THE JOURNAL OF SYMBOLIC LOGIC
Volume 88, Number 1, March 2023

ON κ-HOMOGENEOUS, BUT NOT κ-TRANSITIVE PERMUTATION GROUPS

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Abstract. A permutation group G on a set A is κ-homogeneous iff for all $X, Y \in [A]^\kappa$ with $|A \setminus X| = |A \setminus Y| = |A|$ there is a $g \in G$ with $g[X] = Y$. G is κ-transitive iff for any injective function f with $\text{dom}(f) \cup \text{ran}(f) \in [A]^{\leq\kappa}$ and $|A \setminus \text{dom}(f)| = |A \setminus \text{ran}(f)| = |A|$ there is a $g \in G$ with $f \subset g$.

Giving a partial answer to a question of P. M. Neumann [6] we show that there is an ω-homogeneous but not ω-transitive permutation group on a cardinal λ provided

(i) $\lambda < \omega_\omega$, or
(ii) $2^\omega < \lambda$, and $\mu^\omega = \mu^+$ and $\Box_\mu$ hold for each $\mu \leq \lambda$ with $\omega = \text{cf}(\mu) < \mu$, or
(iii) our model was obtained by adding $(2^\omega)^+$ many Cohen generic reals to some ground model.

For $\kappa > \omega$ we give a method to construct large κ-homogeneous, but not κ-transitive permutation groups. Using this method we show that there exist κ^+-homogeneous, but not κ^+-transitive permutation groups on κ^{+n} for each infinite cardinal κ and natural number $n \geq 1$ provided $V = L$.

§1. Introduction. Denote by $\text{S}(A)$ the group of all permutations of the set A. The subgroups of $\text{S}(A)$ are called *permutation groups on A*.

Let A be a set and $\kappa \leq |A|$ be a cardinal. We say that a permutation group G on A is κ-homogeneous iff for all $X, Y \in [A]^\kappa$ with $|A \setminus X| = |A \setminus Y| = |A|$ there is a $g \in G$ with $g[X] = Y$.

We say that a permutation group G on A is κ-transitive iff for any injective function f with $\text{dom}(f) \cup \text{ran}(f) \in [A]^{\leq\kappa}$ and $|A \setminus \text{dom}(f)| = |A \setminus \text{ran}(f)| = |A|$ there is a $g \in G$ with $f \subset g$.

In this paper we give a partial answer to the following question which was raised by P. M. Neumann in [6, Question 3]:

> *Suppose that $\kappa < \lambda$ are infinite cardinals. Does there exist a permutation group on λ that is κ-homogeneous, but not κ-transitive?*

In Section 2 we show that there exist ω-homogeneous, but not ω-transitive permutation groups on $\lambda < \omega_\omega$ in ZFC, and on any infinite λ if $V = L$ (see Theorem 2.5).

In Section 3 we develop a general method to obtain large κ-homogeneous, but not κ-transitive permutation groups for arbitrary $\kappa \geq \omega$ (see Theorem 3.2). Applying our method we show that if $\kappa^\omega = \kappa$, $\lambda = \kappa^{+n}$ for some $n < \omega$, and $\Box_\nu$ holds for each $\kappa \leq \nu < \lambda$, then there is a κ-homogeneous, but not κ-transitive permutation group on λ (Corollary 3.12).

Received March 10, 2021.
2020 *Mathematics Subject Classification.* Primary 03E35, 20B22.
Key words and phrases. permutation group, transitive, homogeneous.

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0022-4812/23/8801-0013
DOI:10.1017/jsl.2021.63

https://doi.org/10.1017/jsl.2021.63 Published online by Cambridge University Press

In Section 4 first we show that if Martin's axiom holds for countable posets, then every subgroup of $\mathrm{S}_\omega(\omega_1)$ with cardinality $<2^\omega$ can be extended to an ω-homogeneous, but not ω-transitive permutation group on ω_1. Based on this theorem we prove that after adding $(2^\omega)^+$ Cohen reals to any ground model in the generic extension for each infinite λ there exist ω-homogeneous, but not ω-transitive permutation groups on λ (Corollary 4.9).

Our notation is standard.

Definition 1.1. If λ is fixed and $f \in S(A)$ for some $A \subset \lambda$, we take

$$f^+ = f \cup (\mathrm{id} \restriction (\lambda \setminus A)) \in S(\lambda).$$

Given a family of functions, $\mathcal{G}$, we say that a function y is $\mathcal{G}$-large iff

$$|y \setminus \bigcup \mathcal{H}| = |y|$$

for each finite $\mathcal{H} \subset \mathcal{G}$.

We say that a permutation group on A is κ-intransitive iff there is a G-large injective function y with $\mathrm{dom}(y) \cup \mathrm{ran}(y) \in [A]^\kappa$ and $|A \setminus \mathrm{dom}(y)| = |A \setminus \mathrm{ran}(y)| = |A|$.

A κ-intransitive group is clearly not κ-transitive.

§2. ω-homogeneous but not ω-transitive.

Definition 2.1. Given a set A we say that a family $\mathcal{A} \subset [A]^\omega$ is *nice on A* iff $\mathcal{A}$ has an enumeration $\{A_\alpha : \alpha < \mu\}$ such that

(N1) $\mathcal{A}$ is cofinal in $\left\langle [A]^\omega, \subset \right\rangle$,

(N2) for each $\beta < \mu$ there is a countable set $I_\beta \in [\beta]^\omega$ such that for all $\alpha < \beta$ there is a finite set $J_{\alpha,\beta} \in [I_\beta]^{<\omega}$ such that

$$A_\alpha \cap A_\beta \subset \bigcup_{\zeta \in J_{\alpha,\beta}} A_\zeta.$$

Theorem 2.2. *Assume that λ is an infinite cardinal, and $\mathcal{A} \subset [\lambda]^\omega$ is a nice family on λ. Then for each $A \in \mathcal{A}$ there is an ordering $\leq_A$ on A such that*

(1) *$tp(A, \leq_A) = \omega$ for each $A \in \mathcal{A}$,*

(2) *if $A, B \in \mathcal{A}$, then there is a partition $\{C_i : i < n\}$ of $A \cap B$ into finitely many subsets such that $\leq_A \restriction C_i = \leq_B \restriction C_i$ for all $i < n$.*

Proof. Fix an enumeration $\{A_\beta : \beta < \mu\}$ of $\mathcal{A}$ witnessing that $\mathcal{A}$ is nice.

We will define $\leq_{A_\beta}$ by induction on $\beta < \mu$.

Assume that $\leq_{A_\alpha}$ is defined for $\alpha < \beta$.

By (N2) we can fix a countable set $I_\beta = \{\beta_i : i < \omega\} \in [\beta]^\omega$ such that for all $\alpha < \beta$ there is $n_\alpha < \omega$ such that

$$A_\alpha \cap A_\beta \subset \bigcup_{i < n_\alpha} A_{\beta_i}.$$

Choose an order $\leq_{A_\beta}$ on A_β such that

https://doi.org/10.1017/jsl.2021.63 Published online by Cambridge University Press

(i) for each $i < \omega$ writing $D_i = A_{\beta_i} \setminus \bigcup_{j<i} A_{\beta_j}$ we have

$$\leq_{A_\beta} \restriction (A_\beta \cap D_i) \quad = \quad \leq_{A_{\beta_i}} \restriction (A_\beta \cap D_i);$$

(ii) $tp(A_\beta, \leq_{A_\beta}) = \omega$.

By induction on β we show that (2) holds for A_α and A_β for each $\alpha < \beta$. Assume that this statement holds for each $\beta' < \beta$. To check for β fix $\alpha < \beta$.

To define $\leq_\beta$ we considered a set $I_\beta = \{\beta_i : i < \omega\} \in [\beta]^\omega$ such that we had $n_\alpha < \omega$ with

$$A_\alpha \cap A_\beta \subset \bigcup_{i<n_\alpha} A_{\beta_i}.$$

For $i < n_\alpha$ let $C_i' = A_\alpha \cap A_\beta \cap D_i$, where $D_i = A_{\beta_i} \setminus \bigcup_{j<i} A_{\beta_j}$. Then $\{C_i' : i < n_\alpha\}$ is a partition of $A_\alpha \cap A_\beta$ and

$$\leq_{A_\beta} \restriction C_i' = \leq_{A_{\beta_i}} \restriction C_i'$$

by (i). By the inductive hypothesis, $A_{\beta_i} \cap A_\alpha$ has a partition into finitely many pieces $\{C_{i,j} : j < k_i\}$ such that $\leq_{A_\alpha} \restriction C_{i,j} = \leq_{A_{\beta_i}} \restriction C_{i,j}$. Then the partition

$$\{C_i' \cap C_{i,j} : i < n, j < k_i\}$$

of $A_\alpha \cap A_\beta$ works for α and β. Indeed,

$$\leq_{A_\alpha} \restriction C_i' \cap C_{i,j} \quad = \quad \leq_{A_{\beta_i}} \restriction C_i' \cap C_{i,j} \quad = \quad \leq_{A_\beta} \restriction C_i' \cap C_{i,j}. \qquad \dashv$$

THEOREM 2.3. *Assume that λ is an infinite cardinal, $\mathcal{A} \subset [\lambda]^\omega$ is a cofinal family, and for each $A \in \mathcal{A}$ we have an ordering $\leq_A$ on A such that*

(1) *$tp(A, \leq_A) = \omega$ for each $A \in \mathcal{A}$,*
(2) *if $A, B \in \mathcal{A}$, then there is a partition $\{C_i : i < n\}$ of $A \cap B$ into finitely many subsets such that $\leq_A \restriction C_i = \leq_B \restriction C_i$ for all $i < n$.*

Then there is a permutation group on λ that is ω-homogeneous and ω-intransitive.

