Proceedings of the Royal Society of Edinburgh, **147A**, 429–447, *2017*
DOI:10.1017/S0308210516000214

Neumann to Steklov eigenvalues: asymptotic and monotonicity results

Pier Domenico Lamberti and **Luigi Provenzano**
Dipartimento di Matematica 'Tullio Levi-Civita',
Università degli Studi di Padova, Via Trieste 63, 35126 Padova, Italy
(lamberti@math.unipd.it; proz@math.unipd.it)

(MS received 17 March 2015; accepted 24 November 2015)

We consider the Steklov eigenvalues of the Laplace operator as limiting Neumann eigenvalues in a problem of mass concentration at the boundary of a ball. We discuss the asymptotic behaviour of the Neumann eigenvalues and find explicit formulae for their derivatives in the limiting problem. We deduce that the Neumann eigenvalues have a monotone behaviour in the limit and that Steklov eigenvalues locally minimize the Neumann eigenvalues.

Keywords: Steklov boundary conditions; eigenvalues; density perturbation; monotonicity; Bessel functions

2010 *Mathematics subject classification:* Primary 35C20
Secondary 35P15; 35B25; 35J25; 33C10

1. Introduction

Let B be the unit ball in $\mathbb{R}^N$, $N \geqslant 2$, centred at zero. We consider the Steklov eigenvalue problem for the Laplace operator

$$\left.\begin{aligned} \Delta u &= 0 && \text{in } B,\\ \frac{\partial u}{\partial \nu} &= \lambda\rho u && \text{on } \partial B, \end{aligned}\right\} \tag{1.1}$$

in the unknowns λ (the eigenvalue) and u (the eigenfunction), where $\rho = M/\sigma_N$, $M > 0$ is a fixed constant and σ_N denotes the surface measure of ∂B.

As is well known, the eigenvalues of problem (1.1) are given explicitly by the sequence

$$\lambda_l = \frac{l}{\rho}, \quad l \in \mathbb{N}, \tag{1.2}$$

and the eigenfunctions corresponding to λ_l are homogeneous harmonic polynomials of degree l. In particular, the multiplicity of λ_l is $(2l + N - 2)(l + N - 3)!/(l!(N - 2)!)$, and only λ_0 is simple, the corresponding eigenfunctions being the constant functions. See [8] for an introduction to the theory of harmonic polynomials.

A classical reference for problem (1.1) is [18]. For a recent survey paper, we refer the reader to [9]; see also [13, 17] for related problems.

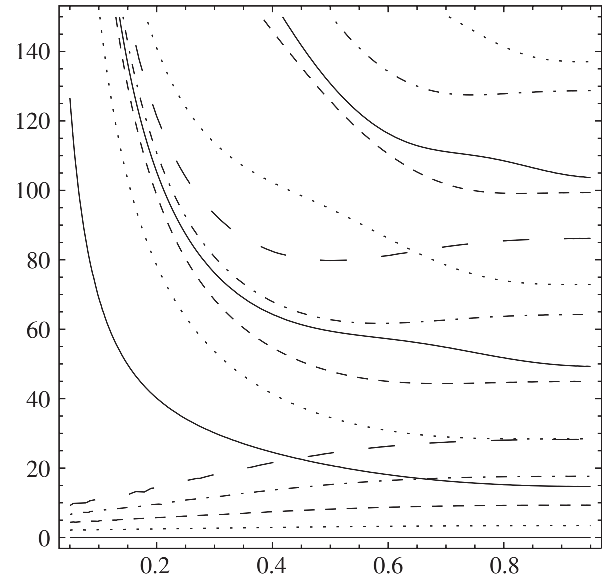

Figure 1. Solution branches of (2.4) with $N = 2$, $M = \pi$ for $(\varepsilon, \lambda) \in]0, 1[\times]0, 150[$. The line style refers to the choice of l in (2.4): continuous line ($l = 0$), dotted line ($l = 1$), short dashed line ($l = 2$), dot-dashed line ($l = 3$), long dashed line ($l = 4$).

It is well known that, for $N = 2$, problem (1.1) provides the vibration modes of a free elastic membrane with total mass M, which is concentrated at the boundary with density ρ (see, for example, [4]). As is pointed out in [17], such a boundary concentration phenomenon can be explained in any dimension $N \geqslant 2$ as follows.

For any $0 < \varepsilon < 1$, we define a 'mass density' ρ_ε in the whole of B by setting

$$\rho_\varepsilon(x) = \begin{cases} \varepsilon & \text{if } |x| \leqslant 1 - \varepsilon, \\ \dfrac{M - \varepsilon\omega_N(1-\varepsilon)^N}{\omega_N(1-(1-\varepsilon)^N)} & \text{if } 1 - \varepsilon < |x| < 1, \end{cases} \tag{1.3}$$

where $\omega_N = \sigma_N/N$ is the measure of the unit ball. Note that for any $x \in B$ we have $\rho_\varepsilon(x) \to 0$ as $\varepsilon \to 0$, and

$$\int_B \rho_\varepsilon \, \mathrm{d}x = M \quad \text{for all } \varepsilon > 0,$$

which means that the 'total mass' M is fixed and concentrates at the boundary of B as $\varepsilon \to 0$. Then we consider an eigenvalue problem for the Laplace operator with Neumann boundary conditions:

$$\left.\begin{aligned} -\Delta u &= \lambda\rho_\varepsilon u && \text{in } B, \\ \frac{\partial u}{\partial \nu} &= 0 && \text{on } \partial B. \end{aligned}\right\} \tag{1.4}$$

We recall that for $N = 2$ (1.4) provides the vibration modes of a free elastic membrane with mass density ρ_ε and total mass M (see, for example, [6]). The eigenvalues

https://doi.org/10.1017/S0308210516000214 Published online by Cambridge University Press

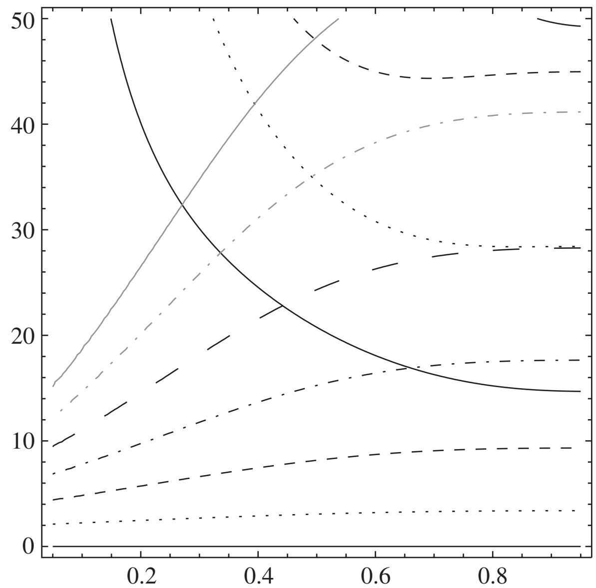

Figure 2. Solution branches of (2.4) with $N = 2$, $M = \pi$ for $(\varepsilon, \lambda) \in]0, 1[\times]0, 50[$. The line style refers to the choice of l in (2.4): solid black line ($l = 0$), dotted line ($l = 1$), short dashed line ($l = 2$), dot-dashed line ($l = 3$), long dashed line ($l = 4$), grey dot-dashed line ($l = 5$), grey line ($l = 6$).

of (1.4) have finite multiplicity and form a sequence

$$\lambda_0(\varepsilon) < \lambda_1(\varepsilon) \leqslant \lambda_2(\varepsilon) \leqslant \cdots,$$

depending on ε, with $\lambda_0(\varepsilon) = 0$.

It is not difficult to prove that, for any $l \in \mathbb{N}$,

$$\lambda_l(\varepsilon) \to \lambda_l \quad \text{as } \varepsilon \to 0 \tag{1.5}$$

(see [2, 3, 17]). (See also [5] for a detailed analysis of an analogous problem for the biharmonic operator.) Thus, the Steklov problem can be considered as a limiting Neumann problem where the mass is concentrated at the boundary of the domain.

In this paper we study the asymptotic behaviour of $\lambda_l(\varepsilon)$ as $\varepsilon \to 0$. Namely, we prove that such eigenvalues are continuously differentiable with respect to ε for $\varepsilon \geqslant 0$ small enough, and that the following formula holds:

$$\lambda_l'(0) = \frac{2l\lambda_l}{3} + \frac{2\lambda_l^2}{N(2l + N)}. \tag{1.6}$$

In particular, for $l \neq 0$, $\lambda_l'(0) > 0$ hence $\lambda_l(\varepsilon)$ is strictly increasing and the Steklov eigenvalues λ_l minimize the Neumann eigenvalues $\lambda_l(\varepsilon)$ for ε small enough.

It is interesting to compare our results with those in [19], where the Neumann Laplacian is considered in the annulus $1-\varepsilon < |x| < 1$ and it is proved that for $N = 2$ the first positive eigenvalue is an increasing function of ε. Note that our analysis concerns all eigenvalues λ_l with arbitrary indices and multiplicity, and that we do

https://doi.org/10.1017/S0308210516000214 Published online by Cambridge University Press

not prove global monotonicity of $\lambda_l(\varepsilon)$, which in fact does not hold for any l (see figures 1 and 2).

