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ON NONCRITICAL GALOIS REPRESENTATIONS

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Abstract We propose a conjecture that the Galois representation attached to every Hilbert modular form is noncritical and prove it under certain conditions. Under the same condition we prove Chida, Mok and Park's conjecture that Fontaine-Mazur L-invariant and Teitelbaum-type L-invariant coincide with each other.

Keywords: Galois representations, L-invariants

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Introduction

Let F be a totally real number field of degree $g=[F:\mathbb{Q}]$ and $\mathfrak{p}$ a prime ideal of F above a fixed prime number p.

0.1. Noncritical Galois representations

The objects of this article are p-adic representations of the Galois group $G_F=\mathrm{Gal}(\overline{F}/F)$. Among them there is a subclass called semistable; roughly speaking, a p-adic Galois representation is called *semistable at* $\mathfrak{p}$ if its restriction to $G_{F_\mathfrak{p}}$ has periods in Fontaine's period ring B_{st}.

Let L be an extension of $F_\mathfrak{p}$ that splits $F_\mathfrak{p}$. Among semistable (but noncrystalline) 2-dimensional L-representations of G_F, there is a subclass, called noncritical, that can be attached to Fontaine-Mazur L-invariants. See Section 1 for its precise definition. The importance of L-invariants is due to the fact that they occur in the exceptional zero conjecture proposed by Mazur, Tate and Teitelbaum [19]. This conjecture was proved by Greenberg and Stevens [16].

When $F_\mathfrak{p}=\mathbb{Q}_p$, all semistable representations are automatically noncritical. However, when $F_\mathfrak{p}$ is different from $\mathbb{Q}_p$, a new phenomenon is that there exist critical semistable noncrystalline 2-dimensional Galois representations.

The main result is the following.

Theorem 0.1. (=*Theorem* 1.2) *Assume that F is a totally real field that satisfies the following condition:*

$$\textit{there is no place other than } \mathfrak{p} \textit{ above } p.$$

Let f_∞ be a Hilbert modular form over F of even weight $(k_1,\cdots,k_g,w)$ and suppose that f_∞ is new at $\mathfrak{p}$ (and another prime ideal if $[F:\mathbb{Q}]$ is odd). Then the p-adic Galois representation attached to f_∞ is semistable and noncritical at $\mathfrak{p}$.

Here, the notion *even weight* means that $k_1,\cdots,k_g$ and w are all even.

Inspired by Theorem 0.1, we propose the following.

Conjecture 0.2. *Let f_∞ be a Hilbert modular form over F that is new at $\mathfrak{p}$. Then the p-adic Galois representation attached to f_∞ is semistable and noncritical at $\mathfrak{p}$.*

The key in the proof of Theorem 0.1 is the Hodge-like decomposition of de Rham cohomology. We state this decomposition below.

Let $\mathcal{H}$ be Drinfeld's upper half plane and Γ an arithmetic Schottky group that is cocompact in $\mathrm{PGL}(2,F_\mathfrak{p})$. Then Γ acts freely on $\mathcal{H}$ and the quotient $X_\Gamma=\Gamma\backslash\mathcal{H}$ is the rigid analytic space associated with a proper smooth curve over $F_\mathfrak{p}$. Let $\mathscr{V}$ be the local system coming from an $L[\Gamma]$-module V, where L is a field that contains $F_\mathfrak{p}$. Fix an embedding $\tau:F_\mathfrak{p}\to L$ and consider V as an $F_\mathfrak{p}[\Gamma]$-module by τ. Let $H^1_{\mathrm{dR},\tau}(X_\Gamma,\mathscr{V})$ be the hypercohomology of the complex $\mathscr{V}\otimes_{\tau,F_\mathfrak{p}}\Omega^\bullet_{X_\Gamma}$. Then we have the following decomposition, called the *Hodge-like decomposition*:

$$H^1_{\mathrm{dR},\tau}(X_\Gamma,\mathscr{V})=H^0(X_\Gamma,\mathscr{V}\otimes_{\tau,F_\mathfrak{p}}\Omega^1_{X_\Gamma})\oplus H^1(\Gamma,V). \tag{0.1}$$

Let $\mathcal{H}^d$ be the d-dimensional Drinfeld p-adic symmetric domain. The Hodge-like decomposition for the de Rham cohomology $H^\bullet_{\mathrm{dR}}(\Gamma\backslash\mathcal{H}^d)$ of certain quotient $\Gamma\backslash\mathcal{H}^d$ of $\mathcal{H}^d$ was conjectured by Schneider [25] and proved by Iovita and Spiess [17]. When $d=2$ – that is, $\mathcal{H}^d$ is the above $\mathcal{H}$ – de Shalit [9] proved the Hodge-like decomposition for certain local systems. However, neither the result of Iovita and Spiess nor the result of de Shalit covers our situation.

We sketch the proof of (0.1). The quotient of $H^1_{\mathrm{dR},\tau}(X_\Gamma,\mathscr{V})$ by $H^1(\Gamma,V)$ is isomorphic to $C^1_{\mathrm{har}}(V)^\Gamma$, the group of Γ-invariant harmonic cocycles on the Bruhat-Tits tree attached to $\mathrm{PGL}(2,F_\mathfrak{p})$. By Amice-Velu and Vishik's method we construct a map

$$C^1_{\mathrm{har}}(V)^\Gamma\to H^0(X_\Gamma,\mathscr{V}\otimes_{\tau,F_\mathfrak{p}}\Omega^1_{X_\Gamma})\quad c\mapsto\omega^\tau_c$$

and show that for each c the image of ω^τ_c by the quotient map $H^1_{\mathrm{dR},\tau}(X_\Gamma,\mathscr{V})\to C^1_{\mathrm{har}}(V)^\Gamma$ is just c. Combining this with a comparing dimensions argument we obtain (0.1).

Now, we sketch the proof of Theorem 0.1. The Galois representation attached to f_∞ comes from the étale cohomology H^1_{et} of some local system on a Shimura curve. The Shimura curve has a p-adic uniformisation; precisely there are some arithmetic Schottky groups Γ_i such that the rigid analytic space attached to the Shimura curve is isomorphic to the union $\cup_i\Gamma_i\backslash\mathcal{H}$.

We will give a precise description of the filtered φ_q-isocrystal, denoted by $\mathscr{V}$, attached to the above local system. In [6], Coleman and Iovita provided a precise description of the monodromy on the de Rham cohomology of $\mathscr{V}$. By their result and the Hodge-like decomposition, we show that the monodromy is injective on $\bigoplus_i H^0(\Gamma_i\backslash\mathcal{H},\mathscr{V}\otimes_{\tau,F_\mathfrak{p}}\Omega^1)$. Our precise description of $\mathscr{V}$ will imply that $\bigoplus_i H^0(\Gamma_i\backslash\mathcal{H},\mathscr{V}\otimes_{\tau,F_\mathfrak{p}}\Omega^1)$ coincides

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with $\mathrm{Fil}^{\frac{w+\min_\tau k_\tau}{2}-1}\bigoplus_i H^1_{\mathrm{dR},\tau}(\Gamma_i\backslash\mathcal{H},\mathscr{V})$. Then we deduce that the monodromy induces an isomorphism

$$\mathrm{Fil}^{\frac{w+\min_\tau k_\tau}{2}-1}\bigoplus_i H^1_{\mathrm{dR},\tau}(\Gamma_i\backslash\mathcal{H},\mathscr{V}) \xrightarrow{\sim} \bigoplus_i H^1_{\mathrm{dR},\tau}(\Gamma_i\backslash\mathcal{H},\mathscr{V})/\mathrm{Fil}^{\frac{w+\min_\tau k_\tau}{2}-1}\bigoplus_i H^1_{\mathrm{dR},\tau}(\Gamma_i\backslash\mathcal{H},\mathscr{V}),$$

which implies Theorem 0.1.

When F has more than one place (say r places) above p, our method of computing filtered φ_q-isocrystals is not valid. To make it work, one may have to consider the Shimura variety studied by Rapoport and Zink [23, Chapter 6] (which is of dimension r) instead of the Shimura curve. Coleman and Iovita's result [6] is valid only for curves and so cannot be applied directly.

0.2. Fontaine-Mazur L-invariants and Teitelbaum-type L-invariants

Because the Galois representation attached to f_∞ is noncritical at $\mathfrak{p}$, we can attach to it the Fontaine-Mazur L-invariant, denoted by $\mathcal{L}_{FM}(f_\infty)$.

Chida, Mok and Park [4] attached to each automorphic form $\mathbf{f}$ over a totally definite quaternion algebra (also of weight $(k_1,\cdots,k_g,w)$) that satisfies the following condition:

$$\text{(CMP)} \qquad \mathbf{f} \text{ is new at } \mathfrak{p} \text{ and } U_{\mathfrak{p}}\mathbf{f} = \mathcal{N}\mathfrak{p}^{w/2}\mathbf{f},$$

another kind of L-invariant $\mathcal{L}_T(\mathbf{f})$, called the Teitelbaum-type L-invariant. Both $\mathcal{L}_{FM}(f_\infty)$ and $\mathcal{L}_T(\mathbf{f})$ are vector valued. See Subsection 1.2 and Subsection 9.2 for their precise definitions. As mentioned previously, the importance of L-invariants is due to the fact that they occur in the exceptional zero conjecture [19]. The readers are invited to consult Colmez's paper [7] for a historical account on the exceptional zero conjecture and L-invariants.

In [4], Chida, Mok and Park conjectured that $\mathcal{L}_{FM}(f_\infty) = \mathcal{L}_T(\mathbf{f})$ when f_∞ and $\mathbf{f}$ are attached to each other by Jacquet-Langlands correspondence. When $F=\mathbb{Q}$, this is already known by Iovita and Spiess [18]. We prove their conjecture under the same assumption as in Theorem 0.1.

Theorem 0.3. (=*Theorem 9.3*) *Assume that F is a totally real number field that satisfies the following condition:*

there is no place other than $\mathfrak{p}$ above p.

Let f_∞ and $\mathbf{f}$ be as above. Then $\mathcal{L}_{FM}(f_\infty) = \mathcal{L}_T(f_\infty)$.

As in [18], we prove Theorem 0.3 by analyzing the relation among the monodromy operator, Coleman integration and Schneider integration.

The article is organised as follows. In Section 1 we recall the notion of noncritical 2-dimensional Galois representations and state the main theorem. Coleman and Iovita's result is recalled in Section 2. Section 3 is devoted to computing the filtered φ_q-isocrystal

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attached to the universal special formal module. We introduce various Shimura curves and study their p-adic uniformisations following Rapoport and Zink in Section 4 and Section 5, respectively. In Section 6 we use the result in Section 3 to determine the filtered φ_q-isocrystals attached to various local systems on Shimura curves. In Section 7 we recall the theory of de Rham cohomology of certain local systems and prove the Hodge-like decomposition theorem. In Section 8, we combine results in Section 2, Section 6 and Section 7 to prove Theorem 0.1. In Section 9 we recall Chida, Mok and Park's construction of Teitelbaum-type L-invariants and prove Theorem 0.3.

Notation

For two $\mathbb{Q}$-algebras A and B, write $A\otimes B$ for $A\otimes_{\mathbb{Q}} B$. For a ring R let $R^{\times}$ denote the multiplicative group of invertible elements in R.

Let F be a totally real number field, $g=[F:\mathbb{Q}]$. Let p be a fixed prime. Suppose that p is inertia in F; that is, there exists exactly one place of F above p, denoted by $\mathfrak{p}$. If q is a power of p, we use $v_p(q)$ to denote $\log_p q$.

Let $\mathbb{A}_f$ denote $\mathbb{Q}\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$ and let $\mathbb{A}_f^p$ denote $\mathbb{Q}\otimes_{\mathbb{Z}}(\prod_{\ell\neq p}\mathbb{Z}_\ell)$. Similarly, for any number field E let $\mathbb{A}_{E,f}$ denote $E\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$, the ring of finite adèles of E.

Fix an algebraic closure of $F_{\mathfrak{p}}$, denoted by $\overline{F_{\mathfrak{p}}}$, and let $\mathbb{C}_p$ be the completion of $\overline{F_{\mathfrak{p}}}$ with respect to the p-adic topology. In this way we have fixed an embedding $F_{\mathfrak{p}}\hookrightarrow\mathbb{C}_p$. The Galois group $G_{F_{\mathfrak{p}}}=\mathrm{Gal}(\overline{F_{\mathfrak{p}}}/F_{\mathfrak{p}})$ can be naturally identified with the group of continuous $F_{\mathfrak{p}}$-automorphisms of $\mathbb{C}_p$.

1. Noncritical Galois representations

1.1. Noncritical Galois representations and Fontaine-Mazur L-invariant

Let $F_{\mathfrak{p}0}$ be the maximal absolutely unramified subfield of $F_{\mathfrak{p}}$, q the cardinal number of the residue field of $F_{\mathfrak{p}}$.

Let $\mathrm{B}_{\mathrm{cris}}$,$\mathrm{B}_{\mathrm{st}}$ and B_{dR} be Fontaine's period rings [15]. As is well known, there are operators φ and N on B_{st} and a descending $\mathbb{Z}$-filtration on B_{dR}; $\mathrm{B}_{\mathrm{cris}}$ is a φ-stable subring of B_{st}, and N vanishes on $\mathrm{B}_{\mathrm{cris}}$. Put $\mathrm{B}_{\mathrm{st},F_{\mathfrak{p}}}:=\mathrm{B}_{\mathrm{st}}\otimes_{F_{\mathfrak{p}0}}F_{\mathfrak{p}}$; $\mathrm{B}_{\mathrm{st},F_{\mathfrak{p}}}$ can be considered as a subring of B_{dR}. We extend the operators $\varphi_q=\varphi^{v_p(q)}$ and N $F_{\mathfrak{p}}$-linearly to $\mathrm{B}_{\mathrm{st},F_{\mathfrak{p}}}$.

Let K be either a finite unramified extension of $F_{\mathfrak{p}}$ or the completion of the maximal unramified extension of $F_{\mathfrak{p}}$ in $\mathbb{C}_p$. By our assumption on K we have

$$(\mathrm{B}_{\mathrm{cris},F_{\mathfrak{p}}})^{G_K}=(\mathrm{B}_{\mathrm{st},F_{\mathfrak{p}}})^{G_K}=(\mathrm{B}_{\mathrm{dR}})^{G_K}=K.$$

Let L be a finite extension of $\mathbb{Q}_p$. For a 2-dimensional L-linear representation V of G_K, we put

$$D_{\mathrm{st},F_{\mathfrak{p}}}(V):=(V\otimes_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{st},F_{\mathfrak{p}}})^{G_K}.$$

This is a finite rank $L\otimes_{\mathbb{Q}_p}K$-module. If V is semistable, then $D_{\mathrm{st},F_{\mathfrak{p}}}(V)$ is a filtered (φ_q,N)-module: the (φ_q,N)-module structure is induced from the operators $\varphi_q=1_V\otimes\varphi_q$ and $N=1_V\otimes N$ on $V\otimes_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{st},F_{\mathfrak{p}}}$; the filtration comes from that on $V\otimes_{\mathbb{Q}_p}\mathrm{B}_{\mathrm{dR}}$. Note that

φ_q and N are $L\otimes_{\mathbb{Q}_p} K$-linear. If V is noncrystalline, then there exists a basis $\{n_0,n_1\}$ of $D_{\mathrm{st},F_\mathfrak{p}}(V)$ over $L\otimes_{\mathbb{Q}_p} K$ such that $Nn_1=n_0$ and $Nn_0=0$.

If L splits $F_\mathfrak{p}$, then $L\otimes_{\mathbb{Q}_p} K$ is isomorphic to $\bigoplus_\sigma L\otimes_{\sigma,F_\mathfrak{p}} K$, where σ runs through all embeddings of $F_\mathfrak{p}$ into L. Here the subscript σ under $\otimes$ indicates that $F_\mathfrak{p}$ is considered as a subfield of L via σ. Let e_σ be the unity of the subring $L\otimes_{\sigma,F_\mathfrak{p}} K$.

If D is a filtered (φ_q,N)-module, for each σ we put $D_\sigma=e_\sigma D$. Let $-k_{2,\sigma}\le -k_{1,\sigma}$ be the Hodge-Tate weights of D_σ. For D to be noncritical, one demands $-k_{2,\sigma}<-k_{1,\sigma}$ for each σ. Then there exists

$$(a_\sigma,b_\sigma)\in (L\otimes_{\sigma,F_\mathfrak{p}} K)\times(L\otimes_{\sigma,F_\mathfrak{p}} K)\backslash\{(0,0)\}$$

such that

$$\mathrm{Fil}^i D_\sigma=\begin{cases} D_\sigma & \text{if } i\le k_{1,\sigma}\\ (L\otimes_{\sigma,F_\mathfrak{p}} K)(a_\sigma n_{1,\sigma}+b_\sigma n_{0,\sigma}) & \text{if } k_{1,\sigma}<i\le k_{2,\sigma}\\ 0 & \text{if } i>k_{2,\sigma},\end{cases}$$

where $n_{1,\sigma}=e_\sigma n_1$ and $n_{2,\sigma}=e_\sigma n_2$. If for each σ, a_σ is invertible, we say that D is *noncritical*. If the filtered (φ_q,N)-module attached to V is noncritical, we say that V is *noncritical*. In this case, we put $\mathcal{L}_{FM,\sigma}(V)=-b_\sigma/a_\sigma$, and we call the vector $\mathcal{L}_{FM}(V)=(\mathcal{L}_{FM,\sigma}(V))_\sigma$ the *Fontaine-Mazur L-invariant* of V.

1.2. Galois representations attached to Hilbert modular forms

Let $\{\tau_1,\cdots,\tau_g\}$ be the set of real embeddings $F\hookrightarrow\mathbb{R}$. Fix a multiweight $\mathrm{k}=(k_1,\cdots,k_g,w)\in\mathbb{N}^{g+1}$ satisfying $k_i\ge 2$ and $k_i\equiv w \mod 2$.

Let $\pi=\otimes_v\pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_F)$ such that for each τ_i, π_{τ_i} is the holomorphic discrete series representation $D_{k_i,2-w}$. See [3] for the definition of $D_{k_i,2-w}$. Let $\mathfrak{n}$ be the level of π.

Carayol [3] attached to such an automorphic representation (under a further condition) an ℓ-adic Galois representation, which is recalled as follows.

Let L be a sufficiently large number field of finite degree over $\mathbb{Q}$ such that the finite part $\pi^\infty=\otimes_{\mathfrak{p}\nmid\infty}\pi_\mathfrak{p}$ of π admits an L-structure π_L^∞. The fixed part $(\pi_L^\infty)^{K_1(\mathfrak{n})}$ is of dimension 1 and generated by an eigenform f_∞. In this case we write π_{f_∞} for π.

The local Langlands correspondence associates to every irreducible admissible representation $\pi_\mathfrak{p}$ of $\mathrm{GL}(2,F_\mathfrak{p})$ defined over L a 2-dimensional L-rational Frobenius semisimple representation $\sigma(\pi_\mathfrak{p})$ of the Weil-Deligne group $WD(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$. Let $\check{\sigma}(\pi_\mathfrak{p})$ denote the dual of $\sigma(\pi_\mathfrak{p})$.

For an ℓ-adic representation ρ of G_F, let $\rho_\mathfrak{p}$ denote its restriction to $G_{F_\mathfrak{p}}$, $\prime\!\rho_\mathfrak{p}$ the Weil-Deligne representation attached to $\rho_\mathfrak{p}$ and $\prime\!\rho_\mathfrak{p}^{\text{F-ss}}$ the Frobenius semisimplification of $\prime\!\rho_\mathfrak{p}$.

Theorem 1.1. *[3] Let f_∞ be an eigenform of multiweight* k *satisfying the following condition:*

If $g=[F:\mathbb{Q}]$ is even, then there exists a finite place $\mathfrak{q}$ such that the $\mathfrak{q}$-factor $\pi_{f_\infty,\mathfrak{q}}$ lies in the discrete series.

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Then for any prime number ℓ and a finite place λ of L above ℓ, there exists a λ-adic representation $\rho = \rho_{f_\infty,\lambda} : G_F \to \mathrm{GL}_{L_\lambda}(V_{f_\infty,\lambda})$ satisfying the following property:

For any finite place $\mathfrak{p} \nmid \ell$ there is an isomorphism

$$'\rho^{\text{F-ss}}_{f_\infty,\lambda,\mathfrak{p}} \simeq \check{\sigma}(\pi_{f_\infty,\mathfrak{p}}) \otimes_L L_\lambda$$

of representations of the Weil-Deligne group $WD(\overline{F}_\mathfrak{p}/F_\mathfrak{p})$.

Saito [24] showed that when $\mathfrak{p} \mid \ell$, $\rho_{f_\infty,\lambda,\mathfrak{p}}$ is potentially semistable.

Now we assume that $\ell = p$, $\mathfrak{p}$ is the prime ideal of F above p and L contains F. Let $\mathfrak{P}$ be a prime ideal of L above $\mathfrak{p}$.

The main result of our article is the following.

Theorem 1.2. *Let f_∞ be as in Theorem 1.1 and of even weight, $\ell = p$ and $\lambda = \mathfrak{P}$. If f_∞ is new at $\mathfrak{p}$ (when $[F:\mathbb{Q}]$ is odd, we demand that f_∞ is new at another prime ideal), then $\rho_{f_\infty,\mathfrak{P},\mathfrak{p}}$ is a noncritical semistable (noncrystalline) representation of $G_{F_\mathfrak{p}}$.*

Remark 1.3. The conditions in Theorem 1.1 and Theorem 1.2 are used to ensure that via the Jacquet-Langlands correspondence f_∞ corresponds to a modular form on the Shimura curve M associated to a quaternion algebra B that splits at exactly one real place; in Theorem 1.2 the quaternion algebra B is demanded to be ramified at $\mathfrak{p}$. See Subsection 4.1 for the construction of M.

Thus, $D_{\mathrm{st},F_\mathfrak{p}}(\rho_{f_\infty,\mathfrak{P},\mathfrak{p}})$ is associated with the Fontaine-Mazur L-invariant. We define the *Fontaine-Mazur L-invariant* of f_∞, denoted by $\mathcal{L}_{FM}(f_\infty)$, to be that of $D_{\mathrm{st},F_\mathfrak{p}}(\rho_{f_\infty,\mathfrak{P},\mathfrak{p}})$.