PROOF. For $A \in \mathcal{A}$ let

$$\mathcal{G}_A = \{f^+ \in S(\lambda) : f \in S(A) \wedge \text{there is a finite partition } \{C_i : i < n\} \text{ of } A \text{ such that } f \restriction C_i \text{ is } \leq_A \text{-order preserving}\}.$$

Let G be the permutation group on λ generated by

$$\bigcup \{\mathcal{G}_A : A \in \mathcal{A}\}.$$

CLAIM 2.3.1. *G is ω-homogeneous.*

Indeed, let $X, Y \in [\lambda]^\omega$ with $|\lambda \setminus X| = |\lambda \setminus Y| = \lambda$. Pick $A \in \mathcal{A}$ such that $X \cup Y \subset A$ and $|A \setminus X| = |A \setminus Y| = \omega$.

Let c be the unique $\leq_A$-monotone bijection between X and Y and d be the unique $\leq_A$-monotone bijection between $A \setminus X$ and $A \setminus Y$. Then taking $g = c \cup d$ we have $g^+ \in \mathcal{G}_A \subset G$ and $g^+[X] = Y$.

CLAIM 2.3.2. *G is ω-intransitive.*

Pick $A \in \mathcal{A}$ and choose $B \in [A]^\omega$ such that $|A \setminus B| = \omega$.

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Let $b_0, b_1, \dots$ be the $\leq_A$-increasing enumeration of B. Define a bijection $y : B \to \omega$ as follows: for $i < \omega$ and $j < 2^i$ let

$$y(b_{2^i+j}) = b_{2^{i+1}-j}.$$

Observe that if c is $\leq_A$-monotone then

$$|\{i < \omega : |\{j < 2^i : c(b_{2^i+j}) = r(b_{2^i+j})\}| \geq 2\}| \leq 1.$$

Indeed, if $|\{j < 2^i : c(b_{2^i+j}) = y(b_{2^i+j})\}| \geq 2$, then c should be $\leq_A$-decreasing, and if $|\{i : \{j < 2^i : c(b_{2^i+j}) = y(b_{2^i+j})\} \neq \emptyset\}| \geq 2$, then y should be $\leq_A$-increasing.

So y cannot be covered by finitely many $\leq_A$-monotone functions. But for any $h \in G$, $h \cap (A \times A)$ can be covered by finitely many $\leq_A$-monotone functions by (2) and by the construction of G.

Thus y is G-large. ⊣

To obtain nice families we recall some topological results. We say that a topological space X is *splendid* (see [2]) iff it is countably compact, locally compact, and locally countable such that $|\overline{A}| = \omega$ for each $A \in [X]^\omega$.

We need the following theorem:

THEOREM (Juhász, Nagy, and Weiss) [2]. *If*

(i) $\kappa < \omega_\omega$, *or*
(ii) $2^\omega < \kappa$, $\operatorname{cf}(\kappa) > \omega$, *and* $\mu^\omega = \mu^+$ *and* $\square_\mu$ *hold for each* $\mu < \kappa$ *with* $\omega = \operatorname{cf}(\mu) < \mu$,

then there is a splendid space X of size κ.

REMARK. In [2, Theorem 11] the authors formulated a bit weaker result: *if $V = L$ and $\operatorname{cf}(\kappa) > \omega$ then there is a splendid space X of size κ*. However, to obtain that results they combined "Lemmas 7, 9, and 16 with the remark after Theorem 8" and their arguments used only the assumptions of the theorem above.

If $\mathcal{A}$ is a family of sets, and X is a set, write

$$\mathcal{A}\lceil X = \{A \cap X : A \in \mathcal{A}\}$$

and

$$\mathcal{A}\lceil^* X = \{\bigcap \mathcal{A}' \cap X : \mathcal{A}' \in [\mathcal{A}]^{<\omega}\}.$$

LEMMA 2.4. *If X is a splendid space, $\mathcal{U}$ is the family of compact open subsets of X, and $Y \subset X$, then $\mathcal{U}\lceil Y$ is nice on Y.*

PROOF. Let $A \in [Y]^\omega$. Then $\overline{A}$ is countable, so it is compact. Since a splendid space is zero-dimensional, A can be covered by finitely many compact open sets, and so A can be covered by an element of $\mathcal{U}$. Thus $\mathcal{U}\lceil Y$ is cofinal in $\left\langle [Y]^\omega, \subset \right\rangle$.

To check (N2) observe that every $U \in \mathcal{U}$ is a countable compact space, so it is homeomorphic to a countable successor ordinal. Thus U has only countably many compact open subsets. Hence $\mathcal{U}\lceil U$ is countable which implies (N2) in the following stronger form:

($N2^+$) for each $\beta < \mu$ there is a set $I_\beta \in [\beta]^\omega$ such that for all $\alpha < \beta$ there is $\zeta_\alpha \in I_\beta$ such that

$$A_\alpha \cap A_\beta = A_{\zeta_\alpha} \cap A_\beta.$$

⊣

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REMARK. By [3, Corollary 2.2], *if* $(\omega_{\omega+1}, \omega_\omega) \to (\omega_1, \omega)$ *holds, then the cardinality of a splendid space is less than* ω_ω. So we need some new ideas if we want to construct arbitrarily large nice families in ZFC.

THEOREM 2.5. *If* λ *is an infinite cardinal, and*

(i) $\lambda < \omega_\omega$, *or*

(ii) $2^\omega < \lambda$, *and* $\mu^\omega = \mu^+$ *and* $\Box_\mu$ *hold for each* $\mu \le \lambda$ *with* $\omega = \mathrm{cf}(\mu) < \mu$,

then there is an ω*-homogeneous and* ω*-intransitive permutation group on* λ.

PROOF. Applying the Juhász–Nagy–Weiss theorem for $\kappa = \lambda$ if $\mathrm{cf}(\lambda) > \omega$, and for $\kappa = \lambda^+$ if $\lambda > \mathrm{cf}(\lambda) = \omega$, we obtain a splendid space on $\kappa \ge \lambda$. So, by Lemma 2.4, we obtain a nice family $\mathcal{A}$ on λ.

Thus, putting together Theorems 2.2 and 2.3 we obtained the desired permutation group on λ. ⊣

§3. κ-homogeneous but not κ-transitive for $\kappa > \omega$.

DEFINITION 3.1. Let $\kappa < \lambda$ be cardinals. We say that a cofinal family $\mathcal{A} \subset [\lambda]^\kappa$ is *locally small* iff $|\mathcal{A}\lceil A| \le \kappa$ for all $A \in \mathcal{A}$.

THEOREM 3.2. *Assume that* $2^\kappa = \kappa^+$ *and there is a cofinal, locally small family* $\mathcal{A} \subset [\lambda]^\kappa$. *Then there is a permutation group G on* λ *which is* κ*-homogeneous, but not* κ*-transitive.*

Before proving this theorem we need some preparation.

DEFINITION 3.3. If X, Y are subsets of ordinals with the same order types, then let $\rho_{X,Y}$ be the unique order preserving bijection between X and Y.

DEFINITION 3.4. If $\mathcal{F}$ is a set of functions, an $\mathcal{F} \cup \{x\}$-term t is a sequence $\langle h_0, \dots, h_{n-1}\rangle$, where $h_i = x$ or $h_i = x^{-1}$ or $h_i = f_i$ or $h_i = f_i^{-1}$ for some $f_i \in \mathcal{F}$. If g is function we use $t[g]$ to denote the function $h'_0 \circ h'_1 \circ \cdots \circ h'_{n-1}$, where

$$h'_i = \begin{cases} f_i & \text{if } h_i = f_i, \\ f_i^{-1} & \text{if } h_i = f_i^{-1}, \\ g & \text{if } h_i = x, \\ g^{-1} & \text{if } h_i = x^{-1}. \end{cases}$$

If $\mathcal{H}$ is a set of $\mathcal{F} \cup \{x\}$-terms, then write

$$\mathcal{H}[g] = \{t[g] : t \in H\}.$$

We say that an $\mathcal{F} \cup \{x\}$-term t is an $\mathcal{F}$-term iff neither x nor x^{-1} appears in t. If t is an $\mathcal{F}$-term, then the function $t[g]$ does not depend on g, so we will write $t[\ ]$ instead of $t[g]$ in that situation.

