

THE IDENTITY OF ARGUMENT-PLACES

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Abstract. Argument-places play an important role in our dealing with relations. However, that does not mean that argument-places should be taken as primitive entities. It is possible to give an account of ‘real’ relations in which argument-places play no role. But if argument-places are not basic, then what can we say about their identity? Can they, for example, be reconstructed in set theory with appropriate urelements? In this article, we show that for some relations, argument-places cannot be modeled in a *neutral* way in $V[A]$, the cumulative hierarchy with basic ingredients of the relation as urelements. We argue that a natural way to conceive of argument-places is to identify them with abstract, structureless points of a derivative structure exemplified by positional frames. In case the relation has symmetry, these points may be indiscernible.

1. Introduction. “Adam” occurs first in “Adam loves Eve,” but Adam does not occur first in Adam’s loving Eve. The order is a representational artifact since there simply is no *intrinsic* order or direction between the arguments in the states of a relation. A more faithful, neutral representation makes use of unordered argument-places like *lover* and *beloved*. It is often assumed that such argument-places are primitive entities. But as convincingly argued by Fine (2000), there is a more basic neutral view on relations in which argument-places do not occur as primitives. In this so-called *antipositionalist* view on relations, a key role is played by the general operation of substitution.

If we should not take argument-places as primitive, what can we say about their identity? Can we (re)construct them in a satisfactory way? These questions are particularly of interest since argument-places play such a prominent role in the way we deal with relations in ordinary life.

In Leo (2008), we showed that for so-called *simple* relations of finite degree, we can construct argument-places or positions that are unique, modulo some equivalence relation. But this result does not seem completely satisfactory from a metaphysical point of view. Could we perhaps also construct them as unique in an absolute sense?

Whether this is possible may depend on the demands we want to impose on such a construction. One demand seems obvious: we do not want to allow arbitrary choices within the construction.

According to Fine, we can transform biased relations, that is, relations in which the arguments are ordered, into unbiased ones by taking a ‘permutation class’ of biased relations and by identifying each argument-place of the unbiased relation with a function that takes each biased relation of the permutation class into a corresponding numerical position (Fine, 2000, p. 15). However, Fine also mentions—without further elaboration—that there are certain complications (Fine, 2000, p. 15, footnote 9).

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We show that Fine's construction only works for relations without strict symmetry. Initially, I guessed it would be possible to develop a similar construction that would work for any relation. But after several attempts to find such a construction, this turned out to be impossible. This suggests that we should look for a radically different approach to define argument-places in a neutral way. We propose to define them as *abstractions* of the positions of positional frames for relations. This may be the most natural view on the identity of relations, although the ontological status of such abstract argument-places may still be a point of discussion.

The outline of this article is as follows. In Section 2, we give an informal explanation of the different views on relations and of ways to reconstruct argument-places. Then, in Section 3, we define mathematical models/frames corresponding with the views on relations. Most of our results are formulated in terms of relational frames.

Fine's construction to transform biased relations into unbiased ones is discussed in Section 4. In Section 5, we introduce a formal notion of *neutrality* of a set with respect to another set, and in Section 6, we show that argument-places cannot always be constructed in a formally neutral sense with respect to the permutation class of biased relational frames. We use this result in Section 7 to prove that for certain *simple* relations, no formally neutral reconstruction of argument-places is possible within the context of ordinary set theory.

In Section 8, we argue that the impossibility of the construction may be due to limitations of ordinary set theory as a modeling medium. In Section 9, we consider the possibility of conceiving argument-places as abstractions of the positions of positional frames for relations. We end in Section 10 with a consideration about the metaphysical relevance of the results.

2. Informal explanation. Fine (2000) presents three views on relations: the standard view, the positionalist view, and the antipositionalist view. For readers not familiar with Fine's article 'Neutral relations' or with my article 'Modeling relations', I start with a very brief characterization of the views.

According to the standard view, the arguments of any relation are ordered. We have, for example, the biased relation *loves* and its converse *is loved by*. But as we remarked in the Introduction, there is no order between the arguments of the state of Adam's loving Eve. So, if this state is a genuine relational complex, it should contain a relation in which the arguments are not ordered.

The positionalist view assumes that any relation comes with positions, as, for example, *lover* and *beloved*. A great advantage of this view is that we can identify a neutral relation for the state of Adam's loving Eve. But an ontological objection against this view is that it regards positions as part of the 'fundamental furniture of the universe'.

The antipositionalist view does not assume any ordering between arguments and any positions. Instead, the states of a relation form a network of states interrelated by *substitutions*. For example, from the state of Adam's loving Eve, we can obtain the state of Clark's loving Lois by substituting Clark for Adam and Lois for Eve.

A proponent of any of the three views may be interested in adequate positional representations for relations. The main question of this article is how positions of a relations can be constructed in an *unbiased* or *neutral* way. Here, we consider this question by discussing a few examples.

Take as starting point the set Φ of the biased relations *loves* and *is loved by*. These relations are permutations of each other. The first (second) numerical position of *loves* corresponds to the second (first) position of *is loved by*. Now we get an unbiased positional

representation of the amatory relation by identifying the position *lover* with the function from Φ to $\{0, 1\}$ that maps *loves* to 0 and *is loved by* to 1 and by identifying the position *beloved* with the function that maps *loves* to 1 and *is loved by* to 0.

Also for certain permutation classes of symmetric relations, we may define positions in a neutral way. Take for example the adjacency relation. Then, the permutation class of this relation consists just of nothing but the adjacency relation itself. Defining positions as the set $\{0, 1\}$ gives an unbiased representation.

Unfortunately, not for every relation with symmetry, we can construct positions as a neutral set. Take the complex relation \mathfrak{R} with $\mathfrak{R}abcd$ the state of *a's loving b, b's hating c, c's loving d, and d's hating a*. We argue in Section 6 that such a relation puts us in a situation similar to that of Buridan's ass. It seems impossible to define positions for such relations as a set in $V[S, O]$, the cumulative hierarchy with states S and objects O of the relation as urelements, in a completely nonarbitrary way.

But if positions cannot be defined in $V[S, O]$ in a neutral way, then there is still a viable alternative. We propose to define positions as abstractions, that is, as structureless places exemplified by representations in $V[S, O]$. Assigning objects to such positions *yields* states. Despite the fact that such positions may be indiscernible, it would in general be wrong to assume that objects *occupy* positions within the states. In the next sections, these ideas are elaborated in more detail.

3. Relational models. In Leo (2008), we defined *frames* for relations to model the logical space of relations, the frames being all of the form $\langle S, O, \dots \rangle$, where S is a nonempty set of states and O a nonempty set of objects. We reserved the word *models* for extensions of the frames with a subset H of S representing the states that obtain. Here, we briefly repeat the main definitions.

3.1. Relational frames. We present three types of frames corresponding with different views on relations as presented by Fine (2000):

- directional frames* \sim *standard view*
- positional frames* \sim *positionalist view*
- substitution frames* \sim *antipositionalist view*.

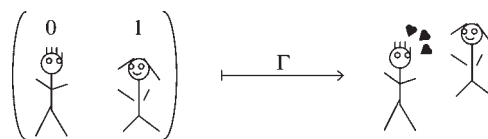
These type of frames are not uniquely defined by the different views. But for our purposes, they adequately model the different views on relations.

3.1.1. Directional frames. A *directional frame* models the logical space of relations in which the arguments are ordered in a specific way.

DEFINITION 3.1. A directional frame is a quadruple $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, α is an ordinal number, and Γ is a function from O^α to S .

We call the cardinality of α the *degree* of the frame. We denote it as $\text{degree}_{\mathcal{F}}$.

For the relation *loves*, we can make a directional frame with Γ depicted as:

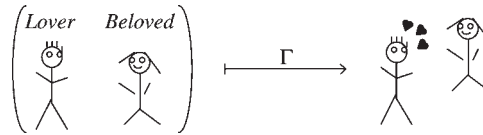


3.1.2. *Positional frames.* A positional frame models the logical space of neutral relations, that is, relations for which the order of the arguments is irrelevant.

DEFINITION 3.2. A positional frame is a quadruple $\mathcal{F} = \langle S, O, P, \Gamma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, P is a set of positions, and Γ is a function from O^P to S .

We call the cardinality of P the degree of the frame. We denote it as $\text{degree}_{\mathcal{F}}$.

For the love relation, we can make a positional frame with Γ depicted as:



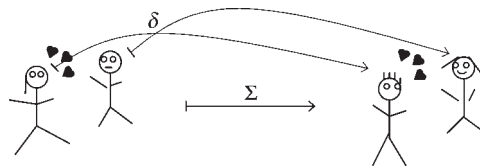
3.1.3. *Substitution frames.* A substitution frame also models the logical space of neutral relations. This type of frame is more abstract and at first sight probably more difficult to appreciate than the two other types. It might be helpful to take a look at Leo (2005, pp. 23–25) where substitution frames are developed in a number of steps.