The proof of our results relies on the use of Bessel functions, which allows us to recast problem (1.4) in the form of an equation $F(\lambda, \varepsilon) = 0$ in the unknowns λ, ε. Then, after some preparatory work, it is possible to apply the implicit function theorem and conclude. We note that, although the idea of the proof is rather simple and is used in other contexts (see, for example, [15]), the rigorous application of this method requires lengthy computations, suitable Taylor's expansions and estimates for the corresponding remainders, as well as recursive formulae for the cross products of Bessel functions and their derivatives.

Importantly, the multiplicity of the eigenvalues, which is often an obstruction in the application of standard asymptotic analysis, does not affect our method.

We note that if the ball B is replaced by a general bounded smooth domain Ω, the convergence of the Neumann eigenvalues to the Steklov eigenvalues when the mass concentrates in a neighbourhood of $\partial\Omega$ still holds. However, the explicit computation of the appropriate formula generalizing (1.6) is not easy and requires a completely different technique, which will be discussed in the forthcoming paper [7].

We also note that an asymptotic analysis of a similar problem is contained in [10, 11], although explicit computations of the coefficients in the asymptotic expansions of the eigenvalues are not provided therein.

It would be interesting to investigate the monotonicity properties of the Neumann eigenvalues in the case of more general families of mass densities ρ_ε. However, we believe that it would be difficult to adapt our method (which is based on explicit representation formulae) even in the case of radial mass densities (note that if ρ_ε is not radial, one could obtain a limiting Steklov-type problem with non-constant mass density; see [3] for a general discussion).

This paper is organized as follows. The proof of formula (1.6) is discussed in § 2. In particular, § 2.1 is devoted to certain technical estimates necessary for the rigorous justification of our arguments. In § 2.2 we consider the case $N = 1$ and prove (1.6) for λ_1, which is the only non-zero eigenvalue of the one-dimensional Steklov problem. In the appendix we establish the required recursive formulae for the cross products of Bessel functions and their derivatives, which are deduced using the standard formulae available in the literature.

2. Asymptotic behaviour of Neumann eigenvalues

It is convenient to use the standard spherical coordinates (r, θ) in $\mathbb{R}^N$, where $\theta = (\theta_1, \dots \theta_{N-1})$. The corresponding transformation of coordinates is

$$\begin{aligned}
x_1 &= r\cos(\theta_1),\\
x_2 &= r\sin(\theta_1)\cos(\theta_2),\\
&\;\;\vdots\\
x_{N-1} &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{N-2})\cos(\theta_{N-1}),\\
x_N &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{N-2})\sin(\theta_{N-1}),
\end{aligned}$$

with $\theta_1, \dots, \theta_{N-2} \in [0, \pi]$, $\theta_{N-1} \in [0, 2\pi[$ (here it is understood that $\theta_1 \in [0, 2\pi[$ if $N = 2$). We denote by δ the Laplace–Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$

of $\mathbb{R}^N$, which can be written in spherical coordinates as

$$\delta = \sum_{j=1}^{N-1} \frac{1}{q_j (\sin\theta_j)^{N-j-1}} \frac{\partial}{\partial\theta_j}\left((\sin\theta_j)^{N-j-1}\frac{\partial}{\partial\theta_j}\right),$$

where

$$q_1 = 1, \qquad q_j = (\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{j-1})^2, \quad j = 2,\dots,N-1$$

(see, for example, [12, p. 40]). To shorten notation, in what follows we shall set

$$a = \sqrt{\lambda\varepsilon}(1-\varepsilon) \quad \text{and} \quad b = \sqrt{\lambda\tilde{\rho}_\varepsilon}(1-\varepsilon),$$

where

$$\tilde{\rho}_\varepsilon = \frac{M - \varepsilon\omega_N(1-\varepsilon)^N}{\omega_N(1-(1-\varepsilon)^N)}.$$

As is customary, we denote by J_ν and Y_ν the Bessel functions of the first and second species of order ν, respectively (recall that J_ν and Y_ν are solutions of the Bessel equation $z^2 y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = 0$).

We begin with the following lemma.

LEMMA 2.1. *Given an eigenvalue λ of problem (1.4), a corresponding eigenfunction u is of the form $u(r,\theta) = S_l(r)H_l(\theta)$, where $H_l(\theta)$ is a spherical harmonic of some order $l \in \mathbb{N}$ and*

$$S_l(r) = \begin{cases} r^{1-N/2} J_{\nu_l}(\sqrt{\lambda\varepsilon}r) & \text{if } r < 1-\varepsilon, \\ r^{1-N/2}(\alpha J_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}r) + \beta Y_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}r)) & \text{if } 1-\varepsilon < r < 1, \end{cases} \tag{2.1}$$

where $\nu_l = \frac{1}{2}(N + 2l - 2)$ and α, β are given by

$$\alpha = \frac{\pi b}{2}\left(J_{\nu_l}(a)Y'_{\nu_l}(b) - \frac{a}{b}J'_{\nu_l}(a)Y_{\nu_l}(b)\right),$$

$$\beta = \frac{\pi b}{2}\left(\frac{a}{b}J_{\nu_l}(b)J'_{\nu_l}(a) - J'_{\nu_l}(b)J_{\nu_l}(a)\right).$$

Proof. Recall that the Laplace operator can be written in spherical coordinates as

$$\Delta = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\delta.$$

In order to solve $-\Delta u = \lambda\rho_\varepsilon u$, we separate the variables so that $u(r,\theta) = S(r)H(\theta)$. Then, using $l(l+N-2)$, $l \in \mathbb{N}$, as the separation constant, we obtain

$$r^2 S'' + r(N-1)S' + r^2\lambda\rho_\varepsilon S - l(l+N-2)S = 0 \tag{2.2}$$

and

$$-\delta H = l(l+N-2)H. \tag{2.3}$$

By setting $S(r) = r^{1-N/2}\tilde{S}(r)$ in (2.2), it follows that $\tilde{S}(r)$ satisfies the Bessel equation:

$$\tilde{S}'' + \frac{\tilde{S}'}{r} + \left(\lambda\rho_\varepsilon - \frac{\nu_l^2}{r^2}\right)\tilde{S} = 0.$$

https://doi.org/10.1017/S0308210516000214 Published online by Cambridge University Press

Since solutions u of (1.4) are bounded on Ω, and $Y_{\nu_l}(z)$ blows up at $z = 0$, it follows that, for $r < 1 - \varepsilon$, $S(r)$ is a multiple of the function $r^{1-N/2} J_{\nu_l}(\sqrt{\lambda\varepsilon} r)$. For $1 - \varepsilon < r < 1$, $S(r)$ is a linear combination of the functions $r^{1-N/2} J_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon} r)$ and $r^{1-N/2} Y_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon} r)$. On the other hand, the solutions of (2.3) are the spherical harmonics of order l. Then u can be written as in (2.1) for suitable values of $\alpha, \beta \in \mathbb{R}$.

Now we compute the coefficients α and β in (2.1). Since the right-hand side of the equation in (1.4) is a function in $L^2(\Omega)$, by standard regularity theory a solution u of (1.4) belongs to the standard Sobolev space $H^2(\Omega)$. Hence, α and β must be chosen in such a way that u and $\partial_r u$ are continuous at $r = 1 - \varepsilon$, i.e.

$$\alpha J_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}(1-\varepsilon)) + \beta Y_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}(1-\varepsilon)) = J_{\nu_l}(\sqrt{\lambda\varepsilon}(1-\varepsilon)),$$

$$\alpha J'_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}(1-\varepsilon)) + \beta Y'_{\nu_l}(\sqrt{\lambda\tilde{\rho}_\varepsilon}(1-\varepsilon)) = \sqrt{\frac{\varepsilon}{\tilde{\rho}_\varepsilon}} J'_{\nu_l}(\sqrt{\lambda\varepsilon}(1-\varepsilon)).$$

Solving the system, we obtain

$$\alpha = \frac{J_{\nu_l}(a)Y'_{\nu_l}(b) - (a/b)J'_{\nu_l}(a)Y_{\nu_l}(b)}{J_{\nu_l}(b)Y'_{\nu_l}(b) - J'_{\nu_l}(b)Y_{\nu_l}(b)}, \qquad \beta = \frac{(a/b)J_{\nu_l}(b)J'_{\nu_l}(a) - J'_{\nu_l}(b)J_{\nu_l}(a)}{J_{\nu_l}(b)Y'_{\nu_l}(b) - J'_{\nu_l}(b)Y_{\nu_l}(b)}.$$

Note that $J_{\nu_l}(b)Y'_{\nu_l}(b) - J'_{\nu_l}(b)Y_{\nu_l}(b)$ is the Wronskian in b, which is known to be $2/\pi b$ (see [1, § 9]). This concludes the proof. □

We are ready to establish an implicit characterization of the eigenvalues of (1.4).