We say that a term t' is a *subterm* of a term $t = \langle h_0, \dots, h_{n-1}\rangle$ iff $t' = \langle h_{i_0}, h_{i_1}, \dots, h_{i_k}\rangle$, where $i_0 < i_1 < \cdots < i_k < n$.

The set of all $\mathcal{F} \cup \{x\}$-terms is denoted by $TERM(\mathcal{F} \cup \{x\})$.

The set of all $\mathcal{F}$-terms is denoted by $TERM(\mathcal{F})$.

LEMMA 3.5. *Assume that*

(1) λ *is a cardinal,* $\mathcal{H}$ *is a finite set of* $S(\lambda) \cup \{x\}$*-terms, and* $\mathcal{H}$ *is closed for subterms,*

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(2) *g is an injective function*, $\operatorname{dom}(g) \cup \operatorname{ran}(g) \subset \lambda$,
(3) $\alpha, \alpha^* \in \lambda$ *such that*

$$\langle \alpha, \alpha^* \rangle \notin \bigcup \mathcal{H}[g],$$

(4) $\zeta_0 \in \lambda \setminus \operatorname{dom}(g)$ *and* $\zeta_1 \in \lambda \setminus \operatorname{ran}(g)$,
(5) $\eta_0 \in \lambda \setminus \operatorname{ran}(g)$ *and* $\eta_1 \in \lambda \setminus \operatorname{dom}(g)$ *such that*

$$\eta_0, \eta_1 \notin \{t[g](\alpha), t[g]^{-1}(\alpha^*) : t \in \mathcal{H}\}.$$

Let $g_0 = g \cup \{\langle \zeta_0, \eta_0 \rangle\}$ *and* $g_1 = g \cup \{\langle \eta_1, \zeta_1 \rangle\}$. *Then*

$$\langle \alpha, \alpha^* \rangle \notin \mathcal{H}[g_0] \cup \mathcal{H}[g_1].$$

Proof. We prove only $\langle \alpha, \alpha^* \rangle \notin \mathcal{H}[g_0]$. The proof of the other statement is similar.

Assume on the contrary that $\langle \alpha, \alpha^* \rangle \in \mathcal{H}[g_0]$.

Pick the shortest term $t = \langle f_0, \dots, f_n \rangle$ from $\mathcal{H}$ such that $t[g_0](\alpha) = \alpha^*$.

Write $\alpha_{n+1} = \alpha$ and $\alpha_i = \langle f_i, \dots, f_n \rangle [g_0](\alpha)$ for $0 \le i \le n$. Hence $\alpha_0 = \alpha^*$.

Let i maximal such that α_i is ζ_0 or η_0. Since $t[g](\alpha)$ cannot be α^* by (3), i is defined.

Since $\alpha_i = \langle f_i, \dots, f_n \rangle [g](\alpha)$, it follows that $\alpha_i \ne \eta_0$ by (5). So $\alpha_i = \zeta_0$.

Let j minimal such that α_j is ζ_0 or η_0. Since

$$\alpha_j = (\langle f_0, \dots, f_{j-1} \rangle [g])^{-1}(\alpha^*),$$

it follows that $\alpha_j \ne \eta_0$ by (5). So $\alpha_j = \zeta_0$ by (5). Thus $\alpha_i = \alpha_j = \zeta_0$, and so

$$\alpha^* = \langle f_0, \dots, f_{j-1}, f_i, \dots, f_n \rangle [g_0](\alpha).$$

Since $j < i$, the term $t' = \langle f_0, \dots, f_{j-1}, f_i, \dots, f_n \rangle$ is shorter than t and still $\alpha^* = t'[g_0](\alpha)$. So the length of t was not minimal. Contradiction. ⊣

Lemma 3.6. *Assume that*

(1) $y \in \mathrm{S}(\kappa)$,
(2) $A \in [\lambda]^{\kappa}$, *and* $B, C \in [A]^{\kappa}$ *such that* $|A \setminus B| = |A \setminus C| = \kappa$,
(3) $\mathcal{F} \in [\mathrm{S}(\lambda)]^{\kappa}$ *such that*

$$|y \setminus \bigcup \mathcal{H}[\,]| = \kappa$$

whenever $\mathcal{H}$ is a finite set of $\mathcal{F}$-terms.

Then there is $g \in \mathrm{S}(A)$ *such that*

(i) $g[B] = C$,
(ii)

$$|y \setminus \mathcal{H}[g^+]| = \kappa$$

whenever $\mathcal{H}$ is a finite set of $\mathcal{F} \cup \{x\}$-terms.

Proof of Lemma 3.6. Write

$$\mathbb{TASK}_0 = A \times \{\mathrm{dom}, \mathrm{ran}\} \text{ and } \mathbb{TASK}_1 = \left[TERM(\mathcal{F} \cup \{x\})\right]^{<\omega} \times \kappa.$$

Let $\{I_0, I_1\} \in \left[[\kappa]^{\kappa}\right]^2$ be a partition of κ, and fix enumerations $\{T_i : i \in I_0\}$ of $\mathbb{TASK}_0$, and $\{T_i : i \in I_1\}$ of $\mathbb{TASK}_1$.

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By transfinite induction, for $i < \kappa$ we will construct a function g_i and if $i = j + 1$ for some $j \in K_1$ then we also pick an ordinal $\alpha_{j+1} \in \kappa$ such that

(a) g_i is an injective function, $\mathrm{dom}(g_i) \cup \mathrm{ran}(g_i) \subset A$;
(b) $g_i[B] \subset C$ and $g_i[A \setminus B] \subset A \setminus C$;
(c) $|g_i| \leq i$;
(d) if $i = j + 1$, $j \in I_0$, and $T_j = \langle \zeta, \mathrm{dom}\rangle$, then $\zeta \in \mathrm{dom}(g_i)$;
(e) if $i = j + 1$, $j \in I_0$, and $T_j = \langle \zeta, \mathrm{ran}\rangle$, then $\zeta \in \mathrm{ran}(g_i)$;
(f) if $i = j + 1$, $j \in I_1$, and $T_j = \langle \mathcal{H}_j, \chi_j\rangle$, then
   (i) $\alpha_{j+1} \in \kappa \setminus \{\alpha_{j'+1} : j' \in I_1 \cap j\}$; and
   (ii) $t[g_i \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1})$ is defined and $t[g_i \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1}) \neq y(\alpha_{j+1})$ for each $t \in \mathcal{H}_j$.

Let $g_0 = \emptyset$.
If i is limit, then let $g_i = \bigcup_{j<i} g_j$.
Assume that $i = j + 1$.

CLAIM 3.6.1.

$$|y \setminus \bigcup \mathcal{H}[g_j \cup \mathrm{id}_{\lambda\setminus A}]| = \kappa, \tag{†}$$

for each finite set $\mathcal{H}$ of $\mathcal{F} \cup \{x\}$-terms.

PROOF OF THE CLAIM. Fix $\mathcal{H}$. We can assume that $\mathcal{H}$ is closed for subterms. By (3) we have $|y \setminus \bigcup \mathcal{H}[\,]| = \kappa$, and

$$y \cap \bigcup \mathcal{H}[\,] = y \cap \bigcup \mathcal{H}[\mathrm{id}_{\lambda\setminus A}], \tag{$\circ$}$$

because $\mathcal{H}$ is closed for subterms. Since $|g_j| < \kappa$, we have

$$|t[\mathrm{g_j} \cup \mathrm{id}_{\lambda\setminus A}] \setminus t[\mathrm{id}_{\lambda\setminus A}]| < \kappa, \tag{$\bullet$}$$

for each $t \in \mathcal{H}$. Putting together $|y \setminus \bigcup \mathcal{H}[\,]| = \kappa$, ($\circ$), and ($\bullet$) we obtain (†). ⊣

CASE 1. $j \in I_0$ *and so* $T_j = \langle \zeta_j, x_j\rangle \in A \times \{\mathrm{dom}, \mathrm{ran}\}$.
Assume first that $x_j = \mathrm{dom}$. If $\zeta_j \in \mathrm{dom}(g_j)$, let $g_i = g_j$. If $\zeta_j \notin \mathrm{dom}(g_j)$, then pick $\eta \in C$ if $\zeta_i \in B$, and pick $\eta \in A \setminus C$ if $\zeta_i \in A \setminus B$ such that and $\eta \notin \mathrm{ran}(g_j)$.
Let $g_i = g_j \cup \langle \zeta_i, \eta\rangle$. Then g_i satisfies (a)–(f).
The case $x_j = \mathrm{ran}$ is similar.
CASE 2. $j \in I_1$ *and so* $T_j = \langle \mathcal{H}_j, \chi_j\rangle \in \left[TERM(\mathcal{F} \cup \{x\})\right]^{<\omega} \times \kappa$.
We can assume that $\mathcal{H}_j$ is closed for subterms.
By Claim 3.6.1, we have

$$|y \setminus \bigcup \mathcal{H}_j[g_j \cup id_{(\lambda\setminus A)}]| = \kappa.$$

So we can pick $\alpha_{j+1} \in \kappa \setminus \{\alpha_{j'+1} : j' \in I_1 \cap j\}$ such that

(*) for each $t \in \mathcal{H}_j$ either $t[g_j \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1})$ is undefined or $t[g_j \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1}) \neq y(\alpha_{j+1})$.

