are autonomous, the choice of the initial time as zero is purely conventional. Consequently, for any $a \in \mathbb{R}$, we shall speak of weak solutions in $[a,a+T]$, $T > 0$, of the problem

$$\left.\begin{array}{rl} u_{tt} - \Delta u = f(x,u) & \text{in } (a,\infty)\times\Omega, \\ u = 0 & \text{on } (a,\infty)\times\Gamma_0, \\ \partial_\nu u = -Q(x,u_t) & \text{on } (a,\infty)\times\Gamma_1, \\ u(a,x) = u_0(x), \quad u_t(a,x) = u_1(x) & \text{in } \Omega, \end{array}\right\} \quad (2.8)$$

when (a)–(d) in definition 2.3 hold true with 0 and T replaced by a and $a+T$, respectively. Moreover, we have the following.

(i) The function u is a weak solution of (1.1) in $[0,T]$ if and only if the time-shifted function $\tau_a u$ defined by

$$(\tau_a u)(t) := u(t-a) \quad (2.9)$$

is a weak solution of (2.8) in $[a,a+T]$.

(ii) Let $b \in \mathbb{R}$, let $0 < T_1 < T_2$, let u_1 be a weak solution in $[b,b+T_1]$ of problem (2.8) with $a = b$ and let u_2 be a weak solution in $[b+T_1,b+T_2]$ of problem (2.8) with $a = b+T_1$. Define u in $[b,b+T_2]$ by $u(t) = u_1(t)$ for $t \in [b,b+T_1]$ and $u(t) = u_2(t)$ for $t \in (b+T_1,b+T_2]$. Then u is a weak solution of (2.8) with $a = b$ in $[b,b+T_2]$ if and only if $u_1(b+T_1) = u_2(b+T_1)$ and $(u_1)_t(b+T_1) = (u_2)_t(b+T_1)$.

We now recall [56, theorem 4].

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Theorem 2.6. *Suppose that (Q1) and (Q2) and (F1) hold, that* $2 < p \leqslant 1 + 2^*/2$, *and* $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$. *Then there is* $T > 0$ *and a unique weak solution of (1.1) in* $[0, T]$. *Moreover,* u *satisfies the energy identity*

$$E(t) - E(s) = -\int_s^t \int_{\Gamma_1} Q(\cdot, u_t) u_t \tag{2.10}$$

for $0 \leqslant s \leqslant t$, *where*

$$E(t) = E(u(t), u_t(t)) = \tfrac{1}{2}\|u_t(t)\|_2^2 + \tfrac{1}{2}\|\nabla u(t)\|_2^2 - \int_\Omega F(\cdot, u(t)) \tag{2.11}$$

and

$$F(x, s) = \int_0^s f(x, \tau)\,\mathrm{d}\tau \quad \text{for } x \in \Omega,\ s \in \mathbb{R}. \tag{2.12}$$

Remark 2.7. Actually, theorem 2.6 was stated in [56] for regular (i.e. C^1) domains, but one can immediately see that Ω can be also disconnected (even if this case is not of particular interest).

As a consequence of the arguments used in the proof of theorem 2.6, we have the following continuation principle, which was used in the quoted paper without an explicit proof. For the sake of clarity, we include its proof here.

Theorem 2.8. *Suppose that (Q1) and (Q2) and (F1) hold, that* $2 < p \leqslant 1 + 2^*/2$, *and* $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$. *Then (1.1) has a unique weak maximal solution* u *in* $[0, T_{\max})$. *Moreover, the following alternatives hold:*

(i) $T_{\max} = \infty$; *or*

(ii) $T_{\max} < \infty$ *and* $\lim_{t \to T_{\max}^-} \|u(t)\|_{H^1_{\Gamma_0}(\Omega)} + \|u_t(t)\|_2 = \infty$.

Proof. By the arguments in the proof of theorem 2.6 it easily follows that the assured existence time T depends on the initial data u_0 and u_1 as a decreasing function of $\|u_0\|^2_{H^1_{\Gamma_0}(\Omega)} + \|u_1\|_2^2$, which is henceforth denoted by

$$T^* = T^*(\|u_0\|^2_{H^1_{\Gamma_0}(\Omega)} + \|u_1\|_2^2).$$

From this remark the statement follows in a standard way. More precisely, we first construct the unique maximal solution u as follows. We set $\mathcal{U}$ to be the set of all weak solutions of (1.1) in right-open intervals $[0, T')$, $T' > 0$.