PROPOSITION 2.2. *The non-zero eigenvalues λ of problem (1.4) are given implicitly as zeros of the equation*

$$\left(1 - \frac{N}{2}\right) P_1(a,b) + \frac{b}{1-\varepsilon} P_2(a,b) = 0, \tag{2.4}$$

where

$$\begin{aligned} P_1(a,b) = {} & J_{\nu_l}(a)\left(Y'_{\nu_l}(b) J_{\nu_l}\left(\frac{b}{1-\varepsilon}\right) - J'_{\nu_l}(b) Y_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)\right) \\ & + \frac{a}{b} J'_{\nu_l}(a)\left(J_{\nu_l}(b) Y_{\nu_l}\left(\frac{b}{1-\varepsilon}\right) - Y_{\nu_l}(b) J_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)\right), \\ P_2(a,b) = {} & J_{\nu_l}(a)\left(Y'_{\nu_l}(b) J'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right) - J'_{\nu_l}(b) Y'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)\right) \\ & + \frac{a}{b} J'_{\nu_l}(a)\left(J_{\nu_l}(b) Y'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right) - Y_{\nu_l}(b) J'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)\right). \end{aligned}$$

Proof. By lemma 2.1, an eigenfunction u associated with an eigenvalue λ is of the form $u(r,\theta) = S_l(r)H_l(\theta)$, where, for $r > 1 - \varepsilon$,

$$\begin{aligned} S_l(r) = \frac{\pi b}{2} r^{1-N/2} \Bigg[& \left(J_{\nu_l}(a)Y'_{\nu_l}(b) - \frac{a}{b} J'_{\nu_l}(a) Y_{\nu_l}(b)\right) J_{\nu_l}\left(\frac{br}{1-\varepsilon}\right) \\ & + \left(\frac{a}{b} J_{\nu_l}(b) J'_{\nu_l}(a) - J'_{\nu_l}(b) J_{\nu_l}(a)\right) Y_{\nu_l}\left(\frac{br}{1-\varepsilon}\right)\Bigg]. \end{aligned}$$

https://doi.org/10.1017/S0308210516000214 Published online by Cambridge University Press

We require that

$$\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial r}\bigg|_{r=1} = 0,$$

which is true if and only if

$$\begin{aligned}\frac{\pi b}{2}\left(1-\frac{N}{2}\right)\bigg[&\left(J_{\nu_l}(a)Y'_{\nu_l}(b)-\frac{a}{b}J'_{\nu_l}(a)Y_{\nu_l}(b)\right)J_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)\\&+\left(\frac{a}{b}J_{\nu_l}(b)J'_{\nu_l}(a)-J'_{\nu_l}(b)J_{\nu_l}(a)\right)Y_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)\bigg]\\+\frac{\pi b^2}{2(1-\varepsilon)}\bigg[&\left(J_{\nu_l}(a)Y'_{\nu_l}(b)-\frac{a}{b}J'_{\nu_l}(a)Y_{\nu_l}(b)\right)J'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)\\&+\left(\frac{a}{b}J_{\nu_l}(b)J'_{\nu_l}(a)-J'_{\nu_l}(b)J_{\nu_l}(a)\right)Y'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)\bigg]=0.\end{aligned}$$

The above equation can be clearly rewritten in the form (2.4). □

We now prove the following.

LEMMA 2.3. *Equation (2.4) can be written in the form*

$$\begin{aligned}\lambda^2\varepsilon\left(\frac{M}{3N\omega_N}-\frac{1}{\nu_l(1+\nu_l)}\right)+\lambda\varepsilon\left(\frac{N}{2}-\nu_l+\frac{(2-N)N\omega_N}{2\nu_l(1+\nu_l)M}\right)-2\lambda\\+\frac{2N\omega_N l}{M}-\frac{2N\omega_N l}{M}\left(\frac{N-1}{2}-\frac{\omega_N}{M}-\nu_l\right)\varepsilon+\mathcal{R}(\lambda,\varepsilon)=0,\end{aligned}\quad(2.5)$$

where $\mathcal{R}(\lambda,\varepsilon)=O(\varepsilon\sqrt{\varepsilon})$ *as* $\varepsilon\to 0$.

Proof. We shall divide the left-hand side of (2.4) by $J'_{\nu_l}(a)$ and analyse the resulting terms using the known Taylor series for Bessel functions. Note that $J'_{\nu_l}(a)>0$ for all ε small enough. We split our analysis into three steps.

STEP 1. We consider the term $P_2(a,b)/J'_{\nu_l}(a)$, that is

$$\begin{aligned}\frac{J_{\nu_l}(a)}{J'_{\nu_l}(a)}\left[Y'_{\nu_l}(b)J'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)-Y'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)J'_{\nu_l}(b)\right]\\+\frac{a}{b}\left[Y'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)J_{\nu_l}(b)-Y_{\nu_l}(b)J'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)\right].\end{aligned}\quad(2.6)$$

Using Taylor's formula, we write the derivatives of the Bessel functions in (2.6) (call them $\mathcal{C}'_{\nu_l}$) as follows:

$$\mathcal{C}'_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)=\mathcal{C}'_{\nu_l}(b)+\mathcal{C}''_{\nu_l}(b)\frac{\varepsilon b}{1-\varepsilon}+\cdots+\frac{\mathcal{C}^{(n)}_{\nu_l}(b)}{(n-1)!}\left(\frac{\varepsilon b}{1-\varepsilon}\right)^{n-1}+o\left(\frac{\varepsilon b}{1-\varepsilon}\right)^{n-1}.\quad(2.7)$$

https://doi.org/10.1017/S0308210516000214 Published online by Cambridge University Press

Then, using (2.7) with $n = 4$ for J'_{ν_l} and Y'_{ν_l} we get

$$\begin{aligned}\frac{J_{\nu_l}(a)}{J'_{\nu_l}(a)}\bigg[&\frac{\varepsilon b}{1-\varepsilon}(Y'_{\nu_l}(b)J''_{\nu_l}(b) - J'_{\nu_l}(b)Y''_{\nu_l}(b)) \\ &+ \frac{\varepsilon^2 b^2}{2(1-\varepsilon)^2}(Y'_{\nu_l}(b)J'''_{\nu_l}(b) - J'_{\nu_l}(b)Y'''_{\nu_l}(b)) \\ &+ \frac{\varepsilon^3 b^3}{6(1-\varepsilon)^3}(Y'_{\nu_l}(b)J^{(\mathrm{iv})}_{\nu_l}(b) - J'_{\nu_l}(b)Y^{(\mathrm{iv})}_{\nu_l}(b)) + R_1(b)\bigg] \\ + \frac{a}{b}\bigg[&(J_{\nu_l}(b)Y'_{\nu_l}(b) - Y_{\nu_l}(b)J'_{\nu_l}(b)) \\ &+ \frac{\varepsilon b}{1-\varepsilon}(J_{\nu_l}(b)Y''_{\nu_l}(b) - Y_{\nu_l}(b)J''_{\nu_l}(b)) \\ &+ \frac{\varepsilon^2 b^2}{2(1-\varepsilon)^2}(J_{\nu_l}(b)Y'''_{\nu_l}(b) - Y_{\nu_l}(b)J'''_{\nu_l}(b)) + R_2(b)\bigg], \end{aligned} \tag{2.8}$$

where

$$R_1(b) = \sum_{k=4}^{+\infty} \frac{\varepsilon^k b^k}{k!(1-\varepsilon)^k}(Y'_{\nu_l}(b)J^{(k+1)}_{\nu_l}(b) - J'_{\nu_l}(b)Y^{(k+1)}_{\nu_l}(b)) \tag{2.9}$$

and

$$R_2(b) = \sum_{k=3}^{+\infty} \frac{\varepsilon^k b^k}{k!(1-\varepsilon)^k}(J_{\nu_l}(b)Y^{(k+1)}_{\nu_l}(b) - Y_{\nu_l}(b)J^{(k+1)}_{\nu_l}(b)). \tag{2.10}$$

Let R_3 be the remainder defined in lemma 2.6. We set

$$\begin{aligned} R(\lambda,\varepsilon) = R_3(a)\bigg[&\frac{\varepsilon b}{1-\varepsilon}(Y'_{\nu_l}(b)J''_{\nu_l}(b) - J'_{\nu_l}(b)Y''_{\nu_l}(b)) \\ &+ \frac{\varepsilon^2 b^2}{2(1-\varepsilon)^2}(Y'_{\nu_l}(b)J'''_{\nu_l}(b) - J'_{\nu_l}(b)Y'''_{\nu_l}(b)) \\ &+ \frac{\varepsilon^3 b^3}{6(1-\varepsilon)^3}(Y'_{\nu_l}(b)J^{(\mathrm{iv})}_{\nu_l}(b) - J'_{\nu_l}(b)Y^{(\mathrm{iv})}_{\nu_l}(b))\bigg] \\ &+ R_1(b)\bigg[\frac{a}{\nu_l} + \frac{a^3}{2{\nu_l}^2(1+\nu_l)}\bigg] + R_2(b)\frac{a}{b} + R_3(a)R_1(b). \end{aligned} \tag{2.11}$$

By lemma 2.7, it turns out that $R(\lambda,\varepsilon) = O(\varepsilon^3)$ as $\varepsilon \to 0$.