Now in finitely many steps, using Lemma 3.5, we can extend the function g_j to a function g_i such that

(*) $t[g_i \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1})$ is defined and $t[g_i \cup \mathrm{id}_{\lambda\setminus A}](\alpha_{j+1}) \neq y(\alpha_{j+1})$ for each $t \in \mathcal{H}_j$.

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Indeed, if $t[g' \cup \mathrm{id}_{\lambda \setminus A}](\alpha_{j+1})$ is not defined, where $t = \langle t_0, \ldots, t_n \rangle$ then there is $i < n$ such that either

$\zeta_i = \langle t_{i+1}, \ldots, t_n \rangle [g' \cup \mathrm{id}_{\lambda \setminus A}](\alpha_{j+1})$ is defined, $t_i = x$, and $\zeta_i \in A \setminus \mathrm{dom}(g')$,

or

$\zeta_i = \langle t_{i+1}, \ldots, t_n \rangle [g' \cup \mathrm{id}_{\lambda \setminus A}](\alpha_{j+1})$ is defined, $t_i = x^{-1}$, and $\zeta_i \in A \setminus \mathrm{ran}(g')$.

In both cases, using Lemma 3.5, we can extend g' to g'' such that $\langle t_i, \ldots, t_n \rangle [g'' \cup \mathrm{id}_{\lambda \setminus A}](\alpha_{j+1})$ is defined and $\langle \alpha_{j+1}, y(\alpha_{j+1}) \rangle \notin \bigcup \mathcal{H}_j[g'' \cup id_{\lambda \setminus A}]$.

After the inductive construction, the function $g = \bigcup_{i<\kappa} g_i$ meets the requirements. ⊣

LEMMA 3.7. *Assume that $2^\kappa = \kappa^+$ and there is a cofinal, locally small subfamily $\mathcal{C} \subset [\lambda]^\kappa$. Then there is a family $\mathcal{D} \subset [\lambda]^\kappa \times [\lambda]^\kappa$ such that*

(1) *if $\langle A, B \rangle \in \mathcal{D}$, then $B \cup \kappa \subset A$ and $|A \setminus B| = \kappa$.*

Moreover, writing $\mathcal{A} = \{A : \langle A, B \rangle \in \mathcal{D}\}$ and $\mathcal{B} = \{B : \langle A, B \rangle \in \mathcal{D}\}$

(2) *$\mathcal{A}$ is a cofinal, locally small subfamily of $[\lambda]^\kappa$,*
(3) *$\mathcal{B}$ is cofinal in $\langle [\lambda]^\kappa, \subset \rangle$,*
(4) *$\{X \subset \kappa : |X| = |\kappa \setminus X| = \kappa\} \subset \mathcal{B}$.*

PROOF OF LEMMA 3.7. Fix a locally small, cofinal subfamily $\mathcal{C} \subset [\lambda]^\kappa$ such that $\mu = |\mathcal{C}|$ is minimal. Then $|\{C \in \mathcal{C} : D \subset C\}| = |\mathcal{C}|$ for all $D \in [\lambda]^\kappa$.

Write $\mathcal{C} = \{C_\alpha : \alpha < \mu\}$. Since $2^\kappa = \kappa^+ \le \lambda \le \mu$ there is a sequence $\langle B_\alpha : \alpha < \mu \rangle \subset [\lambda]^\kappa$ such that

(a) $\{B_\alpha : \alpha < \kappa^+\} \supset \{X \subset \kappa : |X| = |\kappa \setminus X| = \kappa\}$,
(b) $\{B_\alpha : \alpha < \mu\} \supset \mathcal{C}$.

Thus $\mathcal{B} = \{B_\alpha : \alpha < \mu\}$ is cofinal in $[\lambda]^\kappa$. Now, for each $\alpha < \mu$ pick $A_\alpha \in \mathcal{C}$ such that $A_\alpha \supset C_\alpha \cup B_\alpha \cup \kappa$ and $|A_\alpha \setminus B_\alpha| = \kappa$.

Then $\mathcal{D} = \{\langle A_\alpha, B_\alpha \rangle : \alpha < \mu\}$ satisfies the requirements. ⊣

After that preparation we prove the main theorem of this section.

PROOF OF THEOREM 3.2. Fix $\mathcal{D}$, $\mathcal{A}$, and $\mathcal{B}$ as in Lemma 3.7.

For $\langle A, B \rangle \in \mathcal{D}$ consider the structure

$$\mathcal{M}_{\langle A,B \rangle} = \langle A, <, B, \{A \cap X : A \in \mathcal{A}\} \rangle .$$

Fix $\mathcal{D}' \in [\mathcal{D}]^{\kappa^+}$ such that writing $\mathcal{A}' = \{A' : \langle A', B' \rangle \in \mathcal{D}'\}$ and $\mathcal{B}' = \{B' : \langle A', B' \rangle \in \mathcal{D}'\}$ we have

(a) $\forall \langle A, B \rangle \in \mathcal{D} \exists \langle A', B' \rangle \in \mathcal{D}'$ such that $\rho_{A,A'}$ is an isomorphism between $\mathcal{M}_{\langle A,B \rangle}$ and $\mathcal{M}_{\langle A',B' \rangle}$.
(b) $\{X \subset \kappa : |X| = |\kappa \setminus X| = \kappa\} \subset \mathcal{B}'$.

Pick $K \in [\kappa]^\kappa$ with $|\kappa \setminus K| = \kappa$. Choose $y \in S(\kappa)$ such that $y(\alpha) \neq \alpha$ for each $\alpha \in \kappa$.

LEMMA 3.8 (Key lemma). *There are functions $\mathcal{F} = \{f_{\langle A,B \rangle} : \langle A, B \rangle \in \mathcal{D}'\}$ such that*

(a) $f_{\langle A,B \rangle} \in \mathrm{S}(A)$,
(b) $f_{\langle A,B \rangle}[B] = K$;

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moreover, taking

$$\mathcal{S} = \{\rho_{C_0,C_1} : \langle A_0, B_0\rangle, \langle A_1, B_1\rangle \in \mathcal{D}', C_0 \in \mathcal{A}\lceil^* A_0, C_1 \in \mathcal{A}\lceil^* A_1, \rho_{C_0,C_1}[\mathcal{A}\lceil C_0] = \mathcal{A}\lceil C_1\},$$

if $\mathcal{H}$ is a finite collection of $\mathcal{F} \cup \mathcal{S}$-terms, then

$$|y \setminus \bigcup \mathcal{H}[\,]| = \kappa.$$

Before proving the Key lemma, we show how the Key Lemma completes the proof of Theorem 3.2.

So assume that the Key lemma holds.

For each $\langle A, B\rangle \in \mathcal{D}$ pick $\langle A', B'\rangle \in \mathcal{D}'$ such that $\rho_{A,A'}$ is an isomorphism between $\mathcal{M}_{\langle A,B\rangle}$ and $\mathcal{M}_{\langle A',B'\rangle}$. We assume that $\langle A', B'\rangle = \langle A, B\rangle$ for $\langle A, B\rangle \in \mathcal{D}'$.

Let

$$g_{\langle A,B\rangle} = \rho_{A',A} \circ f_{\langle A',B'\rangle} \circ \rho_{A,A'} \in S(A).$$

Let G be the permutation group on λ generated by

$$\mathcal{G} = \{g_{\langle A,B\rangle}{}^+ : \langle A, B\rangle \in \mathcal{D}\}.$$

LEMMA 3.9. *G is κ-homogeneous.*

PROOF OF LEMMA 3.9. It is enough to show that for each $X \in [\lambda]^\kappa$ there is $g \in G$ with $g[X] = K$.

So fix $X \in [\lambda]^\kappa$. Pick $\langle A, B\rangle \in \mathcal{D}$ such that $X \subset B$.

Then

$$\begin{aligned} Z = g_{\langle A,B\rangle}[X] \subset g_{\langle A,B\rangle}[B] &= (\rho_{A',A} \circ f_{\langle A',B'\rangle} \circ \rho_{A,A'})[B] \\ &= (\rho_{A',A} \circ f_{\langle A',B'\rangle})[B'] = \rho_{A',A}[K] = K. \end{aligned}$$

Since $|Z| = |\kappa \setminus Z| = \kappa$, there is C such that $\langle C, Z\rangle \in \mathcal{D}'$. Then $f_{\langle C,Z\rangle}[Z] = K$. Thus $g_{\langle C,Z\rangle}{}^+[Z] = K$ because $\langle C', Z'\rangle = \langle C, Z\rangle$ and so $f_{\langle C,Z\rangle} = g_{\langle C,Z\rangle}$.