Then we claim that for any couple u, v of elements of $\mathcal{U}$, weak solutions respectively in $[0, T_u)$ and $[0, T_v)$, $u = v$ in the intersection $[0, T)$ of their domains. To prove our claim we set

$$t_0 := \sup\{t \in [0, T) \colon u(s) = v(s) \text{ for all } s \in [0, t)\}, \tag{2.13}$$

so $t_0 \leqslant T$. Now we suppose by contradiction that $t_0 < T$. Since

$$u, v \in C([0, t_0]; H^1_{\Gamma_0}(\Omega)) \cap C^1([0, t_0]; L^2(\Omega))$$

we easily get that $u(t_0) = v(t_0) := v_0$ and $u_t(t_0) = v_t(t_0) := v_1$. Now since u, v are weak solutions (see remark 2.5) of (2.8) with $a = t_0$ and initial data v_0, v_1, we see

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that $\tau_{-t_0}u$ and $\tau_{-t_0}v$ (defined in (2.9)) are both weak solutions in $[0, T-t_0)$ of (1.1) with initial data v_0 and v_1. Hence, by the uniqueness assertion in theorem 2.6 we get that $\tau_{-t_0}u = \tau_{-t_0}v$ in $[0, T'']$, $T'' = T^*(\|v_0\|^2_{H^1_{\Gamma_0}(\Omega)} + \|v_1\|_2^2) > 0$. Consequently, $u = v$ in $[0, t_0 + T'']$, contradicting (2.13). Hence, $t_0 = T$, proving our claim. To construct the maximal weak solution we define u to coincide with any element of $\mathcal{U}$ in the union of the domains.

We now need to prove the alternative statement. We suppose, by contradiction, that

$$T_{\max} < \infty \quad \text{and} \quad \liminf_{t\to T^-_{\max}} (\|u(t)\|_{H^1_{\Gamma_0}(\Omega)} + \|u_t(t)\|_2) < \infty. \tag{2.14}$$

Then there is a sequence $t_n \to T^-_{\max}$ such that $\|u(t_n)\|_{H^1_{\Gamma_0}(\Omega)}$ and $\|u_t(t_n)\|_2$ are bounded, so

$$M := \sup_n (\|u(t_n)\|^2_{H^1_{\Gamma_0}(\Omega)} + \|u_t(t_n)\|_2^2) < \infty.$$

By theorem 2.6 and the monotonicity of T^* asserted earlier for each $n \in \mathbb{N}$, the problem (1.1) with initial data $u(t_n)$ and $u_t(t_n)$ has a unique weak solution v_n in $[0, T_1]$, $T_1 = T^*(M)$. Hence, for each $n \in \mathbb{N}$, $w_n = \tau_{t_n}v_n$ is a weak solution of (2.8) in $[t_n, t_n + T_1]$ with $a = t_n$ and initial data $u(t_n)$ and $u_t(t_n)$. It follows (see remark 2.5) that u can be extended to a weak solution of (1.1) in $[0, t_n + T_1]$, contradicting the maximality of u for n large enough. □

We now recall from [56] the additional assumption on f needed to set up the potential-well theory.

(F2) There exists $c_3 > 0$ such that

$$F(x, u) \leqslant \frac{c_3}{p}|u|^p$$

for all $x \in \Omega$ and $u \in \mathbb{R}$, where F is the primitive of f defined in (2.12).

Remark 2.9. Recalling remark 2.2, it is clear that f_0 in (2.6) satisfies (F1) and (F2) when $2 \leqslant q < p$, $a \leqslant 0$ and $b \in \mathbb{R}$.

For $2 < p \leqslant 2^*$, we set

$$K_0 = \sup_{u\in H^1_{\Gamma_0}(\Omega), u\neq 0} \frac{\int_\Omega F(\cdot, u)}{\|\nabla u\|_2^p}. \tag{2.15}$$

By (F1) and (F2), we have $0 \leqslant K_0 \leqslant p^{-1}c_3B_1^p$, where B_1 is the optimal constant of the Sobolev embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^p(\Omega)$, i.e.

$$B_1 = \sup_{u\in H^1_{\Gamma_0}(\Omega), u\neq 0} \frac{\|u\|_p}{\|\nabla u\|_2}. \tag{2.16}$$

We define[3]

$$\lambda_1 = \left(\frac{1}{pK_0}\right)^{1/(p-2)}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)\lambda_1^2 \tag{2.17}$$

[3]This is the correct form of the equation for λ_1, the unique positive maximum point of the function $\frac{1}{2}\lambda^2 - K_0\lambda^p$; in [56] the definition contains a typographical error.

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when $K_0 > 0$, while $\lambda_1 = E_1 = +\infty$ when $K_0 = 0$, and

$$W = \{(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) : E(u_0, u_1) < E_1 \text{ and } \|\nabla u_0\|_2 > \lambda_1\}, \tag{2.18}$$

where, in accordance with (2.11),

$$E(u_0, u_1) := \tfrac{1}{2}\|u_1\|_2^2 + \tfrac{1}{2}\|\nabla u_0\|_2^2 - \int_\Omega F(\cdot, u_0). \tag{2.19}$$

Clearly, when $K_0 = 0$ then $W = \emptyset$, so what follows is of interest only when $K_0 > 0$. On the other hand, when $K_0 = 0$ all weak solutions are global (see [56, p. 389]). We recall the following result [56, lemma 2 (ii)].

LEMMA 2.10. *Suppose that the assumptions of theorem 2.6, together with (F2), hold true. Let u be the maximal solution of (1.1). Assume moreover that $(u_0, u_1) \in W$. Then there is $\lambda_2 > \lambda_1$ such that $\|\nabla u(t)\|_2 \geqslant \lambda_2$ and $\|u(t)\|_p \geqslant (pK_0/c_3)^{1/p}\lambda_2$ for all $t \in [0, T_{\max})$.*

Our final assumptions are as follows.

