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Some sharp results about the global existence and blowup of solutions to a class of pseudo-parabolic equations

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In this paper we are interested in a sharp result about the global existence and blowup of solutions to a class of pseudo-parabolic equations. First, we represent a unique local weak solution in a new integral form that does not depend on any semigroup. Second, with the help of the Nehari manifold related to the stationary equation, we separate the whole space into two components S^+ and S^- via a new method, then a sufficient and necessary condition under which the weak solution blows up is established, that is, a weak solution blows up at a finite time if and only if the initial data belongs to S^-. Furthermore, we study the decay behaviour of both the solution and the energy functional, and the decay ratios are given specifically.

Keywords: pseudo-parabolic equation; global existence; blowup; exponential decay; Nehari manifold

2010 *Mathematics subject classification:* Primary 35K70; 35A01; 35B09; 35K44

1. Introduction

The aim of this paper is to study the initial boundary-value problem (IBVP) for a class of pseudo-parabolic equations

$$\left.\begin{aligned} u_t - \Delta u_t - \Delta u + u &= |u|^{p-2}u, \quad && (x,t) \in \Omega \times (0,\infty), \\ u(x,0) &= u_0(x), && x \in \Omega, \\ u(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,\infty), \end{aligned}\right\} \tag{1.1}$$

where $p \in (2, 2^*)$ and either $\Omega = \mathbb{R}^N$ $(N \geqslant 3)$ or Ω is a smooth bounded domain in $\mathbb{R}^N$. Here, $2^* = 2N/(N-2)$ for $N \geqslant 3$. When $\Omega = \mathbb{R}^N$, the boundary condition that $u(x,t) = 0$ for all $(x,t) \in \partial\Omega \times (0,\infty)$ is ineffective.

In physics, (1.1) with a general nonlinear term $f(u)$ describes a variety of physical phenomena such as the unidirectional propagation of nonlinear, dispersive, long waves [1] and the aggregation of populations [18]. Moreover, (1.1) is often used in the analysis of non-stationary processes in semiconductors in the presence of sources, where $\Delta u_t - u_t$ stands for the free electron density rate, Δu stands for the linear dissipation of free charge current, while $|u|^{p-1}u$ describes a source of free

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electron current [11]. Furthermore, (1.1) is also regarded as a Sobolev-type equation or a Sobolev–Galpern-type equation, which were introduced in [16].

Since the work of Ting [22, 23], pseudo-parabolic equations have been paid a large amount of attention by mathematicians; see [1, 2, 4–7, 12, 14, 17, 20, 21, 25, 26] and the references therein. Among this literature, Colton [5] studied pseudo-parabolic equations in one space variable. He constructed the Riemann function for the pseudo-parabolic equation and reduced the solution of the first IBVP for the pseudo-parabolic equation to that of a one-dimensional Volterra integral equation. Di Benedetto and Pierre established a comparison theorem for (1.1) in [7], and in [2] Cao *et al*. obtained the critical global existence exponent and the critical Fujita exponent for (1.1) by way of the integral representation and the contraction mapping principle. It should be pointed out that the semigroup theory was used in [2] to give the integral representation. In this paper we represent the weak solution to IBVP (1.1) in a different integral form that does not contain any semigroup.

Furthermore, as in the study of the parabolic equation, the blowup of solutions to pseudo-parabolic equations has also been extensively investigated; see, for example, [2,12,14,17,25]. For example, it can be derived from [14, theorem II] that for IBVP (1.1) the negative initial energy ($J(u_0) < 0$) is a sufficient condition for finite time blowup of the solution, and in [17] Meyvaci obtained sufficient conditions for the blowup of solutions of the IBVP for a class of nonlinear pseudo-parabolic equations involving a nonlinear convective term. He proved that if the initial data u_0 has a large norm in some suitable space, then the solution blows up at a finite time. Moreover, the establishment of a sharp criteria for the blowup and global existence of weak solutions to IBVP (1.1) is also followed with interest. To achieve such a criterion, it is well known that the Nehari functional I, the energy functional J and the ground state energy d of the stationary equation related to IBVP (1.1) play an important role, where the quantities I, J and d will be explained concretely in the latter part of this section. Recently, when Ω is a smooth bounded domain in $\mathbb{R}^N$ and $J(u_0) \leqslant d$, Xu and Su [25] achieved a criterion for the global existence and finite time blowup for (1.1) via the potential wells method introduced by Payne and Sattinger [19]. Under the assumption that $J(u_0) \leqslant d$, they proved that if $I(u_0) > 0$, then the solution is global and if $I(u_0) < 0$, then the solution blows up at a finite time. In view of the above conclusion derived in [25], a natural question for IBVP (1.1) is whether or not the condition $I(u_0) < 0$ is still sufficient for finite time blowup when $J(u_0) > d$. The same question also arises in the study of the parabolic IBVP (see [10, § 1, pp. 963–964])

$$\left.\begin{array}{ll} u_t - \Delta u = |u|^{p-2}u, & (x,t) \in \Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{array}\right\} \tag{1.2}$$

where $p \in (2, 2^*)$ and Ω is a smooth bounded domain in $\mathbb{R}^N$, and recently Dickstein *et al*. [8] proved that the answer is negative and there exist solutions converging to any given steady state, with initial Nehari energy $I(u_0)$ either negative or positive. In other words, they proved that $I(u_0) < 0$ is not a sufficient condition for finite time blowup of solutions to parabolic IBVP (1.2). However, as for the pseudo-parabolic case, in this paper we will prove that the answer to the above question is positive, that is, $I(u_0) < 0$ is a sufficient condition for finite time blowup of solutions to IBVP (1.1) whether Ω is smooth bounded or is equal to $\mathbb{R}^N$ (see theorems 5.1 and 1.4). This surprising phenomena indicates again the difference between the pseudo-parabolic and parabolic equations. Moreover, to the best of our knowledge, when $J(u_0) > d$ the conclusions about the global existence and blowup of weak solutions to IBVP (1.1) are very few, and there is no sharp criterion for the global existence and blowup of weak solutions without the assumption that $J(u_0) \leqslant d$.

In order to state our main results specifically, we shall introduce some definitions and notation as follows.

Throughout this paper, we use $\|\cdot\|$ and $(\cdot,\cdot)$ as the norm and the associated inner product on $H_0^1(\Omega)$, respectively, that is

$$\|u\| = \left(\int_\Omega [|\nabla u|^2 + u^2]\right)^{1/2} \quad \text{and} \quad (u,v) = \int_\Omega [\nabla u \cdot \nabla v + uv]$$

for $u, v \in H_0^1(\Omega)$. Note that $H_0^1(\mathbb{R}^N) = H^1(\mathbb{R}^N)$. Then we also use $H_0^1(\Omega)$ as the working space for the case in which $\Omega = \mathbb{R}^N$. Moreover, we denote by S_p the best Sobolev constant for the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ for $p \in [2, 2^*]$, that is,

$$S_p = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{|u|_p^2} = \inf_{u \in H_0^1(\Omega),\, |u|_p = 1} \|u\|^2, \tag{1.3}$$

where $|\cdot|_p$ stands for the norm on $L^p(\Omega)$, and we use C as a different constant on different lines.

First we give the definition of a weak solution to IBVP (1.1).

Definition 1.1. A function $u \in L^2_{\mathrm{loc}}([0,T), H_0^1(\Omega))$ with $u' \in L^2_{\mathrm{loc}}([0,T), H_0^1(\Omega))$ is a weak solution of IBVP (1.1) on $\Omega \times [0,T)$ provided that

(i) for almost every (a.e.) $t \in [0,T)$, the identity

$$(u'(t), v) + (u(t), v) = \int_\Omega |u(t)|^{p-2} u(t) v, \quad v \in H_0^1(\Omega),$$

holds; and

(ii) $u(0) = u_0$.

Since $u \in L^2_{\mathrm{loc}}([0,T), H_0^1(\Omega))$ and $u' \in L^2_{\mathrm{loc}}([0,T), H_0^1(\Omega))$, we have that $u \in H^1_{\mathrm{loc}}((0,T), H_0^1(\Omega))$ [3, definition 1.4.33, p. 13]. According to [3, corollary 1.4.36, p. 14], we have that $u \in C([0,T), H_0^1(\Omega))$. Then u is continuous at 0 and definition 1.1(ii) is valid. Next we will define the maximal existence time $T = T(u)$ for a weak solution u of IBVP (1.1).