2. Local systems and the associated filtered φ_q-isocrystals

Let X be a p-adic formal $\mathscr{O}_{F_\mathfrak{p}}$-scheme. Suppose that X is analytically smooth over $\mathscr{O}_{F_\mathfrak{p}}$; that is, the generic fibre X^{an} of X is smooth. Here, by a formal $\mathscr{O}_{F_\mathfrak{p}}$-scheme, we mean a formal $\mathscr{O}_{F_\mathfrak{p}}$-scheme locally of finite type.

The filtered convergent φ-isocrystals attached to local systems are studied in [14, 6]. It is more convenience for us to compute the filtered convergent φ_q-isocrystals attached to the local systems that we will be interested in. From now on, we will ignore 'convergent' in the notion.

Filtered φ_q-isocrystal is a natural analogue of filtered φ-isocrystal. To define it one needs the notion of $F_\mathfrak{p}$-enlargement. An *$F_\mathfrak{p}$-enlargement* of X is a pair (T, x_T) consisting of a flat formal $\mathscr{O}_{F_\mathfrak{p}}$-scheme T and a morphism of formal $\mathscr{O}_{F_\mathfrak{p}}$-scheme $x_T : T_0 \to X$, where T_0 is the reduced closed subscheme of T defined by the ideal $\pi\mathscr{O}_T$.

An *isocrystal* $\mathscr{E}$ on X consists of the following data:

- for every $F_\mathfrak{p}$-enlargement (T, x_T) a coherent $\mathscr{O}_T \otimes_{\mathscr{O}_{F_\mathfrak{p}}} F_\mathfrak{p}$-module $\mathscr{E}_T$,
- for every morphism of $F_\mathfrak{p}$-enlargements $g : (T', x_{T'}) \to (T, x_T)$ an isomorphism of $\mathscr{O}_{T'} \otimes_{\mathscr{O}_{F_\mathfrak{p}}} F_\mathfrak{p}$-modules $\theta_g : g^*(\mathscr{E}_T) \to \mathscr{E}_{T'}$.

The collection of isomorphisms $\{\theta_g\}$ is required to satisfy certain cocycle condition. If T is an $F_\mathfrak{p}$-enlargement of X, then $\mathscr{E}_T$ may be interpreted as a coherent sheaf E^{an}_T on the rigid space T^{an}.

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Because X is analytically smooth over $\mathscr{O}_{F_\mathfrak{p}}$, there is a natural integrable connection

$$\nabla_X : E_X^{\mathrm{an}} \to E_X^{\mathrm{an}} \otimes \Omega^1_{X^{\mathrm{an}}}.$$

Note that an isocrystal on X depends only on X_0, the reduced closed subscheme of X defined by the ideal $\pi\mathscr{O}_X$. Let φ_q denote the absolute q-Frobenius of X_0. A *φ_q-isocrystal* on X is an isocrystal $\mathscr{E}$ on X together with an isomorphism of isocrystals $\varphi_q : \varphi_q^*\mathscr{E} \to \mathscr{E}$. A *filtered φ_q-isocrystal* on X is a φ_q-isocrystal $\mathscr{E}$ with a descending $\mathbb{Z}$-filtration on E_X^{an}.

The following result compares the de Rham cohomology of a filtered φ_q-isocrystal $\mathscr{E}$ and the étale cohomology of the $\mathbb{Q}_p$-local system $\mathcal{E}$ over the general fibre $X_{\overline{F}_\mathfrak{p}}$ associated to it. Let us explain what that $\mathscr{E}$ and $\mathcal{E}$ are attached to each other means. The question is local, so we may assume that there exists a scheme $\mathbf{X}$ over $\mathscr{O}_{F_\mathfrak{p}}$ whose special fibre is isomorphic to X_0 and whose completion along the special fibre is isomorphic to X. When $\mathrm{Spec}(R) \subset \mathbf{X}$ is a sufficiently small affine subscheme, one may form a certain filtered ring $B(R)$. Evaluate $\mathscr{E}$ on it to get $\mathscr{E}(B(R))$, which admits a Galois action and a filtration. That $\mathscr{E}$ and $\mathcal{E}$ are associated to each other means that functorially in R one has $\mathscr{E}(B(R)) \cong B(R) \otimes \mathcal{E}$ respecting Galois actions and filtrations. See [12, 13] for details.

Theorem 2.1. *[14, Theorem 3.2] Suppose that X is a semistable proper curve over $\mathscr{O}_{F_\mathfrak{p}}$. Let $\mathscr{E}$ be a filtered φ_q-isocrystal over X and $\mathcal{E}$ be a $\mathbb{Q}_p$-local system over $X_{\overline{F}_\mathfrak{p}}$ that are attached to each other. Then the Galois representation $H^i_{\mathrm{et}}(X_{\overline{F}_\mathfrak{p}}, \mathcal{E})$ of $G_{F_\mathfrak{p}}$ is semistable and the associated filtered (φ_q, N)-module $D_{\mathrm{st}, F_\mathfrak{p}}(H^i_{\mathrm{et}}(X_{\overline{F}_\mathfrak{p}}, \mathcal{E}))$ is isomorphic to $H^i_{\mathrm{dR}}(X^{\mathrm{an}}, \mathscr{E})$.*

Now let X be a connected, smooth and proper curve over $F_\mathfrak{p}$ with a regular semistable model $\mathcal{X}$ over $\mathscr{O}_{F_\mathfrak{p}}$ such that all irreducible components of its special fibre $\overline{\mathcal{X}}$ are smooth. For a subset U of $\overline{\mathcal{X}}$ let $]U[$ denote the tube of U in X^{an}. We associate to $\overline{\mathcal{X}}$ a graph $\mathrm{Gr}(\overline{\mathcal{X}})$. Let $\mathrm{n} : \overline{\mathcal{X}}^{\mathrm{n}} \to \overline{\mathcal{X}}$ be the normalisation of $\overline{\mathcal{X}}$. The vertices $\mathrm{V}(\overline{\mathcal{X}})$ of $\mathrm{Gr}(\overline{\mathcal{X}})$ are irreducible components of $\overline{\mathcal{X}}$. For every vertex v let C_v be the irreducible component corresponding to v. The edges $\mathrm{E}(\overline{\mathcal{X}})$ of $\mathrm{Gr}(\overline{\mathcal{X}})$ are ordered pairs $\{x,y\}$ where x and y are two different liftings in $\overline{\mathcal{X}}^{\mathrm{n}}$ of a singular point. Let τ be the involution on $\mathrm{E}(\overline{\mathcal{X}})$ such that $\tau\{x,y\} = \{y,x\}$. Below, for a module M on which τ acts, set $M^{\pm} = \{m \in M : \tau(m) = \pm m\}$.

Let $\mathscr{E}$ be a filtered φ_q-isocrystal over X. For any $e = \{x,y\} \in \mathrm{E}(\overline{\mathcal{X}})$, let $H^i_{\mathrm{dR}}(]e[, \mathscr{E})$ denote $H^i_{\mathrm{dR}}(]\mathrm{n}(x)[, \mathscr{E})$. Then τ exchanges $H^i_{\mathrm{dR}}(]e[, \mathscr{E})$ and $H^i_{\mathrm{dR}}(]\bar{e}[, \mathscr{E})$ where $\bar{e} = \{y,x\}$. Note that $\{C_v\}_{v \in \mathrm{V}(\overline{\mathcal{X}})}$ is an admissible covering of X^{an}. From the Mayer-Vietoris exact sequence with respect to this admissible covering, we obtain the following short exact sequence:

$$0 \longrightarrow \Big(\bigoplus_{e \in \mathrm{E}(\overline{\mathcal{X}})} H^0_{\mathrm{dR}}(]e[, \mathscr{E})\Big)^{-} / \text{the image of } \bigoplus_{v \in \mathrm{V}(\overline{\mathcal{X}})} H^0_{\mathrm{dR}}(]C_v[, \mathscr{E}) \xrightarrow{\ \iota\ } H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathscr{E})$$

$$\longrightarrow \ker\Big(\bigoplus_{v \in \mathrm{V}(\overline{\mathcal{X}})} H^1_{\mathrm{dR}}(]C_v[, \mathscr{E}) \to \Big(\bigoplus_{e \in \mathrm{E}(\overline{\mathcal{X}})} H^1_{\mathrm{dR}}(]e[, \mathscr{E})\Big)^{+}\Big) \longrightarrow 0. \tag{2.1}$$

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For any $e\in \mathrm{E}(\overline{\mathcal{X}})$ there is a residue map $\mathrm{Res}_e : H^1_{\mathrm{dR}}(]e[,\mathscr{E}) \to H^0_{\mathrm{dR}}(]e[,\mathscr{E})$ [6, Section 4.1]. These residue maps induce a map

$$\bigoplus_{e\in \mathrm{E}(\overline{\mathcal{X}})} \mathrm{Res}_e : \quad \Big(\bigoplus_{e\in \mathrm{E}(\overline{\mathcal{X}})} H^1_{\mathrm{dR}}(]e[,\mathscr{E})\Big)^+ \to \Big(\bigoplus_{e\in \mathrm{E}(\overline{\mathcal{X}})} H^0_{\mathrm{dR}}(]e[,\mathscr{E})\Big)^-.$$

Proposition 2.2. *[6, Theorem 2.6, Remark 2.7] The monodromy operator N on $H^1_{\mathrm{dR}}(X^{\mathrm{an}},\mathscr{E})$ coincides with the composition*

$$\iota\circ\Big(\bigoplus_{e\in \mathrm{E}(\overline{\mathcal{X}})} \mathrm{Res}_e\Big)\circ\Big(H^1_{\mathrm{dR}}(X^{\mathrm{an}},\mathscr{E}) \to \Big(\bigoplus_{e\in \mathrm{E}(\overline{\mathcal{X}})} H^1_{\mathrm{dR}}(]e[,\mathscr{E})\Big)^+\Big)$$

where $H^1_{\mathrm{dR}}(X^{\mathrm{an}},\mathscr{E}) \to \Big(\bigoplus_{e\in \mathrm{E}(\overline{\mathcal{X}})} H^1_{\mathrm{dR}}(]e[,\mathscr{E})\Big)^+$ is the restriction map and ι is the connecting homomorphism appeared in (2.1).

3. The universal special formal module

3.1. Special formal modules and Drinfeld's moduli theorem

Let $B_{\mathfrak{p}}$ be the quaternion algebra over $F_{\mathfrak{p}}$ with invariant $1/2$. So $B_{\mathfrak{p}}$ is isomorphic to $F_{\mathfrak{p}}^{(2)}[\Pi]$; $\Pi^2=\pi$ and $\Pi a = \bar{a}\Pi$ for all $a\in F_{\mathfrak{p}}^{(2)}$. Here, π is a fixed uniformiser of $F_{\mathfrak{p}}$, $F_{\mathfrak{p}}^{(2)}$ is the unramified extension of $F_{\mathfrak{p}}$ of degree 2 and $a\mapsto \bar{a}$ denotes the nontrivial $F_{\mathfrak{p}}$-automorphism of $F_{\mathfrak{p}}^{(2)}$.

Let $\mathscr{O}_{B_{\mathfrak{p}}}$ be the ring of integers in $B_{\mathfrak{p}}$. Let k be the residue field of $F_{\mathfrak{p}}$ and $F_{\mathfrak{p}0}^{(2)}$ the unramified extension of $F_{\mathfrak{p}0}$ of degree 2.

Let $\mathscr{O}^{\mathrm{ur}}$ denote the maximal unramified extension of $\mathscr{O}_{F_{\mathfrak{p}}}$ and $\widehat{\mathscr{O}^{\mathrm{ur}}}$ its π-adic completion. Fix an algebraic closure $\bar{k}$ of k. We identify $\widehat{\mathscr{O}^{\mathrm{ur}}}/\pi\widehat{\mathscr{O}^{\mathrm{ur}}}$ with $\bar{k}$. Then $\mathrm{W}(\bar{k})\otimes_{\mathscr{O}_{F^0_{\mathfrak{p}}}}\mathscr{O}_{F_{\mathfrak{p}}} \cong \widehat{\mathscr{O}^{\mathrm{ur}}}$. Let $\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}$ be the fractional field of $\widehat{\mathscr{O}^{\mathrm{ur}}}$.

We use the notion of special formal $\mathscr{O}_{B_{\mathfrak{p}}}$-module in [11].

We fix a special formal $\mathscr{O}_{B_{\mathfrak{p}}}$-module over $\bar{k}$, Φ, as in [23, (3.54)]. Let ι denote the natural embedding of $F_{\mathfrak{p}0}$ into $\mathrm{W}(\bar{k})[1/p]$. Then all embeddings of $F_{\mathfrak{p}0}$ into $\mathrm{W}(\bar{k})[1/p]$ are $\varphi^j\circ\iota$ $(0\le j\le v_p(q)-1)$. We have the decomposition

$$\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k}) = \prod_{j=0}^{v_p(q)-1}\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathscr{O}_{F^0_{\mathfrak{p}}},\varphi^j\circ\iota}\mathrm{W}(\bar{k}).$$

Let $u\in \mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k})$ be the element whose $\varphi^j\circ\iota$-component with respect to this decomposition is

$$u_{\varphi^j\circ\iota} = \begin{cases} \Pi\otimes 1 & \text{if } j=0,\\ 1\otimes 1 & \text{if } j=1,\dots,v_p(q)-1.\end{cases}$$

Let $\widetilde{\mathrm{F}}$ be the $1\otimes\varphi$-semilinear operator on $\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k})$ defined by

$$\widetilde{\mathrm{F}}x = (1\otimes\varphi)x\cdot u,\quad x\in \mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k}).$$

Let $\widetilde{\mathrm{V}}$ be the $1\otimes\varphi^{-1}$-semilinear operator on $\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k})$ such that $\widetilde{\mathrm{F}}\widetilde{\mathrm{V}}=p$. Then

$$(\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k}),\widetilde{\mathrm{V}},\widetilde{\mathrm{F}})$$

is a Dieudonne module over $\mathrm{W}(\bar{k})$ with an action of $\mathscr{O}_{B_{\mathfrak{p}}}$ by the left multiplication. Let Φ be the special formal $\mathscr{O}_{B_{\mathfrak{p}}}$-module over $\bar{k}$ whose contravariant Dieudonne crystal is $(\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k}),\widetilde{\mathrm{V}},\widetilde{\mathrm{F}})$. [1]

Let ι_0 and ι_1 be the extensions of ι to $F_{\mathfrak{p}0}^{(2)}$. Then

$$\varphi^j\iota_0,\ \varphi^j\iota_1\quad(0\le j\le v_p(q)-1)$$

are all embeddings of $F_{\mathfrak{p}0}^{(2)}$ into $\mathrm{W}(\bar{k})[1/p]$. We have

$$\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k})=\prod_{j=0}^{v_p(q)-1}\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathscr{O}_{F_{\mathfrak{p}0}^{(2)}},\varphi^j\circ\iota_0}\mathrm{W}(\bar{k})\times\prod_{j=0}^{v_p(q)-1}\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathscr{O}_{F_{\mathfrak{p}0}^{(2)}},\varphi^j\circ\iota_1}\mathrm{W}(\bar{k}),$$

where $\mathscr{O}_{B_{\mathfrak{p}}}$ is considered an $\mathscr{O}_{F_{\mathfrak{p}}^{(2)}}$-module by the left multiplication. Let X be the element of $\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k})$ whose $\varphi^j\circ\iota_0$-component $(0\le j\le v_p(q)-1)$ is $1\otimes1$ and whose $\varphi^j\circ\iota_1$-component $(0\le j\le v_p(q)-1)$ is $\Pi\otimes1$. Similarly, let Y be the element whose $\varphi^j\circ\iota_0$-component $(0\le j\le v_p(q)-1)$ is $\Pi\otimes1$ and whose $\varphi^j\circ\iota_1$-component $(0\le j\le v_p(q)-1)$ is $\pi\otimes1$. Then $\{X,Y\}$ is a basis of $\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k})$ over $\mathscr{O}_{F_{\mathfrak{p}}^{(2)}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k})$.

Note that $\mathrm{GL}(2,F_{\mathfrak{p}})=(\mathrm{End}^0_{\mathscr{O}_{B_{\mathfrak{p}}}}\Phi)^{\times}$ [23, Lemma 3.60]. We normalise the isomorphism such that the action on the φ-module

$$(\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k}),\widetilde{\mathrm{F}})[1/p]=(B_{\mathfrak{p}}\otimes_{\mathbb{Q}_p}\mathrm{W}(\bar{k})[1/p],\widetilde{\mathrm{F}})$$

is given by $\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]^{-1}X=(a\otimes1)X+(c\otimes1)Y$ and $\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]^{-1}Y=(b\otimes1)X+(d\otimes1)Y$. We can also let $\mathrm{GL}(2,F_{\mathfrak{p}})$ act on the φ-module on the right-hand side by $X\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]=(a\otimes1)X+(c\otimes1)Y$ and $Y\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]=(b\otimes1)X+(d\otimes1)Y$.

Let $\widetilde{\mathrm{D}}_0$ denote the φ_q-module

$$(B_{\mathfrak{p}}\otimes_{\mathbb{Q}_p}\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}},\widetilde{\mathrm{F}}^{v_p(q)})$$

coming from the φ-module $(\mathscr{O}_{B_{\mathfrak{p}}}\otimes_{\mathbb{Z}_p}\mathrm{W}(\bar{k}),\widetilde{\mathrm{F}})[1/p]$.

We describe Drinfeld's moduli problem. Let Nilp be the category of $\widehat{\mathscr{O}^{\mathrm{ur}}}$-algebras on which π is nilpotent. For any $A\in\mathrm{Nilp}$, let ψ be the homomorphism $\bar{k}\to A/\pi A$; let

[1]The Dieudonne crystal in [23, (3.54)] is exactly the covariant Dieudonne crystal of Φ. The duality between the contravariant Dieudonne crystal and the covariant Dieudonne crystal is induced by the trace map

$$<\cdot,\cdot>:\mathscr{O}_{B_{\mathfrak{p}}}\times\mathscr{O}_{B_{\mathfrak{p}}}\to\mathbb{Z}_p,(x,y)\mapsto\mathrm{tr}_{F_{\mathfrak{p}}/\mathbb{Q}_p}\Big(\delta^{-1}_{F_{\mathfrak{p}}/\mathbb{Q}_p}\mathrm{tr}_{B_{\mathfrak{p}}/F_{\mathfrak{p}}}(xy^t)\Big),$$

where $\delta_{F_{\mathfrak{p}}/\mathbb{Q}_p}$ is the difference of $F_{\mathfrak{p}}$ over $\mathbb{Q}_p$, $\mathrm{tr}_{B_{\mathfrak{p}}/F_{\mathfrak{p}}}$ is the reduced trace map and $y\mapsto y^t$ is the involution of $B_{\mathfrak{p}}$ such that $\Pi^t=\Pi$ and $a^t=\bar{a}$ if $a\in F_{\mathfrak{p}}^{(2)}$. Then we have $<b\cdot x,y>=<x,b^t\cdot y>$ for any $b\in\mathscr{O}_{B_{\mathfrak{p}}}$.

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$\mathrm{SFM}(A)$ be the set of pairs (G,ρ) where G is a special formal $\mathscr{O}_{B_\mathfrak{p}}$-module over A and $\rho : \Phi_{A/\pi A} = \psi_*\Phi \to G$ is a quasi-isogeny of height zero.

We state a part of Drinfeld's theorem [11] as follows. In [1] Boutot and Carayol provided more details for [11].

Let $\mathcal{H}$ be the Drinfeld upper half plane over $F_\mathfrak{p}$; that is, the rigid analytic $F_\mathfrak{p}$-variety whose $\mathbb{C}_p$-points are $\mathbb{C}_p - F_\mathfrak{p}$.

Theorem 3.1. *The functor* SFM *is represented by the Deligne formal scheme* $\widehat{\mathcal{H}}\hat{\otimes}\widehat{\mathscr{O}^{\mathrm{ur}}}$ *over* $\widehat{\mathscr{O}^{\mathrm{ur}}}$ *whose generic fibre is* $\mathcal{H}_{\widehat{F_\mathfrak{p}^{\mathrm{ur}}}} = \mathcal{H}\hat{\otimes}\widehat{F_\mathfrak{p}^{\mathrm{ur}}}$.

See [1, Chapter I] for a precise description of $\widehat{\mathcal{H}}\hat{\otimes}\widehat{\mathscr{O}^{\mathrm{ur}}}$. It is closely related to the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}(2,F_\mathfrak{p})$. Each edge e (respectively vertex v) of $\mathcal{T}$ is assigned an affine formal scheme of finite type $\mathrm{Spf}(A_e)$ (respectively $\mathrm{Spf}(A_v)$). Then $\widehat{\mathcal{H}}\hat{\otimes}\widehat{\mathscr{O}^{\mathrm{ur}}}$ is the union of these $\mathrm{Spf}(A_e)$. If e and e' have a common vertex v, then $\mathrm{Spf}(A_e)\cap\mathrm{Spf}(A_{e'})$ is $\mathrm{Spf}(A_v)$. Otherwise, $\mathrm{Spf}(A_e)\cap\mathrm{Spf}(A_{e'}) = \emptyset$.

Let $\mathcal{G}$ be the universal special formal $\mathscr{O}_{B_\mathfrak{p}}$-module over $\widehat{\mathcal{H}}\hat{\otimes}\widehat{\mathscr{O}^{\mathrm{ur}}}$. There is an action of $\mathrm{GL}(2,F_\mathfrak{p})$ on $\mathcal{G}$ (see [1, Chapter II (9.2)]): The group $\mathrm{GL}(2,F_\mathfrak{p})$ acts on the functor SFM by $g\cdot(\psi;G,\rho) = (\psi\circ\mathrm{Frob}^{-n};G,\rho\circ\psi_*(g^{-1}\circ\mathrm{Frob}^n))$ if $v_\mathfrak{p}(\det g) = n$. Here, $v_\mathfrak{p}$ is the valuation of $\mathbb{C}_p$ normalised such that $v_\mathfrak{p}(\pi) = 1$.

3.2. The filtered φ_q-isocrystal attached to the universal special formal module

It is rather difficult to describe $\mathcal{G}$ precisely.[2] However, we can determine the filtered φ_q-isocrystal $\mathcal{M}$ attached to the local system $V_p\mathcal{G}$, the Tate module of $\mathcal{G}$ tensoring with $\mathbb{Q}$.

For every $A \in \mathrm{Nilp}$ and each pair $(G,\rho)\in\mathrm{SFM}(A)$, G admits a universal extension E_G by a vector group. Considering tangent spaces we obtain a homomorphism

$$M_G \to \mathrm{Lie}_G$$

that is functorial in A, where M_G and Lie_G are the Lie algebras of E_G and G, respectively.