(Q3) There exists a $c_4 > 0$ such that

$$Q(x, v)v \geqslant c_4\alpha(x)(|v|^\mu + |v|^m), \quad 1 < \mu \leqslant m,$$

for all $x \in \Gamma_1$, $v \in \mathbb{R}$.

(F3) There is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ there exists a $c_5 = c_5(\varepsilon) > 0$ such that

$$f(x, u)u - (p - \varepsilon)F(x, u) \geqslant c_5|u|^p$$

for all $x \in \Omega$, $u \in \mathbb{R}$.

REMARK 2.11. Clearly, Q_0 given in (2.1) also satisfies (Q3) with $c_4 = 1$, as well as (Q1) and (Q2) (as noted in remark 2.1). Moreover, (Q3) immediately follows from (2.3) when $m = \mu$, while it is not a consequence of (Q1) and (Q2) when $\mu < m$. Next, in addition to satisfying (F1) and (F2) (see remark 2.9), f_0 given in (2.6) satisfies (F3) when $a \leqslant 0$ and $b > 0$, with $\varepsilon_0 = p - q > 0$ and $c_5(\varepsilon) = b\varepsilon/p$. Next (F3) implies the standard growth condition

$$f(x, u)u \geqslant pF(x, u) \quad \text{for all } x \in \Omega,\ u \in \mathbb{R}. \tag{2.20}$$

Finally, observe that (F1), (F2) and (2.20) cannot be responsible of a blow-up phenomenon, since $f \equiv 0$ satisfies them and blow-up does not occur in this case.

3. Main result

This section is devoted to stating and proving our main result. We start with a key estimate.

LEMMA 3.1. *Let $1 < m \leqslant 1 + p/2$ and $2 < p \leqslant 2^*$. Then there is a positive constant $C_1 = C_1(m, p, \Omega, \Gamma_0)$ such that*

$$\|u\|_{m,\Gamma_1}^m \leqslant C_1\|u\|_p^{m-1}\|\nabla u\|_2 \quad \textit{for all } u \in H^1_{\Gamma_0}(\Omega). \tag{3.1}$$

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Proof. We first consider the auxiliary non-homogeneous Neumann problem

$$\left.\begin{aligned} -\Delta w + w = 0 \quad \text{in } \Omega, \\ \partial_\nu w = 1 \quad \text{on } \Gamma. \end{aligned}\right\} \tag{3.2}$$

By the Riesz–Fréchet theorem, problem (3.2) has a unique weak solution, i.e. $w \in H^1(\Omega)$ such that

$$\int_\Omega \nabla w \nabla \phi + \int_\Omega w\phi = \int_\Gamma \phi \quad \text{for all } \phi \in H^1(\Omega). \tag{3.3}$$

Moreover, since Ω is bounded and $C^{1,1}$, by the Agmon–Douglis–Nirenberg regularity estimate (here used in the form stated in [23, theorem 2.4.2.7, p. 126]), we have $w \in W^{2,q}(\Omega)$ for all $q > 1$. It follows, by Morrey's theorem [8, corollary 9.15, p. 285], that $w \in C^1(\bar{\Omega})$[4].

Now let $u \in H^1(\Omega)$. We claim that $|u|^m \in W^{1,1}(\Omega)$. Since $m \leqslant 2^*$, by Sobolev embedding theorem we have $|u|^m \in L^1(\Omega)$. Moreover, by using the chain rule for Sobolev function (see [42, theorem 2.2]), we get that $|u|^m$ possesses a weak gradient $\nabla(|u|^m) = m|u|^{m-2}u\nabla u$. Since $m \leqslant 1 + 2^*/2$, using Sobolev embedding theorem again, we have $|u|^{m-2}u \in L^2(\Omega)$; hence, by the Hölder inequality we get that $\nabla(|u|^m) \in [L^1(\Omega)]^n$ and

$$\|\nabla(|u|^m)\|_1 \leqslant m\biggl(\int_\Omega |u|^{2(m-1)}\biggr)^{1/2} \|\nabla u\|_2.$$

Since $2(m-1) \leqslant p$ and Ω is bounded, it follows that

$$\|\nabla(|u|^m)\|_1 \leqslant m|\Omega|^{1/2-(m-1)/p}\|u\|_p^{m-1}\|\nabla u\|_2, \tag{3.4}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Our claim is then proved. Consequently (see [8, corollary 9.8 p. 277]), there is a sequence $(\phi_n)_n$ in $C_c^\infty(\mathbb{R}^N)$ such that $\phi_n|_\Omega \to |u|^m$ in $W^{1,1}(\Omega)$. By the trace theorem it follows that $\phi_n|_\Gamma \to |u|^m|_\Gamma$ in $L^1(\Gamma)$. Since, in particular, $\phi_n \in H^1(\Omega)$, (3.3) holds with $\phi = \phi_n$ for $n \in \mathbb{N}$. Since $w, |\nabla w| \in L^\infty(\Omega)$ we can pass to the limit as $n \to \infty$ to get

$$\int_\Omega \nabla w \nabla(|u|^m) + \int_\Omega w|u|^m = \int_\Gamma |u|^m. \tag{3.5}$$