Definition 1.2.

(i) If u is a weak solution of IBVP (1.1) on $\Omega \times [0,\infty)$, then we define the maximal existence time $T = \infty$, that is, u is a global solution.

(ii) If there exists a $T_0 \in (0,\infty)$ such that u is a weak solution of IBVP (1.1) on $\Omega \times [0,T_0)$ but, for any $\delta > 0$, u cannot be expanded into a weak solution on $\Omega \times [0, T_0 + \delta)$, then we define the maximal existence time $T = T_0$.

Our first result is the following theorem.

THEOREM 1.3. *Assume that $u_0 \in H_0^1(\Omega)$. Then IBVP (1.1) admits a unique weak solution $u \in C^1([0,T), H_0^1(\Omega))$, where T is its maximal existence time, and u can be represented in the integral form*

$$u(t) = u_0 + \int_0^t [(I-\Delta)^{-1}|u(s)|^{p-2}u(s) - u(s)]\,\mathrm{d}s, \quad t \in [0,T),$$

or

$$u(t) = \mathrm{e}^{-t}u_0 + \int_0^t \mathrm{e}^{-(t-s)}(I-\Delta)^{-1}|u(s)|^{p-2}u(s)\,\mathrm{d}s, \quad t \in [0,T).$$

Moreover, if $T < \infty$, then $\limsup_{t\to T^-} \|u(t)\| = \infty$.

Let $u = u(x,t;u_0)$ be the unique weak solution to IBVP (1.1) obtained by theorem 1.3 whose maximal existence time is T. Then, as a consequence of $u \in C^1([0,T), H_0^1(\Omega))$, definition 1.1(i) holds for all $t \in [0,T)$ and $v \in H_0^1(\Omega)$. Moreover, we can claim that if $u_0 \neq 0$, then $u(t) \neq 0$ for all $t \in [0,T)$ (see remark 2.4). Therefore, in this paper, we assume that $u_0 \neq 0$.

To account for our sharp result about the global existence and the blowup phenomenon of weak solutions to IBVP (1.1), we also introduce the following functionals and sets. It is well known that the energy functional related to the stationary equation of IBVP (1.1) is

$$J(u) = \tfrac{1}{2}\|u\|^2 - \frac{1}{p}|u|_p^p, \quad u \in H_0^1(\Omega). \tag{1.4}$$

We introduce the set

$$K = \{u \in H_0^1(\Omega)\colon J'(u) = 0\},$$

i.e. K is a set consisting of all stationary solutions of (1.1), and, for $c \in \mathbb{R}$,

$$K_c = \{u \in K\colon J(u) = c\}.$$

Let

$$I(u) = \langle J'(u), u\rangle = \|u\|^2 - |u|_p^p, \quad u \in H_0^1(\Omega).$$

Then the Nehari manifold associated with the steady state of IBVP (1.1) is

$$N = \{u \in H_0^1(\Omega) \setminus \{0\}\colon I(u) = 0\}.$$

We also define two sets $N_\pm$ related to the Nehari manifold N as follows:

$$N_+ = \{u \in H_0^1(\Omega)\colon I(u) > 0\}, \qquad N_- = \{u \in H_0^1(\Omega)\colon I(u) < 0\}.$$

Then the whole space is the union of N, $N_\pm$ and $\{0\}$,

$$H_0^1(\Omega) = N \cup N_+ \cup N_- \cup \{0\}, \tag{1.5}$$

and N_- will be proved to be an invariant set in § 5, that is, if $u_0 \in N_-$, then $u(t) \in N_-$ for all $t \in [0,T)$.

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Since, for all $u \in H_0^1(\Omega)$ and $t \in [0,\infty)$,

$$I(tu) = t^2(\|u\|^2 - t^{p-2}|u|_p^p),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[J(tu)] = t(\|u\|^2 - t^{p-2}|u|_p^p),$$

we have that for each $u \in H_0^1(\Omega) \setminus \{0\}$ there exists a unique $t_u > 0$ such that $I(t_u u) = 0$, $I(tu) > 0$ for $t \in (0, t_u)$ and $I(tu) < 0$ for $t \in (t_u, \infty)$, and so $J(tu)$ achieves its maximum at t_u on $[0,\infty)$. This implies that the sets N and $N_\pm$ are not empty.

For $u \in N$, by the Sobolev embedding, we can derive that $\|u\|^2 = |u|_p^p \leqslant S_p^{-p/2}\|u\|^p$. This implies that $\|u\|^{p-2} \geqslant S_p^{p/2}$, and then

$$J(u) = \frac{p-2}{2p}\|u\|^2 \geqslant \frac{p-2}{2p}S_p^{p/(p-2)}.$$

Thus,

$$d = \inf_{u\in N} J(u) \tag{1.6}$$

is well defined and

$$d \geqslant \frac{p-2}{2p}S_p^{p/(p-2)}.$$

Moreover, in § 3 we prove that the identity in the above inequality holds no matter whether $\Omega = \mathbb{R}^N$ or Ω is a smooth bounded domain in $\mathbb{R}^N$.

In the remainder of this section, we always assume that $u_0 \in H_0^1(\Omega)$ and $u = u(x,t;u_0)$ is the weak solution of IBVP (1.1) whose maximal existence time is T. Using a flow decided by the weak solution u and the decomposition (1.5), we can separate the whole space into two new parts. Specifically, we first introduce sets

$$S^+ = \{u_0 \in H_0^1(\Omega)\colon u(t) \notin N_-,\ t \in [0,T)\}, \tag{1.7}$$
$$S^- = \{u_0 \in H_0^1(\Omega)\colon \text{there is a } t_0 \in [0,T) \text{ such that } u(t_0) \in N_-\},$$

where the character S stands for the word 'source'. Then we have the decomposition

$$H_0^1(\Omega) = S^+ \cup S^-,$$

and the fact that $N_- \cup (N \setminus K) \subset S^-$ is confirmed in § 5. In particular, for the case in which Ω is a smooth bounded domain in $\mathbb{R}^N$ we can prove that S^- is strictly larger than N_- by illustrating that $N \setminus K \neq \emptyset$ (see remark 5.4).

Now we state our sharp results about the global existence and the blowup phenomenon of weak solutions to the IBVP (1.1) as the following theorem.

Theorem 1.4.

(i) *$T < \infty$ if and only if $u_0 \in S^-$. In other words, $T < \infty$ if and only if there exists a $t_0 \in [0,T)$ such that $I(u(t_0)) < 0$. Moreover, if $T < \infty$, then $\lim_{t\to T^-}\|u(t)\| = \infty$.*

(ii) *Zero is the unique stable equilibrium solution of IBVP (1.1).*

REMARK 1.5. Firstly, different from the previous literature, theorem 1.4(i) gives a criterion for the blowup of weak solutions to IBVP (1.1) that is not really related to the initial data but depends on the corresponding trajectory of the solution. Moreover, in § 5, we also obtain that N_-, S^- are invariant under the flow decided by the solution u. In this sense, the blowup of a solution can be linked with some invariant sets related to itself. This is unusual and interesting.

Secondly, it is known that the ground state equilibrium solution of IBVP (1.1) is unstable. Now, from theorem 1.4(ii), we can also deduce that all the non-zero equilibrium solutions are unstable. Therefore, our result is more complete than those obtained to this point.

In the proof of theorem 1.4, we also derive the following interesting conclusions.

THEOREM 1.6.

