

# Multipath Mitigation by Global Robust Estimation

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In the last decades, robust estimation has been widely applied to overcome the presence of gross errors in observations. Recently it has been shown that robust estimation, if computed appropriately, is able to cope with a larger amount of gross errors and also capable of providing reliable estimations in situations where systematic errors are present. It is the purpose of this paper to propose the use of global robust estimation for computing baselines strongly affected by multipath effects.

## KEY WORDS

1. robust estimation.
2. least squares.
3. multipath.
4. global optimization.

1. INTRODUCTION. Signal multipath is the result of signal reflections from objects in the environment (Braash 1996). This effect degrades the signal and may be able to impede ambiguity resolution. Consequently, many efforts have been devoted to avoid multipath, the most obvious being the selection of a location free from potential reflections and antenna design improvements.

However, for many practical situations, especially in surveying engineering works, multipath effects remain after observation thus entering into the estimation stage. Many multipath mitigation techniques have been developed to act in the estimation process, to cite just a few: analysis of signal-to-noise ratio (Axelrad et al 1996), wavelet analysis (Souza and Monico 2004, Satirapod and Rizos 2005, Zhong et al 2007, Aram et al 2007), modelling by means of harmonic functions (Amiri-Simkooei and Tiberius 2007), multipath reconstruction (Lau and Cross 2007) and a very long etcetera.

On the other hand, robust estimation foundations were developed by Andrews et al (1972), Huber (1981) and others, and then widely applied to geodesy and surveying (Fuchs 1982, Harvey 1993, Yang et al 1999, and Qingming and Jinshan 1999, for instance) to provide a reliable estimation in an adjustment process in which gross errors participate. In particular, robust estimation was successfully applied to GPS positioning in a pioneer work by Wieser and Brunner (2002). Their conclusion was that “*if the data contain outliers the result of robust estimation is far better than that of standard least squares. Otherwise the results are equal.*” However, they computed the estimator by the usual means of an iteratively reweighted least squares (IRLS) algorithm. Recently Baselga (2007) has shown that the process of computing a robust

estimator by means of an IRLS algorithm highly undermines its estimation capabilities: IRLS robust estimation turns out to be only a local optimization around the initial least squares solution. Since this initial solution may be highly contaminated by the presence of undesirable errors and the correct solution may be far from the initial one, a process of global optimization was proposed to compute the robust estimator. Furthermore, this Global Robust Estimation (GRE) scheme proved to be capable of avoiding not only gross errors but also systematic errors, and it was successfully applied to the estimation of single frequency GPS baselines affected by a strong ionospheric delay which were unsolvable by classic methods (Baselga and García-Asenjo 2008).

This paper will show how baseline solutions free from multipath influences can be obtained by GRE. First we will present the adjustment problem, i.e. the mathematical model to be considered along with a suitable minimizing function (the robust estimator), then the computing strategy to be applied will be presented and finally we will consider an illustrative example in which GRE succeeds whereas classic methods fail in avoiding multipath effects.

**2. MATHEMATICAL MODEL.** Let us restrict the formulation, just for simplicity, to the case of single frequency phase observations ( $L_1$  phases). Note, however, that the use of additional observation types would not modify the essentials of the estimation process that we are proposing.

Let us consider observed single phases of receivers  $i$  and  $j$  from satellites  $r$  and  $s$ . The double differenced phase observing equation may be written as

$$\nabla\Delta\phi_{ij}^{rs} = \nabla\Delta\rho_{ij}^{rs} - \lambda\nabla\Delta N_{ij}^{rs} + \nabla\Delta T_{ij}^{rs} - \nabla\Delta I_{ij}^{rs} + \varepsilon_{ij}^{rs} \tag{1}$$

where subscripts denote receivers and superscripts satellites,  $\nabla\Delta\phi_{ij}^{rs}$  is the observed double differenced phase,  $\nabla\Delta\rho_{ij}^{rs}$  the double differenced geometric range,  $\nabla\Delta N_{ij}^{rs}$  the double differenced ambiguity for wavelength  $\lambda$  and  $\nabla\Delta T_{ij}^{rs}$  and  $\nabla\Delta I_{ij}^{rs}$  respectively the double differenced tropospheric delay, which is usually modelled, and the double differenced ionospheric delay, which is usually neglected for short baselines (or which cancels out in the dual frequency case). Residuals  $\varepsilon_{ij}^{rs}$  are assumed to follow a Gaussian distribution centred on zero, which happens if no significant systematic errors are present in the functional model, i.e. if Equation (1) accurately represents the observation reality.

