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GROWTH OF INTEGRAL TRANSFORMS AND EXTINCTION IN CRITICAL GALTON–WATSON PROCESSES

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Abstract

The mean time to extinction of a critical Galton–Watson process with initial population size k is shown to be asymptotically equivalent to two integral transforms: one involving the kth iterate of the probability generating function and one involving the generating function itself. Relating the growth of these transforms to the regular variation of their arguments, immediately connects statements involving the regular variation of the probability generating function, its iterates at 0, the quasistationary measures, their partial sums, and the limiting distribution of the time to extinction. In the critical case of finite variance we also give the growth of the mean time to extinction, conditioned on extinction occurring by time n.

Keywords: Critical Galton–Watson process; branching process; integral transform; regular variation; mean time to extinction; Tauberian theorem

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1. Background

Critical Galton–Watson processes with regularly varying probability generating functions (PGFs) were originally studied in [13], [14], and [15], where the authors considered a critical process with PGF $f(s)$ satisfying

$$f(s) = s + (1-s)^{1+\rho} L(1-s), \tag{1}$$

where $\rho \in (0, 1]$ and L is a slowly varying function at 0, i.e. for any $t > 0$, $\lim_{s\to\infty} L(st)/L(s) = 1$. (Hereafter, all the functions mentioned are assumed to be *Baire*, i.e. belonging to the smallest class of functions that includes continuous functions on $\mathbb{R}$ and is closed under pointwise limits—see, e.g. [6, pp. 104–106]. Unless specified otherwise, L is assumed to be slowly varying at 0.) These works have been extensively cited and the process with condition (1) has been explored in numerous papers in the field (see, e.g. [4], [5], [7], and [10]).

Let $\{Z(n)\}_{n=0}^{\infty}$ denote a Galton–Watson process, $Z := Z(1)$, and let us assume that $Z(0) = 1$ unless specified otherwise. For noncritical processes, the condition $\mathrm{E}\, Z \log Z < \infty$ separates a range of qualitatively different behaviour, whereas for critical processes, the appropriate criterion is $\sigma^2 = \mathrm{E}\, Z^2 - 1 < \infty$ (which is equivalent to $f''(1) < \infty$) and condition (1) is a natural extension of it since it controls the rate at which $f''(s)$ grows at 1. This growth and that of the PGF itself can be expected to be related to the growth of the functional iterates $f_{(n)}(s)$

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of the PGF, which are the PGFs of the process at time n as well as their appropriate transforms that describe the various conditioned processes (see, e.g. [4]).

The following lemma shows what condition (1) means in terms of moments.

Lemma 1. *If* $\{Z\}_{i=0}^{\infty}$ *is a critical process satisfying (1) then the following statements hold.*

(a) $\rho \in [0, 1]$.

(b) *If* $\sigma^2 < \infty$ *then* $\rho = 1$ *and* $L(0) = f''(1)/2$.

(c) $\mathrm{E}\, Z^{1+1/\varepsilon} < \infty$ *if* $\varepsilon > 1/\rho$ *and* $\mathrm{E}\, Z^{1+1/\varepsilon} = \infty$ *if* $\varepsilon < 1/\rho$.

In particular, if $\rho = 0$ *then* $\mathrm{E}\, Z^{1+1/\varepsilon} = \infty$ *for any* $\varepsilon > 0$.

Proof. Using Lamperti's rule (see, e.g. [14]), $f'(s) - 1$ and $f''(s)$ can be easily seen to be regularly varying at 1, i.e. $f'(s) - 1 \sim (1+\rho)(1-s)^{\rho} L(1-s)$ and $f''(s) = \rho(1+\rho)(1-s)^{\rho-1}L(1-s)$. Since $f'(1) = 1 < \infty$ and (b) is equivalent to $0 < f''(1) < \infty$, (a) and (b) must follow.