Such an assignment exists even for complete flat $\widehat{\mathscr{O}^{\mathrm{ur}}}$-algebra of finite type A. Indeed, this follows from the crystalline property of the Dieudonne crystal of $G\otimes_A A/pA$ [20, Chapter V (1.6)]. Tensoring with $\mathbb{Q}$ we obtain $M_G\otimes\mathbb{Q}\to\mathrm{Lie}_G\otimes\mathbb{Q}$. Let $\mathrm{Fil}^1(M_G\otimes\mathbb{Q})$ be the kernel of this morphism.

We apply it to $\mathcal{G}|_{\mathrm{Spf}(A_e)}$ and $\mathcal{G}|_{\mathrm{Spf}(A_v)}$. Patching them, we obtain the filtered φ_q-isocrystal $\mathcal{M}$ attached to $V_p\mathcal{G}$. From these data we obtain a period map of $\mathcal{H}^{\mathrm{an}}$, the general fibre of $\widehat{\mathcal{H}}$, that is defined by the filtration. See [23, 3.29 and 5.18] for a more precise construction of this period map.

Taking dual, we get the filtered φ_q-isocrystal $\mathcal{D}$ attached to the dual of $V_p\mathcal{G}$. Precisely, the filtration on $\mathcal{D}$ is defined in the way that

$\mathrm{Fil}^j\mathcal{D}$ and $\mathrm{Fil}^{2-j}\mathcal{M}$ are annihilators of each other.

In the following, we write $\mathscr{O}_{\mathcal{H},\widehat{F_\mathfrak{p}^{\mathrm{ur}}}}$ for $\mathscr{O}_{\mathcal{H}\hat{\otimes}\widehat{F_\mathfrak{p}^{\mathrm{ur}}}}$ and $\Omega_{\mathcal{H},\widehat{F_\mathfrak{p}^{\mathrm{ur}}}}$ for the differential sheaf $\Omega_{\mathcal{H}\hat{\otimes}\widehat{F_\mathfrak{p}^{\mathrm{ur}}}}$.

[2]See [26] for some information about $\mathcal{G}$ and [29] for a higher rank analogue.

Lemma 3.2. *$\mathcal{D}$ is naturally isomorphic to the φ_q-isocrystal*

$$\widetilde{\mathrm{D}}_0 \otimes_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}} \mathscr{O}_{\mathcal{H},\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$$

with the q-Frobenius being $\widetilde{\mathrm{F}}^{v_p(q)} \otimes \varphi_{q,\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}} 1$ and the connection being

$$1 \otimes \mathrm{d} : \widetilde{\mathrm{D}}_0 \otimes_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}} \mathscr{O}_{\mathcal{H},\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}} \to \widetilde{\mathrm{D}}_0 \otimes_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}} \Omega_{\mathcal{H},\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}.$$

Proof. What we need to show is that $\mathcal{D}$ is constant except for the filtration. The same property for $\mathcal{M}$ is mentioned in [14, Section 5] without providing details. It follows from the rigidity of quasi-isogeny [23, Proposition 3.62] and the Grothendieck-Messing theorem [20, Chapter V (1.6)]. We sketch the proof for the reader's convenience.

For any formal $\widehat{\mathscr{O}^{\mathrm{ur}}}$-scheme (of finite type) T and a morphism $x_T : T \to \widehat{\mathcal{H}}\hat{\otimes}\widehat{\mathscr{O}^{\mathrm{ur}}}$, $x_T^*\mathcal{G}$ is a special formal $\mathscr{O}_{B_{\mathfrak{p}}}$-module over T, denoted by G_T. Let T_0 be the closed subscheme of T defined by π and T_0' the closed subscheme defined by p. Then both T_0 and T_0' are $\bar{k}$-schemes. By definition of Drinfeld's functor, $G_{T_0} = G_T \times_T T_0$ is quasi-isogenous to Φ_{T_0}, the pullback of Φ via $T_0 \to \mathrm{Spec}(\bar{k})$. By [23, Proposition 3.62], this quasi-isogeny uniquely extends to a quasi-isogeny $G_{T'} := G_T \times_T T_0' \to \Phi_{T_0'}$. Let $\mathbb{D}$ be the covariant Dieudonne crystal functor. By the Grothendieck-Messing theorem we have

$$M_G = \mathbb{D}(G_{T_0'})_T \doteq \mathbb{D}(\Phi_{T_0'})_T = \mathscr{O}_T \otimes_{\mathrm{W}(\bar{k})} \mathbb{D}(\Phi)_{\mathrm{W}(\bar{k})}. \tag{3.1}$$

Here, $\doteq$ means that the equalty holds after tensoring with $\mathbb{Q}$. See also [23, Proposition 5.15].

If $f : S \to T$ is a morphism of formal $\widehat{\mathscr{O}^{\mathrm{ur}}}$-schemes, put $x_S = x_T \circ f$ and $G_S = x_S^*\mathcal{G}$. We form S_0 and S_0' in the same way. Then we have a commutative diagram

$$\begin{array}{ccc} G_{S_0'} & \longrightarrow & (f_0')^* G_{T_0'} \\ \downarrow & & \downarrow \\ \Phi_{S_0'} & \longrightarrow & (f_0')^* \Phi_{T_0'}, \end{array}$$

where the vertical arrows are quasi-isogenies, the horizontal arrows are natural isomorphisms and $f_0' : S_0' \to T_0'$ is the morphism induced from f. This implies that the isomorphism (3.1) is functorial. Hence, the isocrystal structure of $\mathcal{M}$ is constant.

Let $\mathbf{F}$ be the absolute Frobenius. From the commutative diagram

$$\begin{array}{ccc} G_{T_0'} & \xrightarrow{\mathrm{F}^{v_p(q)}} & G_{T_0'} \\ \downarrow & & \downarrow \\ \Phi_{T_0'} & \xrightarrow{\mathrm{F}^{v_p(q)}} & \Phi_{T_0'} \end{array}$$

we obtain the constancy of the φ_q-module structure of $\mathcal{M}$. □

Next, we determine the filtration on $\widetilde{\mathrm{D}}_0 \otimes_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}} \mathscr{O}_{\mathcal{H},\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$.

For any $F_{\mathfrak{p}}$-algebras K and L, $L \otimes_{\mathbb{Q}_p} K$ is isomorphic to $L \otimes_{F_{\mathfrak{p}}} K \oplus (L \otimes_{\mathbb{Q}_p} K)_{\mathrm{non}}$, where $(L \otimes_{\mathbb{Q}_p} K)_{\mathrm{non}}$ is the kernel of the homomorphism $L \otimes_{\mathbb{Q}_p} K \to L \otimes_{F_{\mathfrak{p}}} K$, $\ell \otimes a \mapsto \ell \otimes a$.

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If L is a field extension of $F_{\mathfrak{p}}$ that splits $F_{\mathfrak{p}}$, then $L\otimes_{\mathbb{Q}_p}K=\bigoplus_{\tau:F_{\mathfrak{p}}\hookrightarrow L}L\otimes_{\tau,F_{\mathfrak{p}}}K$, and $(L\otimes_{\mathbb{Q}_p}K)_{\mathrm{non}}$ corresponds to the nonnatural embeddings. We apply this to $L=F_{\mathfrak{p}}$ and $K=\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}$; consider $\widetilde{\mathrm{D}}_0=B_{\mathfrak{p}}\otimes_{\mathbb{Q}_p}\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}$ as an $F_{\mathfrak{p}}\otimes_{\mathbb{Q}_p}\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}$-module. Then $\widetilde{\mathrm{D}}_0$ splits into two parts: one is the canonical part that corresponds to the natural embedding $\mathrm{id}:F_{\mathfrak{p}}\hookrightarrow F_{\mathfrak{p}}$ and the other is the noncanonical part that corresponds to the nonnatural embeddings. Correspondingly, $\widetilde{\mathrm{D}}_0\otimes_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}\mathscr{O}_{\mathcal{H},\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$ splits into two parts, the canonical part $B_{\mathfrak{p}}\otimes_{F_{\mathfrak{p}}}\mathscr{O}_{\mathcal{H},\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$ and the noncanonical part. Because $F_{\mathfrak{p}}$ acts on the Lie algebra of any special formal $\mathscr{O}_{B_{\mathfrak{p}}}$-module through the natural embedding, the filtration on the noncanonical part is trivial.

The filtration on the canonical part is precisely described by Drinfeld's period morphism. Let us recall the definition of Drinfeld's period morphism. We will use the notations in [29, Section 2.2].

Let $M(\Phi)$ be the Cartier module of Φ, a $\mathbb{Z}/2\mathbb{Z}$-graded module. The $\mathbb{Z}/2\mathbb{Z}$-grading depends on a choice of $F_{\mathfrak{p}}$-embedding of $F_{\mathfrak{p}}^{(2)}$ into $\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}$. We choose the one, $\tilde{\iota}_0$, that restricts to ι_0 and denote the other $F_{\mathfrak{p}}$-embedding by $\tilde{\iota}_1$. We fix a graded V-basis $\{g^0,g^1\}$ of $M(\Phi)$ such that $\mathrm{V}g^0=\Pi g^0$ and $\mathrm{V}g^1=\Pi g^1$. Then $\{g^0,g^1,\mathrm{V}g^0,\mathrm{V}g^1\}$ is a basis of $M(\Phi)[1/p]$ over $\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}$; $F_{\mathfrak{p}}^{(2)}\subset B_{\mathfrak{p}}$ acts on $\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}g^0\oplus\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}\mathrm{V}g^1$ by $\tilde{\iota}_0$ and acts on $\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}\mathrm{V}g^0\oplus\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}g^1$ by $\tilde{\iota}_1$. See [11] for the definition of Cartier module and the meaning of graded V-basis.

Let R be any flat π-adically complete $\mathscr{O}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$-algebra. Drinfeld constructed for each $(\psi;G,\rho)\in\mathrm{SFM}(R)$ a quadruple (η,T,u,ρ). Let $M=M(G)$ be the Cartier module of G, $N(M)$ the auxiliary module that is the quotient of $M\oplus M$ by the submodule generated by elements of the form $(\mathrm{V}x,-\Pi x)$ and β_M the quotient map $M\oplus M\to N(M)$. For $(x_0,x_1)\in M\oplus M$, we write $((x_0,x_1))$ for $\beta_M(x_0,x_1)$. Then we have a map $\varphi_M:N(M)\to N(M)$. See [29, Definition 4] for its definition. Put

$$\eta_M:=N(M)^{\varphi_M},\quad T_M:=M/\mathrm{V}M;$$

both η_M and T_M are $\mathbb{Z}/2\mathbb{Z}$-graded. Note that T_M is exactly the tangent sheaf of G (see [1, Subsection II.8]).

Let $u_M:\eta_M\to T_M$ be the $\mathscr{O}_{F_{\mathfrak{p}}}[\Pi]$-linear map of degree 0 that is the composition of the inclusion $\eta_M\hookrightarrow N(M)$ and the map

$$N(M)\to M/\mathrm{V}M,\qquad ((x_0,x_1))\mapsto x_0 \bmod \mathrm{V}M.$$

Then $\eta_{M(\Phi)}$ is a free $\mathscr{O}_{F_{\mathfrak{p}}}$-module of rank 4 with a basis

$$\{((g^0,0)),\ ((g^1,0)),((\mathrm{V}g^0,0)),((\mathrm{V}g^1,0))\},$$

where $((g^0,0))$ and $((\mathrm{V}g^1,0))$ are in degree 0 and $((g^1,0))$ and $((\mathrm{V}g^0,0))$ are in degree 1. The quasi-isogeny $\rho:\psi_*\Phi\to G_{R/\pi R}$ induces an isomorphism

$$\rho:\eta^0_{M(\Phi)}\otimes_{\mathscr{O}_{F_{\mathfrak{p}}}}F_{\mathfrak{p}}\xrightarrow{\sim}\eta^0_{M(G)}\otimes_{\mathscr{O}_{F_{\mathfrak{p}}}}F_{\mathfrak{p}}.$$

The Drinfeld period of (G,ρ) is defined by

$$z(G,\rho)=\frac{u'_M\circ\rho((\mathrm{V}g^1,0))}{u'_M\circ\rho((g^0,0))},\tag{3.2}$$

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where u'_M is the map $\eta^0_{M(G)} \otimes_{\mathscr{O}_{F_\mathfrak{p}}} F_\mathfrak{p} \to T^0_M \otimes_R R[1/p]$ induced by u_M. By [23, Subsection 5.49] Drinfeld's period map coincides with the period map defined by filtration.

By [23, (3.55)], $M(\Phi)$ is isomorphic to the caonical part of the covariant Dieudonne module attached to Φ. In [23] the Cartier module is called τ-$W_F(L)$-crystal. So, as a φ_q-module, $M(\Phi)[1/p]$ is the dual of $B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \widehat{F^{\mathrm{ur}}_\mathfrak{p}}$, the canonical part of $\widetilde{\mathrm{D}}_0$. Let $\{v_0, v_1, v_2, v_3\}$ be the basis of $B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \widehat{F^{\mathrm{ur}}_\mathfrak{p}}$ over $\widehat{F^{\mathrm{ur}}_\mathfrak{p}}$ dual to $\{\pi g^1, g^0, \mathrm{V}g^0, \mathrm{V}g^1\}$.

Lemma 3.3. *We have*

$$\begin{aligned}
\mathrm{Fil}^0 B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \mathscr{O}_{\mathcal{H},\widehat{F^{\mathrm{ur}}_\mathfrak{p}}} &= B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \mathscr{O}_{\mathcal{H},\widehat{F^{\mathrm{ur}}_\mathfrak{p}}} \\
\mathrm{Fil}^1 B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \mathscr{O}_{\mathcal{H},\widehat{F^{\mathrm{ur}}_\mathfrak{p}}} &= \textit{the } \mathscr{O}_{\mathcal{H},\widehat{F^{\mathrm{ur}}_\mathfrak{p}}}\textit{-submodule generated by} \\
&\qquad \widehat{F^{\mathrm{ur}}_\mathfrak{p}} \cdot (v_1 + z v_3) \oplus \widehat{F^{\mathrm{ur}}_\mathfrak{p}} \cdot (z v_0 + v_2) \\
\mathrm{Fil}^2 B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \mathscr{O}_{\mathcal{H},\widehat{F^{\mathrm{ur}}_\mathfrak{p}}} &= 0.
\end{aligned}$$

Here, z is the canonical coordinate on $\mathcal{H}_{\widehat{F^{\mathrm{ur}}_\mathfrak{p}}}$.

Proof. Let R and $(\psi; G, \rho) \in \mathrm{SFM}(R)$ be as above. Via ρ, as a φ_q-module, $M(G)[1/p]$ is isomorphic to $M(\Phi)[1/p] \otimes_{\widehat{F^{\mathrm{ur}}_\mathfrak{p}}} R[1/p]$ and thus

$$M(G)[1/p] = R[1/p] \cdot \pi g^1 \oplus R[1/p] \cdot g^0 \oplus R[1/p] \cdot \mathrm{V}g^1 \oplus R[1/p] \cdot \mathrm{V}g^0.$$

Let z be the Drinfeld period of (G, ρ). Because Drinfeld's period map coincides with the period map defined by filtration, we have

$$\mathrm{Fil}^1 M(G)[1/p] = R[1/p](\mathrm{V}g^1 - z g^0) \oplus R[1/p](z\mathrm{V}g^0 - \pi g^1).$$

Here, we note that $\pi g^1 - z\mathrm{V}g^0 = \mathrm{V}(\mathrm{V}g^1 - z g^0)$.

Taking dual, we obtain the desired filtration structure on $\mathcal{D}$. □

We decompose $B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \widehat{F^{\mathrm{ur}}_\mathfrak{p}}$ into two direct summands:

$$B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \widehat{F^{\mathrm{ur}}_\mathfrak{p}} = B_\mathfrak{p} \otimes_{F^{(2)}_\mathfrak{p},\, \tilde{\iota}_0} \widehat{F^{\mathrm{ur}}_\mathfrak{p}} \oplus B_\mathfrak{p} \otimes_{F^{(2)}_\mathfrak{p},\, \tilde{\iota}_1} \widehat{F^{\mathrm{ur}}_\mathfrak{p}},$$

where $B_\mathfrak{p}$ is considered as an $F^{(2)}_\mathfrak{p}$-module by left multiplication. Let e_0 and e_1 denote the projection to the first summand and that to the second, respectively. We may choose g^0 and g^1 such that $v_0 = e_0 X$, $v_1 = e_1 Y$, $v_2 = e_0 Y$ and $v_3 = e_1 X$. Thus,

$$\begin{aligned}
\mathrm{Fil}^0 B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \mathscr{O}_{\mathcal{H},\widehat{F^{\mathrm{ur}}_\mathfrak{p}}} &= B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \mathscr{O}_{\mathcal{H},\widehat{F^{\mathrm{ur}}_\mathfrak{p}}}, \\
\mathrm{Fil}^1 B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \mathscr{O}_{\mathcal{H},\widehat{F^{\mathrm{ur}}_\mathfrak{p}}} &= \text{the } F^{(2)}_\mathfrak{p} \otimes_{F_\mathfrak{p}} \mathscr{O}_{\mathcal{H},\widehat{F^{\mathrm{ur}}_\mathfrak{p}}}\text{-submodule generated by } zX + Y, \quad \text{and} \\
\mathrm{Fil}^2 B_\mathfrak{p} \otimes_{F_\mathfrak{p}} \mathscr{O}_{\mathcal{H},\widehat{F^{\mathrm{ur}}_\mathfrak{p}}} &= 0.
\end{aligned}$$

Finally, we note that the induced action of $\mathrm{GL}(2, F_\mathfrak{p})$ on $\mathcal{H}$ is given by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] z = \frac{az+b}{cz+d}$.

4. Shimura curves

Fix a real place τ_1 of F. Let B be a quaternion algebra over F that splits at τ_1 and is ramified at other real places $\{\tau_2,\cdots,\tau_g\}$ and $\mathfrak{p}$.

4.1. Shimura curves M, M' and M''

We will use three Shimura curves studied by Carayol [2] and recall their constructions in this subsection (see also [24]).

Let G be the reductive algebraic group over $\mathbb{Q}$ such that $G(R)=(B\otimes R)^\times$ for any $\mathbb{Q}$-algebra R. Let Z be the center of G; it is isomorphic to $T=\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$. Let $\nu:G\to T$ be the morphism obtained from the reduced norm $\mathrm{Nrd}_{B/F}$ of B. The kernel of ν is G^{der}, the derived group of G, and thus we have a short exact sequence of algebraic groups

$$1\longrightarrow G^{\mathrm{der}}\longrightarrow G\xrightarrow{\ \nu\ }T\longrightarrow 1.$$

Let X be the $G(\mathbb{R})$-conjugacy class of the homomorphism

$$\begin{aligned} h: \quad & \mathbb{C}^\times \to G(\mathbb{R})=\mathrm{GL}_2(\mathbb{R})\times\mathbb{H}^\times\times\cdots\times\mathbb{H}^\times \\ z=x+\sqrt{-1}y\mapsto \quad & \left(\begin{bmatrix} x & y\\ -y & x\end{bmatrix}^{-1},\ 1,\ \cdots,\ 1\right), \end{aligned}$$

where $\mathbb{H}$ is the Hamilton quaternion algebra. The conjugacy class X is naturally identified with the union of upper and lower half planes. Let $M=M(G,X)=(M_U(G,X))_U$ be the canonical model of the Shimura variety attached to the Shimura pair (G,X); the canonical model is defined over F, the reflex field of (G,X). There is a natural right action of $G(\mathbb{A}_f)$ on $M(G,X)$. Here and in what follows, by abuse of terminology we call a projective system of varieties simply a variety.

Take an imaginary quadratic field $E_0=\mathbb{Q}(\sqrt{-a})$ (a a square-free positive integer) such that p splits in E_0. Put $E=FE_0$ and $D=B\otimes_F E=B\otimes_{\mathbb{Q}}E_0$. We fix a square root ρ of $-a$ in $\mathbb{C}$. Then the prolonging of τ_i to E by $x+y\sqrt{-a}\mapsto\tau_i(x)+\tau_i(y)\rho$ (respectively $x+y\sqrt{-a}\mapsto\tau_i(x)-\tau_i(y)\rho$) is denoted by τ_i (respectively $\bar{\tau}_i$).

Let T_E be the torus $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ and T_E^1 the subtorus of T_E such that $T_E^1(\mathbb{Q})=\{z\in E: z\bar{z}=1\}$. We consider the amalgamate product $G''=G\times_Z T_E$ and the morphism $G''=G\times_Z T_E\xrightarrow{\nu''}T''=T\times T_E^1$ defined by $(g,z)\mapsto(\nu(g)z\bar{z},z/\bar{z})$. Consider the subtorus $T'=\mathbb{G}_m\times T_E^1$ of T'' and let G' be the inverse image of T' by the map ν''. The restriction of ν'' to G' is denoted by ν'. Both the derived group of G' and that of G'' are identified with G^{der}, and we have two short exact sequences of algebraic groups

$$1\longrightarrow G^{\mathrm{der}}\longrightarrow G'\xrightarrow{\ \nu'\ }T'\longrightarrow 1$$

and

$$1\longrightarrow G^{\mathrm{der}}\longrightarrow G''\xrightarrow{\ \nu''\ }T''\longrightarrow 1.$$

The complex embeddings $\tau_1,\cdots,\tau_g$ of E identify $G''(\mathbb{R})$ with $\mathrm{GL}_2(\mathbb{R})\cdot\mathbb{C}^\times\times\mathbb{H}^\times\cdot\mathbb{C}^\times\times\cdots\times\mathbb{H}^\times\cdot\mathbb{C}^\times$. We consider the $G'(\mathbb{R})$-conjugacy class X' (respectively $G''(\mathbb{R})$-conjugacy class X'') of the homomorphism

$$\begin{array}{llr} h': & \mathbb{C}^\times \to G'(\mathbb{R}) \subset & \\ & & G''(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})\cdot\mathbb{C}^\times \times \mathbb{H}^\times\cdot\mathbb{C}^\times \times\cdots\times \mathbb{H}^\times\cdot\mathbb{C}^\times \\ z = x+\sqrt{-1}y \mapsto & & \left(\left[\begin{smallmatrix} x & y \\ -y & x\end{smallmatrix}\right]^{-1}\otimes 1,\ \ 1\otimes z^{-1},\ \ \cdots,\ \ 1\otimes z^{-1}\right). \end{array}$$

Let $M' = M(G',X')$ and $M'' = M(G'',X'')$ be the canonical models of the Shimura varieties defined over their reflex field E. There are natural right actions of $G'(\mathbb{A}_f)$ and $G''(\mathbb{A}_f)$ on M' and M'', respectively.