DEFINITION 3.3. A substitution frame is a triple $\mathcal{F} = \langle S, O, \Sigma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, and Σ is a function from $S \times O^O$ to S such that:

1. $\Sigma(s, \text{id}_O) = s$,
2. $\Sigma(s, \delta' \circ \delta) = \Sigma(\Sigma(s, \delta), \delta')$.

For convenience, we often write $s \cdot_{\mathcal{F}} \delta$ or $s \cdot \delta$ for $\Sigma(s, \delta)$, and $f \cdot g$ for $g \circ f$.

For the love relation, we can make a substitution frame with Σ depicted as:



We define for a substitution frame the *objects* of its states and the *degree* of its states and of the frame itself as follows.

DEFINITION 3.4. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. We call $A \subseteq O$ an object-domain of $s \in S$ if for every $\delta, \delta' : O \rightarrow O$:¹

$$\delta =_A \delta' \Rightarrow s \cdot \delta = s \cdot \delta'.$$

We define the core of s as:

$$\text{Core}_{\mathcal{F}}(s) = \bigcap \{A \mid A \text{ is an object-domain of } s\}.$$

If $\text{Core}_{\mathcal{F}}(s)$ is an object-domain, then we call this set the objects of s . We denote this set as $\text{Ob}_{\mathcal{F}}(s)$. If $\text{Core}_{\mathcal{F}}(s)$ is not an object-domain, then we leave $\text{Ob}_{\mathcal{F}}(s)$ undefined.

We often write $\text{Core}(s)$ and $\text{Ob}(s)$ for $\text{Core}_{\mathcal{F}}(s)$ and $\text{Ob}_{\mathcal{F}}(s)$.

¹ We say that $f =_X g$ if $f \upharpoonright X = g \upharpoonright X$, that is, f restricted to X is equal to g restricted to X .

DEFINITION 3.5. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. For a state s in S , we define the degree of s as:

$$\text{degree}_{\mathcal{F}}(s) = \text{glb} \{|A| \mid A \text{ is an object-domain of } s\}.$$

The degree of \mathcal{F} we define as:

$$\text{degree}_{\mathcal{F}} = \text{lub} \{\text{degree}_{\mathcal{F}}(s) \mid s \in S\}.$$

Here, $|A|$ denotes as usual the cardinality of A , glb denotes the greatest lower bound, and lub denotes the least upper bound.

3.2. Permutations and positional variants. Directional frames have *permutations*.

DEFINITION 3.6. A directional frame $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ is a permutation of a frame $\mathcal{F}' = \langle S', O', \alpha', \Gamma' \rangle$ if $S = S'$, $O = O'$, $\alpha = \alpha'$, and there is a bijection $\pi : \alpha \rightarrow \alpha'$ such that for each $f \in O^\alpha$, $\Gamma(f) = \Gamma'(f \circ \pi)$.

We denote \mathcal{F} as $\pi(\mathcal{F}')$ and define the permutation class of \mathcal{F} as:

$$\Phi_{\mathcal{F}} = \{\pi(\mathcal{F}) \mid \pi \in \text{Perm}(\alpha)\},$$

with $\text{Perm}(\alpha)$ the bijections from α to α .

We say that \mathcal{F} has strict symmetry if there is a bijection $\pi : \alpha \rightarrow \alpha$ with $\pi \neq \text{id}_{\alpha}$ such that $\mathcal{F} = \pi(\mathcal{F})$.

Note that if \mathcal{F} is a permutation of \mathcal{F}' , then $\Phi_{\mathcal{F}} = \Phi_{\mathcal{F}'}$.

In a similar way, we define for positional frames the notion of *positional variants*.

DEFINITION 3.7. A positional frame $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ is a positional variant of a frame $\mathcal{F}' = \langle S', O', P', \Gamma' \rangle$ if $S = S'$, $O = O'$, and there is a bijection $\pi : P' \rightarrow P$ such that for each $f \in O^P$, $\Gamma(f) = \Gamma'(f \circ \pi)$.

We denote \mathcal{F} as $\pi(\mathcal{F}')$.

3.3. Corresponding frames. Directional and positional frames may correspond in an obvious way.

DEFINITION 3.8. A directional frame $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ and a positional frame $\mathcal{G} = \langle S', O', P, \Gamma' \rangle$ correspond if $S = S'$, $O = O'$, and there is a bijection $\mu : P \rightarrow \alpha$ such that for each $f \in O^\alpha$, $\Gamma(f) = \Gamma'(f \circ \mu)$.

We denote \mathcal{F} as $\mu(\mathcal{G})$.

Note that directional frames corresponding to a positional frame \mathcal{G} are not necessarily permutations of each other. This is a consequence of our somewhat arbitrary choice to demand in the definition of permutations of directional frames that $\alpha = \alpha'$.

For substitution frames and directional/positional frames, we define correspondence as follows.

DEFINITION 3.9. A substitution frame $\mathcal{F} = \langle S, O, \Sigma \rangle$ and a directional/positional frame $\mathcal{G} = \langle S', O', X, \Gamma \rangle$ correspond if:

1. $S = S' = \text{im } \Gamma$,
2. $O = O'$,
3. $\Gamma(f) \cdot_{\mathcal{F}} \delta = \Gamma(f \cdot \delta)$.

As said at the beginning of Section 3, relational models can be defined by extending the frames with a subset H of S representing the states that obtain. But also more luxurious

models can be considered by taking into account possible worlds where states can obtain. In our analysis, however, only the logical space of relations plays a role.

4. Fine’s construction. In this section, we show that directional frames without strict symmetry, that is, frames for which $\mathcal{F} = \pi(\mathcal{F})$ only if $\pi = \text{id}_\alpha$, have corresponding positional frames that are uniquely determined by $\Phi_{\mathcal{F}}$, the permutation class of \mathcal{F} . The construction of the positional frames is essentially Fine’s construction to transform biased relations into unbiased ones (Fine, 2000, p. 15).

THEOREM 4.1. *Let $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ be a directional frame without strict symmetry, and let $\mathcal{G}_{\mathcal{F}} = \langle S, O, P, \bar{\Gamma} \rangle$ be defined by:*

$$P = \{p : \Phi_{\mathcal{F}} \rightarrow \alpha \mid \forall \pi \in \text{Perm}(\alpha) (p(\pi(\mathcal{F})) = \pi(p(\mathcal{F})))\},$$

$$\bar{\Gamma}(f) = \Gamma(f \circ \mu) \text{ with } \mu : \alpha \rightarrow P \text{ such that } \forall i \in \alpha (\mu(i)(\mathcal{F}) = i).$$

Then, $\mathcal{G}_{\mathcal{F}}$ is a positional frame corresponding to \mathcal{F} , and if \mathcal{F}' is a permutation of \mathcal{F} , then $\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{\mathcal{F}'}$.

Proof. We show that (1) $\mathcal{G}_{\mathcal{F}}$ is a well-defined positional frame; (2) $\mathcal{G}_{\mathcal{F}}$ corresponds to \mathcal{F} ; (3) if $\mathcal{F}' = \tau(\mathcal{F})$, then $\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{\mathcal{F}'}$.

1. To prove that $\bar{\Gamma}$ is well defined, it is sufficient to show that there is exactly one function $\mu \in P^\alpha$ such that $\forall i \in \alpha (\mu(i)(\mathcal{F}) = i)$. We assumed that \mathcal{F} has no strict symmetry. So, $\pi(\mathcal{F}) = \pi'(\mathcal{F}) \Leftrightarrow \pi = \pi'$. Therefore, for every $i \in \alpha$, there is exactly one $p \in P$ with $p(\mathcal{F}) = i$.

2. To prove that $\mathcal{G}_{\mathcal{F}}$ corresponds to \mathcal{F} , it is sufficient to show that μ is bijective. Now μ is clearly injective because for every $i \in \alpha$, $\mu(i)(\mathcal{F}) = i$. Furthermore, let p be an arbitrary element of P . Then, $\mu(p(\mathcal{F}))(\mathcal{F}) = p(\mathcal{F})$, from which it follows that $\mu(p(\mathcal{F})) = p$. So, μ is also surjective.

3. Let $\mathcal{F}' = \langle S, O, \alpha, \Gamma' \rangle$ be another frame in $\Phi_{\mathcal{F}}$, say $\mathcal{F}' = \tau(\mathcal{F})$, and let $\mathcal{G}_{\mathcal{F}'} = \langle S, O, P', \bar{\Gamma}' \rangle$. To prove that $P' = P$, let p be an element of P . Then,

$$\begin{aligned} p(\pi(\mathcal{F}')) &= p(\pi(\tau(\mathcal{F}))) \\ &= p((\pi \circ \tau)(\mathcal{F})) \\ &= (\pi \circ \tau)(p(\mathcal{F})) \\ &= \pi(\tau(p(\mathcal{F}))) \\ &= \pi(p(\tau(\mathcal{F}))) \\ &= \pi(p(\mathcal{F}')). \end{aligned}$$

Thus, p also belongs to P' . So, $P \subseteq P'$ and, mutatis mutandis, $P' \subseteq P$.