Combining (3.4) and (3.5) we have

$$\|u\|_{m,\Gamma}^m \leqslant \|w\|_\infty\|u\|_m^m + m\|\nabla w\|_\infty|\Omega|^{1/2-(m-1)/p}\|u\|_p^{m-1}\|\nabla u\|_2$$

for all $u \in H^1(\Omega)$. Since $m \leqslant p \leqslant 2^*$ and Ω is bounded, by using the Hölder inequality again we get

$$\|u\|_{m,\Gamma}^m \leqslant (\|w\|_\infty|\Omega|^{1-m/p}\|u\|_p + m\|\nabla w\|_\infty|\Omega|^{1/2-(m-1)/p}\|\nabla u\|_2)\|u\|_p^{m-1}.$$

By now restricting to $u \in H^1_{\Gamma_0}(\Omega)$, we use the Poincaré-type inequality recalled above to get (3.1), where C_1 is given by

$$C_1 = \|w\|_\infty|\Omega|^{1-m/p}B_1 + m\|\nabla w\|_\infty|\Omega|^{1/2-(m-1)/p},$$

[4]We recall, for the reader's convenience, the definition of $C^k(\bar{\Omega})$ used in [8] for any $k \in \mathbb{N}$, i.e. $C^k(\bar{\Omega}) := \{u \in C^k(\Omega) \colon D^\alpha u$ has a continuous extension on $\bar{\Omega}$ for all α with $|\alpha| \leqslant k\}$.

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where B_1 is the positive constant defined in (2.16). Since w depends only on Ω, the proof is complete. □

We can finally state our main result.

THEOREM 3.2. *Suppose that (Q1)–(Q3) and (F1)–(F3) hold, that* $\alpha \in L^\infty(\Gamma_1)$,

$$2 < p \leqslant 1 + 2^*/2, \quad 1 < m < 1 + p/2$$

and $(u_0, u_1) \in W$, *where, recalling the definition (2.18),*

$$W = \{(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \colon E(u_0, u_1) < E_1 \text{ and } \|\nabla u_0\|_2 > \lambda_1\}.$$

Then for any solution of (1.1) we have $T_{\max} < \infty$ *and* $\|u(t)\|_p \to \infty$ *(so also* $\|u(t)\|_\infty \to \infty$ *and* $\|\nabla u(t)\|_2 \to \infty$*) as* $t \to T^-_{\max}$.

Proof. The proof is a variant of the proof of [56, theorem 7], where we use lemma 3.1 instead of the estimate [56, (50)]. Nevertheless, since the proof of [56, theorem 7] was itself a variant of the proof of [52, theorem 2], in the following we give a self-contained proof for clarity.

We first claim that our statement reduces to proving that problem (1.1) cannot have global weak solutions, i.e. weak solutions in the whole of $[0, \infty)$. Indeed, once this fact is proved, then we must have, by theorem 2.8, that $T_{\max} < \infty$ and

$$\|u(t)\|_{H^1_{\Gamma_0}(\Omega)} + \|u_t(t)\|_2 \to \infty \quad \text{as } t \to T^-_{\max}. \tag{3.6}$$

Hence, to prove our claim, we have to show only that also $\|u(t)\|_p \to \infty$ as $t \to T^-_{\max}$. We first note that, by (2.3) and (2.10), the energy function E (defined in (2.11)) is decreasing. Hence, by (2.11),

$$\tfrac{1}{2}\|\nabla u(t)\|_2^2 + \tfrac{1}{2}\|u_t(t)\|_2^2 - \int_\Omega F(x, u(t)) \leqslant E_0 \tag{3.7}$$

for $t \in [0, T_{\max})$, where $E_0 := E(u_0, u_1)$. Hence, by (F2), we have

$$\tfrac{1}{2}\|\nabla u(t)\|_2^2 + \tfrac{1}{2}\|u_t(t)\|_2^2 - \frac{c_3}{p}\|u(t)\|_p^p \leqslant E_0 \tag{3.8}$$

for $t \in [0, T_{\max})$. Consequently, by (3.6), we get that $\|u(t)\|_p \to \infty$ too, thus concluding the proof of our claim.

We now have to prove that problem (1.1) cannot have global solutions. We suppose by contradiction that $T_{\max} = \infty$. We fix $E_2 \in (E_0, E_1)$ and we set

$$\mathcal{H}(t) = \mathcal{H}(u(t), u_t(t)) = E_2 - E(u(t), u_t(t)). \tag{3.9}$$

Since, as noted before, E is decreasing, the function $\mathcal{H}$ is increasing and $\mathcal{H}(t) \geqslant \mathcal{H}_0 := \mathcal{H}(0) = E_2 - E_0 > 0$. In the proof below we shall omit, for simplicity, the explicit time dependence of u and u_t in the notation. By lemma 2.10 we have

$$\mathcal{H}(t) \leqslant E_2 - \tfrac{1}{2}\|\nabla u\|_2^2 + \int_\Omega F(\cdot, u) \leqslant E_1 - \tfrac{1}{2}\lambda_1^2 + \int_\Omega F(\cdot, u)$$

and then, by (2.17) and (F3),

$$\mathcal{H}(t) \leqslant \int_\Omega F(\cdot, u) \leqslant \frac{c_3}{p}\|u\|_p^p. \tag{3.10}$$