Put $T_{E_0} = \mathrm{Res}_{\mathbb{Q}}^{E_0}\mathbb{G}_m$. Using the complex embeddings $\tau_1,\cdots,\tau_g$ of E, we identify $T_E(\mathbb{R})$ with $\mathbb{C}^\times\times\cdots\times\mathbb{C}^\times$; similarly, via the embedding $x+y\sqrt{-a}\to x+y\rho$, we identify $T_{E_0}(\mathbb{R})$ with $\mathbb{C}^\times$. Consider the homomorphisms

$$\begin{aligned} h_E &: \mathbb{C}^\times \to T_E(\mathbb{R}) = \mathbb{C}^\times\times\cdots\times\mathbb{C}^\times, \quad z\mapsto (z^{-1},1,\cdots,1),\\ h_{E_0} &: \mathbb{C}^\times \to T_{E_0}(\mathbb{R}) = \mathbb{C}^\times, \qquad\qquad\qquad z\mapsto z^{-1}. \end{aligned}$$

Let $N_E = M(T_E,h_E)$ and $N_{E_0} = M(T_{E_0},h_{E_0})$ be the canonical models attached to the pairs (T_E,h_E) and (T_{E_0},h_{E_0}), respectively. Then N_E is defined over E and N_{E_0} is defined over E_0.

Consider the homomorphism $\alpha : G\times T_E \to G''$ of algebraic groups inducing

$$B^\times \times E^\times \to G''(\mathbb{Q}) \subset (B\otimes_{\mathbb{Q}} E)^\times, \quad (b,e)\mapsto b\otimes \mathrm{N}_{E/E_0}(e)e^{-1}$$

on $\mathbb{Q}$-valued points and the homomorphism $\beta : G\times T_E \to T_{E_0}$ inducing

$$\mathrm{N}_{E/E_0}\circ \mathrm{pr}_2 : B^\times\times E^\times \to E_0^\times$$

on $\mathbb{Q}$-valued points. Here, N_{E/E_0} denotes the norm map $E^\times \to E_0^\times$. Because $h' = \alpha\circ(h\times h_E)$ and $h_{E_0} = \mathrm{N}_{E/E_0}\circ h_E$, α and β induce morphisms of Shimura varieties $M\times N\to M''$ and $M\times N_E \to N_{E_0}$, again denoted by α and β, respectively. We have the following diagram:

$$\begin{array}{ccccccc} M & \xleftarrow{\mathrm{pr}_1} & M\times N_E & \xrightarrow{\alpha} & M'' & \longleftarrow & M' \\ & & \Big\downarrow{\scriptstyle\beta} & & & & \\ & & N_{E_0}. & & & & \end{array}$$

4.2. Connected components of M, $M\times N_E$, M' and M''

We write $\widetilde{G}$ for $G\times T_E$ and write $\widetilde{M}$ for $M\times N_E$. For $\natural = \widetilde{\ }, \emptyset, {}', {}''$, because B is ramified at $\mathfrak{p}$, there exists a unique maximal compact open subgroup $U^\natural_{p,0}$ of $G^\natural(\mathbb{Q}_p)$. We have $U'_{p,0} = U''_{p,0}\cap G'(\mathbb{Q}_p)$ and $U''_{p,0} = \alpha(\widetilde{U}_{p,0})$.

If $U^\natural$ is a subgroup of $G^\natural(\mathbb{A}_f)$ of the form $U^\natural_{p,0}U^{\natural,p}$ where $U^{\natural,p}$ is a compact open subgroup of $G^\natural(\mathbb{A}_f^p)$, we will write $M^\natural_{0,U^{\natural,p}}$ for $M^\natural_{U^\natural}$. Let $M^\natural_0$ denote the projective system $(M^\natural_{0,U^{\natural,p}})_{U^{\natural,p}}$; this projective system admits a natural right action of $G^\natural(\mathbb{A}_f^p)$.

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Lemma 4.1.

(a) *For any sufficiently small $U^{\natural,p}$, each geometrically connected component of $M^{\natural}_{0,U^{\natural,p}}$ is defined over a field that is unramified at all places above p.*

(b) *Let $\widetilde{U}^p$ be a sufficiently small compact open subgroup of $\widetilde{G}(\mathbb{A}_f^p)$. Then the morphism*

$$\widetilde{M}_{0,\widetilde{U}^p} \to M''_{0,\alpha(\widetilde{U}^p)}$$

induced by α is an isomorphism onto its image when restricted to every geometrically connected component.

Proof. When $U^{\natural,p}$ is sufficiently small, $M^{\natural}_{0,U^{\natural,p}}$ is smooth. Let $\pi_0(M^{\natural}_{0,U^{\natural,p}})$ denote the set of geometrically irreducible components of $M^{\natural}_{0,U^{\natural,p}}$ over $\overline{\mathbb{Q}}$. Then $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ acts on $\pi_0(M^{\natural}_{0,U^{\natural,p}})$. This action is explicitly described by Deligne [8, Theorem 2.6.3], from which we deduce (a).

Because α induces an isomorphism from the derived group of $\widetilde{G}$ to that of G'', by [8, Remark 2.1.16] or [21, Proposition II.2.7], we obtain (b). □

4.3. Modular interpolation of M'

Let $\ell \mapsto \bar{\ell}$ be the involution on $D = B \otimes_{\mathbb{Q}} E_0$ that is the product of the canonical involution on B and the complex conjugate on E_0. Choose an invertible symmetric element $\delta \in D$ ($\delta = \bar{\delta}$). Then we have another involution $\ell \mapsto \ell^* := \delta^{-1}\bar{\ell}\delta$ on D.

Let V denote D considered as a left D-module. Let ψ be the nondegenerate alternating form on V defined by $\psi(x,y) = \mathrm{Tr}_{E/\mathbb{Q}}(\sqrt{-a}\ \mathrm{Trd}_{D/E}(x\delta y^*))$, where $\mathrm{Tr}_{E/\mathbb{Q}}$ is the trace map and $\mathrm{Trd}_{D/E}$ is the reduced trace map. For $\ell \in D$ put

$$t(\ell) = \mathrm{tr}(\ell; V_{\mathbb{C}}/\mathrm{Fil}^0 V_{\mathbb{C}}),$$

where $\mathrm{Fil}^{\bullet}$ is the Hodge structure defined by h'. We have

$$t(\ell) = (\tau_1 + \bar{\tau}_1 + 2\tau_2 + \cdots + 2\tau_g)(\mathrm{tr}_{D/E}(\ell))$$

for $\ell \in D$. The subfield of $\mathbb{C}$ generated by $t(\ell)$, $\ell \in D$, is exactly E.

Choose an order $\mathscr{O}_D$ of D, T the corresponding lattice in V. With a suitable choice of δ, we may assume that $\mathscr{O}_D$ is stable by the involution $\ell \mapsto \ell^*$ and that ψ takes integer values on T. Put $\hat{\mathscr{O}}_D := \mathscr{O}_D \otimes \hat{\mathbb{Z}}$ and $\hat{T} := T \otimes \hat{\mathbb{Z}}$.

In Section 5 when we consider the p-adic uniformisation of the Shimura curves, we need to make the following assumption.

Assumption 4.2. We assume that $\mathscr{O}_D$ and δ are chosen such that $\hat{T}$ is stable by $U'_{p,0}$.

If U' is a sufficiently small compact open subgroup of $G'(\mathbb{A}_f)$ that keeps $\hat{T}$, then $M'_{U'}$ represents the following functor $\mathcal{M}_{U'}$ [24, Section 5]:

For any E-algebra R, $\mathcal{M}_{U'}(R)$ is the set of isomorphism classes of quadruples (A,ι,θ,κ) where

• A is an isomorphism class of abelian schemes over R with an endomorphism $\iota : \mathscr{O}_D \to \mathrm{End}(A)$ such that $\mathrm{tr}(\iota(\ell), \mathrm{Lie} A) = t(\ell)$ for all $\ell \in \mathscr{O}_D$.

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- θ is a polarisation $A \to \check{A}$ whose associated Rosati involution sends $\iota(\ell)$ to $\iota(\ell^*)$.
- κ is a U'-orbit of $\mathscr{O}_D \otimes \hat{\mathbb{Z}}$-linear isomorphisms $\hat{T}(A) := \prod_{\ell} T_\ell(A) \to \hat{T}$ such that there exists a $\hat{\mathbb{Z}}$-linear isomorphism $\kappa' : \hat{T}(1) \to \hat{\mathbb{Z}}$ making the diagram

$$\begin{array}{ccccc} \hat{T}(A) \times \hat{T}(A) & \xrightarrow{(1,\theta_*)} & \hat{T}(A) \times \hat{T}(\check{A}) & \longrightarrow & \hat{T}(1) \\ \downarrow{\scriptstyle \kappa\times\kappa} & & & & \downarrow{\scriptstyle \kappa'} \\ \hat{T} \times \hat{T} & & \xrightarrow{\psi\otimes\hat{\mathbb{Z}}} & & \hat{\mathbb{Z}} \end{array}$$

commutative.

Let $\mathcal{A}_{U'}$ be the universal $\mathscr{O}_D$-abelian scheme over $M'_{U'}$.

5. p-adic Uniformisations of Shimura curves

5.1. Preliminaries

We provide two simple facts that will be useful later.

(I) Let X be a scheme with a discrete action of a group C on the right-hand side and let Z be a group that contains C as a normal subgroup of finite index. Fix a set of representatives $\{g_i\}_{i\in C\backslash Z}$ of $C\backslash Z$ in Z. We define a scheme $X *_C Z$ with a right action of Z below. As a scheme, $X *_C Z$ is $\coprod_{C\backslash Z} X^{(g_i)}$, where $X^{(g_i)}$ is a copy of X. For any $g \in Z$ and $x^{(g_i)} \in X^{(g_i)}$, if $g_i g = h g_k$ with $h \in C$, then $x^{(g_i)} \cdot g = (x \cdot h)^{(g_k)}$. It is easy to show that up to isomorphism $X *_C Z$ and the right action of Z are independent of the choice of $\{g_i\}_{i\in C\backslash Z}$.

(II) Let X_1 and X_2 be two schemes whose connected components are all geometrically connected. Suppose that each of X_1 and X_2 has an action of an abelian group Z; Z acts freely on the set of components of X_1 (respectively X_2). Let C be a closed subgroup of Z. Then the Z-actions on X_1 and X_2 induce Z/C-actions on X_1/C and X_2/C.

Lemma 5.1. *If there exists a Z/C-equivariant isomorphism $\gamma : X_1/C \to X_2/C$, then there exists a Z-equivariant isomorphism $\tilde{\gamma} : X_1 \to X_2$ such that the following diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{\tilde{\gamma}} & X_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ X_1/C & \xrightarrow{\gamma} & X_2/C \end{array}$$

is commutative, where π_1 and π_2 are the natural projections.

Proof. We identify X_1/C with X_2/C by γ and write Y for it. The condition on Z-actions implies that the action of Z/C on the set of connected components of Y is free and that the morphism π_1 (respectively π_2) maps each connected component of X_1 (respectively X_2) isomorphically to its image.

We choose a set of representatives $\{Y_i\}_{i\in I}$ of the Z/C-orbits of components of Y. Then $\{\bar{g}Y_i : \bar{g} \in Z/C, i \in I\}$ are all different connected components of Y. For each $i \in I$ we choose a connected component $\tilde{Y}_i^{(1)}$ (respectively $\tilde{Y}_i^{(2)}$) of X_1 (respectively X_2) that is a

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lifting of Y_i. Then $\{g\tilde{Y}_i^{(1)} : g \in Z, i \in I\}$ (respectively $\{g\tilde{Y}_i^{(2)} : g \in Z, i \in I\}$) are all different connected components of X_1 (respectively X_2).

Because $\pi_1|_{\tilde{Y}_i^{(1)}} : \tilde{Y}_i^{(1)} \to Y_i$ and $\pi_2|_{\tilde{Y}_i^{(2)}} : \tilde{Y}_i^{(2)} \to Y_i$ are isomorphisms, there exists an isomorphism $\tilde{\gamma}_i : \tilde{Y}_i^{(1)} \to \tilde{Y}_i^{(2)}$ such that $\pi_1|_{\tilde{Y}_i^{(1)}} = \pi_2|_{\tilde{Y}_i^{(2)}} \circ \tilde{\gamma}_i$. We define the morphism $\tilde{\gamma} : X^{(1)} \to X^{(2)}$ as follows: $\tilde{\gamma}$ maps $g\tilde{Y}_i^{(1)}$ to $g\tilde{Y}_i^{(2)}$ and $\tilde{\gamma}|_{g\tilde{Y}_i^{(1)}} = g \circ \tilde{\gamma}_i \circ g^{-1}$. Then $\tilde{\gamma}$ is a Z-equivariant isomorphism and $\pi_1 = \pi_2 \circ \tilde{\gamma}$. □

5.2. Some Notation

Fix an isomorphism $\mathbb{C} \cong \mathbb{C}_p$. Combining the isomorphism $\mathbb{C} \cong \mathbb{C}_p$ and the inclusion $E_0 \hookrightarrow \mathbb{C}$, $x + y\sqrt{-a} \mapsto x + y\rho$, we obtain inclusions $E_0 \hookrightarrow \mathbb{Q}_p$ and $E \hookrightarrow F_{\mathfrak{p}}$. Thus, $D \otimes \mathbb{Q}_p$ is isomorphic to $B_{\mathfrak{p}} \oplus B_{\mathfrak{p}}$.

Note that $G(\mathbb{Q}_p)$ is isomorphic to $B_{\mathfrak{p}}^\times$, $G'(\mathbb{Q}_p)$ is isomorphic to the subgroup

$$\{(a,b) : a,b \in B_{\mathfrak{p}}^\times, \bar{a}b \in \mathbb{Q}_p^\times\}$$

of $B_{\mathfrak{p}}^\times \times B_{\mathfrak{p}}^\times$ and $G''(\mathbb{Q}_p)$ is isomorphic to

$$\{(a,b) : a,b \in B_{\mathfrak{p}}^\times, \bar{a}b \in F_{\mathfrak{p}}^\times\},$$

where $a \mapsto \bar{a}$ is the canonical involution on B. Note that $T_E(\mathbb{Q}_p)$ is isomorphic to $F_{\mathfrak{p}}^\times \times F_{\mathfrak{p}}^\times$ and $T_{E_0}(\mathbb{Q}_p)$ is isomorphic to $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. We normalise these isomorphisms such that $G'(\mathbb{Q}_p) \hookrightarrow G''(\mathbb{Q}_p)$ becomes the natural inclusion

$$\{(a,b) : a,b \in B_{\mathfrak{p}}^\times, \bar{a}b \in \mathbb{Q}_p^\times\} \hookrightarrow \{(a,b) : a,b \in B_{\mathfrak{p}}^\times, \bar{a}b \in F_{\mathfrak{p}}^\times\},$$

$\alpha : G(\mathbb{Q}_p) \times T_E(\mathbb{Q}_p) \to G''(\mathbb{Q}_p)$ becomes

$$\begin{aligned} B_{\mathfrak{p}}^\times \times (F_{\mathfrak{p}}^\times \times F_{\mathfrak{p}}^\times) &\to \{(a,b) : a,b \in B_{\mathfrak{p}}^\times, \bar{a}b \in F_{\mathfrak{p}}^\times\} \\ (a,(t_1,t_2)) &\mapsto \left(a\frac{\mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(t_1)}{t_1}, a\frac{\mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(t_2)}{t_2}\right) \end{aligned}$$

and $\beta : G(\mathbb{Q}_p) \times T_E(\mathbb{Q}_p) \to T_{E_0}(\mathbb{Q}_p)$ becomes

$$\begin{aligned} B_{\mathfrak{p}}^\times \times (F_{\mathfrak{p}}^\times \times F_{\mathfrak{p}}^\times) &\to \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \\ (a,(t_1,t_2)) &\mapsto (\mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(t_1), \mathrm{N}_{F_{\mathfrak{p}}/\mathbb{Q}_p}(t_2)). \end{aligned}$$

Let $\bar{B}$ be the quaternion algebra over F such that

$$\mathrm{inv}_v(\bar{B}) = \begin{cases} \mathrm{inv}_v(B) & \text{if } v \neq \tau_1, \mathfrak{p}, \\ \frac{1}{2} & \text{if } v = \tau_1, \\ 0 & \text{if } v = \mathfrak{p}. \end{cases}$$

With $\bar{B}$ instead of B we can define analogues of G, G' and G'', denoted by $\bar{G}$, $\bar{G}'$ and $\bar{G}''$, respectively. For $\natural = \emptyset, ', ''$ we have $\bar{G}^\natural(\mathbb{A}_f^p) = G^\natural(\mathbb{A}_f^p)$; $\bar{G}(\mathbb{Q}_p)$ is isomorphic to $\mathrm{GL}(2, F_{\mathfrak{p}})$, $\bar{G}'(\mathbb{Q}_p)$ is isomorphic to the subgroup

$$\left\{\left(\left[\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right]\right) : a_i, b_i, c_i, d_i \in F_{\mathfrak{p}},\ \left[\begin{smallmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{smallmatrix}\right]\left[\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right] \in \mathbb{Q}_p^\times\right\}$$

of $\mathrm{GL}(2,F_{\mathfrak{p}})\times\mathrm{GL}(2,F_{\mathfrak{p}})$ and $\bar{G}''(\mathbb{Q}_p)$ is isomorphic to

$$\{(\left[\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right],\left[\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right]) : a_i,b_i,c_i,d_i \in F_{\mathfrak{p}},\ \left[\begin{smallmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{smallmatrix}\right]\left[\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right] \in F_{\mathfrak{p}}^{\times}\}.$$

If $\natural=\emptyset$, let $\bar{G}(\mathbb{Q}_p)$ act on $\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$ as in Section 3. If $\natural=\tilde{}$, let $\tilde{\bar{G}}=\bar{G}\times T_E$ act on $\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$ by the projection to the first factor. If $\natural='$ or $''$, let $\bar{G}^{\natural}(\mathbb{Q}_p)$ act on $\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$ by the first factor. Let $\bar{G}^{\natural}(\mathbb{Q})$ act on $\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$ via its embedding into $\bar{G}^{\natural}(\mathbb{Q}_p)$.

The center of $\bar{G}^{\natural}$, $Z(\bar{G}^{\natural})$, is naturally isomorphic to the center of $G^{\natural}$, $Z(G^{\natural})$; we denote both of them by $Z^{\natural}$.

5.3. The p-adic uniformisations

Let $\natural$ be either $\tilde{}$, $'$ or $''$. For any compact open subgroup $U^{\natural,p}$ of $G^{\natural}(\mathbb{A}_f^p)$, let $X^{\natural}_{U^{\natural,p}}$ denote $M^{\natural}_{0,U^{\natural,p}}\times_{\mathrm{Spec}(F_{\mathfrak{p}})}\mathrm{Spec}(\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}})$.

Proposition 5.2. *Suppose that Assumption 4.2 holds.*

(a) *Assume that* $\natural=\tilde{}\,,'$ *or* $''$. *For any sufficiently small compact open subgroup* $U^{\natural,p}$ *of* $G^{\natural}(\mathbb{A}_f^p)$, *writing* $U^{\natural}=U^{\natural}_{p,0}U^{\natural p}$, *we have a* $Z^{\natural}(\mathbb{Q})\backslash Z^{\natural}(\mathbb{A}_f)/(Z^{\natural}(\mathbb{A}_f)\cap U^{\natural})$*-equivariant isomorphism*

$$X^{\natural}_{U^{\natural,p}} \cong \bar{G}^{\natural}(\mathbb{Q})\backslash(\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}\times G^{\natural}(\mathbb{A}_f)/U^{\natural}). \tag{5.1}$$

Here, $\bar{G}^{\natural}(\mathbb{Q})$ *acts on* $\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$ *as mentioned above and acts on* $G^{\natural}(\mathbb{A}_f^p)/U^{\natural,p}$ *by the embedding* $\bar{G}^{\natural}(\mathbb{Q})\hookrightarrow\bar{G}^{\natural}(\mathbb{A}_f^p)\xrightarrow{\simeq}G^{\natural}(\mathbb{A}_f^p)$; *in the case of* $\natural='$ *or* $''$, *if* $g\in\bar{G}^{\natural}(\mathbb{Q})$ *satisfies* $g_p=(a,b)$ *with* $a,b\in\mathrm{GL}(2,F_{\mathfrak{p}})$, *then* g *acts on* $G^{\natural}(\mathbb{Q}_p)/U^{\natural}_{p,0}$ *via the left multiplication by* $(\Pi^{v_{\mathfrak{p}}(\det a)},\Pi^{v_{\mathfrak{p}}(\det b)})$, *whereas in the case of* $\natural=\tilde{}$, $\tilde{g}=(g,t)\in\tilde{\bar{G}}(\mathbb{Q})$ $(g\in\bar{G}(\mathbb{Q}),t\in T_E(\mathbb{Q}))$ *acts on* $\tilde{G}(\mathbb{Q}_p)/\tilde{U}_{p,0}$ *via the left multiplication by* $(\Pi^{v_{\mathfrak{p}}(\det g_p)},t_p)$ *and* $Z^{\natural}(\mathbb{Q})\backslash Z^{\natural}(\mathbb{A}_f)/(Z^{\natural}(\mathbb{A}_f)\cap U^{\natural})$ *acts on the right-hand side of* (5.1) *by right multiplications on* $\bar{G}^{\natural}(\mathbb{A}_f)$.