To prove that $\bar{\Gamma}' = \bar{\Gamma}$, let $\mu' \in P^\alpha$ be such that $\forall i \in \alpha (\mu'(i)(\mathcal{F}') = i)$. We have to prove that for any $f \in O^P$, $\Gamma'(f \circ \mu') = \Gamma(f \circ \mu)$. Because $\mathcal{F}' = \tau(\mathcal{F})$, we have $\Gamma'(f) = \Gamma(f \circ \tau)$. So, in particular, $\Gamma'(f \circ \mu') = \Gamma(f \circ \mu' \circ \tau)$. So, it is sufficient to prove that $\mu' \circ \tau = \mu$.

$$\begin{aligned}
 (\mu' \circ \tau)(i)(\mathcal{F}) &= \mu'(\tau(i))(\mathcal{F}) \\
 &= \mu'(\tau(i))(\tau^{-1}(\mathcal{F}')) \\
 &= \tau^{-1}(\mu'(\tau(i))(\mathcal{F}')) \\
 &= \tau^{-1}(\tau(i)) \\
 &= i.
 \end{aligned}$$

Because there is exactly one $\bar{\mu} \in P^\alpha$ such that $\forall i \in \alpha (\bar{\mu}(i)(\mathcal{F}) = i)$, we see that $\mu' \circ \tau = \mu$. □

Unfortunately, if \mathcal{F} has strict symmetry, then the construction of $\mathcal{G}_{\mathcal{F}}$ in Theorem 4.1 fails because then not for every $i \in \alpha$, there is a $p \in P$ with $p(\mathcal{F}) = i$. In such cases, P may contain less than $\text{card}(\alpha)$ elements. For example, if \mathcal{F} has complete strict symmetry, then P is empty. And if \mathcal{F} is a ternary frame with $\mathcal{F} = \pi(\mathcal{F})$ iff $\pi(1) = 1$, then P does not contain three functions—as we would like to have—but only one.

5. Neutrality. The positional frame constructed in Theorem 4.1 is in an intuitive sense *neutral* with respect to the permutation class of \mathcal{F} . In this section, we give a formal definition of neutrality for set theory with atoms or urelements. We use this notion in the next section to show that the permutation classes of some directional frames have no neutral corresponding positional frame in the cumulative hierarchy with the elements of S and O as urelements.

Let $V[A]$ be the cumulative hierarchy with atoms A . Any function $u : A \rightarrow A$ can be lifted to a function $\tilde{u} : V[A] \rightarrow V[A]$ in an obvious way:

- $\tilde{u}(a) = u(a)$ for any $a \in A$,
- $\tilde{u}(X) = \{\tilde{u}(x) \mid x \in X\}$.

We may regard $\tilde{u}(X)$ as the result of a transformation where for each $a \in A$, all its occurrences in X are substituted by $u(a)$.

We treat any function f as a set of ordered pairs. Thus, we may speak about $\tilde{u}(f)$ as the image of this set.

We have the following elementary properties for \tilde{u} .

LEMMA 5.1. *Let $u : A \rightarrow A$ be lifted to $\tilde{u} : V[A] \rightarrow V[A]$. Then:*

1. $\widetilde{u \circ v} = \tilde{u} \circ \tilde{v}$.
2. If $u = \text{id}_A$, then $\tilde{u} = \text{id}_{V[A]}$.
3. u is injective iff \tilde{u} is injective.
4. If u is bijective, then $\widetilde{u^{-1}} = \tilde{u}^{-1}$.
5. If u is injective, then \tilde{u} maps any function $f : X \rightarrow Y$ with $X, Y \in V[A]$ to a function $\tilde{u}(f) : \tilde{u}(X) \rightarrow \tilde{u}(Y)$ with

$$(\tilde{u}(f))(\tilde{u}(x)) = \tilde{u}(f(x)).$$

6. If u is injective, then \tilde{u} is an endo-functor on $V[A]$, that is,

(a) For any $X \in V[A]$, $\tilde{u}(\text{id}_X) = \text{id}_{\tilde{u}(X)}$

(b) For any function $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ with $X, Y, Z \in V[A]$,

$$\tilde{u}(f \cdot g) = \tilde{u}(f) \cdot \tilde{u}(g).$$

Proof. We prove Property 1 by \in -induction: (i) If $x \in A$, then $\widetilde{u \circ v}(x) = u \circ v(x) = u \circ \widetilde{v}(x) = \widetilde{u} \circ \widetilde{v}(x)$. (ii) Let $x \in \mathbb{V}[A]$ and assume $\widetilde{u \circ v}(z) = \widetilde{u} \circ \widetilde{v}(z)$ for every $z \in x$. Then, $\widetilde{u \circ v}(x) = \{\widetilde{u \circ v}(z) \mid z \in x\} = \{\widetilde{u} \circ \widetilde{v}(z) \mid z \in x\} = \widetilde{u}(\{\widetilde{v}(z) \mid z \in x\}) = \widetilde{u} \circ \widetilde{v}(x)$. So, by \in -induction, $\widetilde{u \circ v} = \widetilde{u} \circ \widetilde{v}$.

Properties 2–4 can be proved in a similar way by \in -induction.

To prove Property 5, assume $u : A \rightarrow A$ is injective. Then, by Property 3, \widetilde{u} is also injective. So, if f is a function, then $\widetilde{u}(f)$ is a function as well. Furthermore, if $f : x \mapsto y$, then $\widetilde{u}(f) : \widetilde{u}(x) \mapsto \widetilde{u}(y)$.

Property 6(a) is trivial and Property 6(b) follows from Property 5:

$$\begin{aligned} (\widetilde{u}(f \cdot g))(\widetilde{u}(x)) &= \widetilde{u}((f \cdot g)(x)) \\ &= \widetilde{u}(g(f(x))) \\ &= (\widetilde{u}(g))(\widetilde{u}(f(x))) \\ &= (\widetilde{u}(g))((\widetilde{u}(f))(\widetilde{u}(x))) \\ &= (\widetilde{u}(f) \cdot \widetilde{u}(g))(\widetilde{u}(x)). \end{aligned} \quad \square$$

Using this lemma, it is easy to prove that if \mathcal{F} and \mathcal{G} are corresponding frames in $\mathbb{V}[S, O]$ and $u : S \cup O \rightarrow S \cup O$ is injective, then $\widetilde{u}(\mathcal{F})$ and $\widetilde{u}(\mathcal{G})$ correspond as well.

We now define what it means for a set in $\mathbb{V}[A]$ to be neutral with respect to another set in $\mathbb{V}[A]$.

DEFINITION 5.2. For $X, Y \in \mathbb{V}[A]$, we say that Y is neutral with respect to X if for any bijection $u : A \rightarrow A$,

$$\widetilde{u}(X) = X \implies \widetilde{u}(Y) = Y.$$

Note that if Y is neutral with respect to X and Z is neutral with respect to Y , then Z is also neutral with respect to X . However, if Y and Z are both neutral with respect to X , then Z is not necessarily neutral with respect to Y .

EXAMPLE 5.3. Let $A = \{a, b\}$ be a set of atoms. Then, $\{a, b\}$ and every set in \mathbb{V} are neutral with respect to $\{a, b\}$, but $\{a\}$ and $\{a, \{b\}\}$ are not. However, every set in $\mathbb{V}[A]$ is neutral with respect to $\{a, \{b\}\}$.

To see the relevance of this formal notion of neutrality for modeling ‘real’ structures, consider a set A of specific entities. Suppose we are given a set $X \in \mathbb{V}[A]$ that models certain connections between the entities in A . Then, for any deterministic construction of a set Y purely on the basis of the structure of X (treating the elements of A as atoms), the set Y must be neutral with respect to X . Thus, if we can show that no member of a certain class of models is neutral with respect to a given model, then this may give us valuable information about the impossibility of certain constructions.

6. Neutrality w.r.t. permutation classes. In this section, we treat the states S and objects O as urelements. Unless mentioned otherwise, we do not assume that S and O are disjoint.

