

ASYMPTOTIC ANALYSIS OF SKOLEM'S EXPONENTIAL FUNCTIONS

ALESSANDRO BERARDUCCI AND MARCELLO MAMINO

Abstract. Skolem (1956) studied the germs at infinity of the smallest class of real valued functions on the positive real line containing the constant 1, the identity function \mathbf{x} , and such that whenever f and g are in the set, $f + g$, fg and f^g are in the set. This set of germs is well ordered and Skolem conjectured that its order type is epsilon-zero. Van den Dries and Levitz (1984) computed the order type of the fragment below 2^{2^x} . Here we prove that the set of asymptotic classes within any Archimedean class of Skolem functions has order type ω . As a consequence we obtain, for each positive integer n , an upper bound for the fragment below 2^{n^x} . We deduce an epsilon-zero upper bound for the fragment below 2^{2^x} , improving the previous epsilon-omega bound by Levitz (1978). A novel feature of our approach is the use of Conway's surreal number for asymptotic calculations.

§1. Skolem problem. Let Sk be the smallest set of functions $f : \mathbb{R}^{>0} \rightarrow \mathbb{R}^{>0}$ containing the constant function 1 and the identity function \mathbf{x} , and such that if $f, g \in Sk$, then also $f + g$, fg and f^g are in Sk . A Skolem function is a function belonging to Sk . Each Skolem function restricts to a function $f : \mathbb{N}^{>0} \rightarrow \mathbb{N}^{>0}$ from positive integers to positive integers and it is determined by its restriction.

We order Sk by $f < g$ if $f(x) < g(x)$ for all large enough x in \mathbb{R} (or equivalently in \mathbb{N}). This defines a total order. Indeed Hardy [17] established the corresponding result for a larger class of functions. The totality of the order also follows from the fact that the structure $\mathbb{R}_{\text{exp}} = (\mathbb{R}, <, +, \cdot, \text{exp})$ is o-minimal [26] and the Skolem functions are definable in \mathbb{R}_{exp} .

In this paper we study the order type of Sk and its fragments. Skolem [25] conjectured that $(Sk, <)$ is a well order and its order type is $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$ (the least ordinal ε such that $\varepsilon = \omega^\varepsilon$). He also exhibited a well ordered subset of order type ε_0 , namely the subset generated from 1 and \mathbf{x} using the operations $+$, \cdot and exponentiation $g \mapsto \mathbf{x}^g$ with base \mathbf{x} . Ehrenfeucht [13], using the tree theorem of Kruskal [18], proved that Sk is indeed well ordered. Levitz [19] showed that its order type is at most equal to the smallest critical epsilon-number (the least ordinal α such that $\alpha = \varepsilon_\alpha$). This improves the earlier bound Γ_0 established by Schmidt [23], where Γ_0 is the Feferman–Schütte ordinal.

Given a well ordered set X , we write $|X|$ for the order type of X . If $f \in Sk$, we let $|f|$ be the order type of the set of Skolem functions less than f . The Skolem functions $< 2^x$ coincide with the nonzero polynomial functions with coefficients in \mathbb{N} , so $|2^x| = |\omega^\omega|$.

Received November 30, 2019.

2020 *Mathematics Subject Classification.* 03C64, 03E10, 16W60, 26A12, 41A58.

Key words and phrases. Skolem problem, surreal numbers, exponentiation, ordinals.

© The Author(s), 2020. Published by Cambridge University Press on behalf of The Association for Symbolic Logic
0022-4812/22/8702-0013
DOI:10.1017/jsl.2020.26

In [19] Levitz introduced the following definition: a regular function is a Skolem function g such that for every Skolem function $f < g$, one has $f^x < g$. The first regular functions are $g_0 = 2$ and $g_1 = 2^{2^x}$ and it is not difficult to show that the regular functions $< 2^{2^x}$ are exactly the functions of the form 2^{n^x} with $2 \leq n \in \mathbb{N}$. Levitz proved that $|g_{1+\alpha}| \leq \varepsilon_\alpha$, where $(g_\alpha)_\alpha$ is a transfinite enumeration of the regular functions, and $(\varepsilon_\alpha)_\alpha$ is an enumeration of the epsilon numbers (i.e. the ordinals ε satisfying $\varepsilon = \omega^\varepsilon$). Levitz's result then yields $|2^{2^x}| \leq \varepsilon_0$, $2^{3^x} \leq \varepsilon_1$ and $|2^{x^x}| \leq \varepsilon_\omega$ (since $g_{1+\omega} = g_\omega = 2^{x^x}$).

In [10] van den Dries and Levitz made a dramatic improvement on Levitz's bound on g_1 by showing that $|2^{2^x}| = \omega^{\omega^\omega}$. Here we prove the following bound on the fragments determined by the first ω regular functions. Let $\omega_0 = 1$ and $\omega_{n+1} = \omega^{\omega_n}$ for $n \in \mathbb{N}$. We have:

THEOREM 14.1. $|2^{n^x}| \leq \omega_{n+1}$ for $n \geq 1$. In particular $|2^{3^x}| \leq \omega_4 = \omega^{\omega^{\omega^\omega}}$.

Theorem 14.1 should be compared with Levitz's bound $|2^{2^x}| \leq \varepsilon_1$. As a consequence we obtain the following upper bound on 2^{x^x} , which improves Levitz's ε_ω bound.

THEOREM 14.2. $|2^{x^x}| \leq \varepsilon_0$.

A novel feature of our approach is the use of Conway's surreal numbers [7] for asymptotic calculations, justified by the fact that the Skolem functions can be embedded in the exponential field of surreal numbers, that is, one can associate a surreal number to each Skolem function preserving the field operations, exponentiation, and ordering. Our main result is as follows.

THEOREM 11.1. Let $c \geq 1$ be a surreal number and let Q be a Skolem function. The set of real numbers $r \in \mathbb{R}$ such that there is a Skolem function h satisfying $(h/Q)^c = r + o(1)$ has no accumulation points in \mathbb{R} .

The case $c = 1$ of the theorem says that, if we fix a Skolem function $Q(x)$, the set of real numbers r such that there is a Skolem function $h(x)$ with $\lim_{x \rightarrow \infty} h(x)/Q(x) = r$, has no accumulation points in \mathbb{R} . This special case is sufficient to obtain the bounds above, and also yields a different proof of the upper bound in [10]. It turns out that, for technical reasons, we need to consider the general case $c \geq 1$ in order to prove the special case $c = 1$.

In the preliminary part of the paper, we prove a result concerning the order type of the set of finite sums $\sum A$ of a well ordered set A of positive elements of an ordered group (**Theorem 4.5**). Unlike the known bounds by Carruth [6] and other authors, our bound takes into account the set of archimedean classes of A .

The equality of two Skolem functions (given the defining expressions) is decidable [22], but it is an open problem whether the order $<$ is decidable. Gurevič [15] established the decidability of $<$ below 2^{x^2} and showed that the decidability of $<$ below 2^{2^x} is Turing equivalent to the decidability of the equality of two "exponential constants," where the exponential constants are the elements in the smallest subset $\mathbb{E}^+ \subset \mathbb{R}$ containing 1 and closed under addition, multiplication, division, and the

real exponential function. In [10] van den Dries and Levitz proved that if the quotient f/g of two Skolem functions smaller than 2^{2^x} tends to a limit in \mathbb{R} , then the limit is in \mathbb{E}^+ . They announced that the result could be extended to the whole class of Skolem functions using the work of [8], where a version of the field of transseries made its first appearance. In the last part of the paper we give a proof of these facts using surreal numbers.

§2. Asymptotic relations. Given f, g in an ordered abelian group, we write $f \preceq g$ if $|f| \leq n|g|$ for some $n \in \mathbb{N}$. We say in this case that f is *dominated* by g . If both $f \preceq g$ and $g \preceq f$ hold, we say that f and g belong to the same *archimedean class*, and we write $f \asymp g$. We say that f is *strictly dominated* by g if we have both $f \preceq g$ and $f \not\asymp g$; we write $f \prec g$ to express this relation. We define $f \sim g$ as $f - g \prec f$ and we say in this case that f is *asymptotic* to g . Notice that \sim is a symmetric relation. Indeed assume $f - g \prec f$ and let us prove that $f - g \prec g$. This is clear if $f \preceq g$. On the other hand if $g \prec f$, then clearly $f - g \asymp f$, contradicting the assumption.

We write $f = o(g)$ if $f \prec g$ and $f = O(g)$ if $f \preceq g$.

The set of germs at $+\infty$ of the Skolem functions generates an ordered field by the results of [17] or [26] cited in the introduction, so we can use the above notations for the Skolem functions. By the cited results, the quotient $f(x)/g(x)$ of two Skolem functions tends to a limit in $\mathbb{R} \cup \{+\infty\}$ for $x \rightarrow +\infty$. We then have $f \prec g$ if f/g tends to 0; $f \sim g$ if $f(x)/g(x)$ tends to 1; and $f \asymp g$ if f/g tends to a nonzero limit in \mathbb{R} . Note that $f \asymp g$ if and only if there is a nonzero real number r such that $f \sim rg$. We will prove as a special case of [Theorem 11.1](#) that, if we fix g and let f vary in Sk , then the corresponding real r ranges in a subset of \mathbb{R} without accumulation points.

§3. Ordinal arithmetic. Let \mathbf{On} be the class of all ordinal numbers. Given $\alpha \in \mathbf{On}$ and $\beta \in \mathbf{On}$, we write $\alpha + \beta$ and $\alpha\beta$ (or sometimes $\alpha \cdot \beta$) for the ordinal sum and product of the given ordinals, and α^β for the ordinal exponentiation. We identify each ordinal with the set of its predecessor and we denote by ω the first infinite ordinal, which can also be thought as the set of all finite ordinals, that is, the set of natural numbers \mathbb{N} .

DEFINITION 3.1. Given a sequence $(\alpha_i)_i$ of ordinals, we define inductively:

- (1) $\sum_{i<0} \alpha_i = 0$;
- (2) $\sum_{i<\beta+1} \alpha_i = \sum_{i<\beta} \alpha_i + \alpha_\beta$;
- (3) $\sum_{i<\lambda} \alpha_i = \sup_{\beta<\lambda} \sum_{i<\beta} \alpha_i$ for λ a limit ordinal.

We recall that every ordinal α can be written in a unique way in the form $\alpha = \sum_{i<n} \omega^{\gamma_i} n_i$ where $n \in \mathbb{N}$, $(\gamma_i)_{i<n}$ is a decreasing sequence of ordinals, and $n_i \in \mathbb{N}^{>0}$ for each $i < n$. This is called the Cantor normal form of α .

We write $\alpha \oplus \beta$ and $\alpha \odot \beta$ for the Hessenberg sum and product [24]. We recall the definitions below.

DEFINITION 3.2. Given $\alpha \in \mathbf{On}$ and $\beta \in \mathbf{On}$, we can find $k \in \mathbb{N}$ and a decreasing finite sequence of ordinals $(\gamma_i)_{i<k}$ such that $\alpha = \sum_{i<k} \omega^{\gamma_i} m_i$ and $\beta = \sum_{i<k} \omega^{\gamma_i} n_i$

with $m_i, n_i < \omega$ (possibly zero). We define

$$\alpha \oplus \beta = \sum_{i < k} \omega^{\gamma_i} (m_i + n_i).$$

DEFINITION 3.3. If $\alpha = \sum_{i < k} \omega^{\alpha_i} m_i$ and $\beta = \sum_{i < l} \omega^{\beta_j} n_j$ are two ordinals in Cantor normal form, their Hessenberg product is defined as

$$\alpha \odot \beta = \bigoplus_{i < k, j < l} \omega^{\alpha_i \oplus \beta_j} m_i n_j.$$

We shall need transfinite iterations of the Hessenberg sum and product.

DEFINITION 3.4. Given a sequence of ordinals $(\alpha_i)_i$ we define inductively:

- (1) $\bigoplus_{i < 0} \alpha_i = 0$,
- (2) $\bigoplus_{i < \beta + 1} \alpha_i = (\bigoplus_{i < \beta} \alpha_i) \oplus \alpha_\beta$,
- (3) $\bigoplus_{i < \lambda} \alpha_i = \sup_{\beta < \lambda} \bigoplus_{i < \beta} \alpha_i$ for λ limit.

The paper [20] contains some comparison results between $\sum_{i < \beta}$ and $\bigoplus_{i < \beta}$. Similarly we define the transfinite iteration of the Hessenberg product.