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We now introduce, as in [20, 40], the main auxiliary function which shows the blow-up properties of u, i.e.

$$\mathcal{Z}(t) = \mathcal{H}^{1-\eta}(t) + \xi \int_\Omega u_t u, \tag{3.11}$$

where $\xi > 0$ and $\eta \in (0,1)$ are constants to be fixed later. In order to estimate the derivative of $\mathcal{Z}$ it is convenient to estimate

$$I_1 := \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u_t u. \tag{3.12}$$

Using definition 2.3 we can take $\varphi = u$ in (2.7) and get

$$I_1 = \|u_t\|_2^2 - \|\nabla u\|_2^2 + \int_\Omega f(\cdot, u)u - \int_{\Gamma_1} Q(\cdot, u_t)u \tag{3.13}$$

almost everywhere in $(0, \infty)$. Now we claim that there are positive constants c_6 and c_7, depending on p and K_0, such that

$$I_1 \geqslant 2\|u_t\|_2^2 + c_6\|u\|_p^p + c_7\|\nabla u\|_2^2 + 2\mathcal{H}(t) - \int_{\Gamma_1} Q(\cdot, u_t)u \tag{3.14}$$

in $[0, \infty)$. Using (2.11) and (3.9) we can write, for any $\varepsilon > 0$, the identity (3.13) in the form

$$\begin{aligned} I_1 = {} & \tfrac{1}{2}(p + 2 - \varepsilon)\|u_t\|_2^2 + \tfrac{1}{2}(p - 2 - \varepsilon)\|\nabla u\|_2^2 \\ & + \int_\Omega [f(\cdot, u)u - (p - \varepsilon)F(\cdot, u)] + (p - \varepsilon)\mathcal{H}(t) - (p - \varepsilon)E_2 - \int_{\Gamma_1} Q(\cdot, u_t)u. \end{aligned} \tag{3.15}$$

Using (F3) for $0 < \varepsilon < \min\{\varepsilon_0, p - 2\}$, we consequently get

$$\begin{aligned} I_1 \geqslant {} & 2\|u_t\|_2^2 + \int_\Omega [f(\cdot, u)u - (p - \varepsilon)F(\cdot, u)] + \tfrac{1}{2}(p - \varepsilon - 2)\|\nabla u\|_2^2 - (p - \varepsilon)E_2 \\ & + (p - \varepsilon)\mathcal{H}(t) - \int_{\Gamma_1} Q(\cdot, u_t)u \\ \geqslant {} & 2\|u_t\|_2^2 + c_5(\varepsilon)\|u\|_p^p + \tfrac{1}{2}(p - \varepsilon - 2)\|\nabla u\|_2^2 - (p - \varepsilon)E_2 + 2\mathcal{H}(t) - \int_{\Gamma_1} Q(\cdot, u_t)u. \end{aligned}$$

By lemma 2.10,

$$\tfrac{1}{2}(p - \varepsilon - 2)\|\nabla u\|_2^2 - (p - \varepsilon)E_2 \geqslant c_7(\varepsilon)\|\nabla u\|_2^2 + c_8(\varepsilon),$$

where

$$c_7(\varepsilon) = \tfrac{1}{2}(p - \varepsilon - 2)(1 - \lambda_1^2/\lambda_2^2) \quad \text{and} \quad c_8(\varepsilon) = \tfrac{1}{2}(p - \varepsilon - 2)\lambda_1^2 - (p - \varepsilon)E_2.$$

Clearly, $c_7(\varepsilon) > 0$ and, as $\varepsilon \to 0^+$,

$$c_8(\varepsilon) \to \tfrac{1}{2}(p - 2)\lambda_1^2 - pE_2 > \tfrac{1}{2}(p - 2)\lambda_1^2 - pE_1 = 0,$$

so, in addition, $c_8(\varepsilon) > 0$ for ε sufficiently small. Fixing a sufficiently small $\varepsilon = \bar{\varepsilon}$ and setting $c_6 = c_5(\bar{\varepsilon})$, $c_7 = c_7(\bar{\varepsilon})$, we conclude the proof of (3.14).

Now, in order to estimate I_1, we estimate the last term in (3.14). Using (Q2), the Hölder inequality (with respect to μ_α) and the assumption that $\alpha \in L^\infty(\Gamma_1)$ we obtain

$$I_2 := \left| \int_{\Gamma_1} Q(\cdot, u_t) u \right| \leqslant c_1 \|\alpha\|_{\infty,\Gamma_1} (\|u_t\|^{\mu-1}_{\mu,\Gamma_1,\alpha} \|u\|_{\mu,\Gamma_1} + \|u_t\|^{m-1}_{m,\Gamma_1,\alpha} \|u\|_{m,\Gamma_1}).$$