Fixing Cartesian geocentric coordinates for one end of the baseline (say  $X_i, Y_i, Z_i$ ), using any approximate coordinates for the other end of the baseline ( $X_{j0}, Y_{j0}, Z_{j0}$ ) and reformulating the problem as the determination of the coordinate differentials for the end of the baseline ( $dX_j, dY_j, dZ_j$ ) along with the double differenced ambiguities ( $\nabla\Delta N_{ij}^{rs}$ ), one can express the above equation, after adding the modelled tropospheric delay to  $\nabla\Delta\rho_{ij0}^{rs}$  and reordering, as

$$\begin{aligned} &\left(\left[\frac{\partial\rho_j^r}{\partial X_j}\right]_0 - \left[\frac{\partial\rho_j^s}{\partial X_j}\right]_0\right)dX_j + \left(\left[\frac{\partial\rho_j^r}{\partial Y_j}\right]_0 - \left[\frac{\partial\rho_j^s}{\partial Y_j}\right]_0\right)dY_j + \left(\left[\frac{\partial\rho_j^r}{\partial Z_j}\right]_0 - \left[\frac{\partial\rho_j^s}{\partial Z_j}\right]_0\right)dZ_j \\ &- \lambda\nabla\Delta N_{ij0}^{rs} = \nabla\Delta\phi_{ij0}^{rs} - \nabla\Delta\rho_{ij0}^{rs} - \varepsilon_{ij0}^{rs} \end{aligned} \tag{2}$$

where subscript 0 denotes particularization for the approximate coordinates ( $X_{j0}, Y_{j0}, Z_{j0}$ ).

Therefore, the system of double differenced phase equations can be expressed, as usual, by

$$\mathbf{Ax} = \mathbf{l} + \boldsymbol{\nu} \quad (3)$$

where  $\mathbf{A}$  is the coefficient matrix,  $\mathbf{x}$  the vector of unknowns – which contains differential corrections to approximate coordinates (float values) and ambiguities (integer values in theory) –  $\mathbf{l}$  is the vector of observed minus approximate values and finally  $\boldsymbol{\nu}$  is the vector of residuals.

This system of equations has to be considered along with weight matrix  $\mathbf{P}$ . A reasonable choice may be taken, for instance, from the SIGMA- $\epsilon$  weight model (Hartinger and Brunner, 1999). After adjustment, baseline components will be obtained by means of final coordinate differences:  $\Delta X = X_{j0} + dX_j - X_i$ ,  $\Delta Y = Y_{j0} + dY_j - Y_i$ ,  $\Delta Z = Z_{j0} + dZ_j - Z_i$ .

**3. ESTIMATION PROCESS.** We are in presence of a dual estimation process, that is, some unknowns (coordinates) are float numbers whereas others (ambiguities) have to be integers. Classically the solution can be obtained by several different methods but all of them involve the application, under one or other scheme, of the least squares estimator, which is highly vulnerable to gross and systematic errors. As mentioned, Wieser and Brunner (2002) proposed the use of robust estimators computed, as usual, by an IRLS scheme. We propose an alternative that according to Baselga (2007) means a significant improvement: the use of global robust estimation to overcome the possibility that robust estimation by IRLS falls only in a *local* optimum in the most complicated cases.

**3.1. Least squares estimation.** The classic least squares estimation process, see e.g. Teunissen 1995, is performed in three steps:

- Least squares solution of the system, Equation (3). This yields real-valued ambiguities, which result in an approximate estimation for the baseline coordinates, namely the “float solution”.
- Integer ambiguity determination. The search for the best set of integer ambiguities is performed in the neighbourhood of the least squares solution found in the previous step.
- Constrained least squares adjustment of the system with the integer ambiguities found in the previous step. The result is known as the baseline “fixed solution”.

Note that if prior assumptions, i.e. the validity of the model and the presence of only random errors in the observations do not hold, then least squares adjustment no longer yields the most likely solution. Such is the case, for example, in the presence of multipath effects. Robust estimation is then a powerful alternative.

**3.2. Robust estimation.** Robust estimation is based on minimizing a function – named estimator – in order to attain a solution maximally resistant to the non-fulfilment of the mathematical model or the appearance of gross or systematic errors in observations, so that it also yields a solution almost identical to that of classic least squares when the model is correct and the observations are affected only by random errors.