DEFINITION 3.5. Given a sequence of ordinals $(\alpha_i)_i$ we define inductively:

- (1) $\bigodot_{i < 0} \alpha_i = 1$,
- (2) $\bigodot_{i < \beta + 1} \alpha_i = (\bigodot_{i < \beta} \alpha_i) \odot \alpha_\beta$,
- (3) $\bigodot_{i < \lambda} \alpha_i = \limsup_{\beta < \lambda} \bigodot_{i < \beta} \alpha_i$ for λ limit.

DEFINITION 3.6. Given two ordinals α and β we define $\alpha^{\odot \beta} = \bigodot_{i < \beta} \alpha$.

PROPOSITION 3.7. If $n < \omega$, then $n^{\odot \gamma} = n^\gamma$ for every $\gamma \in \mathbf{On}$.

PROOF. We can assume $n > 1$. Write $\gamma = \omega\beta + k$ with $\beta \in \mathbf{On}$ and $k < \omega$. Since $n < \omega$, $n^\omega = \omega$, and therefore $n^{\omega\beta + k} = \omega^\beta n^k$. On the other hand by [1, Lemma 3.6] we have $n^{\odot \omega\beta + k} = \omega^\beta n^k = n^{\omega\beta + k}$, thus concluding the proof. \dashv

LEMMA 3.8. If $\alpha \geq \beta$, then $\alpha \oplus \beta \leq \alpha + \beta 2$.

PROOF. We can assume $\beta > 0$. Let $\alpha = \sum_{i < k} \omega^{\alpha_i} m_i$ and $\beta = \sum_{j < l} \omega^{\beta_j} n_j$ be Cantor normal forms. For some $i_0 \leq k$, $\alpha \oplus \beta$ has the form $\sum_{i < i_0} \omega^{\alpha_i} m_i + \omega^{\beta_0} n_0 + \rho$ with $\rho < \omega^{\beta_0} \leq \beta$. Since $\omega^{\beta_0} n_0 \leq \beta$, we obtain $\alpha \oplus \beta \leq \alpha + \beta + \beta = \alpha + \beta 2$. \dashv

LEMMA 3.9. $\alpha\beta \leq \bigoplus_{i < \beta} \alpha \leq \alpha 2\beta$.

PROOF. By induction on β based on Lemma 3.8. The case when β is zero or a limit ordinal follows at once from the induction hypothesis. If $\beta = \gamma + 1$, then $\alpha(\gamma + 1) \leq \bigoplus_{i < \gamma + 1} \alpha = (\bigoplus_{i < \gamma} \alpha) \oplus \alpha \leq \alpha 2\gamma \oplus \alpha \leq \alpha 2\gamma + \alpha 2 = \alpha 2(\gamma + 1)$, where we used Lemma 3.8 and the induction hypothesis. \dashv

COROLLARY 3.10. If λ is limit, then $\bigoplus_{i < \lambda} \alpha = \alpha\lambda$.

PROOF. If β is limit, then $2\beta = \beta$, so we can conclude by Lemma 3.9. \dashv

Let α be an ordinal. We say that α is *additively closed* if the sum of two ordinals less than α is less than α . Similarly, α is *multiplicatively closed* if the product of two ordinals less than α is less than α . We obtain an equivalent definition using the Hessenberg sum and product. The additively closed ordinals > 0 are the ordinals of the form ω^δ for some δ ; the multiplicatively closed ordinals > 1 are the ordinals of the form ω^{ω^δ} for some δ [24].

PROPOSITION 3.11. *If $\alpha \in \mathbf{On}$ and λ is a limit ordinal, then $\alpha^{\odot\lambda} = \alpha^\lambda$. Moreover α^λ is additively closed.*

PROOF. The case $\alpha < \omega$ follows from Proposition 3.7. Assume $\alpha \geq \omega$ and consider first the special case $\alpha = \omega^\gamma$. For every β it is easy to verify by induction that $(\omega^\gamma)^{\odot\beta} = \odot_{i<\beta} \omega^\gamma = \omega^{\oplus_{i<\beta} \gamma}$. Now take $\beta = \lambda$. Since λ is limit, by Corollary 3.10, $\oplus_{i<\lambda} \gamma = \gamma\lambda$, so $(\omega^\gamma)^{\odot\lambda} = \omega^{\gamma\lambda}$.

For a general $\alpha \geq \omega$, let $\delta > 0$ be such that $\omega^\delta \leq \alpha < \omega^{\delta+1}$. Since λ is limit, $(\delta + 1)\lambda = \delta\lambda$. The result now follows from the inequalities $\alpha^{\odot\lambda} \leq (\omega^{\delta+1})^{\odot\lambda} = \omega^{(\delta+1)\lambda} = \omega^{\delta\lambda} \leq \alpha^\lambda \leq \alpha^{\odot\lambda}$. \dashv

COROLLARY 3.12. *For $\beta \in \mathbf{On}$, let $\beta = \lambda + k$ with λ a limit ordinal or zero and $k < \omega$. Then $\alpha^{\odot\beta} = \alpha^\lambda \odot \alpha^{\odot k}$.*

§4. Well ordered subsets of ordered groups. The Hessenberg sum and product can be characterized as follows. Consider disjoint well ordered sets A and B of order type α and β respectively. By [6] or [12] the Hessenberg sum $\alpha \oplus \beta$ is the sup of all ordinals γ such that one can extend the given partial order on $A \cup B$ to a total order of order type γ ; the Hessenberg product $\alpha \odot \beta$ is the sup of all ordinals γ such that one can extend the componentwise partial order on $A \times B$ to a total order of order type γ . By the cited papers, the sups are achieved. An immediate consequence of the above characterization is the following:

FACT 4.1. *Let $X = (X, <)$ be a totally ordered set and let $A, B \subseteq X$ be well ordered subsets. We have:*

- (1) $A \cup B$ is well ordered and $|A \cup B| \leq |A| \oplus |B|$.
- (2) Let $f : X \times X \rightarrow X$ be a binary function which is weakly increasing in both arguments and let $f(A, B) := \{f(a, b) : a \in A, b \in B\}$. Then $f(A, B)$ is well ordered and $|f(A, B)| \leq |A| \odot |B|$.

Given two sets A and B of Skolem functions we write: $A + B$ for the set of all sums $f + g$ with $f \in A$ and $g \in B$; AB for the set of all products fg with $f \in A$ and $g \in B$; A^B for the set of all functions of the form f^g with $f \in A$ and $g \in B$. We write A/\asymp for the ordered set of all \asymp classes of elements of A , and similarly for A/\sim .

COROLLARY 4.2. *Let A and B be sets of Skolem functions. Then:*

- (1) $A \cup B$ has of order type $\leq |A| \oplus |B|$.
- (2) $A + B$, AB and A^B have order type $\leq |A| \odot |B|$.
- (3) AB/\asymp has order type $\leq |A/\asymp| \odot |B/\asymp|$.

PROOF. The first two points are immediate from Fact 4.1. To prove point (3) we use again Fact 4.1 together with the observation that the \asymp -class of fg depends only on the respective \asymp -classes of f and g , and this dependence is weakly increasing in both arguments. ⊖

Given a subset $A \subset Sk$, we write $\sum A$ for the set of finite nonempty sums of elements from A . We want to give an upper bound on $|\sum A|$. The definition of $\sum A$ can be given more generally for a subset A of an ordered abelian group G , so it is convenient to work in this context. If A is a well ordered subset of $G^{>0}$, $\sum A$ is well ordered and Carruth [6] gave an upper bound on its order type in terms of the order type of A . In Theorem 4.5 we obtain a different bound which takes into account the set of archimedean classes of A .

LEMMA 4.3. *Let $(G, +, <)$ be an ordered abelian group and let $A \subseteq G^{>0}$ be a well ordered subset of order type α . Suppose all the elements of A belong to distinct archimedean classes. Then the order type of $\sum A$ is $\leq \omega^\alpha$.*

PROOF. Let $(a_i : i < \alpha)$ be an increasing enumeration of A . Let $x \in \sum A$. Then x can be written uniquely in the form $x = \sum_{i < \alpha} a_i n_i$ where $n_i \in \mathbb{N}$ and $n_i = 0$ for all but finitely many i . We associate to x the ordinal $\bigoplus_{i < \alpha} \omega^i n_i$. This defines an increasing map from $\sum A$ to ω^α yielding the desired result. ⊖

LEMMA 4.4. *Let $(G, +, <)$ be an ordered abelian group and let $A \subseteq G^{>0}$ be a well ordered subset of order type $\alpha \geq 2$. Suppose all the elements of A belong to the same archimedean class. Then*

$$|\sum A| \leq \alpha^\omega.$$

(If $|A| \leq 1$, clearly $|\sum A| \leq \omega$.)

PROOF. Let $b \in \sum A$, let $(\sum A)^{<b}$ be the set of elements less than b in $\sum A$. Since all elements of A belong to the archimedean class of its least element, there exists $m \in \mathbb{N}$, depending on b , such that every element of $(\sum A)^{<b}$ is the sum of at most m elements of A . By induction on $i \leq m$ using Corollary 4.2, the set of sums of i elements of A has order type $\leq \alpha^{\odot i}$. By the same corollary it then follows by induction on m that $|(\sum A)^{<b}| \leq \bigoplus_{i=1}^m \alpha^{\odot i}$. Now for each $i \leq m$, $\alpha^{\odot i} < \alpha^{\odot \omega}$ and $\alpha^{\odot \omega} = \alpha^\omega$ is additively closed (Proposition 3.11). It follows that $|(\sum A)^{<b}| < \alpha^\omega$. Since this holds for every $b \in \sum A$, we can conclude that $|\sum A| \leq \alpha^\omega$. ⊖

THEOREM 4.5. *Let $(G, +, <)$ be an ordered abelian group, let $A \subseteq G^{>0}$ be a well ordered set of order type $\alpha \geq 2$ and let $\beta = |A/\asymp|$ be the order type of the set of archimedean classes of A . Then the order type of $\sum A$ is $\leq (\alpha^\omega)^{\odot \beta}$.*

PROOF. Let $B \subseteq A$ be a set of representatives for the archimedean classes of A and let $(b_i : i < \beta)$ be an increasing enumeration of B . We reason by induction on β . The case $\beta = 1$ is Lemma 4.4.

Case β limit. For $b \in G^{>0}$, let $A^{\preceq b}$ be the subset of A consisting of the elements $\preceq b$ and let $A^{\succ b}$ be the set of elements of A which are $\succ b$. Then $\sum A = \bigcup_{\gamma < \beta} \sum(A^{\preceq b_\gamma})$. The sets in the union are pairwise initial segments of one another. It follows that the order type of the union is the sup of the respective order types. By induction $|\sum A| \leq \sup_{\gamma < \beta} (\alpha^\omega)^{\odot \gamma} = (\alpha^\omega)^{\odot \beta}$.

Case $\beta = \gamma + 1$. We have $\sum A = \sum(A^{\preceq b_\gamma}) + \sum(A^{\succ b_\gamma})$. By the induction hypothesis $|\sum(A^{\preceq b_\gamma})| \leq (\alpha^\omega)^{\odot \gamma}$. The elements of $A^{\succ b_\gamma}$ live in a single archimedean class, so $|\sum(A^{\succ b_\gamma})| \leq \alpha^\omega$. It follows that $|\sum A| \leq (\alpha^\omega)^{\odot \gamma} \odot \alpha^\omega = (\alpha^\omega)^{\odot (\gamma+1)}$. \dashv

We define a sequence of countable ordinals as follows.

DEFINITION 4.6. Let $\omega_0 = 1$ and, inductively, $\omega_{n+1} = \omega^{\omega_n}$.

REMARK 4.7. For all $n \in \mathbb{N}$, ω_n is multiplicatively closed.

PROOF. Clearly the product of two ordinals < 1 is < 1 , so the property holds for $n = 0$. For $n \geq 1$, ω_n has the form ω^{ω^δ} (e.g. $\omega_1 = \omega = \omega^{\omega^0}$ and $\omega_2 = \omega^\omega = \omega^{\omega^1}$), so it is multiplicatively closed. \dashv

For our applications we need the following lemma.

LEMMA 4.8. Let $2 \leq n < \omega$. If $\alpha < \omega_{n+1}$ and $\beta < \omega_n$, then $(\alpha^\omega)^{\odot \beta} < \omega_{n+1}$.