We also set

$$\begin{aligned} f(\varepsilon) &= b_1^2(\varepsilon)a_1^3(\varepsilon)f_1(\varepsilon), \\ g(\varepsilon) &= b_1^2(\varepsilon)a_1(\varepsilon)g_1(\varepsilon) + a_1^3(\varepsilon)g_2(\varepsilon), \\ h(\varepsilon) &= a_1(\varepsilon)h_1(\varepsilon) + \varepsilon^2\frac{a_1^3(\varepsilon)}{b_1^2(\varepsilon)}h_2(\varepsilon), \\ k(\varepsilon) &= \frac{a_1(\varepsilon)}{b_1^2(\varepsilon)}k_1(\varepsilon), \end{aligned}$$

where

$$a_1(\varepsilon) = \frac{a}{\sqrt{\lambda\varepsilon}} = (1-\varepsilon),$$

$$b_1(\varepsilon) = b\sqrt{\frac{\varepsilon}{\lambda}},$$

$$f_1(\varepsilon) = \frac{1}{6{\nu_l}^2(1+\nu_l)(1-\varepsilon)^3},$$

$$g_1(\varepsilon) = \frac{1}{3\nu_l(1-\varepsilon)^3},$$

$$g_2(\varepsilon) = -\frac{1}{{\nu_l}^2(1+\nu_l)(1-\varepsilon)} + \frac{\varepsilon}{2{\nu_l}^2(1+\nu_l)(1-\varepsilon)^2} - \frac{\varepsilon^2(3+2{\nu_l}^2)}{6{\nu_l}^2(1+\nu_l)(1-\varepsilon)^3},$$

$$h_1(\varepsilon) = -\frac{2}{\nu_l(1-\varepsilon)} + \frac{\varepsilon}{\nu_l(1-\varepsilon)^2} - \frac{\varepsilon^2(3+2{\nu_l}^2)}{3\nu_l(1-\varepsilon)^3} - \frac{\varepsilon}{(1-\varepsilon)^2},$$

$$h_2(\varepsilon) = \frac{1}{(1+\nu_l)(1-\varepsilon)} - \frac{3\varepsilon}{2(1+\nu_l)(1-\varepsilon)^2} + \frac{\varepsilon^2({\nu_l}^4+11{\nu_l}^2)}{6{\nu_l}^2(1+\nu_l)(1-\varepsilon)^3},$$

$$k_1(\varepsilon) = 2 + \frac{2\varepsilon\nu_l}{1-\varepsilon} - \frac{3\varepsilon^2\nu_l}{(1-\varepsilon)^2} + \frac{\varepsilon^3({\nu_l}^4+11{\nu_l}^2)}{3\nu_l(1-\varepsilon)^3} - \frac{2\varepsilon}{1-\varepsilon} + \frac{\varepsilon^2(2+{\nu_l}^2)}{(1-\varepsilon)^2}.$$

Note that functions f, g, h, k are continuous at $\varepsilon = 0$ and $f(0), g(0), h(0), k(0) \neq 0$.

Using in (2.8) the explicit formulae for the cross products of Bessel functions given by lemma A.2 and corollary A.3, (2.6) can be written as

$$\frac{1}{\sqrt{\lambda}\pi}\varepsilon\sqrt{\varepsilon}k(\varepsilon) + \frac{\sqrt{\lambda}}{\pi}\varepsilon\sqrt{\varepsilon}h(\varepsilon) + \frac{\lambda\sqrt{\lambda}}{\pi}\varepsilon^2\sqrt{\varepsilon}g(\varepsilon) + \frac{\lambda^2\sqrt{\lambda}}{\pi}\varepsilon^3\sqrt{\varepsilon}f(\varepsilon) + R(\lambda,\varepsilon). \quad (2.12)$$

Step 2. We consider the quantity $P_1(a,b)/J'_{\nu_l}(a)$, i.e.

$$\frac{J_{\nu_l}(a)}{J'_{\nu_l}(a)}\left[Y'_{\nu_l}(b)J_{\nu_l}\left(\frac{b}{1-\varepsilon}\right) - J'_{\nu_l}(b)Y_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)\right] + \frac{a}{b}\left[J_{\nu_l}(b)Y_{\nu_l}\left(\frac{b}{1-\varepsilon}\right) - Y_{\nu_l}(b)J_{\nu_l}\left(\frac{b}{1-\varepsilon}\right)\right]. \quad (2.13)$$

Proceeding as in step 1 and setting

$$\tilde{f}(\varepsilon) = -\frac{a_1^3(\varepsilon)b_1(\varepsilon)}{2\pi{\nu_l}^2(1+\nu_l)(1-\varepsilon)^2},$$

$$\tilde{g}(\varepsilon) = \frac{a_1^3(\varepsilon)}{b_1(\varepsilon)}\left(\frac{1}{\pi{\nu_l}^2(1+\nu_l)} + \frac{\varepsilon^2}{2\pi(1+\nu_l)(1-\varepsilon)^2}\right) - \frac{a_1(\varepsilon)b_1(\varepsilon)}{\nu_l\pi(1-\varepsilon)^2},$$

$$\tilde{h}(\varepsilon) = \frac{a_1(\varepsilon)}{b_1(\varepsilon)}\left(\frac{2}{\nu_l\pi} + \frac{2\varepsilon}{\pi(1-\varepsilon)} + \frac{(\nu_l-1)}{\pi(1-\varepsilon)^2}\varepsilon^2\right),$$

(2.13) can be written as

$$\varepsilon\tilde{h}(\varepsilon) + \lambda\varepsilon^2\tilde{g}(\varepsilon) + \lambda^2\varepsilon^3\tilde{f}(\varepsilon) + \hat{R}(\lambda,\varepsilon), \quad (2.14)$$

where $\hat{R}(\lambda,\varepsilon) = O(\varepsilon^2\sqrt{\varepsilon})$ as $\varepsilon \to 0$ (see lemma 2.7).

STEP 3. We combine (2.12) and (2.14) and rewrite (2.4) in the form

$$\varepsilon\left(1-\frac{N}{2}\right)\tilde{h}(\varepsilon)+\varepsilon\frac{b_1(\varepsilon)k(\varepsilon)}{\pi(1-\varepsilon)}+\lambda\varepsilon^2\left(1-\frac{N}{2}\right)\tilde{g}(\varepsilon)+\lambda\varepsilon\frac{b_1(\varepsilon)h(\varepsilon)}{\pi(1-\varepsilon)} + \lambda^2\varepsilon^3\left(1-\frac{N}{2}\right)\tilde{f}(\varepsilon)+\lambda^2\varepsilon^2\frac{b_1(\varepsilon)g(\varepsilon)}{\pi(1-\varepsilon)}+\lambda^3\varepsilon^3\frac{b_1(\varepsilon)f(\varepsilon)}{\pi(1-\varepsilon)}+\mathcal{R}_0(\lambda,\varepsilon)=0, \quad (2.15)$$

where

$$\mathcal{R}_0(\lambda,\varepsilon)=\frac{\sqrt{\lambda}b_1(\varepsilon)}{(1-\varepsilon)\sqrt{\varepsilon}}R(\lambda,\varepsilon)+\left(1-\frac{N}{2}\right)\hat{R}(\lambda,\varepsilon).$$

Note that $\mathcal{R}_0(\lambda,\varepsilon)=O(\varepsilon^2\sqrt{\varepsilon})$ as $\varepsilon\to 0$. Dividing by ε in (2.15) and setting $\mathcal{R}_1(\lambda,\varepsilon)=\mathcal{R}_0(\lambda,\varepsilon)/\varepsilon$, we obtain

$$\left(1-\frac{N}{2}\right)\tilde{h}(\varepsilon)+\frac{b_1(\varepsilon)k(\varepsilon)}{\pi(1-\varepsilon)}+\lambda\varepsilon\left(1-\frac{N}{2}\right)\tilde{g}(\varepsilon)+\lambda\frac{b_1(\varepsilon)h(\varepsilon)}{\pi(1-\varepsilon)} + \lambda^2\varepsilon^2\left(1-\frac{N}{2}\right)\tilde{f}(\varepsilon)+\lambda^2\varepsilon\frac{b_1(\varepsilon)g(\varepsilon)}{\pi(1-\varepsilon)}+\lambda^3\varepsilon^2\frac{b_1(\varepsilon)f(\varepsilon)}{\pi(1-\varepsilon)}+\mathcal{R}_1(\lambda,\varepsilon)=0. \quad (2.16)$$

We now multiply (2.16) by $\pi\nu_l(1-\varepsilon)/b_1(\varepsilon)$, which is a positive quantity for all $0<\varepsilon<1$. Taking into account the definitions of functions g, h, k, $\tilde{g}$ and $\tilde{h}$, we can finally rewrite (2.16) in the form

$$\lambda^2\varepsilon\left(\frac{\hat{\rho}(\varepsilon)}{3}-\frac{1}{\nu_l(1+\nu_l)}\right)+\lambda\varepsilon\left(\frac{N}{2}-\nu_l+\frac{2-N}{2\nu_l(1+\nu_l)\hat{\rho}(\varepsilon)}\right)-2\lambda + \frac{2l(1+\varepsilon\nu_l)}{\hat{\rho}(\varepsilon)}+\mathcal{R}(\lambda,\varepsilon)=0, \quad (2.17)$$

where

$$\hat{\rho}(\varepsilon)=\varepsilon\tilde{\rho}(\varepsilon)=(M-\omega_N\varepsilon(1-\varepsilon)^N)\left(\omega_N\left(N-\frac{N}{2}(N-1)\varepsilon-\sum_{k=3}^{N}\binom{N}{k}(-1)^k\varepsilon^{k-1}\right)\right)^{-1},$$

and $\mathcal{R}(\lambda,\varepsilon)=O(\varepsilon\sqrt{\varepsilon})$ as $\varepsilon\to 0$. The formulation in (2.5) can easily be deduced by observing that

$$\hat{\rho}_\varepsilon=\frac{M}{N\omega_N}+2\frac{M}{N\omega_N}\left(\frac{N-1}{4}-\frac{\omega_N}{2M}\right)\varepsilon+O(\varepsilon^2)\quad\text{as } \varepsilon\to 0.$$

□

We are now ready to prove our main result.