Let us apply one of the most successful robust estimators that is widely used in literature (Fuchs 1982, Harvey 1993, etc.): the  $L_1$ -norm. Its solution is the one that minimizes the sum of absolute values of the residuals

$$\min \sum |v_i| \quad (4)$$

Typically, the routine to compute such a problem is an IRLS scheme, that is, the estimator minimization (4) is performed by successive least squares adjustments computing for each iteration adapted weights based on the previous adjustment residuals. The equivalent weight function for the  $L_1$ -norm estimator to be used within least squares is

$$\omega_i = \frac{1}{|v_i|} \quad (5)$$

Therefore, minimizing the sum of squared residuals with weights (5) yields the  $L_1$ -norm minimization.

$$\min \sum v_i^2 \omega_i = \min \sum v_i^2 \frac{1}{|v_i|} = \min \sum |v_i| \quad (6)$$

However, the problem with robust estimation as IRLS is a computational one, for if the initial least squares solution lies far away from the correct solution then the iterative process may attain only a local optimum. It seems quite obvious that if one has decided to abandon least squares estimation in favour of robust estimation, then solving robust estimation by means of an iteratively re-weighted least squares process may not be the most desirable solution. In Baselga (2007) it was shown that, in fact, robust estimation by means of an iteratively re-weighted least squares process provides only a sort of “local robust estimation”, whereas robust estimation’s best capabilities are only achieved if it is dealt with as a global optimization problem, thus forgetting all reference to least squares. Below, we present a possible scheme for robust estimation as a global optimization problem, GRE in short.

3.3. *Global robust estimation.* Having decided to focus on the  $L_1$ -norm estimator, the estimation problem can be stated as

$$\begin{cases} \mathbf{Ax} = \mathbf{I} + v \\ \min \sum |v_i| \end{cases} \quad (7)$$

For convenience we have transformed the general system of correlated observations (3) into a system of uncorrelated observations (expressed here with the same notation) by means of singular value decomposition of the original weight matrix  $\mathbf{P}$  and subsequent transformation of (3) by pre-multiplication with the eigenvectors matrix, a procedure through which the vector of unknowns  $\mathbf{x}$  remains invariant. The resulting system of equations has also been reduced to unit weight.

We will now conduct a global optimization process in the search for the best vector  $\mathbf{x}$  in the sense of least  $L_1$ -norm. Several successful global optimization techniques have been developed up to the moment, among them Genetic Algorithms (GA) and the Simulated Annealing (SA) method. We will only sketch the algorithm for the use of SA, a method that tries to emulate the process of crystalline networks’ self-construction provided a sufficiently slow decrease in temperature occurs. In any

case, the reader is referred to Baselga (2007) and Berné and Baselga (2004) for further details. The computational scheme may be as follows:

1. Start with an initial random solution  $\mathbf{x}_0$  within the search domain  $D$ .

In the present case the search domain may be a sufficiently wide area centred in the initial least-squares float solution with boundaries  $x_i \pm k\sigma_{x_i}$  (for a sufficiently large  $k$ , 30 or 50 for instance). Note the different requirements for the solution vector elements: some need to be real values (coordinate increments) and the rest integer values (double differenced ambiguities).

2. Obtain an increment to the previous solution and thus a new solution (provided it is in the search domain)  $\mathbf{x}_{i+1} = \mathbf{x}_i + \Delta\mathbf{x}_i$ .

Displacement  $\Delta\mathbf{x}_i$  has to be generated at random, usually from a Gauss distribution whose standard deviation  $\sigma_i$  decreases with iterations. For example, one may start with an initial  $\sigma_{i0} = k\sigma_{x_i}/2$  so that the search domain can be easily inspected and apply a cooling factor of  $0.999999^n$ , where  $n$  is the iteration number. The affordable cooling speed depends on the problem complexity. Pay attention again to the different requirements for the solution vector elements: some are real values (coordinate increments) whilst other have to be integers (ambiguities).

3. Accept the new solution if it yields a better value for the score function – in our case the robust estimator (4) – otherwise: accept it with probability  $p$  (a small figure such as 0.01) or reject it.
4. Return to step 2 until typical displacement sizes  $\sigma_{x_i}$  are below the required precision (for example 0.001 m for coordinates).

The solution obtained is guaranteed to be the *global* optimum in probabilistic terms, i.e. if the cooling process has been sufficiently slow. At any rate, a *sufficient condition* for the global optimum attainment can be easily checked: repeated executions of the algorithm must provide the same result.

**4. EXAMPLE.** Three very short baselines forming a triangle were observed by GNSS techniques. One of its vertex, point  $A$ , was selected to be in an environment potentially affected by strong multipath, see Figure 1.