(i) *If $I(u_0) > 0$ and $J(u_0) \leqslant d$, then $T = \infty$ and $u_0 \in S^+$. Moreover, for any $t^* \in [0, \infty)$ with $t^* + [d - J(u_0)] > 0$, the decay estimates*

$$\begin{aligned} \|u(t)\|^2 &\leqslant \|u(t^*)\|^2 \mathrm{e}^{-\omega^*(t-t^*)}, & t &\in [t^*, \infty), \\ J(u(t)) &\leqslant C^*/t, & t &\in (t^*, \infty), \end{aligned}$$

always hold, where ω^* and C^* are given by the specific form

$$\omega^* = 2(1-\delta^*), \qquad C^* = \frac{(p-2\delta^*)\|u(t^*)\|^2}{4p(1-\delta^*)} + t^* J(u(t^*)),$$

and

$$\delta^* = \left[\frac{J(u(t^*))}{d}\right]^{(p-2)/2}.$$

(ii) *If $T = \infty$, then $u, u' \in L^\infty([0,\infty), H_0^1(\Omega))$. Moreover, for the case in which Ω is a smooth bounded domain in $\mathbb{R}^N$, there exists $c \in [0, \infty)$ such that $K_c \neq \emptyset$, and there exist $\{t_n\}$ with $t_n \to \infty$ and $u^* \in K_c$ such that*

$$\lim_{n\to\infty} \|u(t_n) - u^*\| = 0.$$

In particular, if u^* is also an isolated equilibrium solution of IBVP (1.1), then $\lim_{t\to\infty} u(t) = u^*$.

REMARK 1.7. The exponential decay obtained in theorem 1.6(i) can also be found in [25]. But, in this paper, we give a new proof with a different and simpler method, and the δ^* that decides the exponential decay ratio is given in a specific form. As for the algebraic decay of the energy functional for the pseudo-parabolic equation and the properties of the global solutions obtained in theorem 1.6(ii), our conclusion may be the first result.

This paper is organized as follows. First, we establish the local existence and uniqueness for a weak solution in § 2, then we consider the case in which $J(u_0) \leqslant d$ in § 3 and, at the same time, we give the proof of theorem 1.6(i). Then the boundedness of global weak solutions and the proof of theorem 1.6(ii) are given in § 4. Finally, in § 5, we prove theorem 1.4.

2. Proof of theorem 1.3

In this part we will consider the local existence and uniqueness of a weak solution to IBVP (1.1). First, we recall a classical lemma that can be derived directly from the Sobolev embedding and Riesz representation theorem.

LEMMA 2.1. *Suppose that Ω is smooth bounded and $q \geqslant 2^*/(2^*-1)$. Then for arbitrary given $f \in L^q(\Omega)$, the elliptic Dirichlet problem*

$$\begin{aligned} -\Delta u + u &= f, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned}$$

admits a unique weak solution $u := (I-\Delta)^{-1} f$ in $H_0^1(\Omega)$ and $\|u\| \leqslant S_{q'}^{-1/2} |f|_q$, where $q' = q/(q-1)$.

Similarly, we can also deduce the following conclusion easily.

COROLLARY 2.2. *Suppose that $q \in [2^*/(2^*-1), 2]$. Then for arbitrary given $f \in L^q(\mathbb{R}^N)$, the equation $-\Delta u + u = f$ has a unique weak solution $u \in H^1(\mathbb{R}^N)$ and $\|u\| \leqslant S_{q'}^{-1/2} |f|_q$, where $q' = q/(q-1)$.*

In addition, we can conclude the following simple lemma.

LEMMA 2.3. *Define a mapping $\Psi\colon H_0^1(\Omega) \to L^q(\Omega)$ by $\Psi(u) = |u|^{p-2}u$, where $q = p/(p-1)$. Then Ψ is locally Lipschitz continuous.*

Proof. For all $u, v \in H_0^1(\Omega)$, by using the mean value theorem, the Hölder inequality and the Sobolev embedding, one can derive that

$$|\Psi(u) - \Psi(v)|_q \leqslant (p-1) S_p^{-(p-1)/2} (\|u\| + \|v\|)^{p-2} \|u - v\|.$$

Thus, the mapping Ψ is locally Lipschitz continuous. □

Next we give some equivalence results. Let u be a weak solution of IBVP (1.1) on $\Omega \times [0,T)$. Then $u \in H_{\text{loc}}^1((0,T), H_0^1(\Omega))$ and so $u \in C([0,T), H_0^1(\Omega))$. According to definition 1.1, lemma 2.1 and corollary 2.2, for a.e. $t \in [0,T)$ and each $v \in H_0^1(\Omega)$ we have that

$$(u'(t), v) + (u(t), v) = \int_\Omega \Psi(u(t)) v = ((I-\Delta)^{-1} \Psi(u(t)), v).$$

Thus, it is obvious that $u \in L_{\text{loc}}^2([0,T), H_0^1(\Omega))$ with $u' \in L_{\text{loc}}^2([0,T), H_0^1(\Omega))$ is a weak solution of IBVP (1.1) if and only if u is a solution of the following initial-value problem in $H_0^1(\Omega)$:

$$\left.\begin{aligned} &u'(t) = (I-\Delta)^{-1} \Psi(u(t)) - u(t) \quad \text{a.e. } t \in [0,T), \\ &u(0) = u_0. \end{aligned}\right\} \tag{2.1}$$

According to lemma 2.1, corollary 2.2 and lemma 2.3, the following integral equation is well defined:

$$u(t) = u_0 + \int_0^t [(I-\Delta)^{-1} \Psi(u(s)) - u(s)] \, \mathrm{d}s, \quad t \in [0,T). \tag{2.2}$$

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It is obvious that u is a solution of the initial-value problem (2.1) in $H_0^1(\Omega)$ if and only if $u \in C([0,T), H_0^1(\Omega))$ is a solution of the integral equation (2.2). Here, it is sufficient to illustrate that the vector-valued function $(I-\Delta)^{-1}\Psi(u)\colon [0,T) \to H_0^1(\Omega)$ is measurable for all $u \in L^2_{\mathrm{loc}}([0,T), H_0^1(\Omega))$. In fact, since the vector-valued function $u\colon [0,T) \to H_0^1(\Omega)$ is measurable, it follows from [3, definition 1.4.1, p. 4] that there exists a set $E \subset [0,T)$ of measure 0 and a sequence $\{u_n\} \subset C_c([0,T), H_0^1(\Omega))$ such that $u_n(t) \to u(t)$ in $H_0^1(\Omega)$ as $n \to \infty$ for all $t \in [0,T) \setminus E$. Thus, by lemma 2.1, corollary 2.2 and lemma 2.3 it holds that $\{(I-\Delta)^{-1}\Psi(u_n)\} \subset C_c([0,T), H_0^1(\Omega))$ and $(I-\Delta)^{-1}\Psi(u_n(t)) \to (I-\Delta)^{-1}\Psi(u(t))$ as $n \to \infty$ for all $t \in [0,T) \setminus E$. This yields that $(I-\Delta)^{-1}\Psi(u)$ is measurable by [3, definition 1.4.1, p. 4].

Note that the mapping $u \mapsto (I-\Delta)^{-1}\Psi(u) - u$ is locally Lipschitz continuous on $H_0^1(\Omega)$, so the initial-value problem (2.1) admits a unique saturated solution in $H_0^1(\Omega)$.

Combining the above results, we can derive the local existence and uniqueness for a weak solution of IBVP (1.1) directly. Moreover, u can be represented in the integral form (2.2), which implies that $u \in C^1([0,T), H_0^1(\Omega))$, where T is the maximal existence time of u.

In order to finish the proof of theorem 1.3, we must also claim that if $T < \infty$, then $\limsup_{t\to T^-} \|u(t)\| = \infty$. Otherwise, suppose that there exists $M > 0$ such that $\|u(t)\| \leqslant M$ for $t \in [0,T)$. Then for $0 < t_1 < t_2 < T$, according to the integral representation (2.2), we have that

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} [(I-\Delta)^{-1}|u(s)|^{p-2}u(s) - u(s)]\,\mathrm{d}s.$$

Hence, it follows from lemma 2.1 and corollary 2.2 that

$$\begin{aligned}\|u(t_2) - u(t_1)\| &\leqslant \int_{t_1}^{t_2} [S_p^{-1/2}||u(s)|^{p-1}|_q + \|u(s)\|]\,\mathrm{d}s \\ &\leqslant \int_{t_1}^{t_2} [S_p^{-p/2}\|u(s)\|^{p-1} + \|u(s)\|]\,\mathrm{d}s \\ &\leqslant C(t_2 - t_1) \to 0, \quad t_2, t_1 \to T^-.\end{aligned}$$

Therefore, there exists some $u^* \in H_0^1(\Omega)$ such that $\lim_{t\to T^-} u(t) = u^*$. Consider the following initial-value problem in $H_0^1(\Omega)$:

$$\left.\begin{aligned} v' = (I-\Delta)^{-1}\Psi(v) - v = g(v),\\ v(T) = u^*. \end{aligned}\right\} \tag{2.3}$$

Since the mapping g is locally Lipschitz continuous, by [13, section 2.1, p. 33] there exists some $\delta > 0$ such that problem (2.3) has a unique solution on $[T, T+\delta] \times H_0^1(\Omega)$. Combining this with (2.2), one can expand the solution u to problem (2.1) into a solution on $\Omega \times [0, T+\delta]$, which contradicts the fact that T is the maximal existence time of u.