Since $\mu \leqslant m$, applying the Hölder inequality again, we get

$$I_2 \leqslant C_2 (\|u_t\|^{\mu-1}_{\mu,\Gamma_1,\alpha} + \|u_t\|^{m-1}_{m,\Gamma_1,\alpha}) \|u\|_{m,\Gamma_1} \tag{3.16}$$

with $C_2 = C_2(\mu, m, c_1, \|\alpha\|_{\infty,\Gamma_1}, \sigma(\Gamma_1)) > 0$. By lemma 3.1 we consequently get

$$I_2 \leqslant C_3 (\|u_t\|^{\mu-1}_{\mu,\Gamma_1,\alpha} + \|u_t\|^{m-1}_{m,\Gamma_1,\alpha}) \|u\|_p^{1-1/m} \|\nabla u\|_2^{1/m}, \tag{3.17}$$

where $C_3 = C_3(\mu, m, p, c_1, \|\alpha\|_{\infty,\Gamma_1}, \Omega, \Gamma_0) > 0$. We define

$$I_3 := \|u_t\|^{\mu-1}_{\mu,\Gamma_1,\alpha} \|u\|_p^{1-1/m} \|\nabla u\|_2^{1/m} \quad \text{and} \quad I_4 := \|u_t\|^{m-1}_{\mu,\Gamma_1,\alpha} \|u\|_p^{1-1/m} \|\nabla u\|_2^{1/m}.$$

It is convenient to write

$$I_3 = \|u_t\|^{\mu-1}_{\mu,\Gamma_1,\alpha} \|\nabla u\|_2^{1/m} \|u\|_p^{p(1/\mu-1/2m)} \|u\|_p^{1-1/m-p(1/\mu-1/2m)}. \tag{3.18}$$

We now apply a weighted Young inequality, for any $\delta > 0$, to the first three multiplicands in the right-hand side of (3.18), with exponents $p_1 = \mu'$, $p_2 = 2m$ and $p_3 = 2m\mu/(2m - \mu)$, so that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$$

(note that trivially $p_1, p_2 > 1$, while $p_3 > 1$ as $1/p_3 = 1/\mu - 1/2m \in (0, 1)$ since $m \geqslant \mu > 1$). Thus, we get the estimate

$$I_3 \leqslant (\delta^{1/(1-\mu)} \|u_t\|^{\mu}_{\mu,\Gamma_1,\alpha} + \delta \|\nabla u\|_2^2 + \delta \|u\|_p^p) \|u\|_p^{1-1/m-p(1/\mu-1/2m)} \tag{3.19}$$

and, by particularizing it to the subcase $m = \mu$, we get

$$I_4 \leqslant (\delta^{1/(1-m)} \|u_t\|^{m}_{m,\Gamma_1,\alpha} + \delta \|\nabla u\|_2^2 + \delta \|u\|_p^p) \|u\|_p^{1-1/m-p/2m}. \tag{3.20}$$

Moreover, by lemma 2.10 we have $\|u\|_p \geqslant [c_3(pK_0)^{2/(p-2)}]^{-1/p}$. Hence, since $\mu \leqslant m$ implies

$$1 - \frac{1}{m} - p\left(\frac{1}{\mu} - \frac{1}{2m}\right) \leqslant 1 - \frac{1}{m} - \frac{p}{2m},$$

we also have

$$\|u\|_p^{1-1/m-p(1/\mu-1/2m)} \leqslant [c_3(pK_0)^{2/(p-2)}]^{1/\mu-1/m} \|u\|_p^{1-1/m-p/2m}. \tag{3.21}$$

By combining (3.17) and (3.19)–(3.21) we get

$$I_2 \leqslant C_4 [S(\delta)(\|u_t\|^{\mu}_{\mu,\Gamma_1,\alpha} + \|u_t\|^{m}_{m,\Gamma_1,\alpha}) + \delta \|\nabla u\|_2^2 + \delta \|u\|_p^p] \|u\|_p^{1-1/m-p/2m}, \tag{3.22}$$

where $S(\delta) = (\delta^{1/(1-\mu)} + \delta^{1/(1-m)})$ and

$$C_4 = C_4(\mu, m, p, c_1, c_3, K_0, \|\alpha\|_{\infty,\Gamma_1}, \Omega, \Gamma_0) > 0.$$

Now we set

$$\bar{\eta} = -\frac{1}{p}\left(1 - \frac{1}{m} - \frac{p}{2m}\right).$$

Since $m < 1 + p/2$, we have $\bar{\eta} > 0$. Moreover,

$$\bar{\eta} = \frac{1}{2m} - \frac{m-1}{pm} < \frac{1}{2m} < 1.$$

By combining (3.22) and (3.10) we get

$$I_2 \leqslant C_5[S(\delta)(\|u_t\|^{\mu}_{\mu,\Gamma_1,\alpha} + \|u_t\|^{m}_{m,\Gamma_1,\alpha}) + \delta\|\nabla u\|_2^2 + \delta\|u\|_p^p]\mathcal{H}^{-\bar{\eta}}(t), \tag{3.23}$$

where $C_5 = C_5(\mu, m, p, c_1, c_3, K_0, \|\alpha\|_{\infty,\Gamma_1}, \Omega, \Gamma_0) > 0$. Since, by (2.10) and (Q3) we have

$$\mathcal{H}'(t) \geqslant c_4(\|u_t\|^{\mu}_{\mu,\Gamma_1,\alpha} + \|u_t\|^{m}_{m,\Gamma_1,\alpha})$$

and $\mathcal{H}(t) \geqslant \mathcal{H}_0$, by (3.23) we get, for any $\eta \in (0, \bar{\eta})$,

$$I_2 \leqslant C_6[S(\delta)\mathcal{H}'(t)\mathcal{H}(t)^{-\eta} + \delta\|\nabla u\|_2^2 + \delta\|u\|_p^p], \tag{3.24}$$

where $C_6 = C_6(\mu, m, p, c_1, c_3, K_0, \|\alpha\|_{\infty,\Gamma_1}, \Omega, \Gamma_0, \mathcal{H}_0) > 0$. By combining (3.14) and (3.24) we have the desired estimate for I_1, i.e.