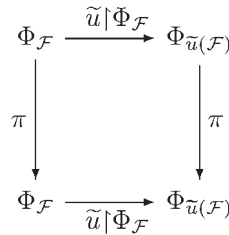
Let us now face the case in which a directional frame $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ may have strict symmetry. Then, it is simple to construct a corresponding positional frame $\mathcal{G} = \langle S, O, P, \overline{\Gamma} \rangle$ in which all frames in $\Phi_{\mathcal{F}}$, the permutation class of \mathcal{F} , are more or less equally well represented:

1. Choose $\Pi \subseteq \text{Perm}(\alpha)$ with $\text{id}_\alpha \in \Pi$ and $\forall \mathcal{F}' \in \Phi_{\mathcal{F}} \exists! \pi \in \Pi (\pi(\mathcal{F}) = \mathcal{F}')$.
2. Define $P = \{p : \Phi_{\mathcal{F}} \rightarrow \alpha \mid \forall \pi \in \Pi (p(\pi(\mathcal{F})) = \pi(p(\mathcal{F})))\}$.
3. Define $\bar{\Gamma}(f) = \Gamma(f \circ \mu)$ with μ such that $\forall i \in \alpha (\mu(i)(\mathcal{F}) = i)$.

We can prove in a similar way as we did for the frames without strict symmetry that \mathcal{G} corresponds to \mathcal{F} . But if \mathcal{F} belongs to $\mathbf{V}[S, O]$, then \mathcal{G} is not necessarily neutral with respect to the permutation class of \mathcal{F} , as a consequence of Theorem 6.7 later in this section.

The next lemma shows that if $u : S \cup O \rightarrow S \cup O$ is injective, then $\tilde{u} \upharpoonright \Phi_{\mathcal{F}}$ is a structure preserving mapping.

LEMMA 6.1. *Let $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle \in \mathbf{V}[S, O]$ be a directional frame. If $u : S \cup O \rightarrow S \cup O$ is injective and $\pi \in \text{Perm}(\alpha)$, then $\tilde{u} \upharpoonright \Phi_{\mathcal{F}} : \Phi_{\mathcal{F}} \rightarrow \Phi_{\tilde{u}(\mathcal{F})}$ is a bijection for which the following diagram commutes:*



Proof. Let $\pi(\mathcal{F}) = \langle S, O, \alpha, \Gamma' \rangle$. Then, by Lemma 5.1,

$$(\tilde{u}(\Gamma'))(\tilde{u}(f)) = \tilde{u}(\Gamma'(f)) = \tilde{u}(\Gamma(f \circ \pi)) = (\tilde{u}(\Gamma))(\tilde{u}(f) \circ \pi).$$

So, $\tilde{u}(\pi(\mathcal{F})) = \pi(\tilde{u}(\mathcal{F}))$, from which it follows that the diagram of the lemma commutes and that $\tilde{u} \upharpoonright \Phi_{\mathcal{F}} : \Phi_{\mathcal{F}} \rightarrow \Phi_{\tilde{u}(\mathcal{F})}$ is surjective. By Property 3 of Lemma 5.1, we also see that $\tilde{u} \upharpoonright \Phi_{\mathcal{F}}$ is injective. □

As we might have expected, if \mathcal{F} has no strict symmetry, then the corresponding positional frame $\mathcal{G}_{\mathcal{F}}$ defined in Theorem 4.1 is neutral with respect to $\Phi_{\mathcal{F}}$.

THEOREM 6.2. *Let $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle \in \mathbf{V}[S, O]$ be a directional frame without strict symmetry, and let $\mathcal{G}_{\mathcal{F}} = \langle S, O, P, \bar{\Gamma} \rangle$ be defined by:*

$$\begin{aligned}
 P &= \{p : \Phi_{\mathcal{F}} \rightarrow \alpha \mid \forall \pi \in \text{Perm}(\alpha) (p(\pi(\mathcal{F})) = \pi(p(\mathcal{F})))\}, \\
 \bar{\Gamma}(f) &= \Gamma(f \circ \mu) \text{ with } \mu \text{ such that } \forall i \in \alpha (\mu(i)(\mathcal{F}) = i).
 \end{aligned}$$

Then, $\mathcal{G}_{\mathcal{F}}$ is neutral with respect to $\Phi_{\mathcal{F}}$.

Proof. Let $u : S \cup O \rightarrow S \cup O$ be a bijection such that $\tilde{u}(\Phi_{\mathcal{F}}) = \Phi_{\mathcal{F}}$. We show that $\tilde{u}(P) = P$ and that $\tilde{u}(\bar{\Gamma}) = \bar{\Gamma}$.

With the use of Lemmas 5.1 and 6.1, it is easy to see that:

$$\tilde{u}(P) = \{p : \tilde{u}(\Phi_{\mathcal{F}}) \rightarrow \alpha \mid \forall \pi \in \text{Perm}(\alpha) (p(\pi(\tilde{u}(\mathcal{F}))) = \pi(p(\tilde{u}(\mathcal{F})))\}.$$

Because $\Phi_{\mathcal{F}} = \tilde{u}(\Phi_{\mathcal{F}})$ and because, by Theorem 4.1, $\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{\tilde{u}(\mathcal{F})}$, it follows that $\tilde{u}(P) = P$. Furthermore, by Lemma 5.1,

$$(\tilde{u}(\bar{\Gamma}))(\tilde{u}(f)) = (\tilde{u}(\Gamma))((\tilde{u}(f)) \circ \tilde{u}(\mu)),$$

with $\tilde{u}(\mu)$ such that $\forall i \in \alpha ((\tilde{u}(\mu))(i)(\tilde{u}(\mathcal{F})) = i)$.

Because $\tilde{u}(P) = P$ and because $\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{\tilde{u}(\mathcal{F})}$, it also follows that $\tilde{u}(\bar{\Gamma}) = \bar{\Gamma}$. □

For the next results, we need the notion of *permutation-based frames*.

DEFINITION 6.3. Let $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ be a directional frame. We define the permutation group of \mathcal{F} as:

$$\text{Perm}_{\mathcal{F}} = \{\pi \in \alpha^\alpha \mid \pi \text{ is a bijection \& } \forall f \in O^\alpha (\Gamma(f \circ \pi) = \Gamma(f))\}.$$

We call \mathcal{F} a permutation-based frame if for every $f, f' \in \alpha^\alpha$,

$$\Gamma(f') = \Gamma(f) \Rightarrow f' = f \circ \pi \text{ for some } \pi \in \text{Perm}_{\mathcal{F}}.$$

Every directional frame \mathcal{F} with complete strict symmetry has a corresponding positional frame that is neutral with respect to $\Phi_{\mathcal{F}}$.

EXAMPLE 6.4. Let $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle \in \mathbf{V}[S, O]$ be a directional frame with complete strict symmetry, that is, $\text{Perm}_{\mathcal{F}} = \text{Perm}(\alpha)$. Then obviously, $\langle S, O, \alpha, \Gamma \rangle$ considered as a positional frame is neutral with respect $\Phi_{\mathcal{F}}$ since \mathcal{F} is the only frame in $\Phi_{\mathcal{F}}$.

Also for certain cyclic frames, we have a positive result.

EXAMPLE 6.5. Let $\mathcal{F} = \langle S, O, 4, \Gamma \rangle \in \mathbf{V}[S]$ be a permutation-based frame with permutation group $\text{Perm}_{\mathcal{F}}$ generated by:

$$\pi_0 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix}.$$

Let $\pi_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix}$. Define $\mathcal{G}_{\mathcal{F}} = \langle S, O, P, \bar{\Gamma} \rangle$ with:

$$P = \{p : \{\mathcal{F}, \pi_1(\mathcal{F})\} \rightarrow \{0, 1, 2, 3\} \mid p(\pi_1(\mathcal{F})) = \pi_1(p(\mathcal{F}))\},$$

$$\bar{\Gamma}(f) = \Gamma(f \circ \mu) \text{ with } \mu \text{ such that } \forall i \in \alpha (\mu(i)(\mathcal{F}) = i).$$

Then, $\mathcal{G}_{\mathcal{F}}$ corresponds to \mathcal{F} and is neutral with respect to $\Phi_{\mathcal{F}}$.

To see this, first note that by Lemma 6.1, if $u : S \rightarrow S$ is a bijection such that $\tilde{u}(\Phi_{\mathcal{F}}) = \Phi_{\mathcal{F}}$, then $\tilde{u}(\mathcal{F}) = \mathcal{F}$ or $\tilde{u}(\mathcal{F}) = \pi_1(\mathcal{F})$. It follows by an analysis similar to the one in Theorem 6.2 that $\tilde{u}(P) = P$ and $\tilde{u}(\bar{\Gamma}) = \bar{\Gamma}$.

In Section 4, we saw how to create a neutral positional frame for the love relation. But now consider relation \mathfrak{R} in which $\mathfrak{R}abcd$ represents the state of a 's loving b and c 's loving d . Let $\mathcal{F} \in \mathbf{V}[S, O]$ be a directional frame for this relation, and let $\mathcal{G} \in \mathbf{V}[S, O]$ be a corresponding positional frame. We claim that \mathcal{G} cannot be neutral with respect to the permutation class $\Phi_{\mathcal{F}}$. We sketch a proof in the next example. Then, in Theorem 6.7, we prove a generalization of the claim.