PROOF. We can write $\beta = \lambda + k$ where λ is a limit ordinal or zero and $k < \omega$. By Corollary 3.12 we have $(\alpha^\omega)^{\odot \beta} = \alpha^{\omega \lambda} \odot (\alpha^\omega)^{\odot k}$. Since $\alpha < \omega_{n+1} = \omega^{\omega_n}$ and ω_n is a limit ordinal, there is some $\gamma < \omega_n$ such that $\alpha \leq \omega^\gamma$. Since ω and λ are $< \omega_n$ and ω_n is multiplicatively closed, we have $\omega \lambda < \omega_n$, hence $\alpha^{\omega \lambda} < \omega_{n+1}$. Similarly, $(\alpha^\omega)^{\odot k} = \alpha^{\omega \odot k} < \omega_{n+1}$. Now since ω_{n+1} is multiplicatively closed, $\alpha^{\omega \lambda} \odot (\alpha^\omega)^{\odot k} < \omega_{n+1}$, as desired. \dashv

COROLLARY 4.9. Let $(G, +, <)$ be an ordered abelian group, let $A \subseteq G^{>0}$ be a well ordered set of order type $< \omega_{n+1}$ whose set of archimedean classes has order type $< \omega_n$. Then the order type of $\sum A$ is $< \omega_{n+1}$ and its set of archimedean classes has order type $< \omega_n$.

PROOF. By Lemma 4.8 and Theorem 4.5, together with the observation that the set of archimedean classes does not change under taking finite sums. \dashv

Another interesting bound on $|\sum A|$ is contained in [9]: if $|A| \leq \alpha$, then $|\sum A| \leq \omega^{\omega \alpha}$. For our purposes we need the bound in Corollary 4.9 which takes into account also the order type of the archimedean classes of A . Note that both bounds imply that if $|A| < \varepsilon_0$, then $|\sum A| < \varepsilon_0$.

§5. Generalized power series. Given an ordered field K , a multiplicative subgroup \mathfrak{M} of $K^{>0}$ is called a group of *monomials* if for each nonzero element x of K there is one and only one $m \in \mathfrak{M}$ such that $x \asymp m$. We assume some familiarity with Hahn’s

field $\mathbb{R}((\mathfrak{M}))$ of generalized power series [16], but we recall a few definitions. An element of $\mathbb{R}((\mathfrak{M}))$ is a formal sum $f = \sum_{i < \alpha} m_i r_i$ where α is an ordinal, $(m_i)_{i < \alpha}$ is a decreasing sequence in \mathfrak{M} , and $0 \neq r_i \in \mathbb{R}$ for each $i < \alpha$. We say that $\{m_i \mid i < \alpha\}$ is the *support* of the series $\sum_{i < \alpha} m_i r_i$. The sum and product of generalized series is defined in the obvious way. We order $\mathbb{R}((\mathfrak{M}))$ by $f = \sum_{i < \alpha} m_i r_i > 0 \iff r_0 > 0$. This makes $\mathbb{R}((\mathfrak{M}))$ into an ordered field with \mathfrak{M} as a group of monomials (where we identify $m \in \mathfrak{M}$ with $m1 \in \mathbb{R}((\mathfrak{M}))$).

A family $(f_i)_{i \in I}$ of elements of $\mathbb{R}((\mathfrak{M}))$ is *summable* if each monomial $m \in \mathfrak{M}$ is contained in the support of finitely many f_i and there is no strictly increasing sequence $(m_n)_{n \in \mathbb{N}}$ of monomials such that each m_n belongs to the support of some f_i . In this case $\sum_{i \in I} f_i \in \mathbf{No}$ is defined adding the coefficients of the corresponding monomials.

To prove that $\mathbb{R}((\mathfrak{M}))$ is a field, we write a nonzero element x of $\mathbb{R}((\mathfrak{M}))$ in the form $rm(1 + \varepsilon)$ with $r \in \mathbb{R}^*$, $m \in \mathfrak{M}$ and $\varepsilon \prec 1$ and observe that $x^{-1} = r^{-1}m^{-1}(\sum_{n \in \mathbb{N}} (-1)^n \varepsilon^n)$ where the summability of $(-1)^n \varepsilon^n$ is ensured by Neumann's Lemma [21]. More generally Neumann's Lemma says that if ε is an infinitesimal element of $\mathbb{R}((\mathfrak{M}))$ and $(r_n)_{n \in \mathbb{N}}$ is any sequence of real numbers, then $(r_n \varepsilon^n)_{n \in \mathbb{N}}$ is summable, so we can evaluate the formal power series $P(X) = \sum_{n \in \mathbb{N}} r_n X^n$ at any infinitesimal element of $\mathbb{R}((\mathfrak{M}))$.

Given f and g in $\mathbb{R}((\mathfrak{M}))$ we say that g is a *truncation* of f if $f = \sum_{i < \alpha} m_i r_i$ and $g = \sum_{i < \beta} m_i r_i \in \mathbb{R}((\mathfrak{M}))$ for some $\beta < \alpha$. If $\mathfrak{G} \subseteq \mathfrak{M}$ is a subset, we write $\mathbb{R}((\mathfrak{G}))$ for the set of all $f \in \mathbb{R}((\mathfrak{M}))$ whose support is contained in \mathfrak{G} .

§6. Surreal numbers. Conway's field \mathbf{No} of surreal numbers [7, 14] is an ordered real closed field extending the field \mathbb{R} of real numbers and containing a copy of the ordinal numbers. In particular \mathbf{No} is a proper class, and admits a group of monomials $\mathfrak{M} \subset \mathbf{No}^{>0}$ which is itself a proper class. We can define generalized power series with monomials in \mathfrak{M} exactly as above, but we denote the resulting field as $\mathbb{R}((\mathfrak{M}))_{\mathbf{On}}$, where the subscript is meant to emphasize that, although \mathfrak{M} is a proper class, the support of a generic element $\sum_{i < \alpha} m_i r_i$ of \mathbf{No} is a set (because α is still assumed to be an ordinal). Conway [7] showed that we can identify \mathbf{No} with $\mathbb{R}((\mathfrak{M}))_{\mathbf{On}}$, where the class $\mathfrak{M} \subset \mathbf{No}$ of monomials is defined explicitly (it coincides with the image of Conway's omega-map).

A surreal $x = \sum_{i < \alpha} m_i r_i \in \mathbb{R}((\mathfrak{M}))_{\mathbf{On}}$ is *purely infinite* if all monomials m_i in its support are > 1 (hence infinite). We write \mathbf{No}^\uparrow for the (nonunitary) ring of purely infinite surreals. We observe that every $x \in \mathbf{No}$ can be written in a unique way in the form $x = x^\uparrow + x^\circ + x^\downarrow$ where $x^\uparrow \in \mathbf{No}^\uparrow$, $x^\circ \in \mathbb{R}$ and $x^\downarrow \prec 1$. This yields a direct sum decomposition of \mathbb{R} -vector spaces

$$\mathbf{No} = \mathbf{No}^\uparrow + \mathbb{R} + o(1),$$

where $o(1)$ is the set of elements $\prec 1$. Gonshor [14] defined an isomorphism of ordered groups $\exp : (\mathbf{No}, +, <) \rightarrow (\mathbf{No}^{>0}, \cdot, <)$ extending the real exponential function and satisfying $\exp(x) \geq 1 + x$ for all $x \in \mathbf{No}$ and $\exp(x) = \sum_{n \in \mathbb{N}} \frac{x^n}{n!}$ for $x \prec 1$ (we need $x \prec 1$ to ensure the summability of the series). Gonshor's \exp is

defined in such a way that $\exp(\mathbf{No}^\uparrow) = \mathfrak{M}$, namely the monomials are the images of the purely infinite numbers. The stated properties are already sufficient to ensure that \mathbf{No} , with Gonshor’s \exp , is a model of the elementary theory T_{\exp} of the real exponential field $\mathbb{R}_{\exp} = (\mathbb{R}, <, +, \cdot, \exp)$; in other words (\mathbf{No}, \exp) satisfies all the property which are true in \mathbb{R}_{\exp} and are expressible by a first-order formula in the ring language and a symbol for the exponential function [9]. A discussion of these issues can also be found in [5], where other fields of generalized power series admitting an exponential map resembling the surreal \exp have been considered.

As long as we are only interested in the elementary theory of \mathbf{No} as an exponential field, both the choice of the monomials $\mathfrak{M} \subset \mathbf{No}$ and the details of the definition of \exp on \mathbf{No}^\uparrow are not important. However they become important for summability issues and the properties of infinite sums, so we need to state a few more facts that are needed in this paper (all of them can be found in [4]). We denote by $\log : \mathbf{No}^{>0} \rightarrow \mathbf{No}$ the compositional inverse of \exp and we also write e^x for $\exp(x)$. It can be shown that if $x \prec 1$, then $\log(1 + x) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} x^n$. An important fact, that depends on the choice of $\mathfrak{M} \subset \mathbf{No}^{>0}$, is that ω is a monomial (where ω is the least infinite ordinal seen as a surreal). More generally, for each $n \in \mathbb{N}$, $\log_n(\omega)$ is an infinite monomial [14], where $\log_0(\omega) = \omega$ and $\log_{n+1}(\omega) = \log(\log_n(\omega))$. This fact is used in [4] to show that \mathbf{No} contains an isomorphic copy of the field \mathbb{T} of transseries as an exponential field (the notation \mathbb{T} is used in [2] and refers to the version of the transseries defined [11] under the name “logarithmic exponential series.”) Moreover \mathbf{No}^\uparrow admits a differential operator $\partial : \mathbf{No} \rightarrow \mathbf{No}$ extending the one on \mathbb{T} [3, 4]. Since \mathbf{No}^\uparrow is closed under multiplication by a real number, any real power $m^r = e^{r \log(m)}$ of a monomial is again a monomial. Moreover, if m is an infinite monomial, e^m is again a monomial (because $\exp(\mathbf{No}^\uparrow) = \mathfrak{M}$).

From the equations $\mathbf{No} = \mathbb{R}((\mathfrak{M}))_{\text{On}}$ and $\mathfrak{M} = e^{\mathbf{No}^\uparrow}$ it follows that every surreal can be written in a unique way in the form $\sum_{i < \alpha} e^{\gamma_i} a_i$ where α is an ordinal, $(\gamma_i)_{i < \alpha}$ is a decreasing sequence in \mathbf{No}^\uparrow and $a_i \in \mathbb{R}^*$ (the empty sum is 0). Following [3], we call this representation *Ressayre form*.

§7. Surreal expansions of Skolem functions. Since the surreal numbers are a model of T_{\exp} there is a unique map from Sk to \mathbf{No} sending the identity function \mathbf{x} into ω and preserving $1, +, \cdot$ and the function $(a, b) \mapsto a^b$ where $a^b = e^{b \log(a)}$. Since ω is greater than any natural number, this map preserves the order, so it is an embedding of ordered semirings endowed with an exponential $(a, b) \mapsto a^b$. We identify a Skolem function $f = f(\mathbf{x}) \in Sk$ with its image $f(\omega) \in \mathbf{No}$ under this embedding, and we define the *Ressayre form of $f \in Sk$* as the Ressayre form of the surreal number $f(\omega)$.

If we identify the transseries \mathbb{T} with a subfield of \mathbf{No} (as in [4]), it is easy to see that the image of the embedding of Sk in \mathbf{No} is contained in \mathbb{T} , but we shall not need this fact.

We can consider the Ressayre form of a Skolem function $f(x)$ as an asymptotic development for $x \rightarrow +\infty$. For example consider the Skolem function $(\mathbf{x} + 1)^\mathbf{x}$ and identify \mathbf{x} with $\omega \in \mathbf{No}$. To find its Ressayre form we write $(\mathbf{x} + 1)^\mathbf{x} = e^{\mathbf{x} \log(1+\mathbf{x})}$ and

we expand $\log(1 + \mathbf{x})$ as follows

$$\begin{aligned} \log(1 + \mathbf{x}) &= \log(\mathbf{x}(1 + \mathbf{x}^{-1})) \\ &= \log(\mathbf{x}) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathbf{x}^{-n} \\ &= \log(\mathbf{x}) + \mathbf{x}^{-1} - \mathbf{x}^{-2}/2 + \dots \end{aligned}$$

Now, using $\mathbf{x}^{\mathbf{x}} = e^{\mathbf{x}\log(\mathbf{x})}$ and $\exp(\varepsilon) = \sum_{n \in \mathbb{N}} \varepsilon^n/n! = 1 + \varepsilon + \dots$ for $\varepsilon \prec 1$, we obtain

$$\begin{aligned} (\mathbf{x} + 1)^{\mathbf{x}} &= \exp(\mathbf{x} \log(\mathbf{x}) + 1 - \mathbf{x}^{-1}/2 + \dots) \\ &= e\mathbf{x}^{\mathbf{x}}(1 - \mathbf{x}^{-1}/2 + \dots) \\ &= e\mathbf{x}^{\mathbf{x}} - e2^{-1}\mathbf{x}^{\mathbf{x}-1} + \dots \end{aligned}$$

Replacing \mathbf{x} with ω we find the Ressayre form of the surreal $(\omega + 1)^{\omega}$.