The results in Table 1 were obtained after an observation time of one hour and a register interval of 30 seconds by means of the *Trimble Geomatics Office 1.62* software. Multipath in point  $A$  is clearly manifest: the first and third baselines, those including  $A$ , have large reference variances and large standard errors, along with low ratios that make us doubt the validity of these determinations. However, the closure computation gives sensible results, which reinforces the validity of the baseline determinations.

Let us now reduce observation times to 8 minutes. An analogous computation by *Trimble Geomatics Office* yields the results in Table 2. In this case, the short observation period has prevented the computation procedure from obtaining a correct solution: the closure computation points towards some incorrect determination and the comparison with the solution obtained for a one hour observing period locates the incorrectness in baseline  $A-B$ . In addition, it is significant how the reference variance and the standard error have increased for this baseline in reducing the observation time. However, note that the ratio for this baseline has increased to a value of 4.2, which, deceptively, could have suggested that this

Table 1. Baseline solutions for an observation time of 1 hour by *TGO*.

Baseline	Ratio	$\sigma_b^2$	RMS (m)	$\Delta X$ (m)	$\Delta Y$ (m)	$\Delta Z$ (m)	Solution type
A–B	1.9	11.85	0.011	29.024	–17.219	–34.267	L <sub>1</sub> fixed
B–C	17.4	2.09	0.005	–25.365	14.902	29.597	L <sub>1</sub> fixed
A–C	2.2	17.65	0.014	3.659	–2.316	–4.665	L <sub>1</sub> fixed
Closure				0.000	–0.001	–0.005	

Table 2. Baseline solutions for an observation time of 8 minutes by *TGO* and component differences with respect to the 1 hour observation solution.

Baseline	Ratio	$\sigma_b^2$	RMS (m)	$\Delta X$ (m)	$\Delta Y$ (m)	$\Delta Z$ (m)	Diff. $\Delta X$ (m)	Diff. $\Delta Y$ (m)	Diff. $\Delta Z$ (m)	Solution type
A–B	4.2	45.22	0.030	29.040	–17.226	–34.398	0.016	–0.007	–0.131	L <sub>1</sub> fixed
B–C	9.4	2.04	0.005	–25.364	14.903	29.595	–0.001	–0.001	0.002	L <sub>1</sub> fixed
A–C	4.0	17.42	0.015	3.660	–2.320	–4.664	–0.001	0.004	–0.001	L <sub>1</sub> fixed
Closure				0.016	–0.003	–0.139				

Figure 1. Point *A* in a potential multipath environment.

incorrect determination was trustworthy. Moreover, a careful inspection of the residuals shows that marginal values (e.g. these exceeding  $2.5\sigma$ ) happen many times more than expected for the Gaussian behaviour assumption.

Finally, let us apply Global Robust Estimation to this example, in a similar fashion as we did in Baselga and García-Asenjo (2008) for coping with ionospheric delays. In this case we have followed the Simulated Annealing method as sketched in Section 3.3. We are trying to benefit from the fact that robust estimation is maximally resistant towards the non-fulfilment of the functional model and the appearance

Table 3. Baseline solutions for an observation time of 8 minutes by GRE (authors software) and component differences with respect to the 1 hour observation solution.

Baseline	$\Delta X$ (m)	$\Delta Y$ (m)	$\Delta Z$ (m)	Diff. $\Delta X$ (m)	Diff. $\Delta Y$ (m)	Diff. $\Delta Z$ (m)	Solution type
A–B	29.001	–17.233	–34.275	–0.023	–0.014	–0.008	L <sub>1</sub> fixed
B–C	–25.352	14.904	29.606	0.013	0.002	0.009	L <sub>1</sub> fixed
A–C	3.677	–2.332	–4.663	0.018	–0.016	0.002	L <sub>1</sub> fixed
Closure	–0.028	0.003	–0.006				

of gross errors, thus exploiting its ability to obtain correct results from largely contaminated samples. In our case, the highly contaminated sample – in which the proportion of reflected signals (i.e. with a multipath term) with respect to direct signals is high – is, in addition, of a short observing time (8 minutes). Results and comparison with the one-hour observation computations are shown in Table 3.

As we can see, contrarily to the classic baseline estimation process, GRE succeeds in obtaining correct baseline components despite the multipath affected observations and the short observation period.

5. CONCLUSION. Robust estimation computed as a global optimization process, or Global Robust Estimation, is presented as a powerful alternative to deal with multipath affected observations. After revising robust estimation basics, both for the classic iteratively re-weighted least squares scheme and for the successful global optimization scheme, we proposed to solve the functional model of double-differenced phases jointly in one step considering the dual nature of unknowns (float-integer) by a global optimization method upon minimization of a robust estimator functional. An example showed GRE advantages versus the classic least squares determination.

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