EXAMPLE 6.6. Let $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle \in \mathbf{V}[S, O]$ be a directional frame with for any $a, b, c, d \in O$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ a & b & c & d \end{pmatrix} \stackrel{\Gamma}{\vdash} a \rightsquigarrow b \ \& \ c \rightsquigarrow d.$$

Let $\mathcal{G} = \langle S, O, P, \bar{\Gamma} \rangle \in \mathbf{V}[S, O]$ be a corresponding positional frame with $P = \{p_0, p_1, p_2, p_3\}$, and

$$\begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ a & b & c & d \end{pmatrix} \stackrel{\bar{\Gamma}}{\vdash} a \rightsquigarrow b \ \& \ c \rightsquigarrow d.$$

Let $u : S \cup O \rightarrow S \cup O$ be such that,

$$u(a \rightsquigarrow b \ \& \ c \rightsquigarrow d) = b \rightsquigarrow c \ \& \ d \rightsquigarrow a,$$

and $u \upharpoonright O = \text{id}_O$. Obviously, $u \circ u = \text{id}_{S \cup O}$, and so, by Lemma 5.1, $\tilde{u} \circ \tilde{u} = \text{id}_{V[S, O]}$. Furthermore, it is not difficult to see that $\tilde{u}(\Phi_{\mathcal{F}}) = \Phi_{\mathcal{F}}$.

Now suppose \mathcal{G} is neutral with respect to $\Phi_{\mathcal{F}}$. Then,

$$\begin{pmatrix} \tilde{u}(p_0) & \tilde{u}(p_1) & \tilde{u}(p_2) & \tilde{u}(p_3) \\ a & b & c & d \end{pmatrix} \xrightarrow{\bar{\Gamma}} b \not\leftrightarrow c \ \& \ d \not\leftrightarrow a.$$

Thus,

$$\tilde{u}(p_0, p_1, p_2, p_3) = (p_1, p_2, p_3, p_0) \text{ or } \tilde{u}(p_0, p_1, p_2, p_3) = (p_3, p_0, p_1, p_2).$$

But then, $\tilde{u} \upharpoonright P \circ \tilde{u} \upharpoonright P \neq \text{id}_P$, contradicting $\tilde{u} \circ \tilde{u} = \text{id}_{V[S, O]}$. Therefore, \mathcal{G} cannot be neutral with respect to $\Phi_{\mathcal{F}}$.

How should we interpret this example? It surely does not say that it is impossible to find a natural positional frame for the disjoint conjunction of two love relations. A quite natural positional frame for it is the frame with positions *Lover*₁, *Beloved*₁, *Lover*₂, and *Beloved*₂. Interestingly, this frame is neutral with respect to the permutation class of \mathcal{F} if \mathcal{F} would have been defined in $V[S_0, O]$, with S_0 being the states of the ordinary love relation, and the conjunction of states s_1 and s_2 would have been modeled as $\{s_1, s_2\}$.

In the example, we could depict each state as four objects equally spaced on a circle such that rotating them by 180° always gives the same state, but rotating them by 90° gives a different state when the objects are not all the same. In the next theorem, we prove that any positional frame $\mathcal{F} \in V[S, O]$ with this property and S, O disjoint has no corresponding positional frame that is neutral with respect to the permutation class of \mathcal{F} . We use this theorem to argue that it is very unlikely that for any ‘real’ relation, we can always make a neutral choice for a corresponding positional frame.

THEOREM 6.7. *Let $\mathcal{F} = \langle S, O, 4, \Gamma \rangle \in V[S, O]$ be a permutation-based frame with $S \cap O = \emptyset$ and permutation group $\text{Perm}_{\mathcal{F}}$ generated by:*

$$\pi_0 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix}.$$

Then, no corresponding positional frame is neutral with respect to $\Phi_{\mathcal{F}}$.

Proof. Let $\pi_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix}$. Define $u : S \cup O \rightarrow S \cup O$ by:

$$u(x) = \begin{cases} \Gamma(f \circ \pi_1) & \text{if } x = \Gamma(f), \\ x & \text{otherwise.} \end{cases}$$

We show that:

1. u is a well-defined bijective function with $u \circ u = \text{id}_{S \cup O}$.
2. $\tilde{u}(\Phi_{\mathcal{F}}) = \Phi_{\mathcal{F}}$.
3. $\tilde{u}(\mathcal{G}) \neq \mathcal{G}$ for any positional frame \mathcal{G} corresponding to \mathcal{F} .

1. Suppose $\Gamma(f') = \Gamma(f)$. Then, because \mathcal{F} is permutation based, $f' = f$ or $f' = f \circ \pi_0$. Assume $f' = f \circ \pi_0$. Then, $f' \circ \pi_1 = f \circ \pi_1 \circ \pi_0$ because $\pi_0 = \pi_1 \circ \pi_1$. So, $\Gamma(f' \circ \pi_1) = \Gamma(f \circ \pi_1)$ because $\pi_0 \in \text{Perm}_{\mathcal{F}}$. It follows that u is well defined. Because $(u \circ u)(\Gamma(f)) = \Gamma(f \circ \pi_1 \circ \pi_1) = \Gamma(f \circ \pi_0) = \Gamma(f)$, we see that $u \circ u = \text{id}_{S \cup O}$. So, u is bijective and, by Lemma 5.1, $\tilde{u} \circ \tilde{u} = \text{id}_{V[S, O]}$.

2. Because $S \cap O = \emptyset$, we have $u \upharpoonright O = \text{id}_O$. Thus, $\tilde{u}(\mathcal{F}) = \pi_1(\mathcal{F})$, and so, because $\Phi_{\mathcal{F}} = \Phi_{\pi_1(\mathcal{F})}$, we have by Lemma 6.1, $\tilde{u}(\Phi_{\mathcal{F}}) = \Phi_{\mathcal{F}}$.

3. Let $\mathcal{G} = \langle S, O, P, \bar{\Gamma} \rangle$ be a positional frame corresponding to \mathcal{F} . Say $\mathcal{G} = \mu_0(\mathcal{F})$ with:

$$\mu_0 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ p_0 & p_1 & p_2 & p_3 \end{pmatrix}.$$

Now suppose $\tilde{u}(\mathcal{G}) = \mathcal{G}$. Then, because $\tilde{u}(P) = P$, for any $f \in O^P$ and any $p \in P$,

$$\begin{aligned} f(\tilde{u}(p)) &= (f \circ \tilde{u} \upharpoonright P)(p) \\ &= \tilde{u}((f \circ \tilde{u} \upharpoonright P)(p)) && \text{because } u \upharpoonright O = \text{id}_O \\ &= (\tilde{u}(f \circ \tilde{u} \upharpoonright P))(\tilde{u}(p)) && \text{by Lemma 5.1, Property 5.} \end{aligned}$$

So, $f = \tilde{u}(f \circ \tilde{u} \upharpoonright P)$. Therefore,

$$\begin{aligned} \bar{\Gamma}(f) &= (\tilde{u}(\bar{\Gamma}))(f) && \text{because } \tilde{u}(\bar{\Gamma}) = \bar{\Gamma} \\ &= (\tilde{u}(\bar{\Gamma}))(\tilde{u}(f \circ \tilde{u} \upharpoonright P)) \\ &= \tilde{u}(\bar{\Gamma}(f \circ \tilde{u} \upharpoonright P)) && \text{by Lemma 5.1, Property 5.} \end{aligned}$$

Because $\bar{\Gamma}(f) = \Gamma(f \circ \mu_0)$ and $\tilde{u}(\Gamma(f)) = u(\Gamma(f)) = \Gamma(f \circ \pi_1)$, we get:

$$\Gamma(f \circ \mu_0) = \Gamma(f \circ \tilde{u} \upharpoonright P \circ \mu_0 \circ \pi_1).$$

Because $\text{Perm}_{\mathcal{F}}$ is generated by π_0 , we have $|O| \geq 2$, and so, because \mathcal{F} is permutation based, $\mu_0 = \tilde{u} \upharpoonright P \circ \mu_0 \circ \pi_1$ or $\mu_0 = \tilde{u} \upharpoonright P \circ \mu_0 \circ \pi_1 \circ \pi_0$. It follows that:

$$\tilde{u} \upharpoonright P = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ p_1 & p_2 & p_3 & p_0 \end{pmatrix} \text{ or } \tilde{u} \upharpoonright P = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ p_3 & p_0 & p_1 & p_2 \end{pmatrix}.$$

But this contradicts that $u \circ u = \text{id}_{S \cup O}$. So, $\tilde{u}(\mathcal{G}) \neq \mathcal{G}$. □

Note that for any O with at least two objects, a frame can be defined that fulfills the conditions of the theorem. Also note that if \mathcal{F} would be in $\mathbf{V}[S]$ and $O \in \mathbf{V}$, then we would get a similar result as in the theorem.