Remark 2.4. Let $u = u(x,t;u_0)$ be a solution to IBVP (1.1) obtained by theorem 1.3, and let its maximal existence time be T. If $u_0 \neq 0$, then $u(t) \neq 0$ for all

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$t \in [0,T)$. Actually, according to definition 1.1(i) and $u \in C^1([0,T), H_0^1(\Omega))$, using $u(t)$ as a test-function, one can obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \|u(t)\|^2 = |u(t)|_p^p \geqslant 0, \quad t \in [0,T).$$

Hence, $(\mathrm{d}/\mathrm{d}t)[\mathrm{e}^{2t}\|u(t)\|^2] \geqslant 0$, which implies that $t \mapsto \mathrm{e}^{2t}\|u(t)\|^2$ is non-decreasing. Furthermore, if there exists some $t_0 \in (0,T)$ such that $u(t_0) = 0$, then $0 < \|u_0\|^2 \leqslant \mathrm{e}^{2t_0}\|u(t_0)\|^2 = 0$, so a contradiction occurs.

3. The case in which $J(u_0) \leqslant d$

In this section we discuss the case in which $J(u_0) \leqslant d$, and give the proof of theorem 1.6(i).

Recall the definitions of J and d defined by (1.4) and (1.6), respectively. We will first prove that d is a critical value of J. When Ω is a smooth bounded domain in $\mathbb{R}^N$, since the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $p \in (2, 2^*)$, it is easy to deduce that d is a critical value by the mountain pass theorem. When $\Omega = \mathbb{R}^N$, since the above compactness fails, the proof becomes more difficult. The following lemma gives a unified proof for these two cases.

Lemma 3.1. *Suppose that $p \in (2, 2^*)$ and d is defined as (1.6). Then d is a critical value of J and*

$$d = \frac{p-2}{2p} S_p^{p/(p-2)}. \tag{3.1}$$

Proof. First, it is well known that S_p defined in (1.3) can be attained. In fact, when Ω is a smooth bounded domain, it follows from the compact embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ for $p \in (2, 2^*)$ that the infimum in (1.3) can be attained. For the case of $\Omega = \mathbb{R}^N$, we only need to use [24, theorem 1.34, p. 22].

Now, we introduce a set

$$S^1 = \{u \in H_0^1(\Omega)\colon |u|_p^p = 1\}.$$

Then there is a one-to-one mapping Φ from S^1 onto N,

$$\Phi(u) = t(u)u,$$

where $t(u) = \|u\|^{2/(p-2)}$. Moreover, for $u \in S^1$, from the definitions of N, I and J we get that

$$J(\Phi(u)) = \frac{p-2}{2p}\|\Phi(u)\|^2 = \frac{p-2}{2p}\|u\|^{2p/(p-2)}.$$

Therefore,

$$d = \inf_{u \in N} J(u) = \inf_{u \in S^1} J(\Phi(u)) = \inf_{u \in S^1} \frac{p-2}{2p}\|u\|^{2p/(p-2)} = \frac{p-2}{2p} S_p^{p/(p-2)}.$$

Since S_p can be attained, d can also be attained. Let $u \in S^1$ and $\|u\|^2 = S_p$. To finish the proof of this lemma, we claim that $\Phi(u)$ is a stationary solution of (1.1). In fact, it follows from the Lagrange multiplier rule that there exists some $\mu \in \mathbb{R}$ such that u is a weak solution of

$$-2\Delta u + 2u = \mu p |u|^{p-2} u.$$

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It is easy to see that $\mu = 2S_p/p$. Thus, u is a weak solution to $-\Delta u + u = S_p|u|^{p-2}u$. Let $v = S_p^{1/(p-2)}u$. Then $v = \Phi(u)$ and v is a weak solution of

$$-\Delta v + v = |v|^{p-2}v.$$

Therefore, $\Phi(u)$ is a stationary solution of (1.1), and d is a critical value of J. □

It follows from the Sobolev embedding and (3.1) that if $\|u\|^2 \leqslant 2pd/(p-2)$, then

$$I(u) \geqslant \|u\|^2(1 - S_p^{-p/2}\|u\|^{p-2}) \geqslant \|u\|^2\left[1 - S_p^{-p/2}\left(\frac{2pd}{p-2}\right)^{(p-2)/2}\right] = 0. \tag{3.2}$$

Furthermore, according to lemma 3.1 we know that d can be attained. Thus, the following assertion holds:

$$V = \{u \in H_0^1(\Omega) \colon I(u) < 0,\ J(u) < d\} \neq \emptyset. \tag{3.3}$$

Indeed, assume that $I(v) = 0$ and $J(v) = d$ for some $v \in H_0^1(\Omega)$. Let $u = tv$ for $t > 1$. Then $u \in V$.

In addition, we prove two important identities that will be used frequently in the remaining part of this paper.

LEMMA 3.2. *Let $u = u(x, t; u_0)$ be the solution to IBVP (1.1) whose maximal existence time is T. Then the identities*

$$J(u(t)) + \int_0^t \|u'(s)\|^2 \,\mathrm{d}s = J(u_0), \qquad t \in [0, T), \tag{3.4}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = (u(t), u'(t)) = -I(u(t)), \quad t \in [0, T), \tag{3.5}$$

hold.

Proof. The energy identity (3.4) follows by testing (1.1) with u' and integrating with respect to t, and through testing (1.1) with u one can also conclude that the identity (3.5) holds. □

It is easy to derive that the energy function $J(u(\cdot))$ is non-increasing from the energy identity (3.4). Furthermore, by the identity (3.5) it also holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = -2I(u(t)) &= (p-2)\|u(t)\|^2 - 2pJ(u(t)) \\ &\geqslant (p-2)\|u(t)\|^2 - 2pJ(u_0), \quad t \in [0, T).\end{aligned} \tag{3.6}$$

Proof of theorem 1.6(i). First we claim that

$$I(u(t)) > 0, \quad J(u(t)) < d, \quad t \in (0, T). \tag{3.7}$$

This means that $u_0 \in S^+$, which is defined as (1.7).

Indeed, since $I(u_0) > 0$, which means that $u_0 \notin K$, i.e. u_0 is not a stationary solution of IBVP (1.1), we have that $J(u(\cdot))$ is decreasing, and then $J(u(t)) < J(u_0) \leqslant d$ for all $t \in (0, T)$. On the other hand, due to the continuity of u and I,

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if there exists $t_0 \in (0, T)$ such that $I(u(t_0)) = 0$ and $I(u(t)) > 0$ for all $t \in [0, t_0)$, then it follows from remark 2.4 and the characterization (1.6) of d that $J(u(t_0)) \geqslant d$ and a contradiction occurs.

As a consequence of (3.7), it holds that

$$d > J(u(t)) \geqslant \frac{p-2}{2p}\|u(t)\|^2, \quad t \in [0, T). \tag{3.8}$$

Therefore, we can derive that $\|u(t)\|^2 < 2pd/(p-2)$ for all $t \in [0, T)$, which yields that $T = \infty$ by virtue of theorem 1.3.