THEOREM 2.4. *All eigenvalues of (1.4) have the following asymptotic behaviour:*

$$\lambda_l(\varepsilon)=\lambda_l+\left(\frac{2l\lambda_l}{3}+\frac{2\lambda_l^2}{N(2l+N)}\right)\varepsilon+o(\varepsilon)\quad\text{as } \varepsilon\to 0, \quad (2.18)$$

where λ_l are the eigenvalues of (1.1).

Moreover, for each $l\in\mathbb{N}$, the function defined by $\lambda_l(\varepsilon)$ for $\varepsilon>0$ and $\lambda_l(0)=\lambda_l$ is continuous in the whole of $[0,1[$ and of class C^1 in a neighbourhood of $\varepsilon=0$.

Proof. By using the min–max principle and related standard arguments, one can easily prove that $\lambda_l(\varepsilon)$ depends on $\varepsilon > 0$ with continuity (see [14,16]). Moreover, by using (1.5), the maps $\varepsilon \mapsto \lambda_l(\varepsilon)$ can be extended by continuity at the point $\varepsilon = 0$ by setting $\lambda_l(0) = \lambda_l$.

In order to prove the differentiability of $\lambda_l(\varepsilon)$ around zero and the validity of (2.18), we consider (2.5) and apply the implicit function theorem. Note that (2.5) can be written in the form $F(\lambda, \varepsilon) = 0$, where F is a function of class C^1 in the variables $(\lambda, \varepsilon) \in \,]0, \infty[\, \times [0, 1[$, with

$$\left.\begin{aligned} F(\lambda,0) &= -2\lambda + \frac{2N\omega_N l}{M}, \qquad F'_\lambda(\lambda,0) = -2,\\ F'_\varepsilon(\lambda,0) &= \lambda^2\left(\frac{M}{3N\omega_N} - \frac{1}{\nu_l(1+\nu_l)}\right) + \lambda\left(\frac{N}{2} - \nu_l + \frac{(2-N)N\omega_N}{2\nu_l(1+\nu_l)M}\right)\\ &\qquad - \frac{2N\omega_N l}{M}\left(\frac{N-1}{2} - \frac{\omega_N}{M} - \nu_l\right). \end{aligned}\right\} \tag{2.19}$$

By (1.2), $\lambda_l = N\omega_N l/M$. Hence, $F(\lambda_l, 0) = 0$. Since $F'_\lambda(\lambda_l, 0) \neq 0$, the implicit function theorem combined with the continuity of the functions $\lambda_l(\cdot)$ allows us to conclude that functions $\lambda_l(\cdot)$ are of class C^1 around zero.

We now compute the derivative of $\lambda_l(\cdot)$ at zero. Using the equality $N\omega_N/M = \lambda_l/l$ and recalling that $\nu_l = l + N/2 - 1$, we get

$$\begin{aligned} F'_\varepsilon(\lambda_l,0) &= \lambda_l^2\left(\frac{l}{3\lambda_l} - \frac{1}{\nu_l(1+\nu_l)}\right) + \lambda_l\left(1 - l + \frac{\lambda_l(2-N)}{2l\nu_l(1+\nu_l)}\right) - 2\lambda_l\left(\frac{1}{2} - l - \frac{\lambda_l}{Nl}\right)\\ &= \lambda_l^2\left(\frac{1}{\nu_l(1+\nu_l)}\left(\frac{2-N}{2l} - 1\right) + \frac{2}{Nl}\right) + \frac{4\lambda_l l}{3}\\ &= \frac{4\lambda_l^2}{N^2 + 2Nl} + \frac{4\lambda_l l}{3}. \end{aligned}$$

Finally, the formula

$$\lambda_l'(0) = -\frac{F'_\varepsilon(\lambda_l,0)}{F'_\lambda(\lambda_l,0)}$$

yields (1.6) and the validity of (2.18). □

Corollary 2.5. *For any $l \in \mathbb{N} \setminus \{0\}$ there exists δ_l such that the function $\lambda_l(\cdot)$ is strictly increasing in the interval $[0, \delta_l[$. In particular, $\lambda_l < \lambda_l(\varepsilon)$ for all $\varepsilon \in \,]0, \delta_l[$.*

2.1. Estimates for the remainders

This subsection is devoted to the proof of a few technical estimates used in the proof of lemma 2.3.

Lemma 2.6. *The function R_3 defined by*

$$\frac{J_\nu(z)}{J'_\nu(z)} = \frac{z}{\nu} + \frac{z^3}{2\nu^2(1+\nu)} + R_3(z), \tag{2.20}$$

is $O(z^5)$ as $z \to 0$.

https://doi.org/10.1017/S0308210516000214 Published online by Cambridge University Press

Proof. Recall the well-known following representation of the Bessel functions of the first species:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{+\infty} \frac{(-1)^j}{j!\Gamma(j+\nu+1)} \left(\frac{z}{2}\right)^{2j}. \tag{2.21}$$

For clarity, we simply write

$$J_\nu(z) = z^\nu(a_0 + a_2 z^2 + a_4 z^4 + O(z^5)). \tag{2.22}$$

Hence,

$$J'_\nu(z) = z^{\nu-1}(\nu a_0 + (\nu+2)a_2 z^2 + (\nu+4)a_4 z^4 + O(z^5)), \tag{2.23}$$

where the coefficients a_0, a_2, a_4 are defined by (2.21). By (2.22), (2.23) and standard computations it follows that

$$\frac{J_\nu(z)}{J'_\nu(z)} = \frac{z}{\nu} - \frac{2a_2}{\nu^2 a_0} z^3 + O(z^5),$$

which gives (2.20) exactly. □

Lemma 2.7. *For any $\lambda > 0$ the remainders $R(\lambda, \varepsilon)$ and $\hat{R}(\lambda, \varepsilon)$ defined in the proof of lemma 2.3 are $O(\varepsilon^3)$ and $O(\varepsilon^2\sqrt{\varepsilon})$, respectively, as $\varepsilon \to 0$. Moreover, the same holds true for the corresponding partial derivatives $\partial_\lambda R(\lambda, \varepsilon)$, $\partial_\lambda \hat{R}(\lambda, \varepsilon)$.*

Proof. First, we consider $R_3(a) = R_3(\sqrt{\lambda\varepsilon}(1-\varepsilon))$, where R_3 is defined in lemma 2.6, and we differentiate it with respect to λ. We obtain

$$\frac{\partial R_3(a)}{\partial \lambda} = \frac{a R'_3(a)}{2\lambda}.$$

Hence, by lemma 2.6 we can conclude that $R_3(a)$ and $\partial R_3(a)/\partial\lambda$ are $O(\varepsilon^2\sqrt{\varepsilon})$ as $\varepsilon \to 0$.

Now consider $R_1(b)$ and $R_2(b)$ defined in (2.9) and (2.10). Since $\lambda > 0$, we have that $b > 0$. Hence, the Bessel functions are analytic in b and we can write

$$\begin{aligned} 2\sqrt{\lambda}\frac{\partial R_1(b)}{\partial\lambda} &= \frac{\varepsilon b_1(\varepsilon)}{\sqrt{\varepsilon}(1-\varepsilon)} \sum_{k=4}^{+\infty} \frac{b^{k-1}\varepsilon^{k-1}}{(k-1)!(1-\varepsilon)^{k-1}} (Y'_\nu(b)J_\nu^{(k+1)}(b) - J'_\nu(b)Y_\nu^{(k+1)}(b)) \\ &\quad + \frac{b_1(\varepsilon)}{\sqrt{\varepsilon}} \sum_{k=4}^{+\infty} \frac{\varepsilon^k b^k}{k!(1-\varepsilon)^k} (Y'_\nu(b)J_\nu^{(k+1)}(b) - J'_\nu(b)Y_\nu^{(k+1)}(b))'. \end{aligned}$$

Here and in the following we write ν instead of ν_l. Using the fact that $b = \sqrt{\lambda/\varepsilon}\, b_1(\varepsilon)$ and lemma A.2, we conclude that all the cross products of the form

$$Y'_\nu(b)J_\nu^{(k+1)}(b) - J'_\nu(b)Y_\nu^{(k+1)}(b)$$

and their derivatives

$$(Y'_\nu(b)J_\nu^{(k+1)}(b) - J'_\nu(b)Y_\nu^{(k+1)}(b))'$$

are $O(\sqrt{\varepsilon})$ and $O(\varepsilon)$, respectively, as $\varepsilon \to 0$. It follows that $R_1(b)$ and $\partial_\lambda R_1(b)$ are $O(\varepsilon^2\sqrt{\varepsilon})$ as $\varepsilon \to 0$.

Similarly,

$$2\sqrt{\lambda}\frac{\partial R_2(b)}{\partial\lambda} = \frac{\varepsilon b_1(\varepsilon)}{\sqrt{\varepsilon}(1-\varepsilon)}\sum_{k=3}^{+\infty}\frac{b^{k-1}\varepsilon^{k-1}}{(k-1)!(1-\varepsilon)^{k-1}}(J_\nu(b)Y_\nu^{(k+1)}(b) - Y_\nu(b)J_\nu^{(k+1)}(b)) + \frac{b_1(\varepsilon)}{\sqrt{\varepsilon}}\sum_{k=3}^{+\infty}\frac{\varepsilon^k b^k}{k!(1-\varepsilon)^k}(J_\nu(b)Y_\nu^{(k+1)}(b) - Y_\nu(b)J_\nu^{(k+1)}(b))'.$$

Hence, $R_2(b)$ and $\partial_\lambda R_2(b)$ are $O(\varepsilon^2)$ as $\varepsilon \to 0$.