What conclusions can we draw from this theorem? Does it follow that some ‘real’ relations do not have a positional frame that is neutral with respect to the permutation class of any reasonable frame for it? Unfortunately, we have not found an example of an atomic ‘real’ relation that could be modeled by frame \mathcal{F} of the theorem. So, let us look at molecular relations.

Suppose we have two binary relations \mathfrak{R}_1 and \mathfrak{R}_2 . Let \mathfrak{R} be the quaternary relation, with $\mathfrak{R}abcd$ being the state of

$$\mathfrak{R}_1ab \ \& \ \mathfrak{R}_2bc \ \& \ \mathfrak{R}_1cd \ \& \ \mathfrak{R}_2da.$$

Then, depending on properties of \mathfrak{R}_1 and \mathfrak{R}_2 , the relation \mathfrak{R} may have a frame \mathcal{F} as in Theorem 6.7. It may even be the case that if \mathcal{F} is defined in $\mathbf{V}[S_1, S_2, O]$, with S_1 being the states of \mathfrak{R}_1 and S_2 being the states of \mathfrak{R}_2 , then there is no corresponding positional frame that is neutral with respect to $\Phi_{\mathcal{F}}$. This is, for example, the case when \mathfrak{R}_1ab is the state of *a's loving b* and \mathfrak{R}_2ab the state of *a's hating b*. Examples like this put us more or less in a similar position as Buridan’s ass. Like the ass cannot choose between two piles of hay, we seem to be unable to make a deliberate choice of sets for the positions of the relation \mathfrak{R} .

7. Neutrality w.r.t. substitution frames. In the previous section, we started with permutation classes of biased frames to create positional frames that are neutral with respect to these classes. But it is not clear that such a class of biased frames itself always gives an unbiased account of the underlying relation. It may be better to start instead with more primitive means, like substitution frames. We investigate that in this section.

We restrict our discussion to *simple substitution frames*.

DEFINITION 7.1. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. We call \mathcal{F} a simple substitution frame if there is a state s_0 such that

$$S = \{s_0 \cdot \delta \mid \delta : O \rightarrow O\}.$$

We call s_0 an initial state.

In Leo (2008), we defined the notion of a *simple relation* in terms of metaphysical principles satisfied by the relation. In this article, we use the term in a more loose sense by calling a relation simple if it can *adequately* be modeled by a simple substitution frame. Furthermore, we say that a simple relation has a neutral positional frame if the positional frame corresponds to a substitution frame for the relation, and it is neutral with respect to this substitution frame.

In Leo (2008), we proved the following theorem about the relationship between substitution and positional frames.

THEOREM 7.2. A substitution frame \mathcal{F} corresponds to some positional frame \mathcal{G} of the same degree iff \mathcal{F} is a simple substitution frame.

Furthermore, if $\text{degree}_{\mathcal{F}}$ is finite, then \mathcal{G} is unique, modulo positional variants.

In the proof of the theorem, we constructed \mathcal{G} as follows:

1. Choose an initial state $s_0 \in S$.
2. Choose an object-domain A of s_0 with $|A| = \text{degree}_{\mathcal{F}}(s_0)$.
3. Define $P = A$.
4. Let f be an arbitrary element of O^P . Let f^+ extend f to $O \rightarrow O$.
Define $\Gamma(f) = s_0 \cdot_{\mathcal{F}} f^+$.

Obviously, we may get in this way many positional frames that are each not neutral with respect to \mathcal{F} .

DEFINITION 7.3. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame of finite degree. We say that \mathcal{F} has strict symmetry if there is a state $s \in S$ and a $\delta \neq \text{id}_{\text{Ob}(s)}$ such that $s \cdot \delta = s$.

For any substitution frame of finite degree without strict symmetry, we can construct a corresponding positional frame that is neutral with respect to it:

EXAMPLE 7.4. Let $\mathcal{F} = \langle S, O, \Sigma \rangle \in \mathbf{V}[S, O]$ be a simple substitution frame of finite degree without strict symmetry. Let S_0 be the set of initial states of \mathcal{F} . Define $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ with:

$$P = \{p : S_0 \rightarrow O \mid \forall s \in S_0 (p(s) \in \text{Ob}(s) \ \& \ \forall \text{bijection } \pi : O \rightarrow O (p(s \cdot_{\mathcal{F}} \pi) = \pi(p(s))))\},$$

$$\Gamma(f) = s_0 \cdot \delta \text{ with } s_0 \in S_0 \text{ and } \forall p \in P (\delta(p(s_0)) = f(p)).$$

It is not difficult to show that \mathcal{G} is neutral with respect to \mathcal{F} .

As an aside, we note that the positions defined in the example can be identified with the *roles* of the substitution frame.

DEFINITION 7.5. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a simple substitution frame of finite object-degree. Then, for any initial state $s_0 \in S$ and $a_0 \in \text{Ob}(s_0)$, we define the role of a_0 in s_0 as:

$$\text{Role}(s_0, a_0) = \{(s, a) \mid \exists \delta (s \cdot \delta = s_0 \ \& \ a \in \text{Ob}(s) \ \& \ \delta(a) = a_0)\}.$$

Furthermore, we define the roles of \mathcal{F} as:

$$\text{Roles}_{\mathcal{F}} = \{\text{Role}(s_0, a_0) \mid s_0 \in S \text{ is an initial state} \ \& \ a_0 \in \text{Ob}(s_0)\}.$$

The next theorem in combination with the results of the previous section shows that there is little hope for a neutral reconstruction of argument-places in ordinary set theory for every relation.

THEOREM 7.6. Let substitution frame $\mathcal{F} = \langle S, O, \Sigma \rangle \in \mathbf{V}[S, O]$ and directional frame $\mathcal{G} = \langle S, O, \alpha, \Gamma \rangle$ be corresponding frames of the same finite degree. Then, \mathcal{F} and $\Phi_{\mathcal{G}}$, the permutation class of \mathcal{G} , are neutral with respect to each other.

Proof. An informal argument to see the correctness of this theorem is that we can construct \mathcal{F} deterministically purely based on the structure of $\Phi_{\mathcal{G}}$ and vice versa. Our formal argument requires a few more steps.

Let $u : S \cup O \rightarrow S \cup O$ be a bijection such that $\tilde{u}(\Phi_{\mathcal{G}}) = \Phi_{\mathcal{G}}$. Then, there is a bijection $\pi : \alpha \rightarrow \alpha$ such that for each $f \in O^\alpha$, $(\tilde{u}(\Gamma))(f) = \Gamma(\pi \cdot f)$. Furthermore, because \mathcal{F} and \mathcal{G} correspond, there is for each $s \in S$, an f such that $s = \Gamma(f)$. So,

$$\begin{aligned} \tilde{u}(s) \cdot_{\tilde{u}(\mathcal{F})} \tilde{u}(\delta) &= \tilde{u}(s \cdot_{\mathcal{F}} \delta) && \text{by Lemma 5.1, Property 5} \\ &= \tilde{u}(\Gamma(f) \cdot_{\mathcal{F}} \delta) \\ &= \tilde{u}(\Gamma(f \cdot \delta)) && \text{because } \mathcal{F} \text{ and } \mathcal{G} \text{ correspond} \\ &= (\tilde{u}(\Gamma))\tilde{u}(f \cdot \delta) && \text{by Lemma 5.1, Property 5} \\ &= \Gamma(\pi \cdot \tilde{u}(f \cdot \delta)) \\ &= \Gamma(\pi \cdot \tilde{u}(f) \cdot \tilde{u}(\delta)) && \text{by Lemma 5.1, Property 6} \\ &= \Gamma(\pi \cdot \tilde{u}(f)) \cdot_{\mathcal{F}} \tilde{u}(\delta) && \text{because } \mathcal{F} \text{ and } \mathcal{G} \text{ correspond} \\ &= \tilde{u}(\Gamma(f)) \cdot_{\mathcal{F}} \tilde{u}(\delta) \\ &= \tilde{u}(s) \cdot_{\mathcal{F}} \tilde{u}(\delta). \end{aligned}$$

It follows that $\tilde{u}(\mathcal{F}) = \mathcal{F}$, and thus that \mathcal{F} is neutral with respect to $\Phi_{\mathcal{G}}$.