Thus $K = (g_{\langle C,Z\rangle}{}^+ \circ g_{\langle A,B\rangle}{}^+)[X]$. ⊣

LEMMA 3.10. *G is not κ-transitive.*

PROOF OF LEMMA 3.10. We prove that $y \not\subset h$ for any $h \in G$.

Assume that

$$h = (g_0^+)^{\ell_0} \circ (g_1^+)^{\ell_1} \circ \cdots \circ (g_{n-1}^+)^{\ell_{n-1}},$$

where $g_i = g_{\langle A_i,B_i\rangle} = \rho_{A_i',A_i} \circ f_{A_i',B_i'} \circ \rho_{A_i,A_i'}$ and $\ell_i \in \{-1, 1\}$ for $i < n$.

Since $g_i^+ \setminus g_i$ is the identity function on $\lambda \setminus A_i$, we have

$$h \subset \bigcup\{(g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \cdots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}} : k < n, i_0 < i_1 < \cdots < i_{k-1} < n\}.$$

Fix $k \leq n$ and $i_0 < i_1 < \cdots < i_{k-1} < n$.

Observe that if $\ell_i = -1$ then

$$(g_i)^{\ell_i} = (\rho_{A_i',A_i} \circ f_{A_i',B_i'} \circ \rho_{A_i,A_i'})^{-1} = \rho_{A_i',A_i} \circ (f_{A_i',B_i'})^{-1} \circ \rho_{A_i,A_i'}.$$

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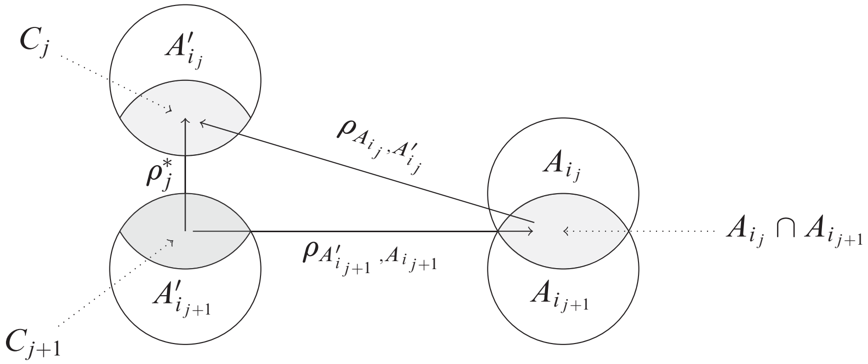

FIGURE 1. The function ρ^*_j.

So

$$(g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \cdots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}}$$
$$= \rho_{A'_{i_0},A_{i_0}} \circ (f_{A'_{i_0},B'_{i_0}})^{\ell_{i_0}} \circ \rho_{A_{i_0},A'_{i_0}} \circ \rho_{A'_{i_1},A_{i_1}} \circ (f_{A'_{i_1},B'_{i_1}})^{\ell_{i_1}} \circ \rho_{A_{i_1},A'_{i_1}} \circ .$$

For $j < k$ let

$$\rho^*_j = \rho_{A_{i_j},A'_{i_j}} \circ \rho_{A'_{i_{j+1}},A_{i_{j+1}}}.$$

Observe that writing

$$C_{j+1} = \rho_{A_{i_{j+1}},A'_{i_{j+1}}}[A_{i_j} \cap A_{i_{j+1}}] \text{ and } C_j = \rho_{A_{i_j},A'_{i_j}}[A_{i_j} \cap A_{i_{j+1}}],$$

we have

$$\rho^*_j = \rho_{C_{j+1},C_j} \in \mathcal{S}$$

(see Figure 1).

Thus

$$(g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \cdots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}}$$
$$= \rho_{A_{i_0},A'_{i_0}} \circ (f_{A'_{i_0},B'_{i_0}})^{\ell_0} \circ \rho^*_0 \circ (f_{A'_{i_1},B'_{i_1}})^{\ell_1} \circ \rho^*_1 \circ \cdots$$
$$\circ (f_{A'_{i_{k-1}},B'_{i_{k-1}}})^{\ell_{i_{k-1}}} \circ \rho_{A'_{i_{k-1}},A_{i_{k-1}}}.$$

Since $\rho_{A_\ell,A'_\ell} \restriction \kappa = \mathrm{id} \restriction \kappa$, we have

$$\left((g_{i_0})^{\ell_{i_0}} \circ (g_{i_1})^{\ell_{i_1}} \circ \cdots \circ (g_{i_{k-1}})^{\ell_{i_{k-1}}}\right) \cap \kappa \times \kappa$$
$$\subset (f_{A'_{i_0},B'_{i_0}})^{\ell_0} \circ \rho^*_0 \circ (f_{A'_{i_1},B'_{i_1}})^{\ell_1} \circ \rho^*_1 \circ \cdots$$
$$\circ (f_{A'_{i_{k-1}},B'_{i_{k-1}}})^{\ell_{i_{k-1}}}.$$

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But $(f_{A'_{i_0},B'_{i_0}})^{\ell_0} \circ \rho_0^* \circ (f_{A'_{i_1},B'_{i_1}})^{\ell_1} \circ \rho_1^* \circ \cdots \circ (f_{A'_{i_{k-1}},B'_{i_{k-1}}})^{\ell_{i_{k-1}}} = t[]$ for the $\mathcal{F} \cup \mathcal{S}$-term $t = \left\langle (f_{A'_{i_0},B'_{i_0}})^{\ell_0}, \rho_0^*, (f_{A'_{i_1},B'_{i_1}})^{\ell_1}, \rho_1^*, \dots, (f_{A'_{i_{k-1}},B'_{i_{k-1}}})^{\ell_{i_{k-1}}} \right\rangle$.

Since there are only finitely many sequences $i_0 < \cdots < i_{k-1} < n$, we obtain that $h \cap \kappa \times \kappa$ is covered by the union of finitely many $\mathcal{F} \cup \mathcal{S}$-terms.

But y is not covered by the union of finitely many $\mathcal{F} \cup \mathcal{S}$-terms. So y witnesses that G is not κ-transitive. ⊣

Proof of the Key Lemma 3.8. Write $\mathcal{D}' = \{\langle A_\alpha, B_\alpha\rangle : \alpha < \kappa^+\}$.

By transfinite induction, we define functions $\{f_\alpha : \alpha < \kappa^+\}$ such that taking

$$\mathcal{F}_{<\beta} = \{f_\gamma : \gamma < \beta\}$$

and

$$\mathcal{S}_{<\beta} = \{\rho_{C_0,C_1} : \delta, \gamma < \beta, C_0 \in \mathcal{A}\lceil^* A_\delta, C_1 \in \mathcal{A}\lceil^* A_\gamma, \\ \rho_{C_0,C_1}[\mathcal{A}\lceil C_0] = \mathcal{A}\lceil C_1\},$$

we have

(i) $f_\alpha \in \mathrm{S}(A_\alpha)$,
(ii) $f_\alpha[B_\alpha] = K$,
(iii) if $\mathcal{H}$ is a finite collection of $\mathcal{F}_{<\alpha+1} \cup \mathcal{S}_{<\alpha+1}$-terms, then

$$|y \setminus \mathcal{H}[\,]| = \kappa.$$

Assume that we have constructed f_β for $\beta < \alpha$. Then we have:

$$\text{if } \mathcal{H} \text{ is a finite collection of } \mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha}\text{-terms, then} |y \setminus \mathcal{H}[\,]| = \kappa. \qquad (*)$$

To continue the construction we need a bit more.

Claim 3.10.1. *If $\mathcal{H}$ is a finite collection of $\mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha+1}$-terms, then*

$$|y \setminus \mathcal{H}[\,]| = \kappa.$$

Proof. First observe that if $\rho_i = \rho_{A_i,A_i^*}$ for $i < 2$, then

$$\rho_1 \circ \rho_0 = \rho_{\rho_0^{-1}[A_0^* \cap A_1], \rho_1[A_0^* \cap A_1]}. \qquad (\dagger)$$

Let

$$t = \langle t_0, t_1, \dots, t_n\rangle$$

be an element of $\mathcal{H}$. Since $\rho_{C_0,C_1} \restriction \kappa = \mathrm{id} \restriction \kappa$, if $t_0 \in \mathcal{S}_{<\alpha+1}$, then $t[\,] \cap \kappa \times \kappa = \langle t_1, \dots, t_n\rangle [\,] \cap \kappa \times \kappa$. So we can assume that $t_0 \in \mathcal{F}_{<\alpha}$. Similar arguments give that we can assume that $t_n \in \mathcal{F}_{<\alpha}$.