Next we consider the decaying behaviour of the solution u. It follows from (1.3), (3.8), the decreasing of $J(u(\cdot))$ and (3.1) that for any $t^* \in [0, \infty)$ with $t^* + [d - J(u_0)] > 0$,

$$\begin{aligned}
|u(t)|_p^p &\leqslant S_p^{-p/2}\|u(t)\|^{p-2}\|u(t)\|^2 \\
&\leqslant S_p^{-p/2}\left[\frac{2p}{p-2}J(u(t))\right]^{(p-2)/2}\|u(t)\|^2 \\
&\leqslant S_p^{-p/2}\left[\frac{2p}{p-2}J(u(t^*))\right]^{(p-2)/2}\|u(t)\|^2 \\
&= \left[\frac{J(u(t^*))}{d}\right]^{(p-2)/2}\|u(t)\|^2 := \delta^*\|u(t)\|^2, \quad t \in [t^*, \infty),
\end{aligned}$$

that is,

$$|u(t)|_p^p \leqslant \delta^*\|u(t)\|^2. \tag{3.9}$$

Note that $J(u(t^*)) < d$. Then it is easy to see that $\delta^* \in (0, 1)$ and that δ^* depends only on $u(t^*)$. Furthermore, by (3.5) we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = 2[|u(t)|_p^p - \|u(t)\|^2] \leqslant -2(1-\delta^*)\|u(t)\|^2, \quad t \in [t^*, \infty).$$

This implies that

$$\|u(t)\|^2 \leqslant \|u(t^*)\|^2 \mathrm{e}^{-2(1-\delta^*)(t-t^*)} := \|u(t^*)\|^2 \mathrm{e}^{-\omega^*(t-t^*)}, \quad t \in [t^*, \infty).$$

Finally, we claim that for the above t^* there exists $C^* > 0$ such that

$$J(u(t)) \leqslant C^*/t, \quad t \in (t^*, \infty). \tag{3.10}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}[tJ(u(t))] \leqslant J(u(t)), \quad t \in [0, \infty). \tag{3.11}$$

Let $a(t) = \|u(t)\|^2$ and $b(t) = |u(t)|_p^p$ for all $t \in [t^*, \infty)$. Then it follows from (3.9) that $c(t) := b(t)/a(t) \in [0, \delta^*]$ for all $t \in [t^*, \infty)$. Since $I(u(t)) > 0$ for all $t \in [t^*, \infty)$, we have

$$\frac{J(u(t))}{I(u(t))} = \frac{a(t)/2 - b(t)/p}{a(t) - b(t)} = \frac{p - 2c(t)}{2p(1 - c(t))}, \quad t \in [t^*, \infty).$$

Noting that the function $d(t) = (p-2t)/(1-t)$ is increasing on $[0, \delta^*]$, we have that $p \leqslant d(t) \leqslant (p - 2\delta^*)/(1 - \delta^*)$ for all $t \in [0, \delta^*]$, and thus

$$\frac{1}{2} \leqslant \frac{J(u(t))}{I(u(t))} \leqslant \frac{p - 2\delta^*}{2p(1-\delta^*)} := M^*, \quad t \in [t^*, \infty). \tag{3.12}$$

Integrating the inequality (3.11) over $[t^*, t]$, we derive from (3.12) and (3.5) that, for $t \in [t^*, \infty)$,

$$\begin{aligned} tJ(u(t)) - t^*J(u(t^*)) &\leqslant \int_{t^*}^{t} J(u(s))\,\mathrm{d}s \\ &\leqslant M^* \int_{t^*}^{t} I(u(s))\,\mathrm{d}s \\ &= -\tfrac{1}{2}M^* \int_{t^*}^{t} \frac{\mathrm{d}}{\mathrm{d}s}\|u(s)\|^2\,\mathrm{d}s \\ &\leqslant \tfrac{1}{2}M^*\|u(t^*)\|^2, \end{aligned}$$

which yields that (3.10) holds for

$$C^* = \tfrac{1}{2}M^*\|u(t^*)\|^2 + t^*J(u(t^*)) = \frac{(p-2\delta^*)\|u(t^*)\|^2}{4p(1-\delta^*)} + t^*J(u(t^*)).$$

The proof is complete. □

Next, we prove a lemma that describes the blowup phenomenon of weak solutions to IBVP (1.1).

Lemma 3.3. *Assume that $u_0 \in H_0^1(\Omega)$ and that $u = u(x, t; u_0)$ is the weak solution of IBVP (1.1) whose maximal existence time is T. Then $T < \infty$ if and only if there exists $t_0 \in [0, T)$ such that $u(t_0) \in V$, which is defined in (3.3). In particular, if $u_0 \in V$, then u blows up at T, that is, $\lim_{t\to T^-}\|u(t)\| = \infty$.*

Proof. Assume first that there exists $t_0 \in [0, T)$ such that $u(t_0) \in V$, and without loss of generality assume that $t_0 = 0$. Similar to the proof of theorem 1.6(i), we can also prove that $u(t) \in V$ for all $t \in [0, T)$. In fact, by (3.4), $J(u(t)) \leqslant J(u_0)$, so we have that $J(u(t)) < d$ for all $t \in [0, T)$. On the other hand, due to the continuity of u and I, if there exists $t_1 \in (0, T)$ such that $I(u(t_1)) = 0$ and $I(u(t)) < 0$ for $t \in [0, t_1)$, then remark 2.4 and the definition of d imply that $J(u(t_1)) \geqslant d$, which is a contradiction.

It follows from the fact that $I(u(t)) < 0$ for all $t \in [0, T)$ and (3.2) that

$$\|u(t)\|^2 > \frac{2pd}{p-2}, \quad t \in [0, T). \tag{3.13}$$

Now, we will prove that $T < \infty$. Suppose that u is a global solution. Then we define a function $F\colon [0, \infty) \to [0, \infty)$ as follows:

$$F(t) = \int_0^t \|u(s)\|^2\,\mathrm{d}s.$$

Through simple calculations and due to (3.5), we obtain that

$$F'(t) = \|u(t)\|^2 > 0, \quad t \in [0, \infty), \tag{3.14}$$

and

$$F''(t) = 2(u(t), u'(t)) = -2I(u(t)) = (p-2)\|u(t)\|^2 - 2pJ(u(t)), \quad t \in [0, \infty).$$

Then it follows from the energy identity (3.4) that

$$F''(t) = (p-2)\|u(t)\|^2 + 2p\int_0^t \|u'(s)\|^2\,\mathrm{d}s - 2pJ(u_0), \quad t\in[0,\infty). \tag{3.15}$$

Noting that

$$\begin{aligned}\bigg(\int_0^t (u(s),u'(s))\,\mathrm{d}s\bigg)^2 &= \bigg(\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\|u(s)\|^2\,\mathrm{d}s\bigg)^2\\ &= \tfrac{1}{4}(\|u(t)\|^2 - \|u_0\|^2)^2\\ &\geqslant \tfrac{1}{4}([F'(t)]^2 - 2\|u_0\|^2F'(t)),\end{aligned}$$

we have that

$$[F'(t)]^2 \leqslant 4\bigg(\int_0^t (u(s),u'(s))\,\mathrm{d}s\bigg)^2 + 2\|u_0\|^2F'(t). \tag{3.16}$$

It then follows from (3.15) and (3.16) that

$$\begin{aligned}F(t)F''(t) &- \frac{p}{2}[F'(t)]^2\\ &\geqslant 2p\bigg[\int_0^t \|u(s)\|^2\,\mathrm{d}s\int_0^t \|u'(s)\|^2\,\mathrm{d}s - \bigg(\int_0^t (u(s),u'(s))\,\mathrm{d}s\bigg)^2\bigg]\\ &\quad + (p-2)F(t)F'(t) - 2pJ(u_0)F(t) - p\|u_0\|^2F'(t).\end{aligned} \tag{3.17}$$

Note that $F(t)\to\infty$ as $t\to\infty$ by (3.13). Then we have that there exists a sufficiently large $t_2>0$ such that

$$F(t)\bigg[\tfrac{1}{2}(p-2) - \frac{pJ(u_0)}{\|u(t)\|^2}\bigg] - p\|u_0\|^2 \geqslant 0, \quad t\in[t_2,\infty). \tag{3.18}$$

In fact, it is easy to see that (3.18) holds for the case in which $J(u_0)\leqslant 0$. If $J(u_0)\in(0,d)$, then it follows from (3.13) that

$$\tfrac{1}{2}(p-2) - \frac{pJ(u_0)}{\|u(t)\|^2} > \tfrac{1}{2}(p-2)\bigg(1-\frac{J(u_0)}{d}\bigg) > 0,$$

so (3.18) holds by the fact that $F(t)\to\infty$ as $t\to\infty$. Furthermore, combining (3.17) with Schwarz's inequality, (3.13), (3.14) and (3.18), we obtain that, for all $t\in[t_2,\infty)$,

$$\begin{aligned}F(t)F''(t) - \tfrac{1}{2}p[F'(t)]^2 &\geqslant (p-2)F(t)F'(t) - 2pJ(u_0)F(t) - p\|u_0\|^2F'(t)\\ &= F(t)[\tfrac{1}{2}(p-2)\|u(t)\|^2 - pJ(u_0)]\\ &\quad + F'(t)\bigg\{F(t)\bigg[\tfrac{1}{2}(p-2) - \frac{pJ(u_0)}{\|u(t)\|^2}\bigg] - p\|u_0\|^2\bigg\}\\ &\geqslant p(d-J(u_0))F(t)\\ &> 0.\end{aligned}$$