Conversely, let $u : S \cup O \rightarrow S \cup O$ be a bijection such that $\tilde{u}(\mathcal{F}) = \mathcal{F}$. Then, for each $s \in S$ and $\delta \in O^O$, $u(s) \cdot_{\mathcal{F}} \tilde{u}(\delta) = u(s \cdot_{\mathcal{F}} \delta)$. Let s_0 be an initial state of \mathcal{F} . Then, $u(s_0)$ is also an initial state. Because \mathcal{F} and \mathcal{G} are of the same finite degree, we have for some injection f_0 , $s_0 = \Gamma(f_0)$, and for some injection f_1 , $u(s_0) = \Gamma(f_1)$. So, because $\text{im } f_1 = \text{Ob}(u(s_0)) = \tilde{u}(\text{Ob}(s_0)) = \tilde{u}(\text{im } f_0)$, there is a bijection $\pi : \alpha \rightarrow \alpha$ such that $f_1 = \pi \cdot \tilde{u}(f_0)$. Furthermore, for each $f \in O^\alpha$, there is a δ such that $f = f_0 \cdot \delta$. So,

$$\begin{aligned}
 (\tilde{u}(\Gamma))(\tilde{u}(f)) &= \tilde{u}(\Gamma(f)) && \text{by Lemma 5.1, Property 5} \\
 &= \tilde{u}(\Gamma(f_0) \cdot_{\mathcal{F}} \delta) && \text{because } \mathcal{F} \text{ and } \mathcal{G} \text{ correspond} \\
 &= \Gamma(f_1) \cdot_{\tilde{u}(\mathcal{F})} \tilde{u}(\delta) && \text{by Lemma 5.1, Property 5} \\
 &= \Gamma(f_1) \cdot_{\mathcal{F}} \tilde{u}(\delta) && \text{because } \tilde{u}(\mathcal{F}) = \mathcal{F} \\
 &= \Gamma(\pi \cdot \tilde{u}(f_0)) \cdot_{\mathcal{F}} \tilde{u}(\delta) \\
 &= \Gamma(\pi \cdot \tilde{u}(f_0) \cdot \tilde{u}(\delta)) && \text{because } \mathcal{F} \text{ and } \mathcal{G} \text{ correspond} \\
 &= \Gamma(\pi \cdot \tilde{u}(f_0 \cdot \delta)) && \text{by Lemma 5.1, Property 5} \\
 &= \Gamma(\pi \cdot \tilde{u}(f)).
 \end{aligned}$$

It follows that $\tilde{u}(\Phi_{\mathcal{G}}) = \Phi_{\mathcal{G}}$ and thus that $\Phi_{\mathcal{G}}$ is neutral with respect to \mathcal{F} . □

So, if the degree of the frames is finite, then it makes for a neutral reconstruction of argument-places no real difference whether we start with a substitution frame or with a permutation class of a corresponding directional frame of the same degree. But what if the degrees are infinite?

In the first part of the proof of the theorem, we did not use any restriction on the degrees of the frames. So, if \mathcal{F} and \mathcal{G} correspond, then \mathcal{F} is always neutral with respect to the permutation class of \mathcal{G} . Interestingly, we need the restriction for the converse.

EXAMPLE 7.7. Let $\mathcal{F}_1 = \langle S_1, O, \Sigma_1 \rangle$ with $O = \omega$, the set of natural numbers, $S_1 = \{s : \omega \rightarrow (\omega \cup \{\infty\}) \mid \exists i (s(i) = \infty)\}$, and Σ_1 defined by:

$$(s \cdot_{\mathcal{F}_1} \delta)(i) = \sum_{\delta(j)=i} s(j),$$

with $i + \infty = \infty + i = \infty + \infty = \infty$.

Let

$$s_0 = [0^\infty, 1, 2, 3, \dots], \text{ that is, } s_0(0) = \infty, \text{ and for } i \geq 1, s_0(i) = 1;$$

$$s'_0 = [0^\infty, 1^\infty, 2, 3, \dots], \text{ that is, } s'_0(0) = s'_0(1) = \infty, \text{ and for } i \geq 2, s'_0(i) = 1.$$

Define

$$\mathcal{G}_1 = \langle S_1, O, \omega, \Gamma_1 \rangle \text{ with } \Gamma_1(f) = s_0 \cdot_{\mathcal{F}_1} f;$$

$$\mathcal{G}'_1 = \langle S_1, O, \omega, \Gamma'_1 \rangle \text{ with } \Gamma'_1(f) = s'_0 \cdot_{\mathcal{F}_1} f.$$

In Leo (2008), we showed that \mathcal{G}_1 and \mathcal{G}'_1 correspond to \mathcal{F}_1 and that \mathcal{G}_1 and \mathcal{G}'_1 are not permutations of each other.

Now define $\mathcal{F} = \langle S, O, \Sigma \rangle$ with $S = S_1 \times S_1$, and

$$(s, s') \cdot_{\mathcal{F}} \delta = (s \cdot_{\mathcal{F}_1} \delta, s' \cdot_{\mathcal{F}_1} \delta).$$

Furthermore, define $\mathcal{G} = \langle S, O, \omega + \omega, \Gamma \rangle$ with:

$$\Gamma(f) = (\Gamma_1(f_1), \Gamma'_1(f'_1)) \text{ with } f_1(\alpha) = f(\alpha) \text{ and } f'_1(\alpha) = f(\omega + \alpha).$$

Obviously, \mathcal{F} is a substitution frame and \mathcal{G} is a corresponding directional frame. Now conceive \mathcal{F} and \mathcal{G} as elements of $\mathbf{V}[S]$. Define $u : S \rightarrow S$ with $u(s, s') = (s', s)$. It is not difficult to verify that $\tilde{u}(\mathcal{F}) = \mathcal{F}$, but $\tilde{u}(\Phi_{\mathcal{G}}) \neq \Phi_{\mathcal{G}}$.

From Theorems 6.7 and 7.6, and the fact that every simple substitution frame has a corresponding positional frame, it follows the following.

COROLLARY 7.8. *Not every simple substitution frame has a corresponding positional frame that is neutral with respect to it.*

Note that by Theorem 6.7, the corollary already applies to frames of degree 4.

8. Alternative constructions. What do the results in the previous sections tell us? Does it follow from the fact that for certain relations, positions cannot be modeled in a neutral way as sets that these relations cannot have argument-places as derived entities?

I think such a conclusion would be too hasty. First of all, we should ask ourselves if we did not perhaps use too limited a notion of neutrality. Maybe we could well-order the objects or states of a relation and exploit this to deterministically construct or select a positional frame for it. The next example shows that for a given well-ordering of the objects, such a selection can be made for simple relations of finite degree.

EXAMPLE 8.1. *Let $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ be a frame of finite degree for a given relation. Suppose we are given a well-ordering $<$ of the objects O . Then, this induces a lexicographical ordering $<$ of O^α . Because α is finite, this ordering of O^α is clearly a well-ordering. So, for any $\mathcal{F}' = \langle S, O, \alpha, \Gamma' \rangle$ and $\mathcal{F}'' = \langle S, O, \alpha, \Gamma'' \rangle$, we may define*

$$\mathcal{F}' < \mathcal{F}'' \text{ :if } \Gamma'(f_0) < \Gamma''(f_0) \text{ with } f_0 \text{ the least element of } \{f : \alpha \rightarrow O \mid \Gamma'(f) \neq \Gamma''(f)\}.$$

It is easy to see that this last relation is a linear order. Because α is finite, $\Phi_{\mathcal{F}}$ is finite. So, the permutation class $\Phi_{\mathcal{F}}$ contains a least element, which we can select as positional frame for the relation by ignoring the order of α .

If α is infinite, then the lexicographical ordering of O^α is not always a well-ordering. So, in that case, the construction of the previous example does not work. If O and α are both \mathbb{N} , the set of natural numbers, then no method for how to well-order O^α is known. Perhaps other constructions might work to uniquely select one frame in $\Phi_{\mathcal{F}}$. For example, if O and S are representable in \mathbf{V} , the cumulative hierarchy, and if $\mathbf{V} = \mathbf{L}$, with \mathbf{L} the constructible universe, then we can use a definable well-ordering of \mathbf{L} to uniquely select an element of $\Phi(\mathcal{F})$.

But how realistic is it to assume that O and S can always be well-ordered in a deterministic way or that they are representable by sets in any model of ZFC, that is, Zermelo-Fraenkel set theory (ZF) with the axiom of choice (AC)? Recall that O and S are *objects* and *states*. We may even have no reason to assume *a priori* that the elements of O and S are always all discernible. But even if we would restrict ourselves to relations with each a finite set of discernible objects, then because of the diversity of objects, it is still questionable whether a single method can be found to deterministically order each set O . Also, it is not very mathematically elegant to require an extrinsic ordering of the objects. In summary, I doubt that looking for specific orderings of the sets of objects and states of a relation is very promising as a general solution.

There is a stronger argument why the conclusion that relations cannot have argument-places as derived entities is too hasty. There is up front no reason why argument-places would have to be sets in \mathbf{V} or $\mathbf{V}[S, O]$. *It could be the case that ordinary set theory as a modeling medium is too limited.* Other modeling media might be more adequate for some relations. We could, for example, model the quaternary relation \mathfrak{R} at the end of Section 6.