Now assume that

$$\langle t_i, \dots, t_j\rangle = \left\langle f_{\alpha_i}, \rho_{C_{i+1},D_{i+1}}, \rho_{C_{i+2},D_{i+2}}, \dots, \rho_{C_{j-1},D_{j-1}}, f_{\alpha_j}\right\rangle.$$

Then, by $(\dagger)$

$$\rho_{C_{i+1},D_{i+1}} \circ \rho_{C_{i+2},D_{i+2}} \circ \cdots \circ \rho_{C_{j-1},D_{j-1}} = \rho_{E_i,E_j},$$

for some $E_i \in \mathcal{A}\lceil C_{i+1}$ and $E_j \in \mathcal{A}\lceil D_{j-1}$.

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Thus we can assume that $j = i + 2$ and

$$\langle t_i, t_{i+1}, t_{i+2}\rangle = \langle f_{\alpha_0}, \rho_{E_0,E_1}, f_{\alpha_1}\rangle.$$

Now

$$f_{\alpha_0} \circ \rho_{E_0,E_1} \circ f_{\alpha_1} = f_{\alpha_0} \circ \rho_{A_{\alpha_0}\cap E_0, A_{\alpha_1}\cap E_1} \circ f_{\alpha_1}$$

and $\rho_{A_{\alpha_0}\cap E_0, A_{\alpha_1}\cap E_1} \in \mathcal{S}_{<\alpha}$.

Thus there is an $\mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha}$-term s_t such that

$$t[\,] \cap (\kappa \times \kappa) = s_t[\,] \cap (\kappa \times \kappa).$$

Since $|y \setminus \bigcup\{s_t[\,] : t \in \mathcal{H}\}| = \kappa$ by (*), the Claim holds. ⊣

Since the claim holds, we can apply Lemma 3.6 for the family $\mathcal{F} = \mathcal{F}_{<\alpha} \cup \mathcal{S}_{<\alpha+1}$ to obtain f_α as g.

So we proved the Key Lemma 3.8. ⊣

So we proved Theorem 3.2. ⊣

The following theorem is hidden in [5]:

THEOREM 3.11. *If $\kappa^\omega = \kappa$, $\lambda = \kappa^{+n}$ for some $n < \omega$, and $\square_\nu$ holds for each $\kappa \le \nu < \lambda$, then there is a cofinal, locally small family in $[\lambda]^\kappa$.*

Indeed, in Section 2.4 of [5] the author defines the *weakly rounded* subsets of $\lambda = \kappa^{+n}$, in Lemma 2.4.1 he shows that the family of weakly rounded sets is cofinal, and finally on page 52 he proves a Claim which clearly implies that the family of weakly rounded sets is locally small.

Putting together Theorems 3.2 and 3.11 we obtain the following corollary.

COROLLARY 3.12. *If $\kappa^\omega = \kappa$, $\lambda = \kappa^{+n}$ for some $n < \omega$, and $\square_\nu$ holds for each $\kappa \le \nu < \lambda$, then there is a κ-homogeneous, but not κ-transitive permutation group on λ.*

§4. ω-homogeneous but not ω-transitive permutation groups in the Cohen model. Let $MA(countable)$ denote the Martin's Axiom restricted to countable partial orderings.

For $f \in \mathrm{S}(\lambda)$ let $\mathrm{supp}(f) = \{\alpha : f(\alpha) \ne \alpha\}$. Write

$$\mathrm{S}_\omega(\lambda) = \{f \in \mathrm{S}(\lambda) : |\mathrm{supp}(f)| \le \omega\}.$$

THEOREM 4.1. *If MA (countable) holds and $H \le \mathrm{S}_\omega(\omega_1)$ is a permutation group with $|H| < 2^\omega$, then there is an ω-homogeneous, but ω-intransitive permutation group $H^* \le \mathrm{S}_\omega(\omega_1)$ with $H^* \supset H$.*

PROOF OF THEOREM 4.1. If $\mathcal{F}$ is a set of functions, let

$$\langle \mathcal{F}\rangle_{gen} = \{f_0 \circ \cdots \circ f_{n-1} : n \in \omega, f_i \in \mathcal{F} \text{ or } f_i^{-1} \in \mathcal{F} \text{ for } i < n\}.$$

LEMMA 4.2. *If $\mathcal{H}$ is a family of functions with $|\mathcal{H}| < 2^\omega$ then some $r \in S(\omega)$ is $\mathcal{H}$-large.*

PROOF. Fix a family $\{r_\alpha : \alpha < 2^\omega\} \subset \mathrm{S}(\omega)$ such that $r_\alpha \cap r_\beta$ is finite for each $\{\alpha, \beta\} \in [2^\omega]^2$.

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Assume on the contrary that for each $\alpha < 2^\omega$ the permutation r_α is not $\mathcal{H}$-large, i.e., there is $\mathcal{H}_\alpha \in [\mathcal{H}]^{<\omega}$ such that $r_\alpha \setminus \bigcup \mathcal{H}_\alpha$ is finite.

Let $\mathcal{U}$ be a non-principal ultrafilter on ω. Then for each $\alpha < 2^\omega$ there is $h(\alpha) \in \mathcal{H}_\alpha$ such that $U_\alpha = \{n \in \omega : r_\alpha(n) = h(\alpha)(n)\} \in \mathcal{U}$.

Since $|\mathcal{H}| < 2^\omega$, there are $\alpha \neq \beta$ such that $h(\alpha) = h(\beta)$. Thus for each $n \in U_\alpha \cap U_\beta$ we have $r_\alpha(n) = h(\alpha)(n) = h(\beta)(n) = r_\beta(n)$. Thus $r_\alpha \cap r_\beta$ is infinite. Contradiction. ⊣

Using Lemma 4.2 fix an H-large $r \in S(\omega)$. Enumerate $[\omega_1]^\omega \times [\omega_1]^\omega$ as $\{\langle A_\alpha, B_\alpha\rangle : \alpha < 2^\omega\}$. By transfinite recursion on $\alpha < 2^\omega$, we will construct permutations $f_\alpha \in \mathrm{S}_\omega(\omega_1)$ such that $f_\alpha[A_\alpha] = B_\alpha$ and writing

$$\mathcal{F}_\delta = \{t[] : t \text{ is a } H \cup \{f_\zeta : \zeta < \delta\}\text{-term}\} = \langle H \cup \{f_\zeta : \zeta < \delta\}\rangle_{gen},$$

the permutation r is $\mathcal{F}_{\alpha+1}$-large.

Since $\mathcal{F}_0 = H$, we know that $r \in S(\omega)$ is $\mathcal{F}_0$-large.

Assume that we have constructed $\langle f_\zeta : \zeta < \alpha\rangle$ such that the function r is $\mathcal{F}_{\zeta+1}$-large for $\zeta < \alpha$. Then r is $\mathcal{F}_\alpha$-large. Next we should construct $f_\alpha \in S(\omega_1)$ such that $f_\alpha[A_\alpha] = B_\alpha$ and r is $\mathcal{F}_{\alpha+1}$-large. We want to apply MA(countable) to construct f_α, but to do so we need some technical lemmas.

Fix first $C_\alpha \in [\omega_1]^\omega$ such that $A_\alpha \cup B_\alpha \subset C_\alpha$ and $C_\alpha \setminus (A_\alpha \cup B_\alpha) = \omega$.

Definition 4.3. Given sets X and Y let us denote by $\mathrm{Bij}_{\mathrm{p}}(X, Y)$ the set of all finite bijections between subsets of X and Y.

For $A, B, C \in [\omega_1]^\omega$ define the poset $\mathcal{P}_{C,A,B} = \langle P_{C,A,B}, \leq\rangle$ as follows. Let

$$P_{C,A,B} = \{p \in \mathrm{Bij}_{\mathrm{p}}(C, C) : p[A] \subset B, p[C \setminus A] \subset C \setminus B\}.$$

Write $p \leq q$ iff $p \supseteq q$.

We want to apply MA(countable) for the countable poset

$$\mathcal{P} = \mathcal{P}_{C_\alpha, A_\alpha, B_\alpha}.$$

Our plan is to define a family $\mathbb{D}$ of dense subsets in P with $|\mathbb{D}| < 2^\omega$ such that if $\mathcal{K}$ is a $\mathbb{D}$-generic filter in P, then $(\bigcup \mathcal{K}) \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}$ works as f_α.

Lemma 4.4. *For $i \in C_\alpha$ the sets $D_i = \{p \in P_{C,A,B} : i \in \mathrm{dom}(p)\}$ and $R_i = \{p \in P_{C,A,B} : i \in \mathrm{ran}(p)\}$ are dense in P.*

Proof. Straightforward. ⊣

Lemma 4.5. *If $M \in \omega$ and $\mathcal{H}$ is a finite set of $\mathcal{F}_\alpha \cup \{x\}$-terms then*

$$E_{\mathcal{H},M} = \{p \in P : \exists m \in \omega \setminus M \; t[p](m) \text{ is defined, but } t[p](m) \neq r(m) \text{ for each } t \in \mathcal{H}\}$$

is dense in P.