Summing all the terms, and using lemma A.1 and corollary A.3, we obtain

$$\begin{aligned} R(\lambda,\varepsilon) = R_3(a)\bigg[&\frac{2\varepsilon}{\pi(1-\varepsilon)}\left(\frac{\nu^2}{b^2}-1\right) + \frac{\varepsilon^2}{\pi(1-\varepsilon)^2}\left(1-\frac{3\nu^2}{b^2}\right) \\ &+ \frac{\varepsilon^3 b^2}{3\pi(1-\varepsilon)^3}\left(\frac{\nu^4+11\nu^2}{b^4} - \frac{3+2\nu^2}{b^2}+1\right)\bigg] \\ &+ R_1(b)\left[\frac{a}{\nu} + \frac{a^3}{2\nu^2(1+\nu)}\right] + R_2(b)\frac{a}{b} + R_3(a)R_1(b). \end{aligned}$$

We conclude that $R(\lambda,\varepsilon)$ is $O(\varepsilon^3)$ as $\varepsilon \to 0$. Moreover, it easily follows that $\partial R(\lambda,\varepsilon)/\partial\lambda$ is also $O(\varepsilon^3)$ as $\varepsilon \to 0$.

The proof of the estimates for $\hat{R}$ and its derivatives is similar and we omit it. □

Remark 2.8. According to standard Landau notation, saying that a function $f(z)$ is $O(g(z))$ as $z \to 0$ means that there exists $C > 0$ such that $|f(z)| \leqslant C|g(z)|$ for any z sufficiently close to zero. Thus, using Landau's notation in the statements of lemmas 2.3 and 2.7 indicates the existence of such constants C, which in principle may depend on $\lambda > 0$. However, a careful analysis of the proofs reveals that, given a bounded interval of the type $[A, B]$ with $0 < A < B$, the appropriate constants C in the estimates can be taken to be independent of $\lambda \in [A, B]$.

2.2. The case $N = 1$

We include here a description of the case $N = 1$ for completeness. In this section, the ball B will be the open interval $]-1, 1[$. Problem (1.1) reads

$$\left.\begin{aligned} u''(x) &= 0 \quad \text{for } x \in\,]-1,1[, \\ u'(\pm 1) &= \pm\tfrac{1}{2}M\lambda u(\pm 1), \end{aligned}\right\} \tag{2.24}$$

in the unknowns λ and u. It is easy to see that the only eigenvalues are $\lambda_0 = 0$ and $\lambda_1 = 2/M$ and they are associated with the constant functions and the function x, respectively. As in (1.3), we define a mass density ρ_ε on the whole of $]-1, 1[$ by

$$\rho_\varepsilon(x) = \begin{cases} \dfrac{M}{2\varepsilon} - 1 + \varepsilon & \text{if } x \in\,]-1,-1+\varepsilon[\,\cup\,]1-\varepsilon,1[, \\ \varepsilon & \text{if } x \in\,]-1+\varepsilon, 1-\varepsilon[. \end{cases}$$

Note that for any $x \in\,]-1, 1[$ we have $\rho_\varepsilon(x) \to 0$ as $\varepsilon \to 0$, and

$$\int_{-1}^{1} \rho_\varepsilon \,\mathrm{d}x = M \quad \text{for all } \varepsilon > 0.$$

https://doi.org/10.1017/S0308210516000214 Published online by Cambridge University Press

Problem (1.4) for $N = 1$ reads

$$\left.\begin{array}{ll} -u''(x) = \lambda\rho_\varepsilon(x)u(x) & \text{for } x \in]-1,1[, \\ u'(-1) = u'(1) = 0. & \end{array}\right\} \tag{2.25}$$

It is well known from Sturm–Liouville theory that problem (2.25) has an increasing sequence of non-negative eigenvalues of multiplicity 1. We denote the eigenvalues of (2.25) by $\lambda_l(\varepsilon)$ with $l \in \mathbb{N}$. For any $\varepsilon \in]0,1[$, the only zero eigenvalue is $\lambda_0(\varepsilon)$, and the corresponding eigenfunctions are the constant functions.

We establish an implicit characterization of the eigenvalues of (2.25).

PROPOSITION 2.9. *The non-zero eigenvalues λ of problem (2.25) are given implicitly as zeros of*

$$\begin{aligned} &2\sqrt{\varepsilon\left(\frac{M}{2\varepsilon}-1+\varepsilon\right)}\cos(2\sqrt{\lambda\varepsilon}(1-\varepsilon))\sin\left(2\varepsilon\sqrt{\lambda\left(\frac{M}{2\varepsilon}-1+\varepsilon\right)}\right) \\ &\quad+\left[-\frac{M}{2\varepsilon}+1+\left(\frac{M}{2\varepsilon}-1+2\varepsilon\right)\cos\left(2\varepsilon\sqrt{\lambda\left(\frac{M}{2\varepsilon}-1+\varepsilon\right)}\right)\right]\sin(2\sqrt{\lambda\varepsilon}(1-\varepsilon))=0. \end{aligned} \tag{2.26}$$

Proof. Given an eigenvalue $\lambda > 0$, a solution of (2.25) is of the form

$$u(x) = \begin{cases} A\cos(\sqrt{\lambda\rho_2}x) + B\sin(\sqrt{\lambda\rho_2}x) & \text{for } x \in]-1,-1+\varepsilon[, \\ C\cos(\sqrt{\lambda\rho_1}x) + D\sin(\sqrt{\lambda\rho_1}x) & \text{for } x \in]-1+\varepsilon,1-\varepsilon[, \\ E\cos(\sqrt{\lambda\rho_2}x) + F\sin(\sqrt{\lambda\rho_2}x) & \text{for } x \in]1-\varepsilon,1[, \end{cases}$$

where

$$\rho_1 = \varepsilon, \rho_2 = \frac{M}{2\varepsilon} - 1 + \varepsilon$$

and A, B, C, D, E, F are suitable real numbers. We impose the continuity of u and u' at the points $x = -1+\varepsilon$ and $x = 1-\varepsilon$ and the boundary conditions, obtaining a homogeneous system of six linear equations of the form $\mathcal{M}v = 0$ in six unknowns, where $v = (A,B,C,D,E,F)$ and $\mathcal{M}$ is the matrix associated with the system. We impose the condition $\det\mathcal{M} = 0$. This yields (2.26). □

Note that $\lambda = 0$ is a solution for all $\varepsilon > 0$. We thus consider only the case of non-zero eigenvalues. Using standard Taylor formulae, we easily prove the following.

LEMMA 2.10. *Equation (2.26) can be rewritten in the form*

$$M - \frac{\lambda M^2}{2} + \frac{\lambda M^2}{6}\left(1+\lambda\left(2+\frac{M}{2}\right)\right)\varepsilon + R(\lambda,\varepsilon) = 0, \tag{2.27}$$

where $R(\lambda,\varepsilon) = O(\varepsilon^2)$ as $\varepsilon \to 0$.

Finally, we can prove the following theorem. Note that (2.28) is the same as (2.18) with $N = 1$, $l = 1$.

Theorem 2.11. *The first eigenvalue of problem (2.25) has the following asymptotic behaviour:*

$$\lambda_1(\varepsilon) = \lambda_1 + \tfrac{2}{3}(\lambda_1 + \lambda_1^2)\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \to 0, \tag{2.28}$$

where $\lambda_1 = 2/M$ *is the only non-zero eigenvalue of problem (2.24). Moreover, for* $l > 1$ *we have that* $\lambda_l(\varepsilon) \to +\infty$ *as* $\varepsilon \to 0$.

Proof. The proof is similar to that of theorem 2.4. It is possible to prove that the eigenvalues $\lambda_l(\varepsilon)$ of (2.25) depend with continuity on $\varepsilon > 0$. We consider (2.27) and apply the implicit function theorem. Equation (2.27) can be written in the form $F(\lambda, \varepsilon) = 0$, with F of class C^1 in $]0, +\infty[\times [0, 1[$ with $F(\lambda, 0) = M - \frac{1}{2}\lambda M^2$, $F'_\lambda(\lambda, 0) = -\frac{1}{2}M^2$ and $F'_\varepsilon(\lambda, 0) = \frac{1}{6}\lambda M^2(1 + \lambda(2 + \frac{1}{2}M))$.

Since $\lambda_1 = 2/M$, $F(\lambda_1, 0) = 0$ and $F'_\lambda(\lambda_1, 0) \neq 0$, the zeros of (2.28) in a neighbourhood of $(\lambda, 0)$ are given by the graph of a C^1-function $\varepsilon \mapsto \lambda(\varepsilon)$ with $\lambda(0) = \lambda_1$. We note that $\lambda(\varepsilon) = \lambda_1(\varepsilon)$ for all ε small enough. Indeed, assuming by contradiction that $\lambda(\varepsilon) = \lambda_l(\varepsilon)$ with $l \geqslant 2$, we would obtain, possibly passing to a subsequence, that $\lambda_1(\varepsilon) \to \bar{\lambda}$ as $\varepsilon \to 0$, for some $\bar{\lambda} \in [0, \lambda_1[$. Then, passing to the limit in (2.27) as $\varepsilon \to 0$, we would obtain a contradiction. Thus, $\lambda_1(\cdot)$ is of class C^1 in a neighbourhood of zero and $\lambda_1'(0) = -F'_\varepsilon(\lambda_1, 0)/F'_\lambda(\lambda_1, 0)$, which yields (2.28).