Let $M(t) = [F(t)]^{-(p-2)/2}$ for all $t \in (0, \infty)$. Then $M(t) > 0$, $M'(t) < 0$ for all $t \in (0, \infty)$ and $M''(t) < 0$ for all $t \in [t_2, \infty)$. This implies that M reaches 0 in a finite time, which contradicts that $M(t) > 0$ for all $t \in (0, \infty)$. Thus, the maximal existence time satisfies $T < \infty$. Since $u(t) \in V$ for all $t \in [0, T)$, $(\|u(t)\|^2)' = -2I(u(t)) > 0$, and so $\|u(t)\|^2$ is increasing on $[0, T)$. From theorem 1.3, we have that $\lim_{t\to T^-} \|u(t)\| = \infty$.

Conversely, assume now that $T < \infty$. Note that

$$\int_0^t \|u'(s)\|^2 \, \mathrm{d}s \geqslant \frac{1}{t}\left(\int_0^t \|u'(s)\| \, \mathrm{d}s\right)^2 \geqslant \frac{1}{t}(\|u(t)\| - \|u_0\|)^2, \quad t \in (0, T),$$

and

$$J(u) = \tfrac{1}{2}I(u) + \left(\frac{1}{2} - \frac{1}{p}\right)|u|_p^p \geqslant \tfrac{1}{2}I(u), \quad u \in H_0^1(\Omega).$$

Then it follows from the energy identity (3.4) that

$$\tfrac{1}{2}I(u(t)) \leqslant J(u(t)) \leqslant J(u_0) - \frac{1}{t}(\|u(t)\| - \|u_0\|)^2, \quad t \in (0, T).$$

Since $\limsup_{t\to T^-} \|u(t)\| = \infty$, we have

$$\lim_{t\to T^-} I(u(t)) = \lim_{t\to T^-} J(u(t)) = -\infty.$$

The proof is complete. □

Remark 3.4. In [25] Xu and Su also proved that if $u_0 \in V$, then $T < \infty$. To achieve the inequality $F(t)F''(t) - p[F'(t)]^2/2 > 0$ for t large enough, they introduced some sets N_δ, W_δ, V_δ and the depth of potential wells $d(\delta)$, whose properties must be verified through a difficult process. Making full use of equality (3.1), we obtain the same inequality more simply than in [25].

4. The boundedness of global weak solutions

In this section we will prove theorem 1.6(ii), and suppose that u is a global weak solution to IBVP (1.1). Without loss of generality, we can assume that

$$d \leqslant J(u(t)) \leqslant J(u_0), \quad t \in [0, \infty). \tag{4.1}$$

In fact, the second inequality follows from the energy identity (3.4). As for the first inequality, we assume that there exists $t_0 \in [0, \infty)$ such that $J(u(t_0)) < d$. It follows from theorem 1.3 and the definition of d that $I(u(t_0)) \neq 0$. If $I(u(t_0)) > 0$, according to theorem 1.6(i), $u \in L^\infty([0, \infty), H_0^1(\Omega))$ and $\lim_{t\to\infty} u(t) = 0$. If $I(u(t_0)) < 0$, u blows up at a finite time (which does not satisfy the assumption $T = \infty$).

Noting that $J(u(\cdot))$ is non-increasing and bounded on $[0, \infty)$ by (4.1), we can also derive that the limit $\lim_{t\to\infty} J(u(t))$ exists. Therefore, it follows from the energy identity (3.4) that

$$\int_0^\infty \|u'(s)\|^2 \, \mathrm{d}s = J(u_0) - \lim_{t\to\infty} J(u(t)).$$

This yields that

$$\int_0^\infty \|u'(t)\|^2 \,\mathrm{d}t < \infty. \tag{4.2}$$

Motivated by [9], we first prove a stability result that plays a crucial role in the following proof.

LEMMA 4.1. *Suppose that u is a global weak solution to IBVP (1.1). Then for every $l > 0$,*

$$\lim_{t\to\infty} \|u(t) - u(t+\tau)\| = 0 \quad \textit{uniformly for } \tau \in [0, l].$$

Proof. Since u is absolutely continuous locally, for any given $l > 0$ it follows from the Hölder inequality that

$$\|u(t) - u(t+\tau)\|^2 = \left\| \int_t^{t+\tau} u'(s)\,\mathrm{d}s \right\|^2 \leqslant l \int_t^{t+l} \|u'(s)\|^2 \,\mathrm{d}s.$$

By (4.2), $\lim_{t\to\infty} \int_t^{t+l} \|u'(s)\|^2 \,\mathrm{d}s = 0$, and the proof is complete. □

Now we are ready to prove theorem 1.6(ii).

Proof of theorem 1.6(ii). Suppose by contradiction that $u \notin L^\infty([0,\infty), H_0^1(\Omega))$. Then due to the continuity of u, we can choose an increasing diverging sequence $\{t_n\}$ such that $\|u(t_n)\| = n + 1$ for $n > \|u_0\|$. By lemma 4.1, it is easy to obtain that

$$\lim_{t\to\infty} \sup\{l > 0 \colon \|u(t) - u(t+\tau)\| < 1,\ \tau \in [0, l]\} = \infty.$$

Let

$$\tau_n = \sup\{l > 0 \colon \|u(t_n) - u(t_n + \tau)\| < 1,\ \tau \in [0, l]\}.$$

Then $\lim_{n\to\infty} \tau_n = \infty$ and, for $n > \|u_0\|$,

$$n \leqslant \|u(t)\| \leqslant n + 2, \quad t \in [t_n, t_n + \tau_n]. \tag{4.3}$$

By integrating (3.6) on the time interval $[t_n, t]$ for each $t \in (t_n, t_n + \tau_n]$, we can derive that

$$\|u(t)\|^2 \geqslant \|u(t_n)\|^2 + \int_{t_n}^t [(p-2)\|u(s)\|^2 - 2pJ(u_0)] \,\mathrm{d}s.$$

Then it follows from (4.3) that for sufficiently large n,

$$\|u(t)\|^2 \geqslant \|u(t_n)\|^2 + \tfrac{1}{2}(p-2) \int_{t_n}^t \|u(s)\|^2 \,\mathrm{d}s, \quad t \in (t_n, t_n + \tau_n].$$

Combining this inequality with (3.6) and (4.3), we obtain that for sufficiently large n and $t \in [t_n, t_n + \tau_n]$,

$$\begin{aligned}
(u(t), u'(t)) &\geqslant \tfrac{1}{2}(p-2)\|u(t)\|^2 - pJ(u_0) \\
&\geqslant \tfrac{1}{2}(p-2)\|u(t_n)\|^2 - pJ(u_0) + \tfrac{1}{4}(p-2)^2 \int_{t_n}^t \|u(s)\|^2 \,\mathrm{d}s \\
&\geqslant C \int_{t_n}^t \|u(s)\|^2 \,\mathrm{d}s,
\end{aligned} \tag{4.4}$$

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where C does not depend on n. On the other hand, it is easy to see that

$$(u(t), u'(t)) \leqslant \tfrac{1}{2}[\|u'(t)\|^2 + \|u(t)\|^2], \quad t \in [t_n, t_n + \tau_n]. \tag{4.5}$$