In [15] it was shown that the distribution of Z is in the domain of the stable law of index ρ and, therefore, $\mathrm{E}\, Z^{1+\varepsilon} < \infty$ if $\varepsilon > 1/\rho$ and is infinite if $\varepsilon > 1/\rho$ (see, e.g. [6, p. 448]).

The case in which $\rho = 0$ has been recently examined in [10].

We write $f(s) \sim g(s)$ as $s \to s_0$ to mean that

$$\lim_{s \to s_0} \frac{f(s)}{g(s)} = 1.$$

Let $f_{(i)}$ denote the ith functional iterate of f, let $P_{ij} := \mathrm{P}(Z(1) = j \mid Z(0) = i)$, and let $\{\eta_i\}$ be the quasistationary measure, i.e.

$$\eta_j = \sum_{i=1}^{\infty} \eta_i P_{ij}.$$

It was shown in [13] and [14] that (1) implies that

$$\sum_{i=1}^{n} \eta_i \sim \frac{1}{\rho\Gamma(1+\rho)} \frac{n^{\rho}}{L(1/n)} \quad \text{as } n \to \infty, \tag{2}$$

$$1 - f_{(n)}(0) \sim n^{-1/\rho} L_*\left(\frac{1}{n}\right), \tag{3}$$

where L and L_* are regularly varying at 0. It was also shown by a number of authors (see [2, p. 38]) that, for subcritical Galton–Watson processes with mean $\mu := \mathrm{E}\, Z$, the following two conditions are equivalent:

$$\mathrm{E}\, Z \log Z < \infty, \tag{4}$$

$$1 - f_{(n)}(0) \sim c\mu^n, \qquad c \in (0, 1]. \tag{5}$$

While in [9] (or [2, p. 89]) it was shown that, for subcritical processes, (5) implies that

$$\sum_{i=1}^{n} \eta_i \sim \frac{\log n}{\log(1/\mu)}. \tag{6}$$

In [1] the growth of the mean time to extinction of subcritical process was given, i.e. if $T^n := \inf\{i : Z(i) = 0 \mid Z(0) = n\}$, (5) implies that

$$\mathrm{E}\, T^n \sim \frac{\log n}{\log(1/\mu)}. \tag{7}$$

The purpose of the present paper is to derive these and other properties as easy corollaries of a general result relating the growth of a function to that of two Mellin-type transforms. In particular, the equivalence of (1), (2), and (3), as well as (for subcritical case) of (4), (5), (6), and (7) are all easily established from our main theorem. We also show the appropriate relationships for subcritical processes when $\mathrm{E}\, Z \log Z = \infty$, give the limiting distribution of the time to extinction of a process started with k particles as $k \to \infty$, and give the growth of the mean time to extinction of a critical process with finite variance, when extinction had occurred by time n. Some related questions, specifically the distribution of extinction time and the paths to extinction, have recently been explored in the subcritical Markov branching case by Jagers *et al.* [8].

2. Main results

The results of this paper rely on the following fact that relates the regular growth of a function at 1 to that of its integral transform of the Mellin type. We defer the proof until the next section.

Theorem 1. *Let $g, h : [0, 1) \to \mathbb{R}^+$, let L be a slowly varying function at* 0*, and let $\rho \in (0, 1)$ ((a), below, remains valid for $\rho = 0$). Define*

$$I(k) := \int_0^1 s^k g(s)\,\mathrm{d}s,$$

$$J(k) := \int_0^1 (1 - s^k) h(s)\,\mathrm{d}s.$$

Then, as $x \to 1$ and $k \to \infty$,

(a) *$g(x) \sim (1 - x)^{-\rho} L(1 - x)$ is equivalent to $I(k) \sim \Gamma(1 - \rho) k^{\rho - 1} L(1/k)$;*

(b) *$h(x) \sim (1 - x)^{-\rho - 1} L(1 - x)$ is equivalent to $J(k) \sim (\Gamma(1 - \rho)/\rho) k^{\rho} L(1/k)$.*

Before we apply the above to Galton–Watson processes, we derive an immediate corollary relating the regular variation of the distribution function to that of the expectation of maxima. The Abelian direction (from distribution function to expectation) follows from a modern treatment of the celebrated theorem of Gnedenko (see, e.g. [3, p. 409]). But we derive both directions as immediate consequences of Theorem 1(a).