$$I_1 \geqslant 2\|u_t\|_2^2 + (c_6 - \delta C_6)\|u\|_p^p + (c_7 - \delta C_6)\|\nabla u\|_2^2 + 2\mathcal{H}(t) - S(\delta)\mathcal{H}'(t)\mathcal{H}^{-\eta}(t). \tag{3.25}$$

By choosing $\delta = \min\{c_6, c_7\}/(2C_6)$, from (3.25) we get

$$I_1 \geqslant 2\|u_t\|_2^2 + \tfrac{1}{2}c_6\|u\|_p^p + \tfrac{1}{2}c_7\|\nabla u\|_2^2 + 2\mathcal{H}(t) - C_7\mathcal{H}'(t)\mathcal{H}^{-\eta}(t), \tag{3.26}$$

where $C_7 = C_7(\mu, m, p, c_1, c_3, K_0, \|\alpha\|_{\infty,\Gamma_1}, \Omega, \Gamma_0, \mathcal{H}_0) > 0$.

By combining (3.11) and (3.26) we get, for any $\eta \in (0, \bar{\eta})$,

$$\mathcal{Z}'(t) \geqslant (1 - \eta - C_7\xi)\mathcal{H}^{-\eta}(t)\mathcal{H}'(t) + 2\xi\mathcal{H}(t) + 2\xi\|u_t\|_2^2 + \tfrac{1}{2}\xi c_6\|u\|_p^p + \tfrac{1}{2}\xi c_7\|\nabla u\|_2^2.$$

We now fix

$$\eta = \min\left\{\frac{\bar{\eta}}{4}, \frac{p-2}{4p}\right\} \in (0, 1)$$

and we restrict to $0 < \xi \leqslant (1 - \eta)/C_7$. Hence, since $\mathcal{H}' \geqslant 0$, from the previous estimate it follows that

$$\mathcal{Z}'(t) \geqslant \xi c_8(\|u_t\|_2^2 + \|\nabla u\|_2^2 + \|u\|_p^p + \mathcal{H}(t)), \tag{3.27}$$

where $c_8 = c_8(p, K_0) > 0$. Next, since

$$\mathcal{Z}(0) = \mathcal{H}_0^{1-\eta} + \xi \int_{\Omega} u_0 u_1,$$

by fixing $\xi = \xi_0 = \xi_0(\mu, m, p, c_1, c_3, K_0, \|\alpha\|_{\infty,\Gamma_1}, \Omega, \Gamma_0, u_0, u_1) > 0$ sufficiently small we have $\mathcal{Z}(0) > 0$, and hence $\mathcal{Z}(t) \geqslant \mathcal{Z}(0) > 0$ by (3.27). Now we define $r = 1/(1-\eta)$ and $\bar{r} = 1/(1-\bar{\eta})$. Since $0 < \eta < \bar{\eta} < 1$, we have $1 < r < \bar{r}$. Now, using the Cauchy–Schwarz inequality as well as the elementary inequality $(A + B)^r \leqslant 2^{r-1}(A^r + B^r)$ for $A, B \geqslant 0$, we have, from (3.11),

$$\mathcal{Z}^r(t) \leqslant \left(\mathcal{H}^{1-\eta}(t) + \xi_0\left|\int_{\Omega} u_t u\right|\right)^r \leqslant 2^{r-1}(\mathcal{H}(t) + \xi_0^r\|u_t\|_2^r\|u\|_2^r).$$

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We now set $q = 2/r = 2(1-\eta)$. Since $\eta < 1/2 - 1/p < 1/2$, it follows that $q > 1$. We can then apply Young's inequality with exponents q and $q' = (1-\eta)/(\frac{1}{2}-\eta)$ to get

$$\mathcal{Z}^r(t) \leqslant 2^{r-1}(\mathcal{H}(t) + \xi_0^2\|u_t\|_2^2 + \|u\|_2^{1/((1/2)-\eta)}).$$