Proof of the lemma. Fix $q \in P$. We can assume that $\mathcal{H}$ is closed for subterms.
We know that $|r \setminus \bigcup \mathcal{H}[\,]| = \omega$ because r is $\mathcal{F}_\alpha$-large.
Since $\mathcal{H}$ is closed for subterms,

$$r \cap \bigcup \mathcal{H}[\,] = r \cap \bigcup \mathcal{H}[\mathrm{id}_{\omega_1 \setminus C_\alpha}].$$

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Since $|q| < \omega$, we have

$$|r \setminus \bigcup \mathcal{H}[q \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}]| = \omega.$$

So we can pick $m \in \omega \setminus M$ such that

(*) for each $t \in \mathcal{H}$ either $t[q \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is undefined or $t[q \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m) \neq r(m)$.

Since $\mathcal{H}$ is finite, we can find $p \leq q$ such that

(*) for each $t \in \mathcal{H}$ either $t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is undefined or $t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m) \neq r(m)$,

($\bullet$) the cardinality of the finite set

$$\{t \in \mathcal{H} \;:\; t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m) \text{ is undefined}\}$$

is minimal.

To show that $p \in E_{\mathcal{H},M}$ we prove that

($\circ$) there is no $t \in \mathcal{H}$ such that $t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is undefined.

Assume on the contrary that this statement is not true.

Fix $t \in \mathcal{H}$ such that $t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is not defined, where $t = \langle t_0, \dots, t_n \rangle$. Thus there is $i < n$ such that

(1) $\langle t_{i+1}, \dots, t_n \rangle\, [p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is defined, but
(2) $\langle t_i, \dots, t_n \rangle\, [p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is not defined.

Then $t' = \langle t_i, \dots, t_n \rangle \in \mathcal{H}$. Let $\zeta_i = \langle t_{i+1}, \dots, t_n \rangle\, [p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$. Then either $t_i = x$ and $\zeta_i \notin \mathrm{dom}(p)$ or $t_i = x^{-1}$ and $\zeta_i \notin \mathrm{ran}(p)$.

In both cases, using Lemma 3.5, we can extend p to p' such that $\langle t_i, \dots, t_n \rangle\, [p' \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m)$ is defined and $\langle m, r(m) \rangle \notin \mathcal{H}[p' \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}]$. Thus $p' \leq q$ and

$$\{t \in \mathcal{H} \;:\; t[p' \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m) \text{ is undefined}\} \subsetneq \{t \in \mathcal{H} \;:\; t[p \cup \mathrm{id}_{\omega_1 \setminus C_\alpha}](m) \text{ is undefined}\},$$

which contradicts ($\bullet$).

So we proved Lemma 4.5. ⊣

Let

$$\mathbb{D} = \{D_i, R_i : i \in C_\alpha\} \cup \{E_{\mathcal{F},M} : M \in \omega,\ \mathcal{F} \text{ is a finite set of } \mathcal{F}_\alpha \cup \{x\}\text{-terms.}\}.$$

Then $\mathbb{D}$ is a family of dense sets in $P_{C_\alpha, A_\alpha, B_\alpha}$ with cardinality $< 2^\omega$. So, by MA(countable), there is a $\mathbb{D}$-generic filter $\mathcal{K}$. Let $f_\alpha = (\bigcup \mathcal{K}) \cup id_{\omega_1 \setminus C_\alpha}$

The assumption $\{D_i, R_j : i \in C_\alpha\} \subset \mathbb{D}$ yields $C_\alpha = \mathrm{dom}(\bigcup \mathcal{K}) = \mathrm{ran}(\bigcup \mathcal{K})$. Since $f_\alpha[A_\alpha] \subset B_\alpha$ and $f_\alpha[C_\alpha \setminus A_\alpha] \subset C_\alpha \setminus B_\alpha$ by the construction of $P_{C_\alpha, A_\alpha, B_\alpha}$ we have $f_\alpha[A_\alpha] = B_\alpha$.

If $\mathcal{F}$ is a finite subset of $\mathcal{F}_{\alpha+1}$, then there is a finite set $\mathcal{H}$ of $\mathcal{F}_\alpha \cup \{x\}$-terms such that

$$\mathcal{F} = \{t[f_\alpha] : t \in \mathcal{H}\}.$$

Then $E_{\mathcal{H},M} \cap \mathcal{K} \neq \emptyset$ implies that there is $m > M$ such that $r(m) \notin \{t[f_\alpha](m) : t \in \mathcal{H}\} = \{f(m) : f \in \mathcal{F}\}$. Thus r is $\mathcal{F}_{\alpha+1}$-large. Hence f_α satisfies the requirements.

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So we carried out the inductive construction, and so we have constructed $\langle f_\alpha : \alpha < 2^\omega \rangle$ such that r is $\mathcal{F}_{2^\omega}$-large. So the group $H^* = \mathcal{F}_{2^\omega}$ satisfies the requirements. This completes the proof of Theorem 4.1. $\dashv$

Next we need a "stepping-up" theorem.

THEOREM 4.6. *Assume that $\lambda \geq \omega_1$ is a cardinal, $G \leq \mathrm{S}(\lambda)$ and $H^* \leq \mathrm{S}(\omega_1)$ are permutation groups such that*

(i) *H^* is ω-homogeneous, but ω-intransitive.*
(ii) *$\forall g \in G\ \forall \delta < \omega_1\ \exists h \in H^*\ g \cap (\delta \times \delta) \subset h$.*
(iii) *$\{g[\omega] : g \in G\}$ is cofinal in $\langle [\lambda]^\omega, \subset \rangle$.*

Then $G^* = \langle G \cup \{h^+ : h \in H\} \rangle_{gen} \leq \mathrm{S}(\lambda)$ is ω-homogeneous, but ω-intransitive.

PROOF OF THEOREM 4.6. First we show that G^* is ω-homogeneous.

Let $X, Y \in \left[\lambda\right]^\omega$ be arbitrary. First, by (iii) we can pick $f, g \in G$ such that $f[\omega] \supset X$ and $g[\omega] \supset Y$. Since H^* is ω-homogeneous, there is $h \in H^*$ such that

$$h[f^{-1}(X)] = g^{-1}(Y).$$

Then $g \circ h^+ \circ f^{-1} \in G^*$ and $(g \circ h^+ \circ f^{-1})[X] = Y$.

Next we show that G^* is ω-intransitive. Fix a countable injective function r with $\mathrm{dom}(r) \cup \mathrm{ran}(r) \in [\omega_1]^\omega$ which is H^*-large. Without loss of generality we can assume that $r \in S(\gamma)$ for some $\gamma < \omega_1$. We will verify that

$$r \text{ is } G^*\text{-large}$$

as well. It is enough to prove the next lemma.

LEMMA 4.7. *For each $g \in G^*$ there is a finite subset H_g of H^* such that*

$$g \cap (\gamma \times \gamma) \subset \bigcup H_g.$$

PROOF OF THE LEMMA. Since $G^* = \langle G \cup H^+ \rangle_{gen}$, where $H^+ = \{h^+ : h \in H^*\}$ and both G and H^+ are subgroups, we can assume that

$$g = e_0 \circ g_0 \circ \cdots \circ e_n \circ g_n,$$

where $g_i \in G$ and $e_i \in H^+$.

For $e \in H^+$, write $e^- = e \restriction \omega_1 \in H^*$.

By finite induction, define countable subsets $A_{n+1}, B_n, A_n, \ldots, B_0, A_0$ of λ as follows: let $A_{n+1} = \gamma$ and $B_i = g_i[A_{i+1}]$ and $A_i = e_i[B_i]$ for $i = n, n-1, \ldots, 0$.

Pick $\delta < \omega_1$ with

$$\bigcup \{A_i, B_i : 0 \leq i \leq n+1\} \cap \omega_1 \subset \delta.$$

For $0 \leq k < m \leq n$ let

$$g_{k,m} = g_k \circ \cdots \circ g_{m-1}.$$

By (ii) we can pick $h_{k,m} \in H^*$ such that $h_{k,m} \supset g_{k,m} \cap (\delta \times \delta)$. Let

$$\mathcal{H}_g = \{e^-_{i_0} \circ h_{i_0,i_1} \circ e^-_{i_1} \circ h_{i_1,i_2} \circ \cdots \circ e^-_{i_\ell} \circ h_{i_\ell,i_{\ell+1}} : \\ 0 \leq i_0 < \cdots < i_\ell < i_{\ell+1} = n\}.$$

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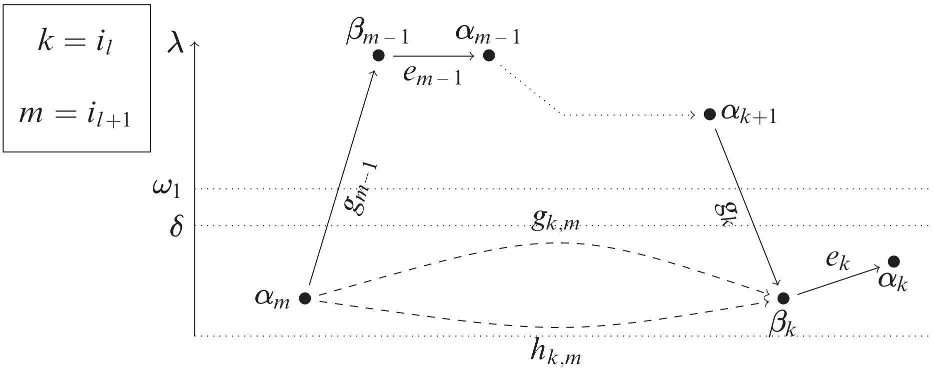

FIGURE 2. The function $h_{k,m}$.