Let $M_n(t) = \int_{t_n}^{t} \|u(s)\|^2 \, \mathrm{d}s$ for all $t \in [t_n, t_n + \tau_n]$. Then, by virtue of (4.4) and (4.5), we derive that for sufficiently large n,

$$M_n'(t) - 2CM_n(t) \geqslant -\|u'(t)\|^2, \quad t \in [t_n, t_n + \tau_n].$$

Multiplying this inequality by e^{-2Ct} and integrating over $[t_n + \frac{1}{2}\tau_n, t_n + \tau_n]$, we obtain that for sufficiently large n,

$$\begin{aligned} M_n(t_n + \tau_n) &\geqslant \mathrm{e}^{C\tau_n} M_n(t_n + \tfrac{1}{2}\tau_n) - \int_{t_n+\tau_n/2}^{t_n+\tau_n} \mathrm{e}^{2C(t_n+\tau_n-s)} \|u'(s)\|^2 \, \mathrm{d}s \\ &\geqslant \mathrm{e}^{C\tau_n} M_n(t_n + \tfrac{1}{2}\tau_n) - \alpha \mathrm{e}^{C\tau_n}, \end{aligned}$$

where $\alpha = \int_0^\infty \|u'(s)\|^2 \, \mathrm{d}s$ is independent of n by (4.2). It follows from (4.3) that for sufficiently large n,

$$M_n(t) \geqslant n^2(t - t_n) \geqslant \tfrac{1}{2} n^2 \tau_n, \quad t \in [t_n + \tfrac{1}{2}\tau_n, t_n + \tau_n]. \tag{4.6}$$

Hence, it is easy to see that for sufficiently large n,

$$\int_{t_n}^{t_n+\tau_n} \|u(s)\|^2 \, \mathrm{d}s = M_n(t_n + \tau_n) \geqslant \tfrac{1}{2} \mathrm{e}^{C\tau_n} (n^2 \tau_n - 2\alpha) \geqslant \tfrac{1}{4} n^2 \tau_n \mathrm{e}^{C\tau_n},$$

where (4.6) has been used. However, according to (4.3), it can also be proved that

$$\int_{t_n}^{t_n+\tau_n} \|u(s)\|^2 \, \mathrm{d}s \leqslant (n+2)^2 \tau_n.$$

This yields that for sufficiently large n,

$$\tfrac{1}{4} n^2 \mathrm{e}^{C\tau_n} \leqslant (n+2)^2,$$

which is absurd since $\lim_{n\to\infty} \tau_n = \infty$. Therefore, $u \in L^\infty([0, \infty), H_0^1(\Omega))$.

Furthermore, for all $t \in [0, \infty)$, through using $u'(t) + u(t)$ as a test-function, the Hölder inequality and the Sobolev embedding, we can also derive that

$$\|u'(t) + u(t)\|^2 = \int_\Omega |u(t)|^{p-2} u(t)[u'(t) + u(t)] \leqslant S_p^{-p/2} \|u(t)\|^{p-1} \|u'(t) + u(t)\|.$$

Thus, it holds that $\|u'(t) + u(t)\| \leqslant S_p^{-p/2} \|u(t)\|^{p-1}$, $t \in [0, \infty)$. Moreover, since $u \in L^\infty([0, \infty), H_0^1(\Omega))$, we have $u' \in L^\infty([0, \infty), H_0^1(\Omega))$.

Now we will prove the second part of theorem 1.6(ii). Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Since u is a global weak solution of IBVP (1.1), it follows from the definitions of J and u that for each $\varphi \in H_0^1(\Omega)$,

$$\langle J'(u(t)), \varphi \rangle = \int_\Omega [\nabla u(t) \cdot \nabla \varphi + u(t)\varphi - |u(t)|^{p-2} u(t) \varphi] = -(u'(t), \varphi), \quad t \in [0, \infty).$$

This implies that

$$\|J'(u(t))\|_{H^{-1}} = \|u'(t)\|, \quad t \in [0, \infty), \tag{4.7}$$

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where H^{-1} is the dual space of $H_0^1(\Omega)$. It thus follows from (4.2) that

$$\liminf_{t\to\infty} \|J'(u(t))\|_{H^{-1}} = \liminf_{t\to\infty} \|u'(t)\| = 0.$$

Therefore, there exists an increasing sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} J'(u(t_n)) = 0$. It follows from assumption (4.1) that the sequence $\{J(u(t_n))\}$ is bounded. Then $\{u(t_n)\}$ is a Palais–Smale (PS) sequence of the energy functional J. Since Ω is a smooth bounded domain in $\mathbb{R}^N$, the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact, and so J satisfies the PS condition [24, lemma 1.20, p. 15]. Thus, there exists $u^* \in H_0^1(\Omega)$ such that $\lim_{n\to\infty} u(t_n) = u^*$ in $H_0^1(\Omega)$ (going to a subsequence if necessary), and it is easy to see that $u^* \neq 0$. Let $c = J(u^*) \geqslant d$. Then $u^* \in K_c$.

Finally, we claim that if the u^* obtained above is an isolated critical point of J, then $\lim_{t\to\infty} u(t) = u^*$. Our proof is motivated by [15]. Let $f(t) = \|u(t) - u^*\|$ for all $t \in [0, \infty)$. Then it is sufficient to prove that $\lim_{t\to\infty} f(t) = 0$. In fact, since u^* is an isolated critical point of J, there exists $r > 0$ such that $J'(v) \neq 0$ for all $v \in \overline{B_{2r}(u^*)} \setminus \{u^*\}$, where $B_{2r}(u^*) = \{v \in H_0^1(\Omega)\colon \|v - u^*\| < 2r\}$. Now, we suppose that the limit $\lim_{t\to\infty} f(t) = 0$ does not hold. Then there exist $\varepsilon_0 > 0$ and an increasing diverging sequence $\{s_n\}$ such that $s_n > t_n$ and $f(s_n) > \varepsilon_0$ for all $n \in \mathbb{N} := \{1, 2, \dots\}$, where $\{t_n\}$ is obtained in the above paragraph and satisfies $\lim_{n\to\infty} f(t_n) = 0$. Let $2\delta_1 = \min\{\varepsilon_0, 2r\}$. Then there must be a subsequence of $\{t_n\}$ (still denoted by $\{t_n\}$) such that $f(t_n) < \delta_1$ for all $n \in \mathbb{N}$. Define $\sup\{t \in (t_n, s_n)\colon f(t) = \delta_1\} := t_n'$ and $\inf\{t \in (t_n', s_n)\colon f(t) = 2\delta_1\} := t_n''$. Then it can be derived from the continuity of f that $f(t) \in (\delta_1, 2\delta_1)$ for all $t \in (t_n', t_n'')$. Moreover, one can also derive that

$$\delta_2 := \inf_{v \in \overline{B_{2\delta_1}(u^*)} \setminus B_{\delta_1}(u^*)} \|J'(v)\| > 0. \tag{4.8}$$

Otherwise, there exists a sequence $\{v_n\} \subset \overline{B_{2\delta_1}(u^*)} \setminus B_{\delta_1}(u^*)$ such that $J'(v_n) \to 0$ as $n \to \infty$. Noting that J is bounded on $\overline{B_{2\delta_1}(u^*)} \setminus B_{\delta_1}(u^*)$, we have that $\{v_n\}$ is a PS sequence of the functional J. Because J satisfies the PS condition, there exist a subsequence of $\{v_n\}$ still denoted by $\{v_n\}$ and $v_0 \in H_0^1(\Omega)$ such that $v_n \to v_0$ in $H_0^1(\Omega)$ as $n \to \infty$. Therefore, $v_0 \in \overline{B_{2\delta_1}(u^*)} \setminus B_{\delta_1}(u^*)$ and $J'(v_0) = 0$ by the continuity of J'. This contradicts the choice of r, and so (4.8) holds. As a consequence of the choice of t_n', t_n'' and the equalities (4.7) and (4.8), it also holds that

$$\|u(t_n') - u^*\| = \delta_1, \quad \|u(t_n'') - u^*\| = 2\delta_1$$

and

$$\|u'(t)\| = \|J'(u(t))\| \geqslant \delta_2, \quad t \in [t_n', t_n''].$$

Furthermore, it follows from (4.7) that

$$\delta_2 \|u'(t)\| \leqslant \|u'(t)\|^2 = -\langle J'(u(t)), u'(t) \rangle = -\frac{\mathrm{d}}{\mathrm{d}t}[J(u(t))], \quad t \in [t_n', t_n''].$$

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Thus, we have that

$$\delta_1 \leqslant \|u(t_n'') - u(t_n')\| \leqslant \int_{t_n'}^{t_n''} \|u'(t)\| \,\mathrm{d}t \leqslant \frac{1}{\delta_2} \int_{t_n'}^{t_n''} -\frac{\mathrm{d}}{\mathrm{d}t}[J(u(t))] \,\mathrm{d}t$$
$$= \frac{1}{\delta_2}[J(u(t_n')) - J(u(t_n''))].$$

This is a contradiction since the limit $\lim_{t\to\infty} J(u(t))$ exists. It therefore follows that $\lim_{t\to\infty} u(t) = u^*$. □

5. Proof of theorem 1.4

In this section we will prove theorem 1.4.