§8. Finer asymptotic relations. The results in this section are stated for **No** but they hold more generally in every model of T_{exp} . We identify Sk as a subset of **No** as discussed in the previous section. In particular $\mathbf{x} = \omega \in \mathbf{No}$.

DEFINITION 8.1. Let $1 \leq c \in \mathbb{K}$. Given two positive surreals f and g we define $f \sim_c g$ if $f^c \sim g^c$ and $f \succ_c g$ if $f^c \succ g^c$.

When $c = 1$, the relations \sim_c and \succ_c become the usual \sim and \succ relations. When $c > 1$ we obtain finer equivalence relations. One of the main ideas of this paper is to try to understand how many classes modulo \sim_c there are inside a class modulo \succ_c . We are primarily interested in the case $c = 1$, but we need to consider the general case to carry out the induction. In our terminology, the paper of [10] deals with the case when c is equal \mathbf{x}^n for some $n \in \mathbb{N}$, but we need to follow a different approach to be able to generalize it. A consequence of our main result (**Theorem 11.1**) is that the set of \sim_c -classes within any class modulo \succ_c has order type $\leq \omega$.

In this section we establish some basic properties of \sim_c and \succ_c . In particular we show that $f \succ_c g \iff c(f - g) \preceq g$ and $f \sim_c g \iff c(f - g) \prec g$, yielding a characterization of these relations which does not depend on the exponential function.

LEMMA 8.2. For any $t \in \mathbb{K}$ we have $t \preceq 1$ if and only if $e^t \succ 1$.

PROOF. We have $t \preceq 1$ if and only if there is $k \in \mathbb{N}$ such that $-k \leq t \leq k$. This happens if and only if $e^{-k} \leq e^t \leq e^k$ for some $k \in \mathbb{N}$, or equivalently $e^t \succ 1$ (because $e^{-k} \succ 1 \succ e^k$). ⊥

LEMMA 8.3. For $t \in \mathbb{K}$ we have $t \prec 1$ if and only if $e^t \sim 1$.

PROOF. We have $t \prec 1$ if and only if $-1/k \leq t \leq 1/k$ for all positive $k \in \mathbb{N}$. This happens if and only if $e^{-1/k} \leq e^t \leq e^{1/k}$ for all positive $k \in \mathbb{N}$, or equivalently $e^t \sim 1$ (because $|e^t - 1| \leq |e^{1/k} - e^{-1/k}|$ and $|e^{1/k} - e^{-1/k}|$ becomes smaller than any positive real for k sufficiently large). ⊥

PROPOSITION 8.4. *Let $c \geq c' \geq 1$ and let $f, g > 0$.*

- (1) *If $f \asymp_c g$, then $f \asymp_{c'} g$.*
- (2) *If $f \sim_c g$, then $f \sim_{c'} g$.*

In particular, if $f \asymp_c g$, then $f \asymp g$.

PROOF. We first observe that, for $z \in \mathbb{K}^{>0}$ and $d \in \mathbb{K}^{\geq 1}$, we have $z < 1 \implies z^d < z$ and $z > 1 \implies z^d > z$, so in any case $|z - 1| \leq |z^d - 1|$. Taking $z = f/g$ and $d = c/c'$, we deduce that $|(f/g)^{c'} - 1| \leq |(f/g)^c - 1|$. Thus if $f \sim_c g$, then $f \sim_{c'} g$. This proves (2).

To prove (1) assume that $f \asymp_c g$ and let $r \in \mathbb{R}^{>0}$ be such that $(f/g)^c \sim r$. Now observe that $(f/g)^{c'}$ is between 1 and $(f/g)^c \sim r > 0$, hence it is asymptotic to a positive real. ⊖

LEMMA 8.5. *For $c \geq 1$ and $z > 0$, we have*

- (1) $z^c \asymp 1 \iff z = 1 + O(1/c)$;
- (2) $z^c \sim 1 \iff z = 1 + o(1/c)$.

PROOF. The case $c \leq 1$ can be reduced the case $c = 1$ using Proposition 8.4. If $c > 1$, we can assume $z \sim 1$, as otherwise both sides of either equivalence are false. We can thus write $z = 1 + \varepsilon$ for some $\varepsilon \prec 1$. The results follow from the following chains of equivalences.

$$\begin{array}{ll}
 z^c \asymp 1 & z^c \sim 1 \\
 \iff e^{c \log(1+\varepsilon)} \asymp 1 & \iff e^{c \log(1+\varepsilon)} \sim 1 \\
 \iff c \log(1 + \varepsilon) \preceq 1 & \iff c \log(1 + \varepsilon) \prec 1 \\
 \iff \log(1 + \varepsilon) \preceq 1/c & \iff \log(1 + \varepsilon) \prec 1/c \\
 \iff \varepsilon \preceq 1/c & \iff \varepsilon \prec 1/c
 \end{array}$$

where in the last step of both columns we used $\log(1 + \varepsilon) \sim \varepsilon$ (which follows from $\varepsilon \prec 1$). ⊖

PROPOSITION 8.6. *For $c \geq 1$ and $f, g > 0$, we have:*

- (1) $f \asymp_c g \iff c(f - g) \preceq g$;
- (2) $f \sim_c g \iff c(f - g) \prec g$.

PROOF. By Lemma 8.5 with $z = f/g$. ⊖

PROPOSITION 8.7. *Let $c > \mathbb{N}$, $z > 0$ and $r \in \mathbb{R}$. Then*

$$z^c \sim e^r \iff z = 1 + r/c + o(1/c).$$

PROOF. Let $z = 1 + r/c + o(1/c)$. Then $z^c = (1 + r/c + o(1/c))^c \sim e^r$. Conversely, assume $z^c \sim e^r$. Then in particular $z^c \asymp 1$. By Lemma 8.5, $z = 1 + O(1/c)$ so we can write $z = 1 + s/c + o(1/c)$ for some $s \in \mathbb{R}$. By the previous part $e^s \sim e^r$, hence $s = r$. ⊖

PROPOSITION 8.8. *Let $c \geq 1$ and $f, g, a, b > 0$.*

- (1) *if $f \asymp_c a$ and $g \asymp_c b$, then $fg \asymp_c ab$ and $f + g \asymp_c a + b$;*
- (2) *If $f \sim_c a$ and $g \sim_c b$, then $fg \sim_c ab$ and $f + g \sim_c a + b$.*

PROOF. Assume $f \asymp_c a$ and $g \asymp_c b$. By Proposition 8.6, $c(f - a) \preceq a$ and $c(g - b) \preceq b$. Since a, b are positive, $c(f - a) \preceq a + b$ and $c(g - b) \preceq a + b$. It follows that $c(f + g - (a + b)) \preceq a + b$, hence $f + g \asymp_c a + b$.

In order to prove $fg \asymp_c ab$ we recall that $f \asymp_c a$ means $f^c \asymp a^c$ and $g \asymp_c b$ means $g^c \asymp b^c$. Multiplying we obtain $(fg)^c \asymp (ab)^c$.

The proof of second part is essentially the same: it suffices to replace \preceq with \prec and \asymp_c with \sim_c . ⊣

§9. The support of a Skolem function. We consider Sk as a substructure of $\mathbf{No} = \mathbb{R}((\mathfrak{M}))_{\mathbf{On}}$ through the embedding induced by the identification $\mathbf{x} = \omega$. Given $f \in Sk$, we can then write $f = \sum_{i < \alpha} m_i r_i$ with $\alpha \in \mathbf{On}$, $m_i \in \mathfrak{M}$ and $r_i \in \mathbb{R}^*$. It thus makes sense to consider the *support of a Skolem function*, that is, the set of monomials m_i which can appear in the above representation. We recall that a surreal number is an *omnific integer* if it belongs to the subring $\mathbf{No}^\uparrow + \mathbb{Z} \subset \mathbf{No}$. We show that every Skolem function is an omnific integer, so it does not have infinitesimal monomials in its support. More generally we prove that a monomial in the support of a Skolem function is either 1 or $\geq \mathbf{x}$ (so it cannot be $\log(\mathbf{x})$ or $\sqrt{\mathbf{x}}$, say). To this aim we first show that every Skolem function belongs to a subfield $\mathbb{K} \subset \mathbf{No}$ which is similar to the field of transseries defined in [11], but unlike the transseries it is not closed under \log , although it is closed under \exp .

DEFINITION 9.1. Let $\mathbf{x} = \omega \in \mathbf{No}$. Working inside \mathbf{No} we define

- (1) $\mathfrak{G}_0 = \mathbf{x}^{\mathbb{Z}}$ and $\mathbb{K}_0 = \mathbb{R}((\mathfrak{G}_0))$;
- (2) $\mathfrak{G}_{n+1} = e^{\mathbb{K}_n^\uparrow \mathbf{x}^{\mathbb{K}_n^\uparrow + \mathbb{Z}}} = e^{\mathbb{K}_n^\uparrow + \log(\mathbf{x})(\mathbb{K}_n^\uparrow + \mathbb{Z})}$ and $\mathbb{K}_{n+1} = \mathbb{R}((\mathfrak{G}_{n+1}))$.

Let $\mathfrak{G} = \bigcup_n \mathfrak{G}_n$ and let $\mathbb{K} = \bigcup_n \mathbb{K}_n \subseteq \mathbb{R}((\mathfrak{G}))$. Finally, let $\mathbb{K}^\uparrow = \mathbb{K} \cap \mathbf{No}^\uparrow$.

We recall that a subfield of \mathbf{No} is *truncation closed* if whenever it contains $\sum_{i < \alpha} m_i r_i$, it also contains its truncations $\sum_{i < \beta} m_i r_i$ for all $\beta < \alpha$. Since \mathbb{K} is an increasing union of the fields $\mathbb{R}((\mathfrak{G}_n))$, it is obviously a subfield of \mathbf{No} closed under truncations.

THEOREM 9.2. *\mathbb{K} is a truncation closed subfield of \mathbf{No} closed under \exp . If f and g are positive elements of the semiring $\mathbb{K}^\uparrow + \mathbb{N} \subset \mathbb{K}$, then $f^g = e^{g \log(f)} \in \mathbb{K}^\uparrow + \mathbb{N}$. It follows that $Sk \subseteq \mathbb{K}^\uparrow + \mathbb{N}$. In particular every Skolem function is an omnific integer.*

PROOF. For each $n \in \mathbb{N}$, \mathfrak{G}_n is a multiplicative group and therefore \mathbb{K}_n is a field. Moreover $\mathfrak{G}_0 \subseteq \mathfrak{G}_1$ and inductively $\mathfrak{G}_n \subseteq \mathfrak{G}_{n+1}$ and $\mathbb{K}_n \subseteq \mathbb{K}_{n+1}$. The fact that \mathbb{K} is a truncation closed subfield of $\mathbb{R}((\mathfrak{G}))$ is clear. To show that \mathbb{K} is closed under \exp , let $x \in \mathbb{K}$ and write $e^x = e^{x^\uparrow} e^{x^\circ} e^{x^\downarrow}$. Now it suffices to observe that $e^{x^\uparrow} \in \mathfrak{G}$, $e^{x^\circ} \in \mathbb{R}$ and $e^{x^\downarrow} = \sum_{n \in \mathbb{N}} (\mathbf{x}^\downarrow)^n / n! \in \mathbb{K}$. More generally \mathbb{K} is closed under the evaluation of a power series at an infinitesimal element. It remains to show that if a, b are positive elements of $\mathbb{K}^\uparrow + \mathbb{N}$, then $a^b \in \mathbb{K}^\uparrow + \mathbb{N}$. ⊣

CLAIM 1. If $m \in \mathfrak{G}$ and $0 < t \in \mathbb{K}^\uparrow$, then $m^t \in \mathfrak{G}$.