Corollary 1. *Let $T_1, T_2, \ldots, T_n$ be nonnegative, independent, identically distributed (i.i.d.) random variables with distribution function F, let $F^{\leftarrow}(s) := \inf\{x : F(x) > s\}$, let $\rho \in [0, 1)$, and let L be regularly varying at* 0*. Then the following statements are equivalent:*

(a) *$F^{\leftarrow}(s) \sim (1 - s)^{-\rho} L(1 - s)$ as $s \to 1$;*

(b) *$\mathrm{E} \max\{T_1, \ldots, T_n\} \sim \Gamma(1 - \rho) n^{\rho} L(1/n)$ as $n \to \infty$;*

(c) *if $\rho \neq 0$, let $b_n = F^{\leftarrow}(1 - 1/n)$, then, for $x \geq 0$,*

$$\lim_{n \to \infty} \mathrm{P}\left(\frac{1}{b_n} \max\{T_1, \ldots, T_n\} < x\right) = \exp(-x^{-1/\rho}).$$

If we also assume that

$$F^{\leftarrow}(s) = \sum_{i=0}^{\infty} a_i s^i \tag{8}$$

with $a_i \geq 0$ *then the above statements are also equivalent to*

(d) $\sum_{i=0}^{n} a_i \sim (n^{\rho}/\Gamma(1+\rho))L(1/n)$ *as* $n \to \infty$.

Furthermore, if we suppose that $\{a_i\}_{i=k}^{\infty}$ *is monotone for some fixed* k *and* $\rho > 0$, *then the above statements are also equivalent to*

(e) $a_n \sim \rho n^{\rho-1} L(1/n)$ *as* $n \to \infty$.

Proof. Recall that if F is the distribution function of a nonnegative random variable T with distribution function F, then T is distributed as $F^{\leftarrow}(R)$ and R is distributed as uniform[0, 1]. Also, if $T_1, \ldots, T_k$ are k i.i.d. copies of T then $\mathrm{P}(\max\{T_1, \ldots, T_k\} < t) = \mathrm{P}(T < t)^k$, and so, for the corresponding expectation,

$$\mathrm{E}\max\{T_1, \ldots, T_k\} = \int_0^1 F^{\leftarrow}(s^{1/k})\,\mathrm{d}s.$$

Substituting $u = x^{1/k}$, we obtain $k\int_0^1 x^{k-1}F^{\leftarrow}(x)\,\mathrm{d}x$, which is in the form of $I(k)$, and the equivalence of statements (a) and (b) follows. The equivalence of statements (a) and (c) follows from Gnedenko's theorem (see, e.g. [11]) once we write $F(s) \sim s^{-1/\rho}\bar{L}(s)$, where $\bar{L}$ is a slowly varying function at ∞ and $\bar{L}(s) = L^{\#}(s^{-1/\rho})$, where $L^{\#}$ is the de Bruijn conjugate of L (see, e.g. [3, p. 29]). Statements (d) and (e) follow immediately from the Tauberian theorem for power series (see, e.g. [6, p. 447]).

Observe that where Corollary 1(a) and condition (9) hold, combining Corollary 1(b) and (d), and using the fact that $\Gamma(1+\rho)\Gamma(1-\rho) = \pi\rho/\sin(\pi\rho)$, we obtain the following asymptotic relationship:

$$\sum_{i=0}^{n} a_i \sim \frac{\sin(\pi\rho)}{\pi\rho}\,\mathrm{E}\max\{T_1, \ldots, T_n\} \sim \frac{F^{\leftarrow}(1-1/n)}{\Gamma(1+\rho)} \quad \text{as } n \to \infty.$$

Now if T^k denotes the time to extinction of a Galton–Watson process with initial population size k then $T^k = \max\{T_1, \ldots, T_k\}$, where T_i are i.i.d. like $T := T^1$, and we have the following corollary.