THEOREM 5.1. *Suppose that* $u_0 \in H_0^1(\Omega)$ *with* $I(u_0) < 0$*, and* $u = u(x, t; u_0)$ *is the solution to IBVP (1.1) whose maximal existence time is* T*. Then* T *is finite and* u *blows up at the time* T*.*

Proof. First we claim that $I(u(t)) < 0$ for all $t \in [0, T)$. In fact, by (3.4) and (3.5), we have that for all $t \in [0, T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}[I(u(t))] = \frac{\mathrm{d}}{\mathrm{d}t}[pJ(u(t)) - \tfrac{1}{2}(p-2)\|u(t)\|^2] = -p\|u'(t)\|^2 + (p-2)I(u(t))$$
$$\leqslant (p-2)I(u(t)). \tag{5.1}$$

Therefore, it follows from the Gronwall inequality that

$$I(u(t)) \leqslant I(u_0)\mathrm{e}^{(p-2)t}, \quad t \in [0, T). \tag{5.2}$$

Furthermore, since $I(u_0) < 0$, it holds that $I(u(t)) < 0$ for all $t \in [0, T)$.

Now we will prove that $T < \infty$. Due to lemma 3.3, if there exists some $t_0 \in [0, T)$ such that $J(u(t_0)) < d$, then $T < \infty$. Therefore, to finish the proof of this theorem, we need only consider the case that $d \leqslant J(u(t)) \leqslant J(u_0)$ for all $t \in [0, T)$. Suppose that $T = \infty$. Then as a result of theorem 1.6(ii), it holds that $u \in L^\infty([0, \infty), H_0^1(\Omega))$. Moreover, it follows from (4.2) that there exists a diverging sequence $\{t_n\}$ such that

$$\lim_{n\to\infty} \|u'(t_n)\| = 0.$$

Thus, on one hand, given (4.7), the fact that $u \in L^\infty([0, \infty), H_0^1(\Omega))$ and the continuity of u, we can obtain that

$$|I(u(t_n))| = |\langle J'(u(t_n)), u(t_n)\rangle| = |(u'(t_n), u(t_n))|$$
$$\leqslant \|u'(t_n)\|\|u(t_n)\| \to 0, \quad n \to \infty.$$

On the other hand, by (5.2) it holds that $|I(u(t_n))| \geqslant -I(u_0)\mathrm{e}^{(p-2)t_n} \to \infty$. So a contradiction occurs. Therefore, $T < \infty$ and it follows from (3.5) and the fact that $I(u(t)) < 0$ for all $t \in [0, T)$ that $\|u(\cdot)\|^2$ is increasing on $[0, T)$. Thus, by theorem 1.3, we can conclude that $\lim_{t\to T^-} \|u(t)\| = \infty$. □

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REMARK 5.2. If $I(u_0) = 0$ and $J'(u_0) \neq 0$, then the conclusion of theorem 5.1 still holds. In fact, since u_0 is not a stationary solution to IBVP (1.1), by (4.7) it holds that $u'(0) \neq 0$ in $H_0^1(\Omega)$. Thus, by (5.1), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}[I(u(t))]|_{t=0} = -p\|u'(0)\|^2 < 0.$$

Therefore, there must be some $t_0 \in (0,T)$ sufficiently small such that $I(u(t_0)) < 0$, that is, $u_0 \in S^-$. Furthermore, through taking t_0 as the initial time and repeating the proof of theorem 5.1, we can derive that $T < \infty$ and the solution blows up at T.

REMARK 5.3. Suppose that there exists some $t_0 \in (0,T)$ such that $I(u(t_0)) = 0$ and $J'(u(t_0)) \neq 0$. Firstly, from remark 5.2 we know that the solution also blows up at its finite maximal existence time. Secondly, since $I(u(\cdot))$ is continuously differentiable and $[I(u(\cdot))]'(t_0) < 0$, there exists some $\delta \in (0, t_0/2)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}[I(u(t))] < 0, \quad t \in (t_0 - \delta, t_0].$$

Therefore, we have that $I(u(t)) > 0$ for all $t \in (t_0 - \delta, t_0)$. This means that the weak solution with the initial data in the set N_+ may also blow up. Thus, describing the blowup phenomenon completely by way of N_- is impossible. Hence, our introduction of $S^\pm$ is valid and essential.

REMARK 5.4. It follows from (5.2) that the set N_- is invariant under the flow decided by the weak solution. Furthermore, by the definition of S^- and remark 5.2 it also holds that $N_- \cup (N \setminus K) \subset S^-$. In particular, if Ω is a smooth bounded domain in $\mathbb{R}^N$, then $N \setminus K \neq \emptyset$, which implies that S^- is strictly larger than N_-. Indeed, suppose that $\lambda_1 > 0$ is the principal eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$ related to the homogeneous Dirichlet boundary condition and $\varphi_1 \in H_0^1(\Omega)$ is the corresponding eigenfunction, which can be chosen as positive. Then it follows from the definition of I that there exists a unique $t_0 > 0$ such that $I(t_0\varphi_1) = 0$, that is, $t_0\varphi_1 \in N$. Now, we claim that $J'(t_0\varphi_1) \neq 0$. Otherwise, $t_0\varphi_1$ is a stationary solution to IBVP (1.1), and so it holds that

$$(\lambda_1 + 1)t_0\varphi_1 = -\Delta(t_0\varphi_1) + t_0\varphi_1 = (t_0\varphi_1)^{p-1}.$$

Therefore, $\varphi_1(x) \equiv (\lambda_1 + 1)^{1/(p-2)}/t_0$ for all $x \in \overline{\Omega}$, which contradicts the fact that $\varphi_1 = 0$ on $\partial\Omega$. Hence, $J'(t_0\varphi_1) \neq 0$ and so $t_0\varphi_1 \in N \setminus K$.

Proof of theorem 1.4. (i) First, theorem 5.1 implies directly the sufficiency of theorem 1.4(i). Next, to obtain the necessity, we will prove that if $u_0 \notin S^-$, then $T = \infty$. Actually, according to remark 5.2 and theorem 1.6(i), we need only prove that if $J(u(t)) > d$ and $I(u(t)) > 0$ for all $t \in [0,T)$, then $T = \infty$.

In what follows we assume that $J(u_0) > d$ and $I(u(t)) > 0$ for all $t \in [0,T)$. By the fact that $J(u(t)) \leqslant J(u_0)$, it holds that

$$\frac{p-2}{2p}\|u(t)\|^2 < \frac{p-2}{2p}\|u(t)\|^2 + \frac{1}{p}I(u(t)) = J(u(t)) \leqslant J(u_0), \quad t \in [0,T).$$

This implies that $\|u(\cdot)\|$ is bounded on $[0,T)$. Thus, according to theorem 1.3, we obtain that $T = \infty$.

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(ii) Now we prove theorem 1.4(ii). Since

$$I(u) \geqslant \|u\|^2[1 - S_p^{-p/2}\|u\|^{p-2}], \quad J(u) \leqslant \tfrac{1}{2}\|u\|^2, \quad u \in H_0^1(\Omega),$$

there exists a positive r_0 with $r_0^2 = \min\{S_p^{p/(p-2)}, 2d\} = 2d$ such that $I(u) > 0$ and $J(u) < d$ for $0 < \|u\| < r_0$. Thus, according to theorem 1.6(i), we obtain that 0 is stable.

Assume that $u_0 \neq 0$ and $J'(u_0) = 0$. Then $I(u_0) = 0$. It follows that

$$I(tu_0) = t^2(1 - t^{p-2})\|u_0\|^2, \quad t \in [0, \infty).$$

Then $I(tu_0) < 0$ for all $t > 1$. Thus, according to theorem 1.4(i), we have that u_0 is unstable. □

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