To prove the claim, write $m = e^\gamma \mathbf{x}^{\theta+n}$ with $\gamma, \theta \in \mathbb{K}^\uparrow$ and $n \in \mathbb{Z}$. Then $m^t = e^{t\gamma} \mathbf{x}^{t(\theta+n)} \in \mathfrak{G}$, as desired.

CLAIM 2. Let a and b be positive elements of $\mathbb{K}^\uparrow + \mathbb{N}$. Then $a^b \in \mathbb{K}^\uparrow + \mathbb{N}$. Moreover if $a \geq 2$ (i.e. $a \neq 1$) and $b > \mathbb{N}$, then $a^b \in \mathbb{K}^\uparrow$.

We can write $b = b^\uparrow + n$ for some $n \in \mathbb{N}$ and $0 < b^\uparrow \in \mathbb{K}^\uparrow$. Since $\mathbb{K}^\uparrow + \mathbb{N}$ is closed under finite products, $a^n \in \mathbb{K}^\uparrow + \mathbb{N}$. It remains to show that $a^{b^\uparrow} \in \mathbb{K}^\uparrow$. This is clear if $a \in \mathbb{N}$. If $a \notin \mathbb{N}$, we can write

$$a = rm(1 + \varepsilon),$$

where $1 < m \in \mathfrak{G}$ is the leading monomial of a , $r \in \mathbb{R}^{>0}$ and $\varepsilon \prec 1$. Then

$$a^{b^\uparrow} = r^{b^\uparrow} m^{b^\uparrow} (1 + \varepsilon)^{b^\uparrow}.$$

By Claim 1 $m^{b^\uparrow} \in \mathfrak{G}$. By definition of \mathfrak{G} we also have $r^{b^\uparrow} = e^{b^\uparrow \log(r)} \in \mathfrak{G}$. The third factor $(1 + \varepsilon)^{b^\uparrow}$ can be written in the form

$$\begin{aligned} (1 + \varepsilon)^{b^\uparrow} &= e^{b^\uparrow \log(1+\varepsilon)} \\ &= e^{(b^\uparrow \log(1+\varepsilon))^\uparrow} e^{(b^\uparrow \log(1+\varepsilon))^\circ} e^{(b^\uparrow \log(1+\varepsilon))^\downarrow}. \end{aligned}$$

Since $\log(1 + \varepsilon) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \varepsilon^n \in \mathbb{K}$ and $b^\uparrow \in \mathbb{K}$, we have $b^\uparrow \log(1 + \varepsilon) \in \mathbb{K}$, so $e^{(b^\uparrow \log(1+\varepsilon))^\uparrow} \in \mathfrak{G}$. Moreover $e^{(b^\uparrow \log(1+\varepsilon))^\circ} \in \mathbb{R}$. The element $\delta = (b^\uparrow \log(1 + \varepsilon))^\downarrow$ is an infinitesimal element of \mathbb{K} and $e^\delta = \sum_{n \in \mathbb{N}} \frac{\delta^n}{n!}$ is a power series in δ , so it belongs to \mathbb{K} . We have thus proved that $(1 + \varepsilon)^{b^\uparrow} \in \mathbb{K}$ and therefore $a^{b^\uparrow} \in \mathbb{K}$.

It remains to show that if $a \geq 2$, then a^{b^\uparrow} is purely infinite. Since $a = rm(1 + \varepsilon)$ is an omnific integer, each monomial in the support of ε is $\geq m^{-1}$. It follows that each monomial in the support of $(1 + \varepsilon)^{b^\uparrow}$ is m^{-n} for some $n \in \mathbb{N}$. Since $a^{b^\uparrow} = r^{b^\uparrow} m^{b^\uparrow} (1 + \varepsilon)^{b^\uparrow}$, it follows that every monomial in the support of a^{b^\uparrow} is $\geq r^{b^\uparrow} m^{b^\uparrow} m^{-n} = r^n (rm)^{b^\uparrow - n}$, which is infinite. We conclude that $a^{b^\uparrow} \in \mathbb{K}^\uparrow$, as desired.

It follows from the claim that the set of positive elements of the semiring $\mathbb{K}^\uparrow + \mathbb{N}$ is closed under the operation $a, b \mapsto a^b$ and therefore it contains Sk .

PROPOSITION 9.3. For every Skolem function f there is a purely infinite surreal number g and some $n \in \mathbb{N}$ such that $f = g + n$. Moreover g is a Skolem function.

PROOF. By Theorem 9.2, $f = g + n$ with $g \in \mathbb{K}^\uparrow$ and $n \in \mathbb{N}$, so we only need to show that $g \in Sk$. We proceed by induction on the formation of the Skolem terms. The case when f is the sum or product of shorter terms is immediate. It remains to consider the case when $f = a^b$ with $a \geq 2$ and $b > \mathbb{N}$. In this case by Theorem 9.2, $a^b \in \mathbb{K}^\uparrow$, so it is purely infinite. ⊖

THEOREM 9.4. The monomial \mathbf{x} is the smallest infinite monomial in \mathbb{K} .

PROOF. We prove by induction on $n \in \mathbb{N}$ that if $1 < m \in \mathfrak{M}_n$, then $m \geq \mathbf{x}$. This is clear for $n = 0$ since $\mathfrak{M}_0 = \mathbf{x}^{\mathbb{Z}}$. Let $1 < m \in \mathfrak{M}_{n+1}$ and assume the result holds for the monomials in \mathfrak{M}_n . By definition $m = e^{\gamma \mathbf{x}^{\theta+k}} = e^{\gamma + \log(\mathbf{x})(\theta+k)}$ with $\gamma, \theta \in \mathbb{K}_n^{\uparrow}$ and $k \in \mathbb{Z}$. By the induction hypothesis, \mathbf{x} is the smallest infinite monomial in \mathbb{K}_n . If for a contradiction $1 < m < \mathbf{x} = e^{\log(\mathbf{x})}$, then

$$0 < \gamma + \log(\mathbf{x})(\theta + k) < \log(\mathbf{x}).$$

Case 1. If $\gamma \asymp \log(\mathbf{x})(\theta + k)$, then $\log(\mathbf{x}) \asymp \frac{\gamma}{\theta+k} \in \mathbb{K}_n$, contradicting the induction hypothesis.

Case 2. If $\gamma \succ \log(\mathbf{x})(\theta + k)$, then $0 < \gamma < 2 \log(\mathbf{x})$, against the induction hypothesis.

Case 3. If $\gamma \prec \log(\mathbf{x})(\theta + k)$, we obtain $0 < \log(\mathbf{x})(\theta + k) < 2 \log(\mathbf{x})$, whence $0 < \theta + k < 2$. Since θ is purely infinite and $k \in \mathbb{Z}$, we obtain $\theta = 0$, hence $\gamma \prec \log(\mathbf{x})$, contradicting the induction hypothesis. \dashv

COROLLARY 9.5. *If m is a monomial in the support of a Skolem function, then either $m = 1$ or $m \geq \mathbf{x}$.*

PROOF. Immediate from Theorem 9.4 and the inclusion $Sk \subset \mathbb{K}^{\uparrow} + \mathbb{N}$ (Theorem 9.2). \dashv

COROLLARY 9.6. *For $f, g \in Sk$ we have:*

- (1) $f \sim g \iff f = g + O(g/\mathbf{x})$.
- (2) $f^{\mathbf{x}} \asymp g^{\mathbf{x}} \iff f \sim g$.

PROOF. The first part follows from Theorem 9.4 and Theorem 9.2, observing that $f - g \in \mathbb{K}$. For (2) we take $z = f/g$ and $c = \mathbf{x}$ in Lemma 8.5 to obtain $f^{\mathbf{x}} \asymp g^{\mathbf{x}}$ if and only if $f = g + O(g/\mathbf{x})$. By the first part this happens if and only if $f \sim g$. \dashv

COROLLARY 9.7. *For any $A \subseteq Sk$, we have $|A^{\mathbf{x}}/\asymp| = |A/\sim|$.*

PROOF. By part (2) of Corollary 9.6. \dashv

§10. Components. Let f be a Skolem function. We say that f is *additively irreducible* if it cannot be written as a sum of two smaller Skolem functions; f is *multiplicatively irreducible* if it cannot be written as a product of two smaller Skolem functions; Following [19] we say that f is a *component* if it is both additively and multiplicatively irreducible.

REMARK 10.1. We can write every Skolem function as a finite sum of finite products of components (not necessarily in a unique way).

PROPOSITION 10.2. *Every component has the form $1, \mathbf{x}$ or f^g . If f^g is a component, then f is multiplicatively irreducible and g is additively irreducible. Every component $> \mathbf{x}$ can be written in the form f^g where $f \geq 2, g \geq \mathbf{x}$, and g is a component.*

PROOF. A Skolem functions $< \mathbf{x}$ is a positive integers, so it is either 1 or additively reducible. It follows that a component is either 1, or \mathbf{x} , or $> \mathbf{x}$. In the latter case it must have the form f^g (because it cannot be of the form $f + g$ or fg). The rest follows at once from the following identities:

- if $f = f_1 f_2$, then $f^g = f_1^g f_2^g$;
- if $g = g_1 + g_2$, then $f^g = f^{g_1} f^{g_2}$;
- if $g = g_1 g_2$, then $f^g = (f^{g_1})^{g_2}$. †

COROLLARY 10.3. *For every Skolem function h one of the following cases holds:*

- (1) $h = f \cdot g$ where $f \geq \mathbf{x}$ and $g \geq \mathbf{x}$;
- (2) $h = f^g$ where $f \geq 2$ and g is a component $\geq \mathbf{x}$;
- (3) $h = f + g$, where $f \asymp h$ and f is a component;
- (4) $h = 1$ or $h = \mathbf{x}$.

§11. Main theorem. We work inside the surreal numbers \mathbf{No} and identify Sk as a subset of \mathbf{No} , with $\mathbf{x} = \omega \in \mathbf{No}$. Our main result is the following.

THEOREM 11.1. *Let $c \geq 1$ be a surreal number and let $Q \in Sk$. The set of real numbers $r \in \mathbb{R}^{>0}$ such that there is $h \in Sk$ satisfying $(h/Q)^c \sim r$, is well ordered and has no accumulation points in \mathbb{R} (hence it has order type $\leq \omega$).*

PROOF. Given Q and c , the set of reals $r \in \mathbb{R}^{>0}$ such that there is $h \in Sk$ with $(h/Q)^c \sim r$ is in order preserving bijection with the set of Skolem functions $\asymp_c Q$, so it is well ordered (as Sk is well ordered). A well ordered subset of $\mathbb{R}^{>0}$ has an accumulation point if and only if it contains a strictly increasing and bounded sequence. Assuming for the sake of a contradiction that the theorem fails, let Q be minimal in the well order of Sk such that there exist a surreal number $c \geq 1$, a strictly increasing and bounded sequence of positive real numbers $(r_n)_{n \in \mathbb{N}}$, and a sequence $(h_n)_{n \in \mathbb{N}}$ of Skolem functions, such that

$$(h_n/Q)^c \sim r_n.$$

By the assumptions, for all $n \in \mathbb{N}$, we have

$$h_n \asymp_c Q$$

which in turn implies $h_n \asymp Q$ (by Proposition 8.4). In other words, all the functions h_n belong to the archimedean class of Q . Let us also notice that, given $c \geq 1$ as above, the minimality property of Q implies that Q is minimal in its \asymp_c -class in Sk (using the fact that if $Q' \asymp_c Q$, there is $s \in \mathbb{R}^{>0}$ with $(h_n/Q')^c \sim sr_n$ for all $n \in \mathbb{N}$).

Now let h be the least Skolem function $\asymp Q$, and note that its multiples nh ($n \in \mathbb{N}$) are cofinal in the archimedean class of Q . There is $N \in \mathbb{N}$ such that $h_n \leq Nh$ for all $n \in \mathbb{N}$, for otherwise the sequence $(r_n)_{n \in \mathbb{N}}$ is unbounded. We call the least such N the *characteristic bound* of the sequence $(h_n)_{n \in \mathbb{N}}$.

We now choose $(h_n)_n$ with the additional property that $(h_n)_n$ has minimal characteristic bound $N \in \mathbb{N}$. Note that the characteristic bound N is only defined

for those sequences $(h_n)_n$ such that there is $c \geq 1$ and a strictly increasing bounded sequence $r_n \sim (h_n/Q)^c$ as above, but it does not depend on the choice of c , so we can minimize N before choosing c . Finally we fix the exponent $c \geq 1$ and we get a strictly increasing bounded sequence $(r_n)_n$ of positive real numbers such that $(h_n/Q)^c \sim r_n$.