(b) *The isomorphisms in* (a) *can be chosen such that, for either* $\sharp=\tilde{}$ *and* $\natural=''$ *or* $\sharp='$ *and* $\natural=''$, *we have a commutative diagram*

$$\begin{array}{ccc} X^{\sharp}_{U^{\sharp,p}} & \longrightarrow & \bar{G}^{\sharp}(\mathbb{Q})\backslash(\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}\times G^{\sharp}(\mathbb{A}_f)/U^{\sharp}) \\ \downarrow & & \downarrow \\ X^{\natural}_{U^{\natural,p}} & \longrightarrow & \bar{G}^{\natural}(\mathbb{Q})\backslash(\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}\times G^{\natural}(\mathbb{A}_f)/U^{\natural}) \end{array}$$

compatible with the $Z^{\sharp}(\mathbb{Q})\backslash Z^{\sharp}(\mathbb{A}_f)/(Z^{\sharp}(\mathbb{A}_f)\cap U^{\sharp})$*-actions on the upper and the* $Z^{\natural}(\mathbb{Q})\backslash Z^{\natural}(\mathbb{A}_f)/(Z^{\natural}(\mathbb{A}_f)\cap U^{\natural})$*-actions on the lower, where the left vertical arrow is induced from the morphism* $M^{\sharp}\to M^{\natural}$ *and the right vertical arrow is induced by the identity morphism* $\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}\to\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$ *and the homomorphism* $\alpha:\tilde{G}=G\times T_E\to G''$ *or the inclusion* $G'\hookrightarrow G''$. *Here, in the case of* $\sharp=\tilde{}$ *and* $\natural=''$, $U^{\natural}=\alpha(U^{\sharp})$; *in the case of* $\sharp='$ *and* $\natural=''$, $U^{\sharp}=U^{\natural}\cap G'(\mathbb{A}_f)$.

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The conclusions of Proposition 5.2, especially (a), are well known [23, 28]. However, the author has no reference for (b), so we provide some detail of the proof.

Proof. Assertion (a) in the case of $\natural ='$ comes from [23, Theorem 6.50].

For the case of $\sharp ='$ and $\natural =''$ we put

$$C = Z'(\mathbb{Q})\backslash Z'(\mathbb{A}_f)/(Z'(\mathbb{A}_f)\cap U')$$

and

$$Z = Z''(\mathbb{Q})\backslash Z''(\mathbb{A}_f)/(Z''(\mathbb{A}_f)\cap U'').$$

Then $X''_{U''^p}$ is Z-equivariantly isomorphic to $X'_{U'^p} *_C Z$, and $\bar{G}''(\mathbb{Q})\backslash(\mathcal{H}_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}} \times G''(\mathbb{A}_f)/U'')$ is Z-equivariantly isomorphic to $\left(\bar{G}'(\mathbb{Q})\backslash(\mathcal{H}_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}} \times G'(\mathbb{A}_f)/U')\right) *_C Z$. So (a) in the case of $\natural =''$ and (b) in the case of $\sharp =', \natural =''$ follow from (a) in the case of $\natural ='$.

Now we consider the remaining cases. Let H be the kernel of the homomorphism $\alpha : \widetilde{G} = G \times T_E \to G''$. Put

$$C = H(\mathbb{Q})\backslash H(\mathbb{A}_f)/(H(\mathbb{A}_f)\cap \widetilde{U}) \ \text{ and } \ Z = \widetilde{Z}(\mathbb{Q})\backslash \widetilde{Z}(\mathbb{A}_f)/(\widetilde{Z}(\mathbb{A}_f)\cap \widetilde{U}).$$

Put $X_1 = \widetilde{X}_{\widetilde{U}^p}$ and $X_2 = \bar{\widetilde{G}}(\mathbb{Q})\backslash(\mathcal{H}_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}} \times \widetilde{G}(\mathbb{A}_f)/\widetilde{U}_{p,0}\widetilde{U}^p)$. By Lemma 4.1 (a), all connected components of X_1 are geometrically connected; it is obvious that all connected components of X_2 are geometrically connected. Thus, Z acts freely on the set of components of X_1 (respectively X_2). Furthermore, X_1/C is isomorphic to $X''_{\alpha(\widetilde{U}^p)}$ and X_2/C is isomorphic to $\bar{G}''(\mathbb{Q})\backslash(\mathcal{H}_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}} \times G''(\mathbb{A}_f)/U''_{p,0}\alpha(\widetilde{U}^p))$. We have already proved that X_1/C is Z/C-equivariantly isomorphic to X_2/C. Applying Lemma 5.1, we obtain (a) in the case of $\natural =\widetilde{}$ and (b) in the case of $\sharp =\widetilde{}, \natural =''$. □

Remark 5.3. By [28] the similar conclusion of Proposition 5.2 (a) holds for the case of $\natural = \emptyset$. We use X_{U^p} to denote $\bar{G}(\mathbb{Q})\backslash(\mathcal{H}_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}} \times G(\mathbb{A}_f)/U_{p,0}U^p)$, where the action of $\bar{G}(\mathbb{Q})$ on $\mathcal{H}_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}} \times G(\mathbb{A}_f)/U_{p,0}U^p$ is defined similarly.

6. Local systems and the associated filtered φ_q-isocrystals on Shimura Curves

6.1. Local systems on Shimura curves

We choose a number field L splitting F and B. We identify $\{\tau_i : F \to L\}$ with $I = \{\tau_i : F \to \mathbb{C}\}$ by the inclusion $L \to \mathbb{C}$. Fix an isomorphism $L \otimes_{\mathbb{Q}} B = M(2,L)^I$. Then we have a natural inclusion $G(\mathbb{Q}) \hookrightarrow \mathrm{GL}(2,L)^I$. Let $\mathfrak{P}$ be a place of L above $\mathfrak{p}$.

For a multiweight $\mathrm{k} = (k_1, \cdots, k_g, w)$ with $k_1 \equiv \cdots k_g \equiv w \mod 2$ and $k_1 \geq 2, \cdots, k_g \geq 2$, we define the morphism $\rho^{(\mathrm{k})} : G \to \mathrm{GL}(n,L)$ $(n = \prod_{i=1}^g (k_i - 1))$ to be the product $\otimes_{i\in I}[(\mathrm{Sym}^{k_i-2} \otimes \det^{(w-k_i)/2}) \circ \check{pr}_i]$. Here $\check{pr}_i$ denotes the contragradient representation of the ith projection $pr_i : \mathrm{GL}(2,L)^I \to \mathrm{GL}(2,L)$. The algebraic group denoted by G^c in [21, Chapter III] is the quotient of G by $\ker(\mathrm{N}_{F/\mathbb{Q}} : F^\times \to \mathbb{Q}^\times)$. Because the restriction of $\rho^{(\mathrm{k})}$ to the center $F^\times$ is the scalar multiplication by $\mathrm{N}_{F/\mathbb{Q}}^{-(w-2)}(\cdot)$, $\rho^{(\mathrm{k})}$ factors through G^c, so we can attach to the representation $\rho^{(\mathrm{k})}$ a $G(\mathbb{A}_f)$-equivariant smooth $L_{\mathfrak{P}}$-sheaf $\mathcal{F}(\mathrm{k})$ on M. We have a pairing

$$\mathcal{F}(\mathrm{k})\times\mathcal{F}(\mathrm{k})\to\mathcal{F}(\det^{w-2}). \tag{6.1}$$

Let $p_2:G''_{E_0}\to G_{E_0}$ be the map induced by the second projection on $(D\otimes_{\mathbb{Q}}E_0)^\times=D^\times\times D^\times$ corresponding to the conjugate $E_0\to E_0$. Because the algebraic representation $\rho''^{(\mathrm{k})}=\rho^{(\mathrm{k})}\circ p_2$ factors through G''^c, we can attach to it a $G''(\mathbb{A}_f)$-equivariant smooth $L_{\mathfrak{P}}$-sheaf $\mathcal{F}''(\mathrm{k})$ on M''. Let $\mathcal{F}'(\mathrm{k})$ be the restriction of $\mathcal{F}''(\mathrm{k})$ to M'.

We define a character $\bar{\chi}:T_0\to\mathbb{G}_m$ such that on $\mathbb{C}$-valued points $\bar{\chi}$ is the inverse of the second projection $T_{0\mathbb{C}}=\mathbb{C}^\times\times\mathbb{C}^\times\to\mathbb{C}^\times$. Let $\mathcal{F}(\bar{\chi})$ be the $L_{\mathfrak{P}}$-sheaf attached to the representation $\bar{\chi}$. By [24] one has the following $G(\mathbb{A}_f)\times T(\mathbb{A}_f)$-equivariant isomorphism of $L_{\mathfrak{P}}$-sheaves:

$$\mathrm{pr}_1^*\mathcal{F}(\mathrm{k})\simeq\alpha^*\mathcal{F}''(\mathrm{k})\otimes\beta^*\mathcal{F}(\bar{\chi}^{-1})^{\otimes(g-1)(w-2)} \tag{6.2}$$

on $M\times N$, where pr_1 is the projection $M\times N\to M$.

Note that $L\otimes_{\mathbb{Q}}D\simeq(\mathrm{M}_2(L)\times\mathrm{M}_2(L))^I$. For each $i\in I$, the first component $\mathrm{M}_2(L)$ corresponds to the embedding $E_0\subset L\subset\mathbb{C}$ and the second $\mathrm{M}_2(L)$ to its conjugate. Let $\mathcal{F}'$ be the local system $R^1g_*L_{\mathfrak{P}}$ where $g:\mathcal{A}\to M'$ is the universal $\mathscr{O}_D$-abelian scheme; it is a sheaf of $L\otimes_{\mathbb{Q}}D$-modules. For each $i\in I$, let $e_i\in L\otimes_{\mathbb{Q}}D$ be the idempotent whose $(2,i)$th component is a rank one idempotent – for example, $\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$ – and whose other components are zero. Let $\mathcal{F}'_i$ denote the e_i-part $e_i\cdot R^1g_*L_{\mathfrak{P}}$. Note that $\mathcal{F}'_i$ does not depend on the choice of the rank one idempotent. By [24] we have an isomorphism of local systems

$$\mathcal{F}'(\mathrm{k})=\bigotimes_{i\in I}\Big(\mathrm{Sym}^{k_i-2}\mathcal{F}'_i\bigotimes(\det\mathcal{F}'_i)^{(w-k_i)/2}\Big).$$

We can define more local systems on M'. For $(\mathrm{k},\mathrm{v})=(k_1,\cdots,k_g;v_1,\cdots,v_g)$, put

$$\mathcal{F}'(\mathrm{k},\mathrm{v})=\bigotimes_{i\in I}\Big(\mathrm{Sym}^{k_i-2}\mathcal{F}'_i\bigotimes(\det\mathcal{F}'_i)^{v_i}\Big). \tag{6.3}$$

6.2. Filtered φ_q-isocrystals associated to the local systems

We use $\tilde{\mathrm{k}}$ uniformly to denote $(\mathrm{k},\mathrm{v})=(k_1,\cdots,k_g;v_1,\cdots,v_g)$ (respectively $\mathrm{k}=(k_1,\cdots,k_g,w)$) in the case of $\natural={}'$ (respectively $\natural=\emptyset,{}',{}''$).

We shall need the filtered φ_q-isocrystal attached to $\mathcal{F}(\tilde{\mathrm{k}})$. However, we do not know how to compute it. Instead, we compute that attached to $\mathrm{pr}_1^*\mathcal{F}(\tilde{\mathrm{k}})$. As a middle step we determine the filtered φ_q-isocrystals associated to $\mathcal{F}'(\tilde{\mathrm{k}})$ and $\mathcal{F}''(\tilde{\mathrm{k}})$.

For any integers k and v with $k\geq 2$ and any inclusion $\sigma:F_{\mathfrak{p}}\to L_{\mathfrak{P}}$, let $V_\sigma(k,v)$ be the space of homogeneous polynomials in two variables X_σ and Y_σ of degree $k-2$ with coefficients in $L_{\mathfrak{P}}$; let $\mathrm{GL}(2,F_{\mathfrak{p}})$ act on $V_\sigma(k,v)$ by

$$\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]^{-1}P(X_\sigma,Y_\sigma)=\sigma(ad-bc)^vP(\sigma(a)X_\sigma+\sigma(b)Y_\sigma,\sigma(c)X_\sigma+\sigma(d)Y_\sigma).$$

For $(\mathrm{k},\mathrm{v})=(k_1,\cdots,k_g;v_1,\cdots,v_g)$ we put

$$V(\mathrm{k},\mathrm{v})=\bigotimes_{\sigma\in I}V_\sigma(k_\sigma,v_\sigma),$$

where the tensor product is taken over $L_{\mathfrak{P}}$.

Let $\bar{G}^\natural$ ($\natural=\emptyset,',''$,$\tilde{}$) be the groups defined in Section 5.2. For $\natural=\tilde{}$, via the projection $\bar{G}^\natural(\mathbb{Q}_p)\to \mathrm{GL}(2,F_{\mathfrak{p}})$, $V(\tilde{\mathrm{k}})$ becomes a $\bar{G}^\natural(\mathbb{Q}_p)$-module. For $\natural=',''$, via the projection of $\bar{G}^\natural(\mathbb{Q}_p)\subset \mathrm{GL}(2,F_{\mathfrak{p}})\times \mathrm{GL}(2,F_{\mathfrak{p}})$ to the second factor, $V(\tilde{\mathrm{k}})$ becomes a $\bar{G}^\natural(\mathbb{Q}_p)$-module. In each case via the inclusion $\bar{G}^\natural(\mathbb{Q})\hookrightarrow \bar{G}^\natural(\mathbb{Q}_p)$, $V(\tilde{\mathrm{k}})$ becomes a $\bar{G}^\natural(\mathbb{Q})$-module. Using the p-adic uniformisation of $X^\natural=X^\natural_{U^{\natural,p}}$ we attach to this $\bar{G}^\natural(\mathbb{Q})$-module a local system $\mathcal{V}^\natural(\tilde{\mathrm{k}})$ on $X^\natural$.

Let $\varphi_{q,\mathrm{k},\mathrm{v}}$ be the operator on $V(\mathrm{k},\mathrm{v})$,

$$\bigotimes_\sigma P_\sigma(X_\sigma,Y_\sigma)\mapsto \prod_\sigma \sigma(-\pi)^{v_\sigma}\cdot\bigotimes_\sigma P_\sigma(Y_\sigma,\sigma(\pi)X_\sigma).$$

For $\mathrm{k}=(k_1,\cdots,k_g,w)$ we put

$$V(\mathrm{k})=V(k_1,\cdots,k_g;(w-k_1)/2,\cdots,(w-k_g)/2)$$

and

$$\varphi_{q,\mathrm{k}}=\varphi_{q,(k_1,\cdots,k_g;(w-k_1)/2,\cdots,(w-k_g)/2)}.$$

Let $\mathscr{F}^\natural(\tilde{\mathrm{k}})$ be the filtered φ_q-isocrystal $\mathcal{V}^\natural(\tilde{\mathrm{k}})\otimes_{\mathbb{Q}_p}\mathscr{O}_{X^\natural}$ on $X^\natural$ with the q-Frobenius $\varphi_{q,\tilde{\mathrm{k}}}\otimes\varphi_{q,\mathscr{O}_{X^\natural}}$ and the connection $1\otimes \mathrm{d}:\mathcal{V}^\natural(\tilde{\mathrm{k}})\otimes_{\mathbb{Q}_p}\mathscr{O}_{X^\natural}\to \mathcal{V}^\natural(\tilde{\mathrm{k}})\otimes_{\mathbb{Q}_p}\Omega^1_{X^\natural}$; the filtration on

$$\mathcal{V}^\natural(\tilde{\mathrm{k}})\otimes_{\mathbb{Q}_p}\mathscr{O}_{X^\natural}=\bigoplus_{\tau:F_{\mathfrak{p}}\hookrightarrow L_{\mathfrak{P}}}\mathcal{V}^\natural(\tilde{\mathrm{k}})\otimes_{\tau,F_{\mathfrak{p}}}\mathscr{O}_{X^\natural} \tag{6.4}$$

is given by

$$\begin{aligned}&\mathrm{Fil}^{j+v_\tau}(\mathcal{V}^\natural(\tilde{\mathrm{k}})\otimes_{\tau,F_{\mathfrak{p}}}\mathscr{O}_{X^\natural})\\ &=\begin{cases}\mathcal{V}^\natural(\tilde{\mathrm{k}})\otimes_{\tau,F_{\mathfrak{p}}}\mathscr{O}_{X^\natural} & \text{if } j\le 0,\\ \text{the } \mathscr{O}_{X^\natural}\text{-submodule locally generated by polynomials} & \\ \qquad\qquad \text{in } V(\tilde{\mathrm{k}}) \text{ divided by } (zX_\tau+Y_\tau)^j & \text{if } 1\le j\le k_\tau-2\\ 0 & \text{if } j\ge k_\tau-1\end{cases}\end{aligned}$$

with the convention that $v_\tau=\frac{w-k_\tau}{2}$ in the case of $\tilde{\mathrm{k}}=(k_1,\cdots,k_g,w)$, where z is the canonical coordinate on $\mathcal{H}_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}}$.

Lemma 6.1. *When $k_1=\cdots=k_{i-1}=k_{i+1}=\cdots=k_g=2$, $k_i=3$ and $v_1=\cdots=v_g=0$, the filtered φ_q-isocrystal attached to $\mathcal{F}'(\mathrm{k},\mathrm{v})$ is isomorphic to $\mathscr{F}'(\mathrm{k},\mathrm{v})$.*

Proof. Let $\tilde{e}_i\in L\otimes_{\mathbb{Q}}D$ be the idempotent whose $(2,i)$th component is $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ and the other component are zero. Let $\mathcal{A}$ be the universal $\mathscr{O}_D$-abelian scheme over M' and $\widehat{\mathcal{A}}$ the formal module on X' attached to $\mathcal{A}$. Note that $\tilde{e}_i(\mathfrak{o}_{L_{\mathfrak{P}}}\otimes_{\mathbb{Z}_p}\widehat{\mathcal{A}})$ is just the pullback of $\mathfrak{o}_{L_{\mathfrak{P}}}\otimes_{\tau_i,\mathfrak{o}_{F_{\mathfrak{p}}}}\mathcal{G}$ by the projection $X'_{U'^p}\to(\bar{G}'(\mathbb{Q})\cap U'^pU'_{p,0})\backslash\mathcal{H}_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}}$ [23, Subsection 6.43],

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where $\mathcal{G}$ is the universal special formal $\mathscr{O}_{B_\mathfrak{p}}$-module (forgetting the information of ρ in Drinfeld's moduli problem).

Because $L_\mathfrak{P}$ splits $B_\mathfrak{p}$, $L_\mathfrak{P}$ contains all embeddings of $F_\mathfrak{p}^{(2)}$. The embedding $\tau_i : F_\mathfrak{p} \hookrightarrow L_\mathfrak{P}$ extends in two ways to $F_\mathfrak{p}^{(2)}$, denoted respectively by $\tau_{i,0}$ and $\tau_{i,1}$. Then,

$$\mathfrak{o}_{L_\mathfrak{P}} \otimes_{\tau_i, \mathfrak{o}_{F_\mathfrak{p}}} \mathfrak{o}_{B_\mathfrak{p}} = \mathfrak{o}_{L_\mathfrak{P}} \otimes_{\tau_{i,0}, \mathfrak{o}_{F_\mathfrak{p}^{(2)}}} \mathfrak{o}_{B_\mathfrak{p}} \bigoplus \mathfrak{o}_{L_\mathfrak{P}} \otimes_{\tau_{i,1}, \mathfrak{o}_{F_\mathfrak{p}^{(2)}}} \mathfrak{o}_{B_\mathfrak{p}}.$$

We decompose $\mathfrak{o}_{L_\mathfrak{P}} \otimes_{\tau_i, \mathfrak{o}_{F_\mathfrak{p}}} \mathcal{G}$ into the sum of two direct summands according to the action of $\mathfrak{o}_{F_\mathfrak{p}^{(2)}} \subset \mathfrak{o}_{B_\mathfrak{p}}$: $\mathfrak{o}_{F_\mathfrak{p}^{(2)}}$ acts by $\tau_{i,0}$ on the first direct summand and acts by $\tau_{i,1}$ on the second. Without loss of generalitym we may assume that e_i in the definition of $\mathcal{F}_i'$ (see Subsection 6.1) is chosen such that e_i is the projection onto the first direct summand. So $e_i(\mathfrak{o}_{L_\mathfrak{P}} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{A}})$ is just the pullback of $\mathfrak{o}_{L_\mathfrak{P}} \otimes_{\tau_{i,0}, \mathfrak{o}_{F_\mathfrak{p}^{(2)}}} \mathcal{G}$ by the projection $X'_{U'^p} \to (\bar{G}'(\mathbb{Q}) \cap U'^p U'_{p,0}) \backslash \mathcal{H}_{\widehat{F_\mathfrak{p}^{\mathrm{ur}}}}$. Now the statement of our lemma follows from the discussion in Subsection 3.2. □

Proposition 6.2. *The filtered φ_q-isocrystal attached to $\mathcal{F}'(\mathrm{k,v})$ is isomorphic to $\mathscr{F}'(\mathrm{k,v})$.*

Proof. Let $\mathscr{F}_i'$ denote the filtered φ_q-isocrystal attached to $\mathcal{F}_i'$. By (6.3), the filtered φ_q-isocrystal attached to $\mathcal{F}'(\mathrm{k,v})$ is isomorphic to

$$\bigotimes_{i\in I} \Big(\mathrm{Sym}^{k_i-2}\mathscr{F}_i' \bigotimes (\det \mathscr{F}_i')^{v_i}\Big). \tag{6.5}$$

By Lemma 6.1, a simple computation implies our conclusion. □

Corollary 6.3. *The filtered φ_q-isocrystal attached to $\mathcal{F}''(\mathrm{k})$ is isomorphic to $\mathscr{F}''(\mathrm{k})$.*

Proof. This follows from Proposition 6.2 and [24, Lemma 6.1]. □

Lemma 6.4. *The filtered φ_q-isocrystal associated to the local system $\mathcal{F}(\bar{\chi})$ over $(N_{E_0,0})_{\widehat{F_\mathfrak{p}^{\mathrm{ur}}}}$ is $\mathcal{F}(\bar{\chi}) \otimes \mathscr{O}_{(N_{E_0,0})_{\widehat{F_\mathfrak{p}^{\mathrm{ur}}}}}$, with the q-Frobenius being $1 \otimes \varphi_{q,(N_{E_0,0})_{\widehat{F_\mathfrak{p}^{\mathrm{ur}}}}}$ and the filtration being trivial.*

Proof. We only need to show that any geometric point of $(N_{E_0,0})_{\widehat{F_\mathfrak{p}^{\mathrm{ur}}}}$ is defined over $\widehat{F_\mathfrak{p}^{\mathrm{ur}}}$.