The divergence of the higher eigenvalues $\lambda_l(\varepsilon)$ with $l > 1$ as $\varepsilon \to 0$ is clearly deduced by the fact that the existence of a converging subsequence of the form $\lambda_l(\varepsilon_n)$, $n \in \mathbb{N}$, would provide the existence of an eigenvalue different from λ_0 and λ_1 for the limiting problem (2.24), which is not admissible. □

Appendix A.

Here we provide explicit formulae for the cross products of the Bessel functions used in this paper.

Lemma A.1. *The following identities hold:*

$$\begin{aligned} Y_\nu(z)J'_\nu(z) - J_\nu(z)Y'_\nu(z) &= -\frac{2}{\pi z}, \\ Y_\nu(z)J''_\nu(z) - J_\nu(z)Y''_\nu(z) &= \frac{2}{\pi z^2}, \\ Y'_\nu(z)J''_\nu(z) - J'_\nu(z)Y''_\nu(z) &= \frac{2}{\pi z}\left(\frac{\nu^2}{z^2} - 1\right). \end{aligned}$$

Proof. It is well known (see [1, § 9]) that

$$J_\nu(z)Y'_\nu(z) - Y_\nu(z)J'_\nu(z) = J_{\nu+1}(z)Y_\nu(z) - J_\nu(z)Y_{\nu+1}(z) = \frac{2}{\pi z},$$

which gives the first identity in the statement. The second identity holds since

$$J_\nu(z)Y''_\nu(z) - Y_\nu(z)J''_\nu(z) = (J_\nu(z)Y'_\nu(z) - Y_\nu(z)J'_\nu(z))' = \left(\frac{2}{\pi z}\right)' = -\frac{2}{\pi z^2}.$$

https://doi.org/10.1017/S0308210516000214 Published online by Cambridge University Press

The third identity holds since

$$\begin{aligned}
&Y_\nu'(z)J_\nu''(z) - J_\nu'(z)Y_\nu''(z)\\
&\quad= Y_\nu'(z)\left(J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z)\right)' - J_\nu'(z)\left(Y_{\nu-1}(z) - \frac{\nu}{z}Y_\nu(z)\right)'\\
&\quad= Y_\nu'(z)J_{\nu-1}'(z) - J_\nu'(z)Y_{\nu-1}'(z) + \frac{\nu}{z^2}(Y_\nu'(z)J_\nu(z) - J_\nu'(z)Y_\nu(z))\\
&\quad= (Y_\nu'(z)\tfrac{1}{2}(J_{\nu-2}(z) - J_\nu(z)) - J_\nu'(z)\tfrac{1}{2}(Y_{\nu-2}(z) - Y_\nu(z))) + \frac{2\nu}{\pi z^3}\\
&\quad= \tfrac{1}{2}(Y_\nu'(z)J_{\nu-2}(z) - J_\nu'(z)Y_{\nu-2}(z))\\
&\qquad\quad - \tfrac{1}{2}(Y_\nu'(z)J_\nu(z) - J_\nu'(z)Y_\nu(z)) + \frac{2\nu}{\pi z^3}\\
&\quad= \tfrac{1}{2}(J_\nu'(z)Y_\nu(z) - Y_\nu'(z)J_\nu(z))\\
&\qquad\quad + \frac{\nu-1}{z}(Y_\nu'(z)J_{\nu-1}(z) - J_\nu'(z)Y_{\nu-1}(z)) - \frac{1}{\pi z} + \frac{2\nu}{\pi z^3}\\
&\quad= \frac{\nu-1}{z}\Bigg(J_{\nu-1}(z)\left(Y_{\nu-1}(z) - \frac{\nu}{z}Y_\nu(z)\right)\\
&\qquad\qquad\qquad - Y_{\nu-1}(z)\left(J_{\nu-1}(z) - \frac{\nu}{z}J_\nu(z)\right)\Bigg) - \frac{2}{\pi z} + \frac{2\nu}{\pi z^3}\\
&\quad= -\frac{\nu(\nu-1)}{z^2}(Y_\nu(z)J_{\nu-1}(z) - J_\nu(z)Y_{\nu-1}(z)) - \frac{2}{\pi z} + \frac{2\nu}{\pi z^3}\\
&\quad= \frac{2}{\pi z}\left(-1 + \frac{\nu^2}{z^2}\right),
\end{aligned}$$

where the first, second and fourth equalities, respectively, follow from the well-known formulae

$$\begin{aligned}
\mathcal{C}_\nu'(z) &= \mathcal{C}_{\nu-1}(z) - \frac{\nu}{z}\mathcal{C}_\nu(z),\\
2\mathcal{C}_\nu'(z) &= \mathcal{C}_{\nu-1}(z) - \mathcal{C}_{\nu+1}(z),\\
\mathcal{C}_{\nu-2}(z) + \mathcal{C}_\nu(z) &= \frac{2(\nu-1)}{z}\mathcal{C}_{\nu-1}(z),
\end{aligned}$$

where $\mathcal{C}_\nu(z)$ stands for both $J_\nu(z)$ and $Y_\nu(z)$ (see [1, § 9]). This proves the lemma. □

Lemma A.2. *The identities*

$$Y_\nu(z)J_\nu^{(k)}(z) - J_\nu(z)Y_\nu^{(k)}(z) = \frac{2}{\pi z}(r_k + R_{\nu,k}(z)), \tag{A 1}$$

$$Y_\nu'(z)J_\nu^{(k)}(z) - J_\nu'(z)Y_\nu^{(k)}(z) = \frac{2}{\pi z}(q_k + Q_{\nu,k}(z)) \tag{A 2}$$

hold for all $k > 2$ and $\nu \geqslant 0$, where $r_k, q_k \in \{0, 1, -1\}$, and $Q_{\nu,k}(z)$, $R_{\nu,k}(z)$ are finite sums of quotients of the form $c_{\nu,k}/z^m$, with $m \geqslant 1$ and $c_{\nu,k}$ a suitable constant depending on ν, k.

Proof. We shall prove (A 1) and (A 2) by induction. Identities (A 1) and (A 2) hold for $k = 1$ and $k = 2$ by lemma A.1. Suppose now that

$$Y_\nu(z)J_\nu^{(k)}(z) - J_\nu(z)Y_\nu^{(k)}(z) = \frac{2}{\pi z}(r_k + R_{\nu,k}(z)),$$
$$Y_\nu'(z)J_\nu^{(k)}(z) - J_\nu'(z)Y^{(k)}(z) = \frac{2}{\pi z}(q_k + Q_{\nu,k}(z))$$

hold for all $\nu \geqslant 0$. First consider

$$Y_\nu'(z)J_\nu^{(k+1)}(z) - J_\nu'(z)Y_\nu^{(k+1)}(z).$$

We use the recurrence relations

$$\mathcal{C}_{\nu+1}(z) + \mathcal{C}_{\nu-1}(z) = \frac{2\nu}{z}\mathcal{C}_\nu(z) \quad \text{and} \quad 2\mathcal{C}'(z) = \mathcal{C}_{\nu-1}(z) - \mathcal{C}_{\nu+1}(z),$$

where $\mathcal{C}_\nu(z)$ stands for both $J_\nu(z)$ and $Y_\nu(z)$ (see [1, § 9]). We have

$$\begin{aligned}
&Y_\nu'(z)J_\nu^{(k+1)}(z) - J_\nu'(z)Y_\nu^{(k+1)}(z)\\
&\quad = Y_\nu'(z)(J_\nu')^{(k)}(z) - J_\nu'(z)(Y_\nu')^{(k)}(z)\\
&\quad = \tfrac{1}{4}[(Y_{\nu-1}(z) - Y_{\nu+1}(z))(J_{\nu-1}(z) - J_{\nu+1}(z))^{(k)}\\
&\qquad\quad - (J_{\nu-1}(z) - J_{\nu+1}(z))(Y_{\nu-1}(z) - Y_{\nu+1}(z))^{(k)}]\\
&\quad = \tfrac{1}{4}[(Y_{\nu-1}(z)J_{\nu-1}^{(k)}(z) - J_{\nu-1}(z)Y_{\nu-1}^{(k)}(z))\\
&\qquad\quad + (Y_{\nu+1}(z)J_{\nu+1}^{(k)}(z) - J_{\nu+1}(z)Y_{\nu+1}^{(k)}(z))\\
&\qquad\quad + (J_{\nu+1}(z)Y_{\nu-1}^{(k)}(z) - Y_{\nu-1}(z)J_{\nu+1}^{(k)}(z))\\
&\qquad\quad + (J_{\nu-1}(z)Y_{\nu+1}^{(k)}(z) - Y_{\nu+1}(z)J_{\nu-1}^{(k)}(z))]\\
&\quad = \frac{1}{4}\bigg[\frac{2}{\pi z}(r_k + R_{\nu-1,k}(z) + r_k + R_{\nu+1,k}(z))\\
&\qquad\quad + \frac{2\nu}{z}(J_\nu(z)Y_{\nu-1}^{(k)} - Y_\nu(z)J_{\nu-1}^{(k)}(z) + J_\nu(z)Y_{\nu+1}^{(k)}(z) - Y_\nu(z)J_{\nu+1}^{(k)}(z))\\
&\qquad\quad - (J_{\nu-1}(z)Y_{\nu-1}^{(k)}(z) - Y_{\nu-1}(z)J_{\nu-1}^{(k)}(z) + J_{\nu+1}(z)Y_{\nu+1}^{(k)}(z) - Y_{\nu+1}J_{\nu+1}^{(k)}(z))\bigg]\\
&\quad = \frac{1}{4}\bigg[\frac{4}{\pi z}(2r_k + R_{\nu-1,k}(z) + R_{\nu+1,k}(z))\\
&\qquad\quad + \frac{2\nu}{z}(J_\nu(z)(Y_{\nu-1}(z) + Y_{\nu+1}(z))^{(k)} - Y_\nu(z)(J_{\nu-1}(z) + J_{\nu+1}(z))^{(k)})\bigg]\\
&\quad = \frac{1}{\pi z}(2r_k + R_{\nu-1,k}(z) + R_{\nu+1,k}(z))\\
&\qquad + \frac{\nu^2}{z}\bigg(J_\nu(z)\bigg(\frac{1}{z}Y_\nu(z)\bigg)^{(k)} - Y_\nu(z)\bigg(\frac{1}{z}J_\nu(z)\bigg)^{(k)}\bigg)\\
&\quad = \frac{2}{\pi z}\bigg[r_k + \tfrac{1}{2}(R_{\nu-1,k}(z) + R_{\nu+1,k}(z)) - \frac{\nu^2}{z}\sum_{j=0}^{k}\frac{k!(-1)^{k-j}}{j!z^{k-j+1}}(r_j + R_{\nu,j}(z))\bigg].
\end{aligned} \tag{A 3}$$