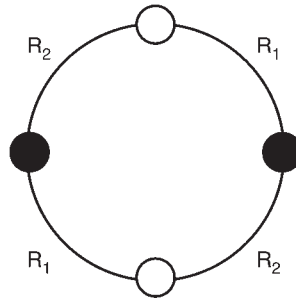


Fig. 1. A neutral graphical representation of argument-places.

graphically in a neutral way, as shown in Figure 1. But I doubt that for every simple relation of finite degree, a nonarbitrary graphical representation is possible. For example, if we have an *atomic* relation with a directional model as in Theorem 6.7, then I do not see how to label the positions in a neutral way. There is, however, a related and more promising approach to define argument-places in a neutral way. The approach will probably appeal to at least the structuralists among us. We discuss it in the next section.

9. Argument-places as abstractions. Let \mathfrak{R} be a simple relation of finite degree, and \mathcal{F} be a substitution frame for it. As we remarked earlier and proved in Leo (2008), a corresponding positional frame \mathcal{G} exists, which is unique modulo positional variants. I propose to define the argument-places of \mathfrak{R} as structureless *places* exemplified by the positions of \mathcal{G} (Figure 2).

This approach gives us argument-places as entities of a structure obtained by abstraction. I have no strong opinion about the question whether we should take them literally as genuine *objects* or that we should regard talk of argument-places as abstractions, as nothing but a convenient way of saying something about the underlying positional frames. I regard this issue as part of a much more comprehensive ontological debate that I want to avoid here (see, e.g., Parsons, 2004; Shapiro, 1997).

Consider the adjacency relation. With the view of argument-places as abstractions, the two argument-places for this relation fulfill exactly the same role. Since the argument-places have no internal structure, the two argument-places are completely indiscernible from each other. More generally, the argument-places of a relation fulfilling the same role are indiscernible. For simple relations of finite degree, the concept of roles as we use it here matches with the roles as defined in Section 7.

Fine (2000, p. 17) gave the following objection against the positionalist view that a relation has argument-places. On this view, the adjacency relation has two argument-places, say *Next* and *Nixt*. Assigning a to *Next* and b to *Nixt* will intuitively give the same state as assigning a to *Nixt* and b to *Next*. Yet, they must be distinct since the argument-places occupied by a and b are distinct. As a consequence, strictly symmetric relations cannot have argument-places.

In Leo (2008, p. 358), I countered Fine's objection by arguing that we could also assume that assigning objects to argument-places *yields* states and that I see no reason to assume that objects *occupy* argument-places within the states. I think the view of argument-places as abstractions gives us additional ammunition against Fine's objection since if the arguments would occupy argument-places, then switching arguments would not give a distinct state if the argument-places are indiscernible.

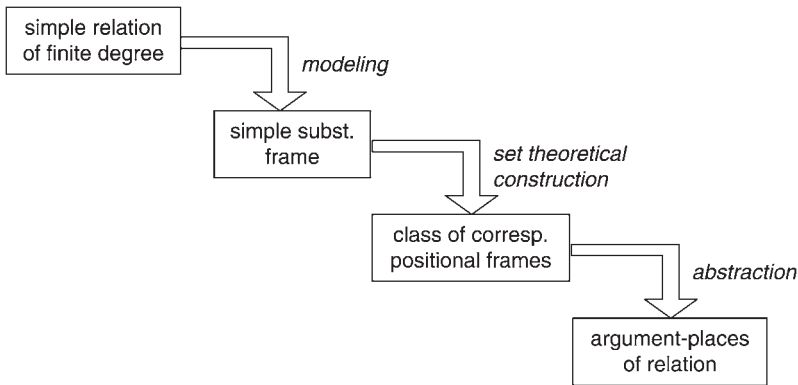


Fig. 2. Construction of abstract argument-places.

Nevertheless, we should not conclude that on the view of argument-places as abstractions for any relation, the arguments may occupy argument-places in the states. For consider the relation \mathfrak{R} with $\mathfrak{R}abc$ being the state of *a's loving b and b's loving c*. Then, the relation has three discernible argument-places, say p_0 , p_1 , and p_2 . If assigning a to p_0 , b to p_1 , and a to p_2 gives the state of *a's loving b and b's loving a*, then we get the same state if we assign b to p_0 , a to p_1 , and b to p_2 . Since in the first case, a is assigned to two argument-places, and in the second case to only one argument-place, it is obviously impossible for the arguments to occupy the argument-places.

On the view of argument-places as abstractions, the internal structure of positions of a positional model is not particularly interesting since on this view, argument-places have by definition no internal structure. The main value that a specific positional model may have from this perspective is that in some cases, it gives a nice or canonical representation for the argument-places of a relation.

We should not, in general, regard the positions of a positional frame as *names* for the argument-places as abstractions. They could only be names if each would uniquely refer to an entity, but this is impossible if the argument-places are indiscernible. What we can do, however, is regard the positions of a positional frame collectively as a *representation* of the abstracted argument-places.

The indiscernibility of argument-places as abstractions might be a problem in the following respect: how can we assign different objects to them when they are indiscernible? For a relation with complete strict symmetry, it might be enough just to say which objects are to be assigned to the argument-places, but for cyclical relations, somehow an order has to be specified. A solution might be to use a representative positional frame: first, we assign objects to its positions, and then, we abstract to get the required assignment.

The construction of argument-places as abstractions is not applicable for every substitution frame. Some substitution frames of infinite degree have corresponding positional frames that are at first sight equally natural but that are nevertheless not positional variants of each other (e.g., \mathcal{F}_1 in Example 7.7). Consequently, for relations with such substitution frames, a construction of argument-places as abstractions seems impossible. Maybe we should not interpret this as a weakness of the construction but rather as a peculiarity of certain exotic relations (if they exist) that they do not have argument-places.

In Shapiro (1997), a theory of structures is sketched, and in Fine (2002), a framework for a general theory of abstraction is given. It would be interesting to investigate whether argument-places as abstractions fit into these theories.

With argument-places as abstractions, we obviously take a step outside the realm of ordinary set theory. However, we may define them in an extension of ZF along the lines sketched in Fine (1998). By taking as urelements not only the states in S and the objects in O but also what Fine calls *variable objects*, positions could be defined as a *system* of variable objects.

10. Conclusions. We started our analysis with the presupposition that argument-places are not primitive entities. I think that argument-places are indeed not primitive since we can give a position-free account of relations based on the notion of substitution, which I regard as more primitive than the notion of argument-places. But to vindicate a position-free account, either a satisfactory construction of argument-places is needed or an argument why the notion of argument-places is problematic.

For a (re)construction of argument-places, we made use of mathematical models for relations. Using a new formal notion of neutrality, we showed that every simple relation without strict symmetry and some other simple relations, like those with complete strict symmetry, have neutral positional frames. But we also showed that it is impossible to construct for every simple relation, even if its degree is finite, a neutral positional frame. As a consequence, it is highly unlikely that for every relation, argument-places can be defined in a canonical way in ordinary set theory.

I consider the positive and the negative results about the existence of neutral positional frames to be primarily of interest from a representational perspective. I do not immediately see what conclusions we should draw about the ontological character of the argument-places for relations that do have a natural and neutral positional frame. Also, with interpreting the negative results, we should be careful. I certainly do not want to conclude from them that the identity of argument-places is problematic. But the negative results do show that for a neutral construction of argument-places of certain relations, we have to go beyond the limits of ordinary set theory.

We showed that for the class of simple relations of finite degree, a unique construction of argument-places as places in a structure exemplified by positional frames is possible. I personally find this view of argument-places as abstractions very natural. However, I do not know whether we should consider argument-places as genuine objects or that we should only talk about them as such. Regarding them as genuine objects is obviously in conflict with Leibniz's Principle of the Identity of Indiscernibles. Since we are dealing with constructed entities, I do not expect that a further analysis of argument-places will itself provide a ground for giving up Leibniz's Principle. We can probably paraphrase all references to argument-places as objects in terms of references to positions of positional frames. But if Leibniz's Principle is denied for whatever reason, then perhaps the assignment of an ontological status to argument-places may be just a matter of choice motivated by what appears to us as the most convenient perspective.

One may question whether the results obtained have any metaphysical significance, for we did not discuss any new metaphysical principles, nor did we reveal any relationship between ontologically fundamental entities. However, by giving a clarification and justification of the notion of argument-places, we contributed in an indirect way to forming and elaborating ideas about the essence of relations. Our ordinary way of using argument-places apparently is pretty much in agreement with treating argument-places as abstractions.

I consider this not only to be support for common sense but also to be confirmation of the antipositionalist view on relations. As such, the results do have metaphysical significance.

11. Acknowledgments. The formulation and proof of the main negative result about the existence of neutral positional frames (Theorem 6.7) became much clearer thanks to Albert Visser's suggestion to reformulate it in terms of set theory with urelements. Furthermore, I am grateful to Kit Fine for helpful comments and to Clemens Grabmayer for many stylistic suggestions.

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