Let h_{E_0} be as in Subsection 4.1 and μ the cocharacter of T_0 defined over E_0 attached to h_{E_0}. Let r be the composition

$$\mathbb{A}_{E_0}^\times \xrightarrow{\mu} T_0(\mathbb{A}_{E_0}) \xrightarrow{\mathrm{N}_{\mathbb{Q}}^{E_0}} T_0(\mathbb{A}).$$

Let

$$\mathrm{art}_{E_0} : \mathbb{A}_{E_0}^\times \twoheadrightarrow \mathrm{Gal}(E_0^{\mathrm{ab}}/E_0)$$

be the reciprocal of the reciprocity map from class field theory. For any compact open subgroup U of $T_0(\mathbb{A}_f)$, $\mathrm{Gal}(\overline{\mathbb{Q}}/E_0)$ acts on $(N_{E_0})_U(\overline{\mathbb{Q}}) = T_0(\mathbb{Q})\backslash T_0(\mathbb{A}_f)/U$ by

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$\sigma(T_0(\mathbb{Q})aU) = T_0(\mathbb{Q})r_f(s_\sigma)aU$, where s_σ is any idèle such that $\mathrm{art}_{E_0}(s_\sigma) = \sigma|E^{\mathrm{ab}}$ and r_f is the composition

$$\mathbb{A}_{E_0}^\times \to T_0(\mathbb{A}) \to T_0(\mathbb{A}_f)$$

of r and the projection map $T_0(\mathbb{A}) \to T_0(\mathbb{A}_f)$. Let $\mathcal{I}$ be the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/E_0)$ consisting of σ such that $s_\sigma \in r_f^{-1}(U)$. Put $\mathcal{K} = \overline{\mathbb{Q}}^{\mathcal{I}}$. Then any geometric point of $(N_{E_0})_U$ is defined over $\mathcal{K}$. Observe that, when U is of the form $U_{p,0}U^p$ with U^p a compact open subgroup of $T_0(\mathbb{A}_f^p)$ and $U_{p,0}$ the maximal compact open subgroup of $T_0(\mathbb{Q}_p)$, $\mathcal{K}$ is unramified over p. Therefore, any geometric point of $(N_{E_0,0})_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$ is already defined over $\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}$. □

Corollary 6.5. *The filtered φ_q-isocrystal attached to* $\mathrm{pr}_1^*\mathcal{F}(\mathrm{k})$ *is* $\mathrm{pr}_1^*\mathscr{F}(\mathrm{k})$.

Proof. By (6.2) the filtered φ_q-isocrystal attached to $\mathrm{pr}_1^*\mathcal{F}(\mathrm{k})$ is the tensor product of the filtered φ_q-isocrystal attached to $\alpha^*\mathcal{F}''(\mathrm{k})$ and that attached to $\beta^*\mathcal{F}(\bar{\chi}^{-1})^{(g-1)(w-2)}$. Namely, it is isomorphic to

$$(\alpha^*\mathcal{V}''(\mathrm{k}) \otimes \beta^*\mathcal{F}(\bar{\chi}^{-1})^{(g-1)(w-2)}) \otimes_{\mathbb{Q}_p} \mathrm{pr}_1^*\mathscr{O}_X \cong \mathrm{pr}_1^*\mathcal{V}(\mathrm{k}) \otimes_{\mathbb{Q}_p} \mathrm{pr}_1^*\mathscr{O}_X;$$

a simple calculation shows that the φ_q-module structure and the filtration on $\mathrm{pr}_1^*\mathcal{V}(\mathrm{k}) \otimes_{\mathbb{Q}_p} \mathrm{pr}_1^*\mathscr{O}_X$ are as desired.

It is rather possible that the filtered φ_q-isocrystal attached to $\mathcal{F}(\mathrm{k})$ is $\mathscr{F}(\mathrm{k})$. But the author does not know how to descend the conclusion of Corollary 6.5 to X_{U^p}.

7. The de Rham cohomology

In this section we prove a Hodge-like decomposition for the de Rham cohomology.

7.1. Covering filtration and Hodge filtration for de Rham cohomology

We fix an arithmetic Schottky group Γ that is cocompact in $\mathrm{PGL}(2,F_{\mathfrak{p}})$. Then Γ acts freely on $\mathcal{H}$, and the quotient $X_\Gamma = \Gamma\backslash\mathcal{H}$ is the rigid analytic space associated with a proper smooth curve over $F_{\mathfrak{p}}$. Here we write $\mathcal{H}$ for $\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$.

We denote by $\widehat{\mathcal{H}}$ the canonical formal model of $\mathcal{H}$ (see Theorem 3.1). The curve X_Γ has a canonical semistable module $\mathcal{X}_\Gamma = \Gamma\backslash\widehat{\mathcal{H}}$; the special fibre $\mathcal{X}_{\Gamma,s}$ of $\mathcal{X}_\Gamma$ is isomorphic to $\Gamma\backslash\widehat{\mathcal{H}}_s$.

The graph $\mathrm{Gr}(\mathcal{X}_{\Gamma,s})$ (cf. Section 2) is closely related to the Bruhat-Tits tree $\mathcal{T}$ for $\mathrm{PGL}(2,F_{\mathfrak{p}})$. The group Γ acts freely on the tree $\mathcal{T}$. Let $\mathcal{T}_\Gamma$ denote the quotient tree. The set of connected components of the special fibre $\mathcal{X}_{\Gamma,s}$ is in one-to-one correspondence to the set $\mathrm{V}(\mathcal{T}_\Gamma)$ of vertices of $\mathcal{T}_\Gamma$. Each component is isomorphic to the projective line over k, the residue field of $F_{\mathfrak{p}}$. Write $\{\mathrm{P}_v^1\}_{v\in \mathrm{V}(\mathcal{T}_\Gamma)}$ for the set of components of $\mathcal{X}_{\Gamma,s}$. The singular points of $\mathcal{X}_{\Gamma,s}$ are ordinary k-rational double singular points; they correspond to (unoriented) edges of $\mathcal{T}_\Gamma$. Two components P_u^1 and P_v^1 intersect if and only if u and v are adjacent; in this case, they intersect at a singular point. For simplicity we will use the edge e joining u and v to denote this singular point. There is a reduction map from

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X_Γ^{an} to $\mathcal{X}_{\Gamma,s}$. For a closed subset U of $\mathcal{X}_{\Gamma,s}$ let $]U[$ denote the tube of U in X_Γ^{an}. Then $\{]\mathrm{P}_v^1[\}_{v\in \mathrm{V}(\mathcal{T}_\Gamma)}$ is an admissible covering of X_Γ^{an}. Clearly $]\mathrm{P}_{o(e)}^1[\cap]\mathrm{P}_{t(e)}^1[=]e[$.

Let L be a field that splits $F_\mathfrak{p}$. Fix an embedding $\tau : F_\mathfrak{p} \hookrightarrow L$.

Let V be an $L[\Gamma]$-module that comes from an algebraic representation of $\mathrm{PGL}(2,F_\mathfrak{p})$ of the form $V(\mathrm{k})$ with $\mathrm{k} = (k_1,\cdots,k_g;2)$. We impose that $w = 2$ because only when $w = 2$ does the action of $\mathrm{GL}(2,F_\mathfrak{p})$ on $V(\mathrm{k})$ factor through $\mathrm{PGL}(2,F_\mathfrak{p})$. We will regard V as an $F_\mathfrak{p}$-vector space by τ. Let $\mathscr{V} = \mathscr{V}(\mathrm{k})$ be the local system on X_Γ associated with V. Let $H^*_{\mathrm{dR},\tau}(X_\Gamma,\mathscr{V})$ be the hypercohomology of the complex $\mathscr{V}\otimes_{\tau,F_\mathfrak{p}}\Omega^\bullet_{X_\Gamma}$.

We consider the Mayer-Vietoris exact sequence attached to $H^*_{\mathrm{dR},\tau}(X_\Gamma,\mathscr{V})$ with respect to the admissible covering $\{]\mathrm{P}_v^1[\}_{v\in \mathrm{V}(\mathcal{T}_\Gamma)}$. As in Section 2, we obtain an injective map

$$\iota^\tau : \Big(\bigoplus_{e\in \mathrm{E}(\mathcal{T}_\Gamma)} H^0_{\mathrm{dR},\tau}(]e[,\mathscr{V})\Big)^-/\text{the image of} \bigoplus_{v\in \mathrm{V}(\mathcal{T}_\Gamma)} H^0_{\mathrm{dR},\tau}(]\mathrm{P}_v^1[,\mathscr{V}) \hookrightarrow H^1_{\mathrm{dR},\tau}(X_\Gamma^{\mathrm{an}},\mathscr{V}).$$

Because $]\mathrm{P}_v^1[$ and $]e[$ are quasi-Stein, a simple computation shows that $H^0_{\mathrm{dR},\tau}(]\mathrm{P}_v^1[,\mathscr{V})$ and $H^0_{\mathrm{dR},\tau}(]e[,\mathscr{V})$ are isomorphic to V. Let $C^0(V)$ be the space of V-valued functions on $\mathrm{V}(\mathcal{T})$ and $C^1(V)$ the space of V-valued functions on $\mathrm{E}(\mathcal{T})$ such that $f(e) = -f(\bar e)$. Let Γ act on $C^i(V)$ by $f \mapsto \gamma\circ f\circ\gamma^{-1}$. Then we have a Γ-equivariant short exact sequence

$$0 \longrightarrow V \longrightarrow C^0(V) \xrightarrow{\ \partial\ } C^1(V) \longrightarrow 0 \tag{7.1}$$

where $\partial(f)(e) = f(o(e)) - f(t(e))$. Observe that

$$\bigoplus_{v\in \mathrm{V}(\mathcal{T}_\Gamma)} H^0_{\mathrm{dR},\tau}(]\mathrm{P}_v^1[,\mathscr{V}) \cong C^0(V)^\Gamma,$$

$$\Big(\bigoplus_{e\in \mathrm{E}(\mathcal{T}_\Gamma)} H^0_{\mathrm{dR},\tau}(]e[,\mathscr{V})\Big)^- \cong C^1(V)^\Gamma$$

and the map

$$\bigoplus_{v\in \mathrm{V}(\mathcal{T}_\Gamma)} H^0_{\mathrm{dR},\tau}(]\mathrm{P}_v^1[,\mathscr{V}) \to \Big(\bigoplus_{e\in \mathrm{E}(\mathcal{T}_\Gamma)} H^0_{\mathrm{dR},\tau}(]e[,\mathscr{V})\Big)^-$$

coincides with ∂. Thus,

$$\Big(\bigoplus_{e\in \mathrm{E}(\mathcal{T}_\Gamma)} H^0_{\mathrm{dR},\tau}(]e[,\mathscr{V})\Big)^-/\text{the image of} \bigoplus_{v\in \mathrm{V}(\mathcal{T}_\Gamma)} H^0_{\mathrm{dR},\tau}(]\mathrm{P}_v^1[,\mathscr{V})$$

is isomorphic to $C^1(V)^\Gamma/\partial C^0(V)^\Gamma$. From (7.1) we get the injective map

$$\delta : C^1(V)^\Gamma/\partial C^0(V)^\Gamma \hookrightarrow H^1(\Gamma,V).$$

Let $C^1_{\mathrm{har}}(V)$ be the space of harmonic forms

$$C^1_{\mathrm{har}}(V) := \{f : \mathrm{Edge}(\mathcal{T}) \to V \mid f(e) = -f(\bar e),\ \forall\ v,\ \sum_{t(e)=v} f(e) = 0\},$$

and put $C^0_{\mathrm{har}}(V) = \partial^{-1}C^1_{\mathrm{har}}(V)$. Then we have an exact sequence

$$0 \longrightarrow V \longrightarrow C^0_{\mathrm{har}}(V) \longrightarrow C^1_{\mathrm{har}}(V) \longrightarrow 0$$

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from which we deduce the following exact sequence:

$$0 \longrightarrow V^{\Gamma} \longrightarrow C^0_{\mathrm{har}}(V)^{\Gamma} \longrightarrow C^1_{\mathrm{har}}(V)^{\Gamma} \longrightarrow H^1(\Gamma, V).$$

In the following we assume that

(coin) the map $V^{\Gamma} \to C^0_{\mathrm{har}}(V)^{\Gamma}$ is an isomorphism.

Fixing some $v \in \mathrm{V}(\mathcal{T})$, let ϵ be the map $C^1_{\mathrm{har}}(V)^{\Gamma} \to H^1(\Gamma, V)$ [4, (2.26)] defined by

$$c \mapsto (\gamma \mapsto \sum_{e: v \to \gamma v} c(e)), \tag{7.2}$$

where the sum runs over the edges joining v and γv; ϵ does not depend on the choice of v. By [4, Appendix A], ϵ is minus the composition

$$C^1_{\mathrm{har}}(V)^{\Gamma} \to C^1(V)^{\Gamma}/\partial C^0(V)^{\Gamma} \xrightarrow{\delta} H^1(\Gamma, V)$$

and is an isomorphism under the condition (coin). Combining this with the injectivity of δ, we obtain that both the natural map $C^1_{\mathrm{har}}(V)^{\Gamma} \to C^1(V)^{\Gamma}/\partial C^0(V)^{\Gamma}$ and δ are isomorphisms. Below, we will identify $C^1_{\mathrm{har}}(V)^{\Gamma}$ with $C^1(V)^{\Gamma}/\partial C^0(V)^{\Gamma}$.

By [10] we have

$$\begin{aligned} H^1_{\mathrm{dR},\tau}(X_{\Gamma}, \mathscr{V}) \cong \{&V\text{-valued differentials of second kind on } X_{\Gamma}\}/ \\ &\{df | f \text{ a } V\text{-valued meromorphic function on } X_{\Gamma}\}. \end{aligned} \tag{7.3}$$

In [10], de Shalit only considered a special case, but his argument is valued for our general case. If ω is a Γ-invariant V-valued differential of the second kind on $\mathcal{H}$, let F_{ω} be a primitive of it [9], which is defined by Coleman's integral [5].[3] Let P^{τ} be the map

$$P^{\tau}: H^1_{\mathrm{dR},\tau}(X_{\Gamma}, \mathscr{V}) \to H^1(\Gamma, V), \quad \omega \mapsto (\gamma \mapsto \gamma(F_{\omega}) - F_{\omega}).$$

Note that $P^{\tau} \circ \iota^{\tau}$ coincides with δ. Thus, P^{τ} splits the inclusion $\iota^{\tau} \circ \delta^{-1}: H^1(\Gamma, V) \to H^1_{\mathrm{dR},\tau}(X_{\Gamma}, \mathscr{V})$.

Let I^{τ} be the map

$$I^{\tau}: H^1_{\mathrm{dR},\tau}(X_{\Gamma}, \mathscr{V}) \to C^1_{\mathrm{har}}(V)^{\Gamma}, \omega \mapsto (e \mapsto \mathrm{Res}_e(\omega)).$$

Proposition 7.1. *Under the condition* (coin) *we have an exact sequence called the covering filtration exact sequence*

$$0 \longrightarrow H^1(\Gamma, V) \xrightarrow{\iota^{\tau} \circ \delta^{-1}} H^1_{\mathrm{dR},\tau}(X_{\Gamma}, \mathscr{V}) \xrightarrow{I^{\tau}} C^1_{\mathrm{har}}(V)^{\Gamma} \longrightarrow 0.$$

Proof. What we need to prove is that the map

$$H^1_{\mathrm{dR},\tau}(X_{\Gamma}, \mathscr{V}) \to H^1(\Gamma, V) \oplus C^1_{\mathrm{har}}(V)^{\Gamma} \quad \omega \mapsto (P^{\tau}(\omega), I^{\tau}(\omega))$$

is an isomorphism. When V is the trivial module, this is already proved in [10]. So we assume that V is not the trivial module. First we prove the injectivity of the above map.

[3]Precisely, we choose a branch of Coleman's integral.

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For this we only need to repeat the argument in [10, Theorem 1.6]. Let ω be a Γ-invariant V-valued differential form of second kind on $\mathcal{H}$ such that $P^\tau([\omega]) = I^\tau([\omega]) = 0$, where $[\omega]$ denotes the class of ω in $H^1_{\mathrm{dR},\tau}(X_\Gamma, \mathscr{V})$. Let F_ω be a primitive of ω. Because $I^\tau(\omega) = 0$, the residues of ω vanish and thus F_ω is meromorphic. Because $P^\tau(\omega) = 0$, we may adjust F_ω by adding a constant vector in V such that it is Γ-invariant. By (7.3) we have $[\omega] = 0$. To show the surjectivity, we only need to compare the dimensions. By [4, Appendix A] we have

$$\dim_{F_\mathfrak{p}} C^1_{\mathrm{har}}(V)^\Gamma = \dim_{F_\mathfrak{p}} H^1(\Gamma, V).$$

By [25, Theorem 1] we have

$$\dim_{F_\mathfrak{p}} H^1_{\mathrm{dR},\tau}(X_\Gamma, \mathscr{V}) = \dim_{F_\mathfrak{p}} H^1(\Gamma, V) + \dim_{F_\mathfrak{p}} H^1(\Gamma, V^*),$$

where $V^* = \mathrm{Hom}_{F_\mathfrak{p}}(V, F_\mathfrak{p})$ is the dual $F_\mathfrak{p}[\Gamma]$-module. Because $w = 2$, being an $F_\mathfrak{p}[\Gamma]$-module, V^* is isomorphic to V. Hence,

$$\dim_{F_\mathfrak{p}} H^1_{\mathrm{dR},\tau}(X_\Gamma, \mathscr{V}) = \dim_{F_\mathfrak{p}} C^1_{\mathrm{har}}(V)^\Gamma + \dim_{F_\mathfrak{p}} H^1(\Gamma, V),$$

as desired. □

7.2. ω_c^τ

We fix an embedding $\tau : F_\mathfrak{p} \hookrightarrow L$.

For each $\sigma : F_\mathfrak{p} \to L$, let $L_\sigma(k, v)$ be the dual of $V_\sigma(k, v)$ with the right action of $\mathrm{GL}(2, F_\mathfrak{p})$: if $g \in \mathrm{GL}(2, F_\mathfrak{p})$, $P' \in L_\sigma(k, v)$ and $P \in V_\sigma(k, v)$, then $\langle P', g \cdot P\rangle = \langle P'|_g, P\rangle$. We realise $L_\sigma(k, v)$ by the same space as $V_\sigma(k, v)$, with the pairing

$$\langle X_\sigma^j Y_\sigma^{k-2-j}, X_\sigma^{j'} Y_\sigma^{k-2-j'}\rangle = \begin{cases} 1 & \text{if } j = j' \\ 0 & \text{if } j \neq j' \end{cases}$$

and the right $\mathrm{GL}(2, F_\mathfrak{p})$-action

$$P|_{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]} = \sigma(ad - bc)^v P(\sigma(a)X_\sigma + \sigma(b)Y_\sigma, \sigma(c)X_\sigma + \sigma(d)Y_\sigma).$$

Put $L(\mathrm{k})^\tau = \bigotimes_{\sigma \in I, \sigma \neq \tau} L_\sigma(k_\sigma, \frac{w-k_\sigma}{2})$. Put $V(\mathrm{k})^\tau = \bigotimes_{\sigma \in I, \sigma \neq \tau} V_\sigma(k_\sigma, \frac{w-k_\sigma}{2})$. Then $L(\mathrm{k})^\tau$ is the dual of $V(\mathrm{k})^\tau$. Assume $w = 2$ below.

Let $\mathrm{LP}^{k_\tau - 2}$ be the space of local polynomials on $\mathrm{P}^1(F_\mathfrak{p})$ of degree $\leq k_\tau - 2$. We define a right action of $\mathrm{GL}(2, F_\mathfrak{p})$ on $\mathrm{LP}^{k_\tau - 2}$ by

$$f|_{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]}(x) = \frac{(cx+d)^{k_\tau - 2}}{(ad - bc)^{\frac{k_\tau}{2} - 1}} f\left(\frac{ax+b}{cx+d}\right).$$

Let c be a nonzero harmonic cocycle in $C^1_{\mathrm{har}}(V(\mathrm{k}))$. We attach to c a $V(\mathrm{k})^\tau$-valued linear functional μ_c^τ of $\mathrm{LP}^{k_\tau - 2}$ such that

$$\langle Q, \int_{U_e} t^j \mu_c^\tau(t)\rangle = \frac{\langle X_\tau^j Y_\tau^{k_\tau - 2 - j} \otimes Q, c(e)\rangle}{\binom{k_\tau - 2}{j}}$$

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for each $Q \in L_{\mathfrak{P}}(\mathrm{k})^\tau$ and $j = 0, \cdots, k_\tau - 2$. By definition, we have

$$\int_{U_e} (t^j|_g)\mu_c^\tau(t) = \int_{gU_e} t^j \mu_{g\cdot c}^\tau(t). \tag{7.4}$$

We say c is *bounded* if for a fixed edge e_0,

$$\sup_{g\in \mathrm{GL}(2,F_{\mathfrak{P}})/B} |g^{-1}(c(ge_0))|$$

exists, where B is the subgroup of $\mathrm{GL}(2, F_{\mathfrak{p}})$ that fixes e_0 and $|\cdot|$ is any norm on $V(\mathrm{k})$. Note that this concept does not depend on the choices of e_0 and $|\cdot|$.

Lemma 7.2. *If c is bounded, then there exists a constant $A > 0$ such that*

$$|\int_{a+\pi^m\mathfrak{o}_{F_\mathfrak{p}}} (t-a)^j \mu_c^\tau(t)| \le A|\pi|^{m(j+1-\frac{k_\tau}{2})}$$

for each $a \in F_\mathfrak{p}$.

Proof. Put $g = \left[\begin{smallmatrix} 1 & -a \\ 0 & \pi^m \end{smallmatrix}\right]$. Then $g(a+\pi^m\mathfrak{o}_{F_\mathfrak{p}}) = \mathfrak{o}_{F_\mathfrak{p}}$ and

$$t^j|_g = \frac{\pi^{m(k_\tau-2)}}{\pi^{m(\frac{k_\tau}{2}-1)}} (\frac{t-a}{\pi^m})^j = \pi^{m(\frac{k_\tau}{2}-1-j)}(t-a)^j.$$

By (7.4) we have

$$\int_{a+\pi^m\mathfrak{o}_{F_\mathfrak{p}}} (t-a)^j \mu_c^\tau(t) = \pi^{m(j+1-\frac{k_\tau}{2})} \int_{\mathfrak{o}_{F_\mathfrak{p}}} t^j \mu_{g\cdot c}^\tau(t).$$

Because c is bounded, this yields our lemma. □

Proposition 7.3. *If c is bounded, then there is a unique $V(\mathrm{k})^\tau$-valued analytic distribution μ_c^τ on $\mathrm{P}^1(F_\mathfrak{p})$ such that*

$$\langle Q, \int_{U_e} t^j \mu_c^\tau(t)\rangle = \frac{\langle X_\tau^j Y_\tau^{k_\tau-2-j} \otimes Q, c(e)\rangle}{\begin{pmatrix} k_\tau - 2 \\ j \end{pmatrix}}, \qquad j = 0, \cdots, k_\tau - 2$$

for each $Q \in L(\mathrm{k})^\tau$.