Now, since $1/(\frac{1}{2}-\eta) < p$, a further application of Young's inequality yields

$$\|u\|_2^{1/((1/2)-\eta)} \leqslant 1 + \|u\|_2^p$$

and then, as Ω is bounded and $\mathcal{H}(t) \geqslant \mathcal{H}_0$, by the Hölder inequality we get

$$\mathcal{Z}^r(t) \leqslant C_8(\mathcal{H}(t) + \|u_t\|_2^2 + \|u\|_p^p), \tag{3.28}$$

where $C_8 = C_8(\mu, m, p, c_1, c_3, K_0, \|\alpha\|_{\infty,\Gamma_1}, \Omega, \Gamma_0, u_0, u_1) > 0$. By combining (3.27) and (3.28), as $r > 1$, we get

$$\mathcal{Z}'(t) \geqslant C_9\mathcal{Z}^r(t) \quad \text{for all } t \in [0,\infty),$$

where $C_9 = C_9(\mu, m, p, c_1, c_3, K_0, \|\alpha\|_{\infty,\Gamma_1}, \Omega, \Gamma_0, u_0, u_1) > 0$. Since $r > 1$, this final estimate gives the desired contradiction. □

4. Proof of theorem 1.1

This section is devoted to showing that theorem 1.1 is a simple corollary of theorem 3.2. We first need to show that, for problem (1.2), E_1 and W, as defined in (2.17) and (2.18), are merely d and W_u (introduced in (1.4), (1.5)). The proof is an adaptation of the proof of [19, lemma 4.1].

Lemma 4.1. *Suppose* $f(x,u) = |u|^{p-2}u$, $2 < p \leqslant 2^*$, $\sigma(\Gamma_0) > 0$. *Then* $E_1 = d$ *and* $W = W_u$.

Proof. When $f(x,u) = |u|^{p-2}u$ we have $K_0 = B_1^p/p$. Hence,

$$\lambda_1 = B_1^{-p/(p-2)} \quad \text{and} \quad E_1 = \left(\frac{1}{2}-\frac{1}{p}\right)B_1^{-2p/(p-2)}. \tag{4.1}$$

An easy calculation shows that for any $u \in H^1_{\Gamma_0}(\Omega) \setminus \{0\}$ we have

$$\max_{\lambda>0} J(\lambda u) = J(\lambda(u)u) = \left(\frac{1}{2}-\frac{1}{p}\right)\left(\frac{\|\nabla u\|_2}{\|u\|_p}\right)^{2p/(p-2)}, \quad \text{where } \lambda(u) = \frac{\|\nabla u\|_2^{2/(p-2)}}{\|u\|_p^{p/(p-2)}}.$$

Hence, by (2.16), $d = (1/2 - 1/p)B_1^{-2p/(p-2)}$. Combining this result with (4.1), we have $d = E_1$.

In order to show that $W = W_u$ we first prove that $W \subseteq W_u$. Let $(u_0, u_1) \in W$ and suppose, by contradiction, that $K(u_0) > 0$. Hence $\|u_0\|_p^p < \|\nabla u_0\|_2^2$ by (1.3). Moreover, $J(u_0) \leqslant E(u_0,u_1) < d = E_1$ and $\|\nabla u_0\|_2 > \lambda_1$. Then it follows that

$$E_1 > E(u_0,u_1) \geqslant J(u_0) > \left(\frac{1}{2}-\frac{1}{p}\right)\|\nabla u_0\|_2^2 > \left(\frac{1}{2}-\frac{1}{p}\right)\lambda_1^2,$$

which contradicts (2.17).

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To prove that $W_u \subseteq W$, we take $(u_0, u_1) \in W_u$. We note that, by (2.16), we have $J(v) \geqslant h(\|\nabla v\|_2)$ for all $v \in H^1_{\Gamma_0}(\Omega)$, where h is defined by

$$h(\lambda) = \frac{\lambda^2}{2} - \frac{B_1^p \lambda^p}{p} \quad \text{for } \lambda \geqslant 0.$$

One may easily verify that $h(\lambda_1) = E_1$. Then, since $J(u_0) \leqslant E(u_0, u_1) < E_1$, we have $\|\nabla u_0\|_2 \neq \lambda_1$. Moreover, since $K(u_0) \leqslant 0$, by (2.16) we have

$$\|\nabla u_0\|_2^2 \leqslant \|u_0\|_p^p \leqslant B_1^p \|\nabla u_0\|_p^p$$

and consequently $\|\nabla u_0\|_2 \geqslant B_1^{-p/(p-2)} = \lambda_1$. Then $\|\nabla u_0\|_2 > B_1^{-p/(p-2)} = \lambda_1$, concluding the proof. □

Remark 4.2. When $f(x, u) = |u|^{p-2}u$, d is also equal to the mountain pass level associated with the elliptic problem

$$\begin{aligned} -\Delta u &= |u|^{p-2}u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \partial_\nu u &= 0 && \text{on } \Gamma_1, \end{aligned}$$

i.e. $d = \inf_{\gamma \in \Lambda} \sup_{t \in [0,1]} J(\gamma(t))$, where

$$\Lambda = \{\gamma \in C([0,1]; H^1_{\Gamma_0}(\Omega)) \colon \gamma(0) = 0,\ J(\gamma(1)) < 0\}.$$

The proof of this remark was given in [53, § 4].

We can now prove theorem 1.1.

Proof of theorem 1.1. By remark 2.11 the nonlinearities involved in problem (1.2) satisfy assumptions (Q1)–(Q3) and (F1)–(F3), so we can apply theorem 3.2. Due to lemma 4.1 we get exactly theorem 1.1. □

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