Corollary 2. *Let* Z *be a Galton–Watson process with PGF* f, *let* $\{\eta_i\}_{i=1}^{\infty}$ *be the quasistationary measure, and let* T^k *be the time to extinction of a population of size* k. *Let* $\mathcal{U}(s)$ *be the quasistationary measure generating function for* Z, *i.e.* $\mathcal{U}(s) = \sum \eta_i s^i$, *and let* $\rho \in [0, 1)$. *Then the following statements are equivalent.*

(a) $\mathcal{U}(s) = (1-s)^{-\rho}L(1-s)$ *as* $s \to 1$.

(b) $1 - f_{(n)}(0) \sim n^{-1/\rho}L^*(n)$ *if* $\rho > 0$, *and* $1 - f_{(n)}(0) \sim L^{\leftarrow}(n)$ *if* $\rho = 0$, *where* $L^*(n)$ *is slowly varying at* ∞ *and* $L^*(n) = L^{\#}(n^{-1/\rho})$.

(c) *If* $\rho > 0$, *let* $b_n = \mathcal{U}^{\leftarrow}(1-1/n)$, *then, for* $t \geq 0$,

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{T^n}{b_n} < t\right) = \exp(-t^{-1/\rho}).$$

(d)

$$\sum_{i=1}^{n} \eta_i \sim \frac{n^\rho}{\Gamma(1+\rho)} L\left(\frac{1}{n}\right) \quad \textit{as } n \to \infty.$$

(e) *If* $\{\eta_i\}_{i=j}^{\infty}$ *is monotone for some* j *and* $\rho > 0$, *then* $\eta_n \sim \rho n^{\rho-1} L(1/n)$ *as* $n \to \infty$.

(f) $\mathrm{E}\, T^n \sim \Gamma(1-\rho) n^\rho L(1/n)$ *as* $n \to \infty$.

Proof. We know (see, e.g. [2, p. 68]) that $\mathcal{U}(f(s)) = \mathcal{U}(s)+1$, and substituting $s = f_{(n)}(0)$, we obtain $f_{(n)}(0) = \mathcal{U}^{-1}(n)$. Hence, setting $a_i = \eta_i$, we let $\mathcal{U} = F^{\leftarrow}$, to apply Corollary 1, to obtain the equivalence of statements (a), (c), (d), and (e). To prove that statement (a) implies statement (b), i.e. in order to translate the statement about $\mathcal{U}$ into the statement about $f_{(n)}(0)$, we use the inversion of regularly varying functions using de Bruijn's conjugate again. To prove that statement (b) implies statement (a), observe that, for any $t \in (0,1)$, we can find an $n \geq 0$ such that $f_{(n)}(0) \leq t \leq f_{(n+1)}(0)$ and, since $\mathcal{U}$ is monotone,

$$(1 - f_{(n+1)}(0))\mathcal{U}(f_{(n)}(0)) \leq (1-t)\mathcal{U}(t) \leq (1 - f_{(n)}(0))\mathcal{U}(f_{(n+1)}(0)).$$

Setting $\rho = 0$ gives the following result.

Corollary 3. *Let* $\{Z(i)\}_{i=1}^{\infty}$ *be a Galton–Watson process with a generating function* $f(s)$. *Let* L *be a slowly varying function at* 0. *Then the following statements are equivalent:*

(a) $1 - f_{(n)}(0) \sim L^{\leftarrow}(n)$;

(b) $\mathrm{E}\, T^n \sim L(1/n)$;

(c) $\sum_{i=1}^{n} \eta_i \sim L(1/n)$.