Proof. This follows from Lemma 7.2 and a standard Amice-Velu and Vishik's argument. □

Now let c be a nonzero harmonic cocycle in $C^1_{\mathrm{har}}(V(\mathrm{k}))^\Gamma$. Then Γ-invariancy of c ensures that c is bounded and so we can attach to c the distribution μ_c^τ. We define a $V(\mathrm{k})^\tau$-valued rigid analytic function g_c^τ, precisely, a global section of $V(\mathrm{k})^\tau \otimes_{\tau, F_\mathfrak{p}} \mathcal{O}_\mathcal{H}$, by

$$g_c^\tau(z) = \int_{\mathrm{P}^1(F_\mathfrak{p})} \frac{1}{z-t} \mu_c^\tau(t)$$

for $z \in \mathcal{H}$.

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Proposition 7.4. *The function* g_c^τ *satisfies the transformation property: for* $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$ *we have*

$$g_c^\tau(\gamma \cdot z) = \frac{(cz+d)^{k_\tau}}{\det(\gamma)^{k_\tau/2}} \gamma \cdot g_c^\tau(z).$$

Proof. This follows from an argument similar to the proof of [27, Theorem 3]. □

Put

$$\omega_c^\tau = g_c^\tau(z)(zX_\tau + Y_\tau)^{k_\tau - 2} \mathrm{d}z.$$

Then ω_c^τ is a Γ-invariant section of $\mathscr{V}(\mathrm{k}) \otimes_{\tau, F_\mathfrak{p}} \Omega^1_{\mathcal{H}}$ that descends to a section of $\mathscr{V}(\mathrm{k}) \otimes_{\tau, F_\mathfrak{p}} \Omega^1_{X_\Gamma}$.

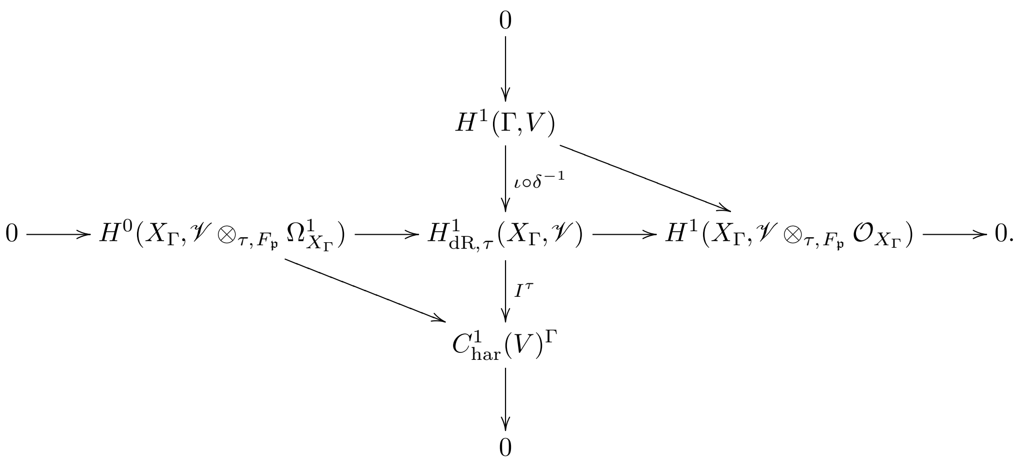
7.3. Hodge-like decomposition

We have also a Hodge filtration exact sequence

$$0 \longrightarrow H^0(X_\Gamma, \mathscr{V} \otimes_{\tau, F_\mathfrak{p}} \Omega^1_{X_\Gamma}) \longrightarrow H^1_{\mathrm{dR},\tau}(X_\Gamma, \mathscr{V}) \longrightarrow H^1(X_\Gamma, \mathscr{V} \otimes_{\tau, F_\mathfrak{p}} \mathcal{O}_{X_\Gamma}) \longrightarrow 0.$$

Thus, we may regard $\omega_c^\tau \in H^0(X_\Gamma, \mathscr{V} \otimes_{\tau, F_\mathfrak{p}} \Omega^1_{X_\Gamma})$ as an element of $H^1_{\mathrm{dR},\tau}(X_\Gamma, \mathscr{V})$.

The Hodge filtration exact sequence and the covering filtration exact sequence fit into the following commutative diagram:



Lemma 7.5. $I^\tau(\omega_c^\tau) = c$.

Proof. The proof is similar to that of [27, Theorem 3]. Recall that

$$g_c^\tau(z) = \int_{\mathrm{P}^1(F_\mathfrak{p})} \frac{1}{z-t} \mu_c^\tau(t).$$

For each edge e of $\mathcal{T}$, let $B(e)$ be the affinoid open disc in $\mathrm{P}^1(\mathbb{C}_p)$ that corresponds to e. Assume that $B(e)$ meets the limits set $\mathrm{P}^1(F_\mathfrak{p})$ in a compact open subset $U(e)$. Put

$$g_{c,e}^\tau(z) = \int_{U(e)} \frac{1}{z-t} \mu_c^\tau(t).$$

Let $a(e)$ be a point in $U(e)$. Expanding $\frac{1}{z-t}$ at $a(e)$ we obtain that

$$g^{\tau}_{c,e}(z) = \sum_{n=0}^{+\infty} \frac{1}{(z-a(e))^{n+1}} \int_{U(e)} (t-a(e))^n \mu^{\tau}_c(t)$$

and thus $g^{\tau}_{c,e}(z)$ converges on the complement of $B(e)$. By the same reason,

$$(g^{\tau}_c - g^{\tau}_{c,e})(z) = \int_{\mathrm{P}^1(F_{\mathfrak{p}})\backslash U(e)} \frac{1}{z-t}\mu^{\tau}_c(t) = -\sum_{n=0}^{+\infty}(z-a(e))^n \int_{\mathrm{P}^1(F_{\mathfrak{p}})\backslash U(e)} \frac{1}{(t-a(e))^{n+1}}\mu^{\tau}_c(t)$$

is analytic on $B(e)$. So, we have

$$I^{\tau}(g^{\tau}_c(zX_\tau+Y_\tau)^{k_\tau-2}\mathrm{d}z)(e) = \mathrm{Res}_e\Big(g^{\tau}_c(zX_\tau+Y_\tau)^{k_\tau-2}\mathrm{d}z\Big) = \mathrm{Res}_e\Big(g^{\tau}_{c,e}(zX_\tau+Y_\tau)^{k_\tau-2}\mathrm{d}z\Big)$$
$$= \mathrm{Res}_e\Big(\int_{U(e)} \frac{(zX_\tau+Y_\tau)^{k_\tau-2}}{z-t}\mu^{\tau}_c(t)\Big) = \int_{U(e)} (tX_\tau+Y_\tau)^{k_\tau-2}\mu^{\tau}_c(t) = c(e),$$

where the fourth equality follows from the fact that Res_e commutes with $\int_{U(e)} \cdot\, \mu^{\tau}_c(t)$. □

Theorem 7.6. *Under the assumption* (coin) *we have the following decomposition:*

$$H^1_{\mathrm{dR},\tau}(X_\Gamma,\mathscr{V}) = H^0(X_\Gamma,\mathscr{V}\otimes_{\tau,F_{\mathfrak{p}}}\Omega^1_{X_\Gamma}) \oplus H^1(\Gamma,V).$$

This decomposition is called the *Hodge-like decomposition*.

Proof. We only need to prove that the composition

$$H^0(X_\Gamma,\mathscr{V}\otimes_{\tau,F_{\mathfrak{p}}}\Omega^1_{X_\Gamma}) \to H^1_{\mathrm{dR},\tau}(X_\Gamma,\mathscr{V}) \to C^1_{\mathrm{har}}(V)^\Gamma$$

is an isomorphism. By Lemma 7.5 this is surjective. So

$$\dim H^0(X_\Gamma,\mathscr{V}\otimes_{\tau,F_{\mathfrak{p}}}\Omega^1_{X_\Gamma}) \geq \dim C^1_{\mathrm{har}}(V)^\Gamma = \dim H^1_{\mathrm{dR},\tau}(X_\Gamma,\mathscr{V})/2.$$

To show the injectivity of the above composition map it suffices to show that

$$\dim H^0(X_\Gamma,\mathscr{V}\otimes_{\tau,F_{\mathfrak{p}}}\Omega^1_{X_\Gamma}) \leq \dim H^1_{\mathrm{dR},\tau}(X_\Gamma,\mathscr{V})/2. \tag{7.5}$$

Note that as an $L[\Gamma]$-module V is dual to itself. Thus, there is a Poincaré pairing on $H^1_{\mathrm{dR},\tau}(X_\Gamma,\mathscr{V})$. For this pairing, $H^0(X_\Gamma,\mathscr{V}\otimes_{\tau,F_{\mathfrak{p}}}\Omega^1_{X_\Gamma})$ is orthogonal to itself, which implies (7.5). □

8. Proof of Theorem 1.2

Let $\mathrm{k} = (k_1,\cdots,k_g,w)$ be a multiweight such that $k_1 \equiv \cdots k_g \equiv w \mod 2$ and $k_1,\cdots,k_g$ are all even and ≥ 2.

Let f_∞ be a (Hilbert) cusp eigenform of weight k as in Theorem 1.2. By the condition in Theorem 1.2 there exists a quaternion algebra B over F that satisfies the condition at the beginning of Section 4 such that by Jacquet-Langlands correspondence f_∞ corresponds to a modular form f_B over the Shimura curve M attached to B. Let $U = U_{p,0}U^p$, a compact open subgroup of $G(\mathbb{A}_f)$, be the level of f_B. Let $\mathfrak{n}^-$ be the ideal of F such that $\mathfrak{p}\mathfrak{n}^-$ is the discriminant of B.

Let L be a (sufficiently large) finite extension of F that splits B and contains all Hecke eigenvalues acting on f_∞, λ a place of L above $\mathfrak{p}$.

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Lemma 8.1. *[24, Lemma 3.1] There is an isomorphism*

$$H^1_{\mathrm{et}}(M_{\overline{F}},\mathcal{F}(\mathrm{k})_\lambda) \simeq \bigoplus_{f'} \pi^\infty_{f',L(f')} \otimes_{L(f')} (\bigoplus_{\lambda'|\lambda} \rho_{f',\lambda'})$$

of representations of $G(\mathbb{A}_f)\times \mathrm{Gal}(\overline{F}/F)$ over L_λ. Here f' runs through the conjugacy classes over L, up to scalars, of eigen newforms of multiweight k that are new at primes dividing $\mathfrak{pn}^-$. The extension of L generated by the Hecke eigenvalues acting on f' is denoted by $L(f')$, and λ' runs through places of $L(f')$ above λ.

Let $\bar{B}$ be as in Subsection 5.2. Put $\widehat{\bar{B}}^\times := (\bar{B}\bigotimes\mathbb{A}_f)^\times$ and $\widehat{\bar{B}}^{p,\times} := (\bar{B}\bigotimes\mathbb{A}^p_f)^\times$. We identify $U^p = \prod_{\mathfrak{l}\neq\mathfrak{p}} U_\mathfrak{l}$ with a subgroup of $\widehat{\bar{B}}^{p,\times}$. Write $\widehat{\bar{B}}^{p,\times} = \sqcup_{i=1}^h \bar{B}^\times x_i U^p$. For each $i = 1,\cdots,h$ we put

$$\widetilde{\Gamma}_i := \{\gamma\in\bar{B}^\times : \gamma_\mathfrak{l}\in(x_i)_\mathfrak{l}U_\mathfrak{l}(x_i)^{-1}_\mathfrak{l} \text{ for } \mathfrak{l}\neq\mathfrak{p}\}.$$

Then X_{U^p} is isomorphic to

$$\bar{B}^\times\backslash(\mathcal{H}_{\widehat{F^{\mathrm{ur}}_\mathfrak{p}}}\times G(\mathbb{Q}_p)/U_{p,0}\times\widehat{\bar{B}}^{p,\times}/U^p)\cong\sqcup_{i=1}^h\widetilde{\Gamma}_i\backslash(\mathcal{H}_{\widehat{F^{\mathrm{ur}}_\mathfrak{p}}}\times\mathbb{Z}).$$

Here we identify $\mathbb{Z}$ with $G(\mathbb{Q}_p)/U_{p,0}$. Note that $\widetilde{\Gamma}_i$ acts transitively on $\mathbb{Z}=G(\mathbb{Q}_p)/U_{p,0}$; for every point in $\mathbb{Z}=G(\mathbb{Q}_p)/U_{p,0}$ it is fixed by $\gamma\in\bar{B}^\times$ if and only if $|\det(\gamma_\mathfrak{p})|_\mathfrak{p}=1$. Put

$$\begin{aligned}\widetilde{\Gamma}_{i,0} &= \{\gamma\in\widetilde{\Gamma}_i : |\det(\gamma_\mathfrak{p})|_\mathfrak{p}=1\}\\ &= \{\gamma\in\bar{B}^\times : \gamma_\mathfrak{l}\in(x_i)_\mathfrak{l}U_\mathfrak{l}(x_i)^{-1}_\mathfrak{l} \text{ for } \mathfrak{l}\neq\mathfrak{p} \text{ and } |\det(\gamma_\mathfrak{p})|_\mathfrak{p}=1\}.\end{aligned}$$

Let $\Gamma_{i,0}$ be the image of $\widetilde{\Gamma}_{i,0}$ in $\mathrm{PGL}(2,F_\mathfrak{p})$. Then we have an isomorphism

$$X_{U^p}\cong\sqcup_{i=1}^h\Gamma_{i,0}\backslash\mathcal{H}_{\widehat{F^{\mathrm{ur}}_\mathfrak{p}}}. \tag{8.1}$$

Now we come to the proof of Theorem 1.2. Twisting f_B by a central character, we may assume that $w=2$.

To show that $\rho_{f_B,\mathfrak{P},\mathfrak{p}}$ is semistable, we only need to prove that $H^1_{\mathrm{et}}((X_{U^p})_{\bar{F}_\mathfrak{p}},\mathcal{F}(\mathrm{k}))$ is semistable, because $\rho_{f_B,\mathfrak{P},\mathfrak{p}}$ is a quotient of $H^1_{\mathrm{et}}((X_{U^p})_{\bar{F}_\mathfrak{p}},\mathcal{F}(\mathrm{k}))$. But this follows from Theorem 2.1 and the fact that X_{U^p} has a semistable reduction.

Being a Shimura variety, N_E is a family of varieties. But in the following we will use N_E to denote any one in this family that corresponds to a level subgroup whose $\mathfrak{p}$-factor is $\mathscr{O}^\times_{E_\mathfrak{p}}$. By the proof of Lemma 6.4, any geometric point of $(N_E)_{\widehat{F^{\mathrm{ur}}_\mathfrak{p}}}$ is defined over $\widehat{F^{\mathrm{ur}}_\mathfrak{p}}$. In other words, $(N_E)_{\widehat{F^{\mathrm{ur}}_\mathfrak{p}}}$ is the product of several copies of $\mathrm{Spec}(\widehat{F^{\mathrm{ur}}_\mathfrak{p}})$. Thus, the $\mathrm{Gal}(\overline{F}_\mathfrak{p}/F^{\mathrm{ur}}_\mathfrak{p})$-representation $H^0_{\mathrm{et}}((N_E)_{\widehat{\overline{F}}_\mathfrak{p}},\mathbb{Q}_p)$ is crystalline and the associated filtered φ_q-module is $H^0_{\mathrm{dR}}((N_E)_{\widehat{F^{\mathrm{ur}}_\mathfrak{p}}},\mathbb{Q}_p)$ with trivial filtration. Let H^0 denote this filtered φ_q-module for simplicity.

Let pr_1 be the projection $X_{U^p}\times(N_E)_{\widehat{F^{\mathrm{ur}}_\mathfrak{p}}}\to X_{U^p}$. Corollary 6.5 tells us that the filtered φ_q-isocrystal attached to $\mathrm{pr}_1^*\mathcal{F}(\mathrm{k})$ is $\mathrm{pr}_1^*\mathscr{F}(\mathrm{k})$. Therefore, the filtered (φ_q,N)-module

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attached to $H^1_{\mathrm{et}}((M_U\times N)_{\overline{F}},\mathrm{pr}_1^*\mathcal{F}(\mathrm{k})_\lambda)$ is

$$H^1_{\mathrm{dR}}(X_{U^p}\times(N_E)_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}},\mathrm{pr}_1^*\mathscr{F}(\mathrm{k}))\cong H^0\otimes_{\mathbb{Q}_p}H^1_{\mathrm{dR}}(X_{U^p},\mathscr{F}(\mathrm{k})).$$

Applying the constructions in Subsection 7.1 to each part $\Gamma_{i,0}\backslash\mathcal{H}_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}}$ of X_{U^p}, we obtain operators ι^τ, P^τ and I^τ. By Proposition 2.2 the restriction of N to $H^1_{\mathrm{dR},\tau}((M_U\times N)_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}},\mathrm{pr}_1^*\mathscr{F}(\mathrm{k}))$ coincides with $\iota^\tau\circ I^\tau$. By [4, Appendix], each $\Gamma_{i,0}$ satisfies (coin). So we can apply Theorem 7.6 to obtain that the restriction of N to $H^0\otimes_{\mathbb{Q}_p}H^0(X_{U^p},\mathscr{F}\otimes_{\mathbb{Q}_p}\Omega^1_{X_{U^p}})$ is injective.

In the proof of Theorem 7.6 we show that each element of $H^0(X_{U^p},\mathscr{F}(\mathrm{k})\otimes_{\tau,F_{\mathfrak{p}}}\Omega^1_{X_{U^p}})$ is of the form $g_c^\tau(z)(zX_\tau+Y_\tau)^{k_\tau-2}\mathrm{d}z$. So,

$$H^0(X_{U^p},\mathscr{F}(\mathrm{k})\otimes_{\tau,F_{\mathfrak{p}}}\Omega^1)\subseteq \mathrm{Fil}^{\frac{w+k_\tau}{2}-1}H^1_{\mathrm{dR},\tau}(X_{U^p},\mathscr{F}(\mathrm{k})).$$

We consider the pullback of the pairing (6.1) to $X_{U^p}\times N_E$, which induces a perfect pairing on $H^0\otimes_{\mathbb{Q}_p}H^1_{\mathrm{dR},\tau}(X_{U^p},\mathscr{F}(\mathrm{k}))$. The filtered φ_q-isocrystal attached to $\mathcal{F}(\det^{w-2})$, denoted by $\mathscr{F}(w-2)$, satisfies that

$$\mathrm{Fil}^iH^1_{\mathrm{dR},\tau}(X_{U^p},\mathscr{F}(w-2))=\begin{cases}H^1_{\mathrm{dR},\tau}(X_{U^p},\mathscr{F}(w-2)) & \text{if } i\le w-1,\\ 0 & \text{if } i\ge w.\end{cases}$$

Hence, with respect to the above pairing,

$$\mathrm{Fil}^{\frac{w-k_\tau}{2}+1}H^0\otimes_{\mathbb{Q}_p}H^1_{\mathrm{dR},\tau}(X_{U^p},\mathscr{F}(\mathrm{k}))$$

is orthogonal to

$$\mathrm{Fil}^{\frac{w+k_\tau}{2}-1}H^0\otimes_{\mathbb{Q}_p}H^1_{\mathrm{dR},\tau}(X_{U^p},\mathscr{F}(\mathrm{k})).$$

Comparing dimensions, we obtain

$$H^0\otimes_{\mathbb{Q}_p}H^0(X_{U^p},\mathscr{F}(\mathrm{k})\otimes_{\tau,F_{\mathfrak{p}}}\Omega^1_{X_{U^p}})=\mathrm{Fil}^{\frac{w+\min_\sigma k_\sigma}{2}-1}H^0\otimes_{\mathbb{Q}_p}H^1_{\mathrm{dR},\tau}(X_{U^p},\mathscr{F}(\mathrm{k})).$$

Therefore, N induces an isomorphism

$$\begin{aligned}&\mathrm{Fil}^{\frac{w+\min_\sigma k_\sigma}{2}-1}H^0\otimes_{\mathbb{Q}_p}H^1_{\mathrm{dR},\tau}(X_{U^p},\mathscr{F}(\mathrm{k}))\\ &\qquad\xrightarrow{\sim}H^0\otimes_{\mathbb{Q}_p}H^1_{\mathrm{dR},\tau}(X_{U^p},\mathscr{F}(\mathrm{k}))/\mathrm{Fil}^{\frac{w+\min_\sigma k_\sigma}{2}-1}H^0\otimes_{\mathbb{Q}_p}H^1_{\mathrm{dR},\tau}(X_{U^p},\mathscr{F}(\mathrm{k}))\end{aligned}$$

for each τ.

Because the filtered (φ_q,N)-module of $\rho_{f_B,\mathfrak{P},\mathfrak{p}}$, denoted by D, is a quotient of $H^0\otimes_{\mathbb{Q}_p}H^1_{\mathrm{dR}}(X_{U^p},\mathscr{F}(\mathrm{k}))$, N induces an isomorphism

$$\mathrm{Fil}^{\frac{w+\min_\sigma k_\sigma}{2}-1}D_\tau\xrightarrow{\sim}D_\tau/\mathrm{Fil}^{\frac{w+\min_\sigma k_\sigma}{2}-1}D_\tau,$$

where D_τ is the τ-component of D. It follows that D is noncritical.

9. Comparing two kinds of L-invariants

9.1. Automorphic forms on totally definite quaternion algebras

We recall the theory of automorphic forms on totally definite quaternion algebras.

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Let $\bar{B}$ be as in Subsection 5.2, which is a totally definite quaternion algebra over F. Let $\Sigma = \prod_{\mathfrak{l}} \Sigma_{\mathfrak{l}}$ be a compact open subgroup of $\widehat{\bar{B}}^{\times}$.

Let $\chi_{F,\mathrm{cyc}} : \mathbb{A}_F^{\times}/F^{\times} \to \mathbb{Z}_p^{\times}$ be the Hecke character obtained by composing the cyclotomic character $\chi_{\mathbb{Q},\mathrm{cyc}} : \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times} \to \mathbb{Z}_p^{\times}$ and the norm map from $\mathbb{A}_F^{\times}$ to $\mathbb{A}_{\mathbb{Q}}^{\times}$.