Along the sequence $(h_n)_n$ there is one of the cases of [Corollary 10.3](#) which holds infinitely often. By taking a subsequence we can thus assume to be in one of the following cases:

- (1) for all $n \in \mathbb{N}$, $h_n = f_n \cdot g_n$ where $f_n \geq \mathbf{x}$ and $g_n \geq \mathbf{x}$;
- (2) for all $n \in \mathbb{N}$, $h_n = f_n^{g_n}$ where $f_n \geq 2$ and $g_n \geq \mathbf{x}$;
- (3) for all $n \in \mathbb{N}$, $h_n = f_n + g_n$, where $f_n \asymp Q$ and f_n is a component;

with $(f_n)_n$ and $(g_n)_n$ weakly increasing (taking advantage of the fact that Sk is well ordered).

We will need the following observation. Define $r \in \mathbb{R}^{>0}$ by $(Q/h_0)^c \sim r$ and let $r'_n = r_n r$. Notice that

$$(h_n/h_0)^c \sim r'_n$$

for all $n \in \mathbb{N}$ and observe that $(r'_n)_n$ is again increasing and bounded.

CASE 1. Suppose $h_n = f_n \cdot g_n$ where $f_n \geq \mathbf{x}$ and $g_n \geq \mathbf{x}$ for all $n \in \mathbb{N}$. By our assumptions $r'_n \sim (h_n/h_0)^c = (f_n/f_0)^c (g_n/g_0)^c$. Both factors in the last expression are ≥ 1 because the sequences $(f_n)_n$ and $(g_n)_n$ are weakly increasing. It then follows that there are real numbers $s_n \geq 1$ and $t_n \geq 1$ such that

$$(f_n/f_0)^c \sim s_n, \quad (g_n/g_0)^c \sim t_n$$

and $r'_n = s_n t_n$. Since $(r'_n)_n$ is bounded, the sequences $(s_n)_{n \in \mathbb{N}}$ and $(t_n)_{n \in \mathbb{N}}$ must also be bounded. Since both f_n and g_n are $\geq \mathbf{x}$ and their product is h_n , they are both $\prec h_n \asymp Q$. In particular f_0 and g_0 are $\prec Q$. By the minimality of Q , the sequences $(s_n)_n$ and $(t_n)_n$ are eventually constant, hence $(r'_n)_n$ is eventually constant, a contradiction.

In the next case we use the full strength of the fact that we work with all the equivalence relations \sim_c and not only with \sim .

CASE 2. Suppose $h_n = f_n^{g_n}$ where $f_n \geq 2$ and $g_n \geq \mathbf{x}$ for all $n \in \mathbb{N}$. Note that $r'_n \sim (h_n/h_0)^c \geq h_n/h_0 = f_n^{g_n}/f_0^{g_0} \geq f_0^{g_n-g_0} \geq 2^{g_n-g_0}$ for $n \in \mathbb{N}$. Since $(r'_n)_n$ is bounded in \mathbb{R} , there is $M \in \mathbb{N}$ such that $g_n - g_0 < M$ for all $n \in \mathbb{N}$. If the difference between two Skolem functions is bounded by a natural number, then it is equal to a natural number ([Proposition 9.3](#)). Since $(g_n)_{n \in \mathbb{N}}$ is weakly increasing, there must be some $k \in \mathbb{N}$ such that $g_n = g_k$ for all $n \geq k$. For $n \geq k$ we have $(h_n/h_k)^c \sim sr'_n$ where $s \in \mathbb{R}^{>0}$ is defined by $s \sim (h_0/h_k)^c$. Taking a subsequence we can assume $k = 0$. Thus $s = 1$ and

$$r'_n \sim (h_n/h_0)^c = (f_n/f_0)^{g_0 c}$$

for all $n \in \mathbb{N}$. Since $f_n \geq 2$ and $g_n \geq \mathbf{x}$, we have $f_n \prec f_n^{g_n} = h_n \asymp Q$ for all $n \in \mathbb{N}$. Since $(f_n/f_0)^{g_0 c} \sim r'_n$ and $f_0 \prec Q$, by the minimality of Q we deduce that $(r'_n)_{n \in \mathbb{N}}$ is eventually constant, a contradiction.

We have shown that a sequence $(h_n)_n$ with minimal characteristic bound falls necessarily under case 3, so it cannot consist entirely of components. It remains to deal with case 3.

CASE 3. Suppose that $h_n = f_n + g_n$ where $f_n \asymp Q$ and f_n is a component for all $n \in \mathbb{N}$. It suffices to consider the cases $c = 1$ and $c > \mathbb{N}$, for if $c \asymp c'$ and $(h_n/Q)^c \sim r_n$, then $(h_n/Q)^{c'} \sim r_n^t$, where $t \in \mathbb{R}^{>0}$ is such that $t \sim c'/c$. Taking a subsequence we can further assume that either $g_n \asymp Q$ for all $n \in \mathbb{N}$, or $g_n \prec Q$ for all $n \in \mathbb{N}$.

CASE $c = 1$. The assumption $(h_n/Q)^c \sim r_n$ becomes $h_n/Q \sim r_n$. Recall that $h_n = f_n + g_n$. Consider first the subcase with $g_n \asymp Q$ for all $n \in \mathbb{N}$. Then all the functions h_n, f_n, g_n are in the archimedean class of Q , so there are positive real numbers $a_n \in \mathbb{R}^{>0}$ and $b_n \in \mathbb{R}^{>0}$ such that

$$a_n \sim f_n/Q \quad \text{and} \quad b_n \sim g_n/Q$$

for all $n \in \mathbb{N}$. It follows that $a_n + b_n = r_n$ for all $n \in \mathbb{N}$. Since $(r_n)_n$ is bounded, it follows that $(a_n)_n$ and $(b_n)_n$ are also bounded. Recall that Q is minimal in its \asymp_c -class. Since $c = 1$, this means that Q is minimal in its archimedean class, so the functions h_n, f_n, g_n are all $\geq Q$. If N is the characteristic bound of $(h_n)_n$, we have $f_n \geq Q$ and $g_n \geq Q$ and $f_n + g_n = h_n \leq NQ$, so both $(h_n)_n$ and $(g_n)_n$ have characteristic bound $\leq N - 1$. By the minimality of N , we deduce that the sequences $(a_n)_n$ and $(b_n)_n$ are eventually constant, hence their sum $(r_n)_n$ is also eventually constant, a contradiction.

Now consider the subcase with $g_n \prec Q$ for all $n \in \mathbb{N}$. Then the functions h_n and f_n are in the archimedean class of Q , but g_n is in a lower archimedean class. It follows that

$$r_n \sim h_n/Q = (f_n + g_n)/Q \sim f_n/Q$$

for all $n \in \mathbb{N}$. The sequence $(f_n)_n$ is then a counterexample to the theorem with the same characteristic bound than $(h_n)_n$ but consisting entirely of components. We have already shown that this cannot happen, so we have a contradiction.

CASE $c > \mathbb{N}$. We are still inside the case $h_n = f_n + g_n$ with h_n a component. By Proposition 8.7 the condition $(h_n/h_0)^c \sim r'_n$ can be rewritten in the form

$$(11.1) \quad h_n/h_0 - 1 = s_n/c + o(1/c)$$

where $s_n = \log(r'_n)$ for $n \in \mathbb{N}$. Note that since $(h_n)_n$ is increasing, we have $r'_n \geq 1$, so $\log(r'_n)$ is well defined and ≥ 0 . Moreover $(s_n)_n$ is strictly increasing. Using $h_n = f_n + g_n$, Equation 11.1 becomes

$$(f_n - f_0) + (g_n - g_0) = s_n(f_0 + g_0)(1/c + o(1/c)).$$

Dividing by f_0 and multiplying by c , it can be rewritten as

$$c \left(\frac{f_n}{f_0} - 1 \right) + \left(\frac{c}{f_0} \right) (g_n - g_0) = s_n \left(1 + \frac{g_0}{f_0} \right) + o(1)$$

Since $g_0 \preceq Q \asymp f_0$ the right-hand-side is finite. The two summands on the left are ≥ 0 and their sum is finite, so they are both finite, that is, they can be written as a real number plus an infinitesimal. This means that we can define $a_n \in \mathbb{R}$ and $b_n \in \mathbb{R}$ by the equations

$$a_n = c \left(\frac{f_n}{f_0} - 1 \right) + o(1) \quad \text{and} \quad b_n = \left(\frac{c}{f_0} \right) (g_n - g_0) + o(1).$$

We can then write

$$(11.2) \quad a_n + b_n = s_n \left(1 + \frac{g_0}{f_0} \right) + o(1).$$

Since $(f_n)_n$ and $(g_n)_n$ are weakly increasing, the sequences of real numbers $(a_n)_n$ and $(b_n)_n$ are weakly increasing. Moreover, since $(s_n)_{n \in \mathbb{N}}$ is bounded and g_0/f_0 does not depend on n , $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are also bounded.

By Proposition 8.7 (and the assumption $c > \mathbb{N}$) the definition of a_n can be rewritten in the form

$$(f_n/f_0)^c \sim e^{a_n}.$$

We claim that $(a_n)_n$ is eventually constant. If $f_0 < Q$ this follows from the minimality property of Q , so we can assume $Q \leq f_0$. We also have $f_0 \leq h_0 \leq h_n \asymp_c Q$, so all the functions f_n are in the \asymp_c -class of Q and therefore there is a real number $s \in \mathbb{R}^{>0}$ such that

$$(f_n/Q)^c \sim se^{a_n}$$

for all $n \in \mathbb{N}$. Assuming for a contradiction that $(a_n)_n$ is not eventually constant, $(f_n)_n$ would be a counterexample to the theorem with a characteristic bound at most equal to that of $(h_n)_n$ (because $f_n \leq h_n$). However $(f_n)_n$ has the additional property that it consists entirely of components and we have already shown that this cannot happen. This contradiction shows that $(a_n)_n$ is indeed eventually constant.

We now claim that $(b_n)_{n \in \mathbb{N}}$ is eventually constant. By our definitions we have $b_n = (c/f_0)(g_n - g_0) + o(1)$ so we can write

$$(11.3) \quad g_n - g_0 = b_n P + o(P)$$

where $P = f_0/c$.

We distinguish three subcases.

Subcase 1. If $g_0 \prec P$, then for all n we have $g_n = b_n P + o(1)$, or equivalently $g_n/P = b_n + o(1)$. Since $P = f_0/c \prec f_0 \asymp Q$, by the minimality of Q we conclude that $(b_n)_{n \in \mathbb{N}}$ is eventually constant (possibly 0), as desired.

Subcase 2. If $g_0 \asymp P$, then there is $r \in \mathbb{R}^{>0}$ such that $g_0 \sim rP$, so $g_n = (b_n + r)P + o(1)$ for all $n \in \mathbb{N}$. Reasoning as above, $(b_n + r)_{n \in \mathbb{N}}$ is eventually constant, hence so is $(b_n)_{n \in \mathbb{N}}$.

Subcase 3. If $g_0 \succ P$, we divide Equation 11.3 by g_0 obtaining $(g_n/g_0 - 1) = b_n(P/g_0) + o(P/g_0)$. Now we multiply by $c' = g_0/P$ to get $c'(g_n/g_0 - 1) = b_n + o(1)$. Since $c' > \mathbb{N}$, by Proposition 8.7 we obtain $\left(\frac{g_n}{g_0}\right)^{c'} \sim e^{b_n}$. If $g_0 < Q$, then by the minimality of Q we conclude that $(b_n)_{n \in \mathbb{N}}$ is eventually constant, as desired. In the opposite case we have $g_0 \asymp_c Q$ (since $Q \leq g_0 \leq h_0 \asymp_c Q$). The new exponent c' is in the same archimedean class of c , because $c' = (g_0/f_0)c$ and $g_0 \asymp f_0$, thus $c/c' = \gamma + o(1)$ for some real $\gamma > 0$. Raising to the power γ both sides of the relation $\left(\frac{g_n}{g_0}\right)^{c'} \sim e^{b_n}$ we then obtain $\left(\frac{g_n}{g_0}\right)^c \sim e^{b_n \gamma}$. Since $g_0 \asymp_c Q$, there is a real $r > 0$ such that $\left(\frac{g_n}{Q}\right)^c \sim re^{b_n \gamma}$. All the functions h_n, f_n, g_n are in the same archimedean class, namely that of Q . Since $h_n = f_n + g_n$, it follows that the characteristic bound of

$(g_n)_n$ is lower than the characteristic bound N of $(h_n)_n$. By the minimality of N , we deduce that $(re^{bn^\gamma})_n$ is eventually constant, hence also $(b_n)_n$ is eventually constant, as desired.