In [12] it was shown that iterates of a PGF of a subcritical process must satisfy

$$1 - f_{(n)}(0) \sim \mu^n L(\mu^n),$$

where $L(s) \to 0$ as $s \to \infty$. Inverting this expression, we obtain $\log_\mu(nL^{\#}(n))$, which is slowly varying, and, hence, Corollary 3 covers subcritical processes. More specifically, we immediately have the equivalence of the following statements:

(i) $1 - f_{(n)}(0) \sim c\mu^n$, $c \in (0, 1]$;

(ii) $\mathrm{E}\, T^n \sim \log n / \log(1/\mu)$;

(iii) $\sum_{i=1}^{n} \eta_i \sim \log n / \log(1/\mu)$.

Note that statement (i) is equivalent to $\mathrm{E}\, Z \log Z < \infty$; this result is due to a number of authors (see [2, p. 38]). To prove that statement (i) implies statement (ii), see [1], and to prove that statement (i) implies statement (iii), see [9].

In order to relate the regular variation of the iterates to that of the PGF, we give an asymptotically equivalent form for $\mathrm{E}\, T^k$ in terms of the PGF using a Riemann sums approximation. We defer the proof untill the next section.

Lemma 2. *Let* $\{Z\}_{t=0}^{\infty}$ *be a critical Galton–Watson process with PGF* $f(s) \neq s$, *and let* T^k *be the time to extinction of* k *individuals of this process. Then*

$$\mathrm{E}\, T^k \sim \int_0^1 \frac{1-s^k}{f(s)-s}\, \mathrm{d}s.$$

This is in the form of $J(k)$ of Theorem 1(b) and we can immediately write the growth of $\mathrm{E}\,T^k$ in terms of f. Hence, we can combine the above observations and obtain the following theorem.

Theorem 2. *Let $\rho \in (0,1)$. For a critical Galton–Watson process with PGF f, the following statements are equivalent.*

(a) $f(s) - s \sim (1-s)^{1+\rho} L(1-s)$ *as* $s \to 1$.

(b) $1 - f_{(n)}(0) \sim n^{-1/\rho} L^*(n)$ *if* $\rho > 0$*, where* $L^*(n)$ *is slowly varying at* ∞ *and* $L^*(n) = L^{\#}(n^{-1/\rho})$.

(c) *Let* $b_n = F^{\leftarrow}(1 - 1/n)$*. Then*

$$\lim_{n\to\infty} \mathrm{P}\left(\frac{T^n}{b_n} < t\right) = \exp(-t^{-1/\rho}).$$

(d)

$$\sum_{i=1}^{n} \eta_i \sim \frac{1}{\rho\Gamma(1+\rho)} \frac{n^\rho}{L(1/n)} \quad \text{as } n \to \infty.$$

(e)

$$\mathrm{E}\,T^n \sim \frac{\Gamma(1-\rho)}{\rho} \frac{n^\rho}{L(1/n)} \quad \text{as } n \to \infty.$$

(f) *If* $\{\eta_i\}_{i=j}^{\infty}$ *is monotone for some* j *then* $\eta_n \sim \rho n^{\rho-1} L(1/n)$ *as* $n \to \infty$.

Note that Theorem 2 is restricted to $\rho \in (0,1)$, i.e. fractional power moments exist beyond the first one, and, of course, $\mathrm{E}\,Z \log Z < \infty$.

3. Proofs of Theorem 1 and Lemma 2

The proof of Theorem 1 relies on a simple substitution and a uniform convergence theorem for slowly varying functions (see, e.g. [3, p. 6]). We prove it for $I(k)$; the treatment for $J(k)$ is entirely analogous.

Proof of Theorem 1. We first prove that $g(s) \sim (1-s)^\rho L(1-s)$ implies $I(k) \sim \Gamma(1-\rho) k^{\rho-1} L(1/k)$. Since $g(s) = (1-s)^\rho L(1-s)$, if we let $s = 1 - x/k$ then $I(k)$ becomes

$$k^{\rho-1} \int_0^k \left(1 - \frac{x}{k}\right)^k x^{-\rho} L\left(\frac{x}{k}\right) \mathrm{d}s.$$