Definition 9.1. An *automorphic form* on $\bar{B}^{\times}$, of *weight* $\mathrm{k} = (k_1, \cdots, k_g, w)$ and *level* Σ, is a function $\mathbf{f} : \widehat{\bar{B}}^{\times} \to V(\mathrm{k})$ that satisfies

$$\mathbf{f}(z\gamma bu) = \chi_{F,\mathrm{cyc}}^{2-w}(z)(u_p^{-1} \cdot \mathbf{f}(b))$$

for all $\gamma \in \bar{B}^{\times}$, $u \in \Sigma$, $b \in \widehat{\bar{B}}^{\times}$ and $z \in \widehat{F}^{\times}$. Denote by $S_{\mathrm{k}}^{\bar{B}}(\Sigma)$ the space of such forms. Remark that our $S_{\mathrm{k}}^{\bar{B}}(\Sigma)$ coincides with $S_{\mathrm{k}',\mathrm{v}}^{\bar{B}}(\Sigma)$ for $\mathrm{k}' = (k_1 - 2, \cdots, k_g - 2)$ and $\mathrm{v} = (\frac{w-k_1}{2}, \frac{w-k_2}{2}, \cdots, \frac{w-k_g}{2})$ in [4].

Observe that a form $\mathbf{f}$ of level Σ is determined by its values on the finite set $\bar{B}^{\times} \backslash \widehat{\bar{B}}^{\times}/\Sigma$. As in Section 8, we write $\widehat{\bar{B}}^{p,\times} = \sqcup_{i=1}^{h} \bar{B}^{\times} x_i \Sigma$; for $i = 1, \cdots, h$, put

$$\widetilde{\Gamma}_i = \{\gamma \in \bar{B}^{\times} : \gamma_{\mathfrak{l}} \in (x_i)_{\mathfrak{l}} \Sigma_{\mathfrak{l}} (x_i)_{\mathfrak{l}}^{-1} \text{ for } \mathfrak{l} \neq \mathfrak{p}\}.$$

Then we have a bijection

$$\sqcup_{i=1}^{h} \widetilde{\Gamma}_i \backslash \mathrm{GL}(2, F_{\mathfrak{p}})/\Sigma_{\mathfrak{p}} \xrightarrow{\sim} \bar{B}^{\times} \backslash \widehat{\bar{B}}^{\times}/\Sigma.$$

The class of g in $\widetilde{\Gamma}_i \backslash \mathrm{GL}(2, F_{\mathfrak{p}})/\Sigma_{\mathfrak{p}}$ corresponds to the class of $x_{i,\mathfrak{p}} g_{\mathfrak{p}}$ in $\bar{B}^{\times} \backslash \widehat{\bar{B}}^{\times}/\Sigma$, where $g_{\mathfrak{p}}$ is the element of $\widehat{\bar{B}}^{\times}$ that is equal to g at the place $\mathfrak{p}$ and equal to the identity at other places. Using this, we can attach to an automorphic form $\mathbf{f}$ of weight k and level Σ an h-tuple of functions $(f_1, \cdots, f_h)$ on $\mathrm{GL}(2, F_{\mathfrak{p}})$ with values in $V(\mathrm{k})$ defined by $f_i(g) = \mathbf{f}(x_{i,\mathfrak{p}} g_{\mathfrak{p}})$. The function f_i satisfies

$$f_i(\gamma_{\mathfrak{p}} g u z) = \chi_{F,\mathrm{cyc}}^{2-w}(z) u^{-1} \cdot f_i(g)$$

for all $\gamma_{\mathfrak{p}} \in \widetilde{\Gamma}_i$, $g \in \mathrm{GL}(2, F_{\mathfrak{p}})$, $u \in \Sigma_{\mathfrak{p}}$ and $z \in F_{\mathfrak{p}}^{\times}$.

For each prime $\mathfrak{l}$ of F such that $\bar{B}$ splits at $\mathfrak{l}$, $\mathfrak{l} \neq \mathfrak{p}$, and $\Sigma_{\mathfrak{l}}$ is maximal, one defines a Hecke operator $T_{\mathfrak{l}}$ on $S_{\mathrm{k}}^{\bar{B}}(\Sigma)$ as follows. Fix an isomorphism $\iota_{\mathfrak{l}} : B_{\mathfrak{l}} \to M_2(F_{\mathfrak{l}})$ such that $\Sigma_{\mathfrak{l}}$ becomes identified with $\mathrm{GL}_2(\mathfrak{o}_{F_{\mathfrak{l}}})$. Let $\pi_{\mathfrak{l}}$ be a uniformiser of $\mathfrak{o}_{F_{\mathfrak{l}}}$. Given a double coset decomposition

$$\mathrm{GL}_2(\mathfrak{o}_{F_{\mathfrak{l}}}) \left[\begin{smallmatrix} 1 & 0 \\ 0 & \pi_{\mathfrak{l}} \end{smallmatrix}\right] \mathrm{GL}_2(\mathfrak{o}_{F_{\mathfrak{l}}}) = \coprod b_i \mathrm{GL}_2(\mathfrak{o}_{F_{\mathfrak{l}}}),$$

we define the Hecke operator $T_{\mathfrak{l}}$ on $S_{\mathrm{k}}^{\bar{B}}(\Sigma)$ by

$$(T_{\mathfrak{l}} \mathbf{f})(b) = \sum_i \mathbf{f}(b b_i).$$

We define $U_{\mathfrak{p}}$ similarly. Let $\mathbb{T}_{\Sigma}$ be the Hecke algebra generated by $U_{\mathfrak{p}}$ and these $T_{\mathfrak{l}}$.

Denote by $\mathfrak{o}_F^{(\mathfrak{p})}$ the ring of $\mathfrak{p}$-integers of F and $(\mathfrak{o}_F^{(\mathfrak{p})})^{\times}$ the group of $\mathfrak{p}$-units of F. We have $\widetilde{\Gamma}_i \cap F^{\times} = (\mathfrak{o}_F^{(\mathfrak{p})})^{\times}$. For $i = 1, \cdots, h$, put $\Gamma_i = \widetilde{\Gamma}_i/(\mathfrak{o}_F^{(\mathfrak{p})})^{\times}$. Consider the following twisted

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action of $\widetilde{\Gamma}_i$ on $V(\mathrm{k})$:

$$\gamma \star v = |\mathrm{Nrd}_{\bar{B}/F}\gamma|_{\mathfrak{p}}^{\frac{w-2}{2}} \gamma_{\mathfrak{p}} \cdot v.$$

Then $(\mathfrak{o}_F^{(\mathfrak{p})})^\times$ is trivial on $V(\mathrm{k})$, so we may consider $V(\mathrm{k})$ as a Γ_i-module via the above twisted action.

9.2. Teitelbaum-type L-invariants

Chida, Mok and Park [4] defined Teitelbaum-type L-invariants for automorphic forms $\mathbf{f} \in S_{\mathrm{k}}^{\bar{B}}(\Sigma)$ satisfying the condition (CMP) given in the Introduction:

$$\mathbf{f} \text{ is new at } \mathfrak{p} \text{ and } U_{\mathfrak{p}}\mathbf{f} = \mathcal{N}\mathfrak{p}^{w/2}\mathbf{f}.$$

We recall their construction below.

We attach to each f_i a Γ_i-invariant $V(\mathrm{k})$-valued cocycle c_{f_i}, where Γ_i acts on $V(\mathrm{k})$ via $\star$. For $e = (s,t) \in \mathrm{E}(\mathcal{T})$, represent s and t by lattices L_s and L_t such that L_s contains L_t with index $\mathcal{N}\mathfrak{p}$. Let $g_e \in \mathrm{GL}(2,F_{\mathfrak{p}})$ be such that $g_e(\mathfrak{o}_{F_{\mathfrak{p}}}^2) = L_s$ and $g_e(\mathfrak{o}_{F_{\mathfrak{p}}} \oplus \mathfrak{p}\mathfrak{o}_{F_{\mathfrak{p}}}) = L_t$. Then we define $c_{f_i}(e) := |\det(g)|_{\mathfrak{p}}^{\frac{w-2}{2}} g_e \star f_i(g_e)$. If $\mathbf{f}$ satisfies (CMP), then c_{f_i} is in $C^1_{\mathrm{har}}(V(\mathrm{k}))^{\Gamma_i}$ [4, Proposition 2.7]. Thus, we obtain a vector of harmonic cocycles $c_{\mathbf{f}} = (c_{f_1}, \cdots, c_{f_h})$.

For each $c \in C^1_{\mathrm{har}}(V(\mathrm{k}))^{\Gamma_i}$ we define κ_c^{sch} to be the following $V(\mathrm{k})$-valued function on Γ_i: fixing some $v \in \mathrm{V}(\mathcal{T})$, for each $\gamma \in \Gamma_i$ we put

$$\kappa_c^{\mathrm{sch}}(\gamma) := \sum_{e:v\to\gamma v} c(e),$$

where e runs over the edges in the geodesic joining v and γv. Because c is Γ_i-invariant, κ_c^{sch} is a 1-cocycle on Γ_i. Furthermore, the class of κ_c^{sch} in $H^1(\Gamma_i, V(\mathrm{k}))$ is independent of the choice of v. Hence, we obtain a map

$$\kappa^{\mathrm{sch}} : \bigoplus_{i=1}^h C^1_{\mathrm{har}}(V(\mathrm{k}))^{\Gamma_i} \to \bigoplus_{i=1}^h H^1(\Gamma_i, V(\mathrm{k})).$$

By [4, Proposition 2.9], κ^{sch} is an isomorphism.

For each harmonic cocycle $c \in C^1_{\mathrm{har}}(V(\mathrm{k}))^{\Gamma_i}$, in Subsection 7.2 we attached to it the $V(\mathrm{k})^\tau$-valued function g_c^τ. We define a $V(\mathrm{k})$-valued cocycle λ_c^τ as follows. Fix a point $z_0 \in \mathcal{H}$. For each $\gamma \in \Gamma_i$ the value $\lambda_c^\tau(\gamma)$ is given by the formula: for $Q \in L_{\mathfrak{P}}(\mathrm{k})^\tau$,

$$\langle X_\tau^j Y_\tau^{k_\tau-2-j} \otimes Q, \lambda_c^\tau(\gamma)\rangle = \binom{k_\tau - 2}{j} \langle Q, \int_{z_0}^{\gamma z_0} z^j g_c^\tau(z)\mathrm{d}z\rangle$$

$(0 \le j \le k_\tau - 2)$, where the integral is the branch of Coleman's integral chosen in Subsection 7.1. Then λ_c^τ is a 1-cocycle on Γ_i and the class of λ_c^τ in $H^1(\Gamma_i, V(\mathrm{k}))$, denoted by $[\lambda_c^\tau]$, is independent of the choice of z_0. This defines a map

$$\kappa^{\mathrm{col},\tau} : \bigoplus_{i=1}^h C^1_{\mathrm{har}}(\mathcal{T}, V(\mathrm{k}))^{\Gamma_i} \to \bigoplus_{i=1}^h H^1(\Gamma_i, V(\mathrm{k})), \quad (c_i)_i \mapsto ([\lambda_{c_i}^\tau])_i.$$

Because κ^{sch} is an isomorphism, for each τ there exists a unique $\ell_\tau \in L_{\mathfrak{P}}$ such that

$$\kappa^{\mathrm{col},\tau}(c_{\mathbf{f}}) = \ell_\tau \kappa^{\mathrm{sch}}(c_{\mathbf{f}}).$$

The *Teitelbaum-type L-invariant* of $\mathbf{f}$, denoted by $\mathcal{L}_T(\mathbf{f})$, is defined to be the vector $(\ell_\tau)_\tau$ [4, Section 3.2]. We also write $\mathcal{L}_{T,\tau}(\mathbf{f})$ for ℓ_τ.

9.3. Comparing L-invariants

Let B, $\bar{B}$, G and $\bar{G}$ be as before. Let $\mathfrak{n}^-$ be the conductor of $\bar{B}$. By our assumption on $\bar{B}$, $\mathfrak{p} \nmid \mathfrak{n}^-$ and the conductor of B is $\mathfrak{p}\mathfrak{n}^-$. Let $\mathfrak{n}^+$ be an ideal of $\mathfrak{o}_F$ that is prime to $\mathfrak{p}\mathfrak{n}^-$ and put $\mathfrak{n} := \mathfrak{p}\mathfrak{n}^+\mathfrak{n}^-$.

For any prime ideal $\mathfrak{l}$ of $\mathfrak{o}_F$, put

$$\bar{R}_{\mathfrak{l}} := \begin{cases} \text{an maximal compact open subgroup of } \bar{B}_{\mathfrak{l}}^\times & \text{if } \mathfrak{l} \text{ is prime to } \mathfrak{n}, \\ \text{the maximal compact open subgroup of } \bar{B}_{\mathfrak{l}}^\times & \text{if } \mathfrak{l} \text{ divides } \mathfrak{n}^-, \\ 1+ \text{ an Eichler order of } \bar{B}_{\mathfrak{l}} \text{ of level } \mathfrak{l}^{\mathrm{val}_{\mathfrak{l}}(\mathfrak{p}\mathfrak{n}^+)} & \text{if } \mathfrak{l} \text{ divides } \mathfrak{p}\mathfrak{n}^+. \end{cases}$$

Let $\bar{\Sigma} = \Sigma(\mathfrak{p}\mathfrak{n}^+,\mathfrak{n}^-)$ be the level $\prod_{\mathfrak{l}} \bar{R}_{\mathfrak{l}}$. Similarly, we put $\Sigma = \Sigma(\mathfrak{n}^+,\mathfrak{p}\mathfrak{n}^-)$, a compact open subgroup of $G(\mathbb{A}_f)$.

Let $\mathrm{k} = (k_1,\cdots,k_g,w)$ be a multiweight such that $k_1 \equiv \cdots k_g \equiv w \mod 2$ and $k_1,\cdots,k_g$ are all even and ≥ 2. We write $S_{\mathrm{k}}^{\bar{B}}(\mathfrak{p}\mathfrak{n}^+,\mathfrak{n}^-)$ for $S_{\mathrm{k}}^{\bar{B}}(\Sigma(\mathfrak{p}\mathfrak{n}^+,\mathfrak{n}^-))$. Let $S_{\mathrm{k}}^{B}(\mathfrak{n}^+,\mathfrak{p}\mathfrak{n}^-)$ be the space of modular forms on the Shimura curve M of weight k and level Σ.

Let f_∞ be a (Hilbert) cusp eigen newform of weight k and level $\mathfrak{n}$. Let $\mathbf{f} \in S_{\mathrm{k}}^{\bar{B}}(\mathfrak{p}\mathfrak{n}^+,\mathfrak{n}^-)$ (respectively $f_B \in S_{\mathrm{k}}^{B}(\mathfrak{n}^+,\mathfrak{p}\mathfrak{n}^-)$) be an eigen newform corresponding to f_∞ by the Jacquet-Langlands correspondence; $\mathbf{f}$ (respectively f_B) is unique up to scalars.

We further assume that $\mathbf{f}$ satisfies (CMP), so that we can attach to $\mathbf{f}$ the Teitelbaum-type L-invariant $\mathcal{L}_T(\mathbf{f})$. We define $\mathcal{L}_T(f_\infty)$ to be $\mathcal{L}_T(\mathbf{f})$. The goal of this section is to compare $\mathcal{L}_{FM}(f_\infty)$ and $\mathcal{L}_T(f_\infty)$.

Let L be a (sufficiently large) finite extension of F that splits B and contains all Hecke eigenvalues acting on f_∞. Let λ be a place of L above $\mathfrak{p}$.

By the strong multiplicity one theorem [22], there exists a primitive idempotent $e_{f_B} \in \mathbb{T}_{\bar{\Sigma}}$ such that $e_{f_B}\mathbb{T}_{\bar{\Sigma}} = Le_{f_B}$ and $e_{f_B} \cdot S_{\mathrm{k}}^{\bar{B}}(\Sigma(\mathfrak{p}\mathfrak{n}^+,\mathfrak{n}^-)) = L \cdot f_B$. Lemma 8.1 tells us that $e_{f_B} \cdot H^1_{\mathrm{et}}(M_{\overline{F}},\mathcal{F}(\mathrm{k})_\lambda)^\Sigma$ is exactly $\rho_{f_B,\lambda}$.

In Section 8 and Subsection 9.1 we associate to $\bar{\Sigma}$ the groups $\widetilde{\Gamma}_{i,0},\widetilde{\Gamma}_i,\Gamma_i$ and $\Gamma_{i,0}$ $(i = 1,\cdots,h)$. By (8.1), X_Σ is isomorphic to $\coprod_i X_{\Gamma_{i,0}}$, where $X_{\Gamma_{i,0}} = \Gamma_{i,0}\backslash\mathcal{H}_{\widehat{F_{\mathfrak{p}}^{\mathrm{ur}}}}$. In Subsection 9.2 we attached to $\mathbf{f} = (f_1,\cdots,f_h)$ an h-tuple $g^\tau = (g_1^\tau,\cdots,g_h^\tau)$ where $g_i^\tau = g^\tau_{c_{f_i}}$. Put

$$\omega_{\mathbf{f}}^\tau := \left(g_i^\tau(z)(zX_\tau + Y_\tau)^{k_\tau - 2}\mathrm{d}z\right)_{1\leq i\leq h},$$

which is an element of $\bigoplus_i H^1_{\mathrm{dR},\tau}(X_{\Gamma_{i,0}},\mathscr{F}(\mathrm{k}))$.

Let P^τ, ι^τ and I^τ be the operators attached to $\mathscr{F}(\mathrm{k})$.

Lemma 9.2. *We have*

$$P^\tau(\omega_{\mathbf{f}}^\tau) = \kappa^{\mathrm{col},\tau}(c_{\mathbf{f}}), \qquad I^\tau(\omega_{\mathbf{f}}^\tau) = c_{\mathbf{f}}.$$

Proof. The first formula comes from the definitions and the second follows from Lemma 7.5. □

Theorem 9.3. *Let f_∞ be as above. Then $\mathcal{L}_{FM}(f_\infty)=\mathcal{L}_T(f_\infty)$.*

Proof. Twisting f_∞ by a central character, we may assume that $w=2$.

We use notations in Section 8. Let H^0 be the filtered φ_q-module $H^0_{\mathrm{dR}}((N_E)_{\widehat{F^{\mathrm{ur}}_{\mathfrak{p}}}},\mathbb{Q}_p)$ and put $D_\tau = H^0\otimes_{\mathbb{Q}_p} e_{f_B}H^1_{\mathrm{dR},\tau}(X_\Sigma,\mathscr{F}(\mathrm{k}))$. Note that the restriction of N to D_τ coincides with $\iota^\tau\circ I^\tau$. Because the kernel of N coincides with the image of $\iota^\tau\circ\delta^{-1}$ and P^τ splits $\iota^\tau\circ\delta^{-1}$, we have $D_\tau=\ker(N)\oplus\ker(P^\tau)$. Write $\omega^\tau_{\mathbf{f}}=x+y$ according to this decomposition. Then

$$\iota^\tau\circ\delta^{-1}\circ P^\tau(\omega^\tau_{\mathbf{f}})=x. \tag{9.1}$$

By the proof of Theorem 1.2, y is nonzero and so $N(y)\neq 0$.

By Lemma 9.2 and the definition of Teitelbaum-type L-invariant, $\mathcal{L}_{T,\tau}(f_\infty)$ is characterised by the property

$$(P^\tau-\mathcal{L}_{T,\tau}(f_\infty)\epsilon\circ I^\tau)\omega^\tau_{\mathbf{f}}=0, \tag{9.2}$$

where ϵ is the map defined by (7.2) that coincides with κ^{sch}. Because $\delta^{-1}\circ\epsilon=-\mathrm{id}$ and $\iota^\tau\circ I^\tau=N$, we have

$$\iota^\tau\circ\delta^{-1}\circ\epsilon\circ I^\tau(\omega^\tau_{\mathbf{f}})=-N(\omega^\tau_{\mathbf{f}}). \tag{9.3}$$

By (9.1), (9.2) and (9.3) we get

$$\mathcal{L}_{T,\tau}(f_\infty)N(\omega^\tau_{\mathbf{f}})+x=0. \tag{9.4}$$

By the definition of Fontaine-Mazur L-invariant, $\mathcal{L}_{FM,\tau}(f_\infty)$ is characterised by the property

$$y-\mathcal{L}_{FM,\tau}(f_\infty)N(y)\in H^0\otimes_{\mathbb{Q}_p}\mathrm{Fil}^{\frac{w+\min_\sigma\{k_\sigma\}}{2}-1}H^1_{\mathrm{dR},\tau}(X_\Sigma,\mathscr{F}(\mathrm{k})). \tag{9.5}$$

Combining (9.4) and (9.5) we obtain

$$\begin{aligned}
&(\mathcal{L}_{FM,\tau}(f_\infty)-\mathcal{L}_{T,\tau}(f_\infty))N(y)\\
&=\mathcal{L}_{FM,\tau}(f_\infty)N(y)-\mathcal{L}_{T,\tau}(f_\infty)N(\omega^\tau_{\mathbf{f}})\\
&\in \omega^\tau_{\mathbf{f}}+H^0\otimes_{\mathbb{Q}_p}\mathrm{Fil}^{\frac{w+\min_\sigma\{k_\sigma\}}{2}-1}H^1_{\mathrm{dR},\tau}(X_\Sigma,\mathscr{F}(\mathrm{k}))\\
&=H^0\otimes_{\mathbb{Q}_p}\mathrm{Fil}^{\frac{w+\min_\sigma\{k_\sigma\}}{2}-1}H^1_{\mathrm{dR},\tau}(X_\Sigma,\mathscr{F}(\mathrm{k})).
\end{aligned}$$

But $N(y)$ is in $\ker(N)$ and is nonzero. Again by the proof of Theorem 1.2,

$$\ker(N)\cap H^0\otimes_{\mathbb{Q}_p}\mathrm{Fil}^{\frac{w+\min_\sigma\{k_\sigma\}}{2}-1}H^1_{\mathrm{dR},\tau}(X_\Sigma,\mathscr{F}(\mathrm{k}))=0.$$

Therefore,

$$\mathcal{L}_{FM,\tau}(f_\infty)-\mathcal{L}_{T,\tau}(f_\infty)=0,$$

as wanted. □

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