https://doi.org/10.1017/S0308210516000214 Published online by Cambridge University Press

We now prove (A 2), as follows:

$$\begin{aligned} &Y_\nu(z)J_\nu^{(k+1)}(z) - J_\nu(z)Y_\nu^{(k+1)}(z) \\ &\qquad = (Y_\nu(z)J_\nu^{(k)}(z) - J_\nu(z)Y_\nu^{(k)}(z))' - (Y'_\nu(z)J_\nu^{(k)}(z) - J'_\nu(z)Y_\nu^{(k)}(z)) \\ &\qquad = \frac{2}{\pi z}\left(-q_k - Q_{\nu,k}(z) - \frac{r_k}{z} - \frac{R_{\nu,k}(z)}{z} + R'_{\nu,k}(z)\right). \end{aligned} \tag{A 4}$$

This concludes the proof. □

Corollary A.3. *The following formulae hold:*

$$\begin{aligned} J_\nu(z)Y_\nu'''(z) - Y_\nu(z)J_\nu'''(z) &= \frac{2}{\pi z}\left(\frac{2+\nu^2}{z^2} - 1\right); \\ Y'_\nu(z)J_\nu'''(z) - J'_\nu(z)Y_\nu'''(z) &= \frac{2}{\pi z^2}\left(1 - \frac{3\nu^2}{z^2}\right); \\ Y'_\nu(z)J_\nu^{(\mathrm{iv})}(z) - J'_\nu(z)Y_\nu^{(\mathrm{iv})}(z) &= \frac{2}{\pi z}\left(1 - \frac{3+2\nu^2}{z^2} + \frac{\nu^4+11\nu^2}{z^4}\right). \end{aligned}$$

Proof. From lemma A.2 (see in particular (A 4)) it follows that

$$\begin{aligned} J_\nu(z)Y_\nu'''(z) - Y_\nu(z)J_\nu'''(z) &= -\frac{2}{\pi z}\left[-q_2 - Q_{\nu,2}(z) - \frac{r_2}{z} - \frac{R_{\nu,2}(z)}{z} + R'_{\nu,2}(z)\right] \\ &= \frac{2}{\pi z}\left(\frac{2+\nu^2}{z^2} - 1\right). \end{aligned}$$

Next we compute

$$\begin{aligned} Y'_\nu(z)J_\nu'''(z) - J'_\nu(z)Y_\nu'''(z) &= \frac{2}{\pi z}\left[r_2 + R_{\nu,2}(z) - \frac{\nu^2}{z}\sum_{j=0}^{2}\frac{2(-1)^{2-j}}{j!z^{2-j+1}}(r_j + R_{\nu,j}(z))\right] \\ &= \frac{2}{\pi z^2}\left(1 - \frac{3\nu^2}{z^2}\right). \end{aligned}$$

Finally, by (A 3) with $k = 3$, we have

$$\begin{aligned} &Y'_\nu(z)J_\nu^{(\mathrm{iv})}(z) - J'_\nu(z)Y_\nu^{(\mathrm{iv})}(z) \\ &\quad = \frac{2}{\pi z}\left[r_3 + \tfrac{1}{2}(R_{\nu-1,3}(z) + R_{\nu+1,3}(z)) - \frac{\nu^2}{z}\sum_{j=0}^{3}\frac{6(-1)^{3-j}}{j!z^{3-j+1}}(r_j + R_{\nu,j}(z))\right] \\ &\quad = \frac{2}{\pi z}\left(1 - \frac{3+2\nu^2}{z^2} + \frac{\nu^4+11\nu^2}{z^4}\right). \end{aligned}$$

□

Acknowledgements

Most of the computations in this paper were performed by L.P. during his PhD under the guidance of P.D.L. The authors acknowledge financial support from the research project CPDA120171/12 'Singular perturbation problems for differential operators', Progetto di Ateneo of the University of Padova and from the research project 'INdAM GNAMPA Project 2015 – Un approccio funzionale analitico per problemi di perturbazione singolare e di omogeneizzazione'. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

1 M. Abramowitz and I. Stegun (eds). *Handbook of mathematical functions with formulas, graphs and Mathematical Tables* (New York: Dover, 1972).

2 J. M. Arrieta and P. D. Lamberti. Spectral stability results for higher-order operators under perturbations of the domain. *C. R. Math.* **351** (2013), 725–730.

3 J. M. Arrieta, A. Jiménez-Casas and A. Rodríguez-Bernal. Flux terms and Robin boundary conditions as limit of reactions and potentials concentrating at the boundary. *Rev. Mat. Ibero.* **24**(1) (2008), 183–211.

4 C. Bandle. *Isoperimetric inequalities and applications.* Monographs and Studies in Mathematics, vol. 7 (Boston, MA: Pitman Advanced Publishing Program, 1980).

5 D. Buoso and L. Provenzano. A few shape optimization results for a biharmonic Steklov problem. *J. Diff. Eqns* **259** (2015), 1778–1818.

6 R. Courant and R. D. Hilbert. *Methods of mathematical physics*, vol. I (New York: Interscience, 1953).

7 M. Dalla Riva and L. Provenzano. On vibrating membranes with mass concentrated near the boundary: an asymptotic analysis, Preprint (2016).

8 G. Folland. *Introduction to partial differential equations*, 2nd edn (Princeton University Press, 1995).

9 A. Girouard and I. Polterovich. Spectral geometry of the Steklov problem. *J. Spectral Theory.* (In the press.)

10 D. Gómez, M. Lobo, S. A. Nazarov and E. Pérez. Spectral stiff problems in domains surrounded by thin bands: asymptotic and uniform estimates for eigenvalues. *J. Math. Pures Appl.* **85** (2006), 598–632.

11 D. Gómez, M. Lobo, S. A. Nazarov and E. Pérez. Asymptotics for the spectrum of the Wentzell problem with a small parameter and other related stiff problems. *J. Math. Pures Appl.* **86** (2006), 369–402.

12 V. A. Kozlov, V. G. Maz′ya and J. Rossmann. *Spectral problems associated with corner singularities of solutions to elliptic equations.* Mathematical Surveys and Monographs, vol. 85 (Providence, RI: American Mathematical Society, 2001).

13 P. D. Lamberti. Steklov-type eigenvalues associated with best Sobolev trace constants: domain perturbation and overdetermined systems. *Complex Variables Elliptic Eqns* **59** (2014), 309–323.

14 P. D. Lamberti and M. Lanza de Cristoforis. A real analyticity result for symmetric functions of the eigenvalues of a domain-dependent Neumann problem for the Laplace operator. *Mediterr. J. Math.* **4** (2007), 435–449.

15 P. D. Lamberti and M. Perin. On the sharpness of a certain spectral stability estimate for the Dirichlet Laplacian. *Eurasian Math. J.* **1** (2010), 111–122.

16 P. D. Lamberti and L. Provenzano. A maximum principle in spectral optimization problems for elliptic operators subject to mass density perturbations. *Eurasian Math. J.* **4**(3) (2013), 70–83.

17 P. D. Lamberti and L. Provenzano. Viewing the Steklov eigenvalues of the Laplace operator as critical Neumann eigenvalues. In *Current Trends in Analysis and Its Applications, Proc. 9th ISAAC Congress, Kraków 2013*, pp. 171–178 (Basel: Birkhäuser, 2015).

18 W. Stekloff. Sur les problémes fondamentaux de la physique mathématique (suite et fin). *Annales Scient. Éc. Norm. Sup.* **19** (1902), 455–490.

19 W.-M. Ni and X. Wang. On the first positive Neumann eigenvalue. *Discrete Contin. Dynam. Syst.* **17** (2007), 1–19.