From Equation (11.2) we can now conclude that $(s_n)_{n \in \mathbb{N}}$ is also eventually constant, against the assumptions. This contradiction concludes the proof. \dashv

COROLLARY 11.2. *Let $1 \leq c \in \mathbf{No}$. The set of \sim_c -classes of Skolem functions within any class modulo \succ_c has order type $\leq \omega$. In particular, the set of asymptotic classes of Skolem functions within any archimedean class has order type $\leq \omega$.*

PROOF. Fix $Q \in Sk$. For every $h \succ_c Q$, the \sim_c -class of h is determined by the real number $r \in \mathbb{R}^{>0}$ defined by $r \sim (h/Q)^c$, so we can apply Theorem 11.1. \dashv

We need the following corollary to obtain an upper bound on the order type of the set of Skolem functions $< 2^{\mathbf{x}}$.

COROLLARY 11.3. *For any $A \subseteq Sk$, $|A^{\mathbf{x}/\succ}| = |A/\sim| \leq \omega|A/\succ|$.*

PROOF. The first equality is Corollary 9.7. The inequality $|A/\sim| \leq \omega|A/\succ|$ follows from Corollary 11.2. \dashv

We give below other consequences of the main theorem.

COROLLARY 11.4. *Let $Q = \sum_{i < \alpha} m_i r_i \in \mathbf{No}$ and let m a monomial smaller than all monomials m_i in the support of Q . Then there is a well ordered subset $D \subseteq \mathbb{R}$ without accumulation points such that for every Skolem function f , if f (seen as an element of \mathbf{No}) has a truncation of the form $Q + rm$, then $r \in D$.*

PROOF. If $Q = 0$ the desired result is an immediate consequence of Theorem 11.1. Assume $Q \neq 0$. We can write $f = Q + rm + o(m)$. Thus $f/Q = 1 + rm/Q + o(m/Q)$. Let $c = Q/m$. Then $c > \mathbb{N}$ and $f/Q = 1 + r/c + o(1/c)$. By Proposition 8.7, $(f/Q)^c \sim e^r$. Let D be the set of possible values of e^r as f varies. Since Sk is well ordered, D is well ordered. Suppose for a contradiction that there is an increasing sequence $e^{r_n} \in D$ with an accumulation point $e^r \in \mathbb{R}$. We can then find $f_n \in Sk$ with $(f_n/Q)^c \sim e^{r_n}$, contradicting Theorem 11.1. \dashv

By Theorem 9.4, given two Skolem functions f, g , the smallest infinite monomial in the support of f/g (seen as a surreal number) is $\mathbf{x} = \omega$. We thus obtain the following result, which extends to the whole class Sk the corresponding result of van den Dries and Levitz [10] for the fragment below $2^{2^{\mathbf{x}}}$.

COROLLARY 11.5. *Let $g \in Sk$. For every finite sequence r_0, \dots, r_k of real numbers (empty if $k = -1$), there is a well ordered subset $R = R(g, r_0, \dots, r_k) \subseteq \mathbb{R}$ without accumulation points such that for every $f \succ g$ in Sk satisfying*

$$f/g = r_0 + r_1/\mathbf{x} + \dots + r_k/\mathbf{x}^k + r_{k+1}/\mathbf{x}^{k+1} + O(1/\mathbf{x}^{k+2})$$

we have $r_{k+1} \in R$.

§12. Levitz's regular functions. We say that $f \in Sk$ is an *additive scale* if the sum of two Skolem functions less than f is less than f . We define f to be a *multiplicative scale* if the product of two Skolem functions less than f is less than f . Clearly every additive scale is additively irreducible and every multiplicative scale is multiplicatively irreducible. It is also easy to see that a multiplicative scale $f \neq 2$ is an additive scale. Indeed if f is not an additive scale, there is $g < f$ with $g + g \geq f$. Since $f \neq 2$, we have $g \neq 1$, so $gg \geq g + g \geq f$, contradicting the fact that f is a multiplicative scale. We have thus proved that a multiplicative scale $\neq 2$ is a component (recall that f is a component if it is both additively and multiplicatively irreducible).

We say that $h \in Sk$ is *regular* if $h \neq 1$ and for all Skolem functions $f < h$ we have $f^x < h$. Regular functions play a crucial role in the work of Levitz [19]. Every regular function is a multiplicative scale, so it is either equal to 2 or a component. In the rest of the sections we characterize the regular functions $\leq 2^{x^x}$.

PROPOSITION 12.1. *The components $< x^x$ are 1, x and p^x with $p \in \mathbb{N}$ prime.*

PROOF. If h is a component $> x$, we can write $h = f^g$ where $f \geq 2$ is multiplicatively irreducible and g is a component $\geq x$ (Proposition 10.2). Since $h < x^x$, we must have $g \leq x$ and $f < x$, so $h = p^x$ with p a prime in \mathbb{N} . \dashv

LEMMA 12.2. *Let $n > 0$. If f is a Skolem function $< 2^{(n+1)^x}$, then there is $k \in \mathbb{N}$ such that $f < 2^{n^x x^k}$. It follows that $f^x < 2^{(n+1)^x}$, hence $2^{(n+1)^x}$ is a regular function.*

PROOF. For a contradiction let $n > 0$ be minimal such that the statement fails. Let $H_n \subset Sk$ be the set of Skolem functions bounded by one of the functions $2^{n^x x^k}$ as k ranges in \mathbb{N} . Let $f < 2^{(n+1)^x}$ be the minimal Skolem function such that $f \notin H_n$. We need to consider the following cases.

- (1) f is not a component;
- (2) f is either 1 or x ;
- (3) f is a component of the form a^x ;
- (4) f is a component of the form a^b with $b > x$.

Case (2) is clearly impossible. Cases (1) and (3) are also impossible by the minimality of f and the fact that H_n is closed under sums, products and exponentiation to the power x . Finally, in case (4), by Proposition 12.1, we can write $b = p^x$ with p prime $< n + 1$. Let $q \in \mathbb{N}$ be minimal such that $qp \geq n + 1$ and notice that $2 \leq q \leq n$. We must have $a < (2^q)^x$, so by the minimality of n we have $a < 2^{(q-1)^x x^k}$ for some $k \in \mathbb{N}$. It follows that $f = a^b < 2^{((q-1)p)^x x^k} \leq 2^{n^x x^k}$. \dashv

PROPOSITION 12.3. *Let $f < 2^{x^x}$ be a Skolem function. Then there is $n \in \mathbb{N}$ such that $f < 2^n$. It follows that 2^{x^x} is the smallest regular function bigger than 2^n for all positive $n \in \mathbb{N}$.*

PROOF. Let $f < 2^{x^x}$ and assume by induction that the proposition holds for all Skolem functions $< f$. If f is not a component, then it is a sum of products of smaller functions, and we conclude observing the the Skolem functions less than 2^n for some $n \in \mathbb{N}$ form an initial segment closed under sums and products. The cases

$f = 1$ or $f = \mathbf{x}$ are trivial. It remains to consider the case when f is a component of the form a^b where $b > 1$ is a component and $a \geq 2$. Since $f = a^b < 2^{\mathbf{x}}$, we have $b < \mathbf{x}$. By Proposition 12.1, either $b = \mathbf{x}$ or $b = p^{\mathbf{x}}$ for some prime $p \in \mathbb{N}$. By the induction hypothesis $a < 2^{m^{\mathbf{x}}}$ for some $m \in \mathbb{N}$, hence in the first case $f = a^b < 2^{(m+1)^{\mathbf{x}}}$ and in the second case $f = a^b < 2^{m^{\mathbf{x}} p^{\mathbf{x}}} = 2^{(mp)^{\mathbf{x}}}$. In either case $f \leq 2^{n^{\mathbf{x}}}$ for a suitable n . ⊣

§13. The fragment of van den Dries and Levitz. Van den Dries and Levitz [10] proved that $|2^{2^{\mathbf{x}}}| = \omega_3 = \omega^{\omega^\omega}$. As a preparation for the results in the next section we give a proof of the inequality $|2^{2^{\mathbf{x}}}| \leq \omega_3$ based on Corollary 11.3. Thanks to the fact that Corollary 11.3 holds for the whole class Sk , we shall then be able to extend the result to bigger fragments with a similar technique.

We recall that given a set $X \subseteq Sk$, $\sum X$ is the set of finite non empty sums of elements of X (we exclude the empty sum because 0 is not a Skolem function). Similarly, we write $\prod X$ for the set of finite products of elements of X , with the convention that the empty product is 1.

THEOREM 13.1 ([10]). $|2^{2^{\mathbf{x}}}| \leq \omega^{\omega^\omega}$. Moreover the set of archimedean classes of the set of Skolem functions $< 2^{2^{\mathbf{x}}}$ has order type $\leq \omega^\omega$.

PROOF. Let A be the set of Skolem functions $< 2^{2^{\mathbf{x}}}$. We need to prove that $|A| \leq \omega_3$ and $|A/\asymp| \leq \omega_2$, where A/\asymp is the set of \asymp -classes of elements of A .

By Lemma 12.2 for $n = 1$ we can write

$$A = \bigcup_{d \in \mathbb{N}} S_d$$

where S_d is set of Skolem functions $< 2^{x^d}$.

By induction on d we show that

$$|S_d| < \omega_3 \quad \text{and} \quad |S_d/\asymp| < \omega_2.$$

Granted this, the supremum over d of these ordinals is $\leq \omega_3$ and $\leq \omega_2$ respectively, yielding the desired bounds $|A| \leq \omega_3$ and $|A/\asymp| \leq \omega_2$.

The case $d = 0$ of the inductive proof is obvious, so assume $d > 0$. Writing a Skolem function as a finite sum of finite products of components, and observing that $g^{\mathbf{x}} \leq 2^{x^d} \implies g < 2^{x^{d-1}}$, we obtain

$$S_d \subseteq \sum \prod (\mathbf{x}^{\mathbb{N}} \cup S_{d-1}^{\mathbf{x}})$$

By the induction hypothesis $|S_{d-1}| < \omega_3$ and $|S_{d-1}/\asymp| < \omega_2$. Now observe that $|S_{d-1}^{\mathbf{x}}| = |S_{d-1}| < \omega_3$. Moreover by Corollary 11.3 we have

$$|S_{d-1}^{\mathbf{x}}/\asymp| \leq \omega |S_{d-1}/\asymp| < \omega_2$$

(because the set of ordinals $< \omega_2$ is closed under multiplication by ω). Letting $X = \mathbf{x}^{\mathbb{N}} \cup S_{d-1}^{\mathbf{x}}$, it follows that $|X| < \omega_3$ and $|X/\asymp| < \omega_2$. Now observe that each element of $\prod X$ is a product of at most 2 elements of X (because $\mathbf{x}^{\mathbb{N}}$ and $S_{d-1}^{\mathbf{x}}$ are closed under finite products). By Corollary 4.2 we then obtain $|\prod X| < \omega_3$ and $|\prod X/\asymp| < \omega_2$.

By Corollary 4.9 we conclude that $|\sum \prod X| < \omega_3$ and $|\sum \prod X/\asymp| < \omega_2$. Since S_d is included in $\sum \prod X$ we get the desired bounds. \dashv

§14. Fragments bounded by larger regular functions. We have seen that $|2^{2^x}| \leq \omega_3$. The following result gives bounds on $|2^{n^x}|$. In particular $|2^{3^x}| \leq \omega_4$, $|2^{4^x}| \leq \omega_5$, and so on.

THEOREM 14.1. *Let $1 \leq n \in \mathbb{N}$. Then $|2^{(n+1)^x}| < \omega_{n+2}$. Moreover the set of archimedean classes of the set of Skolem functions $< 2^{(n+1)^x}$ has order type $\leq \omega_{n+1}$.*

PROOF. Let A_n be the set of all Skolem functions $< 2^{(n+1)^x}$. We prove by induction on n that $|A_n| \leq \omega_{n+2}$ and $|A_n/\asymp| \leq \omega_{n+1}$.

For $n = 1$, A_n is the set of Skolem functions $< 2^{2^x}$ so we can apply Theorem 13.1.