CLAIM 4.7.1. $g \cap (\gamma \times \gamma) \subset \bigcup \mathcal{H}_g$.

PROOF OF THE CLAIM. Let $\alpha \in \gamma$ be arbitrary with $g(\alpha) \in \gamma$. Write $\alpha_{n+1} = \alpha$, $\beta_i = g_i(\alpha_{i+1})$, and $\alpha_i = e_i(\beta_i)$ for $i = n, n-1, \dots, 0$. So $\alpha_0 = g(\alpha) \in \gamma$.

Let $i_0 = 0 < \cdots < i_s = n+1$ be the enumeration of the set $I = \{i \leq n+1 : \alpha_i \in \omega_1\} = \{i \leq n+1 : \alpha_i \in \delta\}$.

Fix $\ell < s$, and write $k = i_\ell$ and $m = i_{\ell+1}$.

If $k+1 = m$, then $\alpha_k, \beta_k, \alpha_m \in \delta$ and so then

$$\alpha_k = e_k(\beta_k) = e_k(g_k(\alpha_m)) = (e_k^- \circ h_{k,m})(\alpha_m).$$

If $k+1 < m$, then

(i) $\alpha_k \in \delta$, $\beta_m \in \delta$, but
(ii) $\alpha_i, \beta_i \in \lambda \setminus \omega_1$ and so $\alpha_i = \beta_i$ for $k < i < m$

(see Figure 2).

Thus

$$\begin{aligned}\beta_k &= (g_k \circ e_k \circ g_{k+1} \circ \cdots \circ e_{m-1} \circ g_{m-1})(\alpha_m) \\ &= (g_k \circ g_{k+1} \circ \cdots \circ g_{m-1})(\alpha_m) = g_{k,m}(\alpha_m) = h_{k,m}(\alpha_m),\end{aligned}$$

and so

$$\alpha_k = e_k(\beta_k) = e_k(h_{k,m}(\alpha_m)) = (e_k^- \circ h_{k,m})(\alpha_m).$$

Hence

$$\begin{aligned}g(\alpha) &= (e_0 \circ g_0 \circ \cdots \circ e_n \circ g_n)(\alpha) \\ &= (e_{i_0}^- \circ h_{i_0,i_1} \circ \cdots \circ e_{i_\ell}^- \circ h_{i_{s-1},i_s})(\alpha)\end{aligned}$$

and $(e_{i_0}^- \circ h_{i_0,i_1} \circ \cdots \circ e_{i_\ell}^- \circ h_{i_{s-1},i_s}) \in \mathcal{H}_g$. ⊣

So we proved the Claim which completes the proof of the Lemma. ⊣

As we observed, the previous lemma implies that r is G^*-large, and so G^* is ω-intransitive which completes the proof of Theorem 4.6. ⊣

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Putting together Theorems 4.1 and 4.6 we can get the following result.

THEOREM 4.8. *Assume that λ is an uncountable cardinal and there is a permutation group $G \leq S_\omega(\lambda)$ such that*

(1) $|\{g \cap (\omega_1 \times \omega_1) : g \in G\}| < 2^\omega$.
(2) $\{g[\omega] : g \in G\}$ *is cofinal in* $\langle [\lambda]^\omega, \subset \rangle$.

If MA (countable) holds, then there is an ω-homogeneous but not ω-transitive permutation group $G^* \leq S_\omega(\lambda)$ with $G^* \supset G$.

PROOF OF THEOREM 4.8. First observe that (2) implies that $|\{g \cap (\omega_1 \times \omega_1) : g \in G\}| \geq \omega_1$, and so $2^\omega > \omega_1$ by (1).

For each countable injective function f with $\mathrm{dom}(f) \cup \mathrm{ran}(f) \subset \omega_1$ pick a permutation $h(f) \in S_\omega(\omega_1)$ with $h(f) \supset f$.

Let

$$H = \langle \{h(g \cap (\alpha \times \alpha)) : g \in G, \alpha < \omega_1\} \rangle_{gen}.$$

Since $2^\omega > \omega_1$, we have

(3) $|H| \leq |\{g \cap (\omega_1 \times \omega_1) : g \in G\}| \cdot \omega_1 < 2^\omega$, and
(4) $\forall g \in G \forall \alpha < \omega_1 \, \exists h \in H$ such that $g \cap (\alpha \times \alpha) \subset h$.

By (3) we can apply Theorem 4.1 and so there is an ω-homogeneous, but ω-intransitive permutation group $H^* \leq S_\omega(\omega_1)$ with $H^* \supset H$.

By (2) and (4) we can apply Theorem 4.6 for G and H^* to show that the permutation group $G^* = \langle G \cup \{h^+ : h \in H^+\} \rangle_{gen} \leq S_\omega(\lambda)$ is ω-homogeneous, but ω-intransitive. ⊣

Given sets X and Y let us denote by $\mathrm{Fin}(X, Y)$ the following poset: its underlying set is the set of all finite functions mapping a finite subset of X into Y, and $p \leq_{\mathrm{Fin}(X,Y)} q$ iff $p \supseteq q$. In particular, $\emptyset$ is the greatest element of $\mathrm{Fin}(X, 2)$.

COROLLARY 4.9. *If $P = \mathrm{Fin}((2^\omega)^+, 2)$ then*

$$V^P \models \text{"for each } \lambda \geq \omega_1 \text{ there is an } \omega\text{-homogeneous, but not } \omega\text{-transitive permutation group on } \lambda.\text{"}$$

REMARK. In Section 2 we showed that if there is a splendid space of cardinality at least λ, then there is an ω-homogeneous but not ω-transitive permutation group on λ. However, it was proved in [3] that it is consistent (modulo some large cardinal assumption), that there is no splendid space of size at least $\aleph_{\omega+1}$ in any c.c.c. generic extension of a certain ZFC model.

PROOF OF COROLLARY 4.9 FROM THEOREM 4.8. We work in V^P. Let $G = S_\omega(\lambda)^V$. Then

$$|\{g \cap \omega_1 \times \omega_1 : g \in G\}| = |\, S_\omega(\omega_1)^V| = (2^\omega)^V < ((2^\omega)^+)^V = (2^\omega)^{V^P}.$$

So (1) holds. Since P is c.c.c., $\{g[\omega] : g \in G\} = [\lambda]^\omega \cap V$ is cofinal in $\langle [\lambda]^\omega, \subset \rangle$. Hence (2) also holds.

So we can apply Theorem 4.8 because it is known that MA(countable) holds after adding $(2^\omega)^+$-many Cohen reals to a ground model (e.g., $cov(\mathcal{M}) = 2^\omega$ in the Cohen model by [1, Table 4], and $cov(\mathcal{M}) = 2^\omega$ implies MA(countable) by [4, Theorem 1]). ⊣

https://doi.org/10.1017/jsl.2021.63 Published online by Cambridge University Press

Acknowledgments. We thank our referee for his or her hard work on our paper, detailed report, and helpful suggestions! The first author was supported by European Research Council, grant no. 338821. Research partially supported by the Israel Science Foundation (ISF) grant no. 1838/19. Research partially supported by NSF grant no. DMS 1833363. Publication Number Sh:1193. The second author was supported by NKFIH grants no. K113047 and K129211.

REFERENCES

[1] A. Blass, *Combinatorial cardinal characteristics of the continuum*, ***Handbook of Set Theory*** (M. Foreman and A. Kanamori, editors), Springer, Dordrecht, 2010, pp. 395–489.

[2] I. Juhász, Z. Nagy, and W. Weiss, *On countably compact, locally countable spaces*. ***Periodica Mathematica Hungarica***, vol. 10 (1979), nos. 2–3, pp. 193–206.

[3] I. Juhász, S. Shelah, and L. Soukup, *More on countably compact, locally countable spaces*. ***Israel Journal of Mathematics***, vol. 62 (1988), no. 3, pp. 302–310.

[4] K. Keremedis, *On the covering and the additivity number of the real line*. ***Proceedings of the American Mathematical Society***, vol. 123 (1995), no. 5, pp. 1583–1590.

[5] R. W. Knight, *A topological application of flat morasses*. ***Fundamenta Mathematicae***, vol. 194 (2007), no. 1, pp. 45–66.

[6] P. M. Neumann, *Homogeneity of infinite permutation groups*. ***The Bulletin of the London Mathematical Society***, vol. 20 (1988), no. 4, pp. 305–312.

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