Assume $n > 1$. For $d \in \mathbb{N}$, let $S_{n,d}$ be the set of Skolem functions $< 2^{n^x \cdot d}$. By Lemma 12.2

$$A_n = \bigcup_{d \in \mathbb{N}} S_{n,d}.$$

By a secondary induction on d we show that

$$|S_{n,d}| < \omega_{n+2} \quad \text{and} \quad |S_{n,d}/\asymp| < \omega_{n+1}.$$

Granted this, the sup over d of these ordinals is $\leq \omega_{n+2}$ and $\leq \omega_{n+1}$ respectively, yielding the desired bounds

$$|A_n| \leq \omega_{n+2} \quad \text{and} \quad |A_n/\asymp| \leq \omega_{n+1}.$$

The case $d = 0$ of the secondary induction follows from

$$S_{n,0} = A_{n-1}$$

applying the primary induction on n .

Assume $d > 0$. We claim that

$$S_{n,d} \subseteq \sum \prod (\mathbf{x}^{\mathbb{N}} \cup S_{n,d-1}^{\mathbf{x}} \cup A_{n-2}^{2^{\mathbf{x}}} \cup \dots \cup A_{n-2}^{n^{\mathbf{x}}}).$$

To prove the claim, it suffices to show that if $h \in S_{n,d}$ is a component, then it belongs to $\mathbf{x}^{\mathbb{N}} \cup S_{n,d-1}^{\mathbf{x}} \cup A_{n-2}^{2^{\mathbf{x}}} \cup \dots \cup A_{n-2}^{n^{\mathbf{x}}}$. We can assume that $h > \mathbf{x}$, so we can write h in the form $h = f^g$ where $f \geq 2$ and g is a component $\geq \mathbf{x}$ (Proposition 10.2). Since $h < 2^{n^x}$, we have $g < n^x$, so either $g = \mathbf{x}$ or $g = p^x$ for some prime $p \in \mathbb{N}$ (Proposition 12.1). Since $h \in S_{n,d}$, we have $h = f^g < 2^{n^x \cdot d}$. So if $g = \mathbf{x}$ we get $f < 2^{n^x \cdot d-1}$ and therefore $h \in S_{n,d-1}^{\mathbf{x}}$. On the other hand if $g = p^x$, then from $f^g < 2^{n^x \cdot d} < 2^{(n+1)^x}$ we obtain $f < 2^{(n-1)^x}$ and $p \leq n$, so $h \in A_{n-2}^{2^{\mathbf{x}}} \cup \dots \cup A_{n-2}^{n^{\mathbf{x}}}$ and the claim is proved.

By the primary induction $|A_{n-2}| \leq \omega_n$. By the secondary induction $|S_{n,d-1}| < \omega_{n+2}$ and $|S_{n,d-1}/\asymp| < \omega_{n+1}$. It follows that $|S_{n,d-1}^{\mathbf{x}}| = |S_{n,d-1}| < \omega_{n+2}$. Moreover by Corollary 11.3 we have

$$|S_{n,d-1}^{\mathbf{x}}/\asymp| \leq \omega |S_{n,d-1}/\asymp| < \omega_{n+1}.$$

We also have $|A_{n-2}^{k^x}| = |A_{n-2}|$ and $|A_{n-2}^{k^x}/\asymp| \leq |A_{n-2}^{k^x}|$. Taking the union of these sets it follows that the set $X = \mathbf{x}^{\mathbb{N}} \cup S_{n,d}^{\mathbf{x}} \cup A_{n-2}^{2^x} \cup \dots \cup A_{n-2}^{n^x}$ satisfies $|X| < \omega_{n+2}$ and $|X/\asymp| < \omega_{n+1}$. The same bounds hold for $\prod X$ because each element of X is a product of at most $n + 1$ elements of X (as X is the union of $n + 1$ sets closed under products). By Corollary 4.9 we conclude that $|\sum \prod X| < \omega_{n+2}$ and $|\sum \prod X/\asymp| < \omega_{n+1}$. Since $S_{n,d}$ is included in $\sum \prod X$ we get the desired bounds. \dashv

THEOREM 14.2. *The set of Skolem functions $< 2^{2^x}$ has order type $\leq \varepsilon_0$.*

PROOF. Immediate from Theorem 14.1 and Proposition 12.3. \dashv

§15. Exponential constants. Let $\mathbb{E}^+ \subseteq \mathbb{R}^{>0}$ be the smallest set of real numbers containing 1 and closed under $+$, \cdot , $^{-1}$ and \exp . Let $\mathbb{E} = \mathbb{E}^+ - \mathbb{E}^+$. Note that \mathbb{E} is a subring of \mathbb{R} , $\exp(\mathbb{E}) \subseteq \mathbb{E}^+$ and $\mathbb{E}^+ \subseteq \mathbb{E}$ (because $1 \in \mathbb{E}$ and $\mathbb{E}^+ \cdot \mathbb{E} \subseteq \mathbb{E}$). The following result is inspired by the final remarks of [10]. The authors gave a detailed proof for the fragment below 2^{2^x} , working with Laurent expansions rather than Ressayre forms, and announced a proof for the whole class Sk using the embryonic form of the transseries in [8].

PROPOSITION 15.1. *Let $f = \sum_{i < \gamma} e^{\gamma_i} c_i \in \mathbf{No}$ be the Ressayre form of a Skolem function f . Then $c_0 \in \mathbb{E}^+$ and $c_i \in \mathbb{E}$ for every $i < \alpha$.*

PROOF. By induction on the formation of f . The cases $f = a + b$ or $f = a \cdot b$ are straightforward, so it suffices to consider the case $f = a^b$ with $a \geq 2$ and $b > \mathbb{N}$ (note that in this case b is purely infinite). By definition

$$a^b = e^{(b \log(a))^\uparrow} e^{(b \log(a))^\circ} e^{(b \log(a))^\downarrow}.$$

We must study the coefficients of the Ressayre form of a^b . Note that $e^{(b \log(a))^\uparrow}$ is a monomial, so it does not contribute to the coefficients. Let us consider the other two factors.

Write $a = \sum_{i < \alpha} e^{\gamma_i} a_i = e^{\gamma_0} a_0 (1 + \varepsilon)$ where $\varepsilon = \sum_{1 \leq i < \alpha} \frac{a_i}{a_0} e^{\gamma_i - \gamma_0}$. Then

$$\log(a) = \gamma_0 + \log(a_0) + \log(1 + \varepsilon).$$

Since b is purely infinite, $(b \log(a))^\circ = (b \log(1 + \varepsilon))^\circ$. Since $\log(1 + \varepsilon)$ is a power series in ε with rational coefficients, and the coefficients $\frac{a_i}{a_0}$ of ε belong to \mathbb{E} , it follows that $(b \log(a))^\circ \in \mathbb{E}$ and therefore $e^{(b \log(a))^\circ} \in \mathbb{E}^+$. This is the leading coefficient of a^b .

The other coefficients of a^b come from the power series expansion of $e^{(b \log(a))^\downarrow}$, so they belong to the ring generated by the coefficients of b and those of ε , which is included in \mathbb{E} . \dashv

Acknowledgements. The first author gave a plenary talk at the XXI congress of the Unione Matematica Italiana, Pavia, September 2–7, 2019 where some of the results of this paper were also mentioned. We thank Vincenzo Mantova and an anonymous

referee for their comments on a preliminary version which helped to improve the exposition.

Both authors were supported by the Italian research project PRIN 2017, “Mathematical logic: models, sets, computability,” Prot. 2017NWTM8RPRIN.

REFERENCES

- [1] H. J. ALTMAN, *Intermediate arithmetic operations on ordinal numbers*. *Mathematical Logic Quarterly*, vol. 63 (2017), no. 3–4, pp. 228–242.
- [2] M. ASCHENBRENNER, L. VAN DEN DRIES, and J. VAN DER HOEVEN, *Asymptotic Differential Algebra and Model Theory of Transseries*, Annals of Mathematics, vol. 195, Princeton University Press, Princeton, NJ, 2017.
- [3] A. BERARDUCCI, S. KUHLMANN, V. MANTOVA, and M. MATUSINSKI, *Exponential fields and Conway's omega-map*. *Proceedings of the American Mathematical Society*, 2018, to appear.
- [4] A. BERARDUCCI and V. MANTOVA, *Surreal numbers, derivations and transseries*. *Journal of the European Mathematical Society*, vol. 20 (2018) no. 2, 339–390.
- [5] ———, *Transseries as germs of surreal functions*. *Transactions of the American Mathematical Society*, vol. 371 (2019), no. 5, 3549–3592.
- [6] P. W. CARRUTH, *Arithmetic of ordinals with applications to the theory of ordered Abelian groups*. *Bulletin of the American Mathematical Society*, vol. 48 (2019), no. 4, 262–271.
- [7] J. H. CONWAY, *On Numbers and Games*, London Mathematical Society Monographs, vol. 6, Academic Press, London, UK, 1976.
- [8] B. I. DAHN, *The limit behaviour of exponential terms*. *Fundamenta Mathematicae*, vol. 124 (1984), no. 2, pp. 169–186.
- [9] L. VAN DEN DRIES and P. EHRLICH, *Fields of surreal numbers and exponentiation*. *Fundamenta Mathematicae*, vol. 167 (2001), no. 2, 173–188, et Erratum, *Fundamenta Mathematicae*, vol. 168, no. 2, pp. 295–297.
- [10] L. VAN DEN DRIES and H. LEVITZ, *On Skolem's exponential functions below 2^{2^x}* . *Transactions of the American Mathematical Society*, vol. 286 (1984), no. 1, 339–349.
- [11] L. VAN DEN DRIES, A. MACINTYRE, and D. MARKER, *Logarithmic-exponential series*. *Annals of Pure and Applied Logic*, vol. 111 (2001), no. 1-2, pp. 61–113.
- [12] D. H. J. DE JONGH and R. PARIKH, *Well-partial orderings and hierarchies*. *Indagationes Mathematicae (Proceedings)*, vol. 80 (1977), no. 3, pp. 195–207.
- [13] A. EHRENFUCHT, *Polynomial functions with exponentiation are well ordered*. *Algebra Universalis*, vol. 3 (1973), no. 1, pp. 261–262.
- [14] H. GONSHOR, *An Introduction to the Theory of Surreal Numbers*, London Mathematical Society Lecture Notes Series, Cambridge University Press, Cambridge, 1986.
- [15] R. GUREVIČ, *Transcendental numbers and eventual dominance of exponential functions*. *Bulletin of the London Mathematical Society*, vol. 18 (1986), no. 6, pp. 560–570.
- [16] H. HAHN, *Über die nichtarchimedischen Grössensysteme*. *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Classe*, vol. 116 (1907), pp. 601–655.
- [17] G. H. HARDY, *Orders of Infinity, the 'Infinitär-calcul' of Paul du Bois-Reymond*, Cambridge University Press, Cambridge, 1910.
- [18] J. B. KRUSKAL, *Well-quasi-ordering, the tree theorem, and Vazsonyi's conjecture*. *Transactions of the American Mathematical Society*, vol. 95 (1960), no. 2, p. 210.
- [19] H. LEVITZ, *An ordinal bound for the set of polynomial functions with exponentiation*. *Algebra Universalis*, vol. 8 (1978), no. 1, pp. 233–243.
- [20] P. LIPPARINI, *Some transfinite natural sums*. *Mathematical Logic Quarterly*, vol. 64 (2018), no. 6, pp. 514–528.
- [21] B. H. NEUMANN, *On ordered division rings*. *Transactions of the American Mathematical Society*, vol. 66 (1949), no. 1, pp. 202–252.
- [22] D. RICHARDSON, *Solution of the identity problem for integral exponential functions*. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 15 (1969), no. 20-22, pp. 333–340.

[23] D. SCHMIDT, *Associative ordinal functions, well partial orderings and a problem of Skolem*. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 24 (1978), no. 19-24, pp. 297–302.

[24] W. SIERPIŃSKI, *Cardinal and Ordinal Numbers*, Polska Akademia Nauk. Monografie matematyczne tom 34, Państwowe Wydawnictwo Naukowe, Warsaw, 1958, 487 pp.

[25] T. SKOLEM, *An ordered set of arithmetic functions representing the least ε -number*. *Det Kongelige Norske Videnskabers Selskab Forhandlinger*, vol. 29 (1956), pp. 54–59.

[26] A. J. WILKIE, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*. *Journal of the American Mathematical Society*, vol. 9 (1996), no. 4, pp. 1051–1094.

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI PISA
LARGO BRUNO PONTECORVO 5, 56127 PISA, ITALY
E-mail: alessandro.berarducci@unipi.it