Let $\varepsilon \in (0,1)$, and let $N > 0$ be a 'large' number (we shall say how 'large' below). Divide $I(k)$ by $k^{\rho-1} L(1/k)$ and split the integral into two parts:

$$\begin{aligned}
&\int_0^N \left(1-\frac{x}{k}\right)^k x^{-\rho} \frac{L(x/k)}{L(1/k)} \,\mathrm{d}x + \int_N^k \left(1-\frac{x}{k}\right)^k x^{-\rho} \frac{L(x/k)}{L(1/k)} \,\mathrm{d}x \\
&\quad = \int_0^N \left(1-\frac{x}{k}\right)^k x^{-\rho} \left(\frac{L(x/k)}{L(1/k)} - 1\right) \mathrm{d}x + \int_0^N \left(1-\frac{x}{k}\right)^k x^{-\rho} \,\mathrm{d}x \\
&\qquad + \int_N^k \left(1-\frac{x}{k}\right)^k x^{-\rho} \frac{L(x/k)}{L(1/k)} \,\mathrm{d}x. \qquad (9)
\end{aligned}$$

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By the monotone convergence theorem for slowly varying functions (see, e.g. [3, p. 6]) for any N_1, we can choose large enough $k_0 = k_0(\varepsilon/3\Gamma(1-\rho))$ so that, for all $x \in [0, N_1]$ and $k > k_0$,

$$\left|\frac{L(x/k)}{L(1/k)} - 1\right| < \frac{\varepsilon}{3\Gamma(1-\rho)}.$$

For the first integral in (9), observe that

$$\int_0^{N_1} \left(1 - \frac{x}{k}\right)^k x^{-\rho}\,\mathrm{d}x \le \int_0^\infty \mathrm{e}^{-x} x^{-\rho}\,\mathrm{d}x = \Gamma(1-\rho);$$

hence, for $k > k_0$,

$$\int_0^{N_1} \left(1 - \frac{x}{k}\right)^k x^{-\rho}\left(\frac{L(x/k)}{L(1/k)} - 1\right)\mathrm{d}x < \frac{\varepsilon}{3}.$$

Choose $N_2 = N_2(\varepsilon/3)$ such that $\int_{N_2}^\infty \mathrm{e}^{-x} x^{-\rho}\,\mathrm{d}x < \varepsilon/3$. Hence, as $k \to \infty$, for $N > N_2$, the second integral in (9) is within $\varepsilon/3$ of $\Gamma(1-\rho)$.

Potter's theorem (see [3, p. 25]) states that, for any $N_3\colon N_3 \le u \le k$ and for any $A > 0$ and $\delta > 0$, there exist large enough k_0 so that, for all $k > k_0$, $L(u/k)/L(1/k) < AN_3^\delta$. Observe that $\int_{N_3}^\infty \mathrm{e}^{-x} x^{-\rho}\,\mathrm{d}x = o(N_3^{-\delta})$ for any $\delta > 0$ (using, e.g. l'Hôpital's rule). Fix A and δ, and take large enough N_3 so that $AN_3^\delta \int_{N_3}^\infty \mathrm{e}^{-x} x^{-\rho}\,\mathrm{d}x < \varepsilon/3$. Taking $N = \max\{N_1, N_2, N_3\}$, we obtain, as $k \to \infty$, $k^{-\rho+1} L(1/k)^{-1} I(k) - \Gamma(1-\rho) \le \varepsilon$ and, hence, the assertion of the theorem follows.

We now prove the converse statement, i.e. that $g(s) \sim (1-s)^\rho L(1-s)$ is implied by

$$\left(k^{\rho-1} L\left(\frac{1}{k}\right)\right)^{-1} \int_0^1 s^k g(s)\,\mathrm{d}s \to \Gamma(1-\rho).$$

Performing the same substitution as above, bringing $(k^{\rho-1}L(1/k))^{-1}$ inside the integral, and using $L(1/k) = L(x/k) - o(1)L(1/k)$ for $x \in [0, N]$, we obtain

$$\begin{aligned}
&\int_0^N \left(1 - \frac{x}{k}\right)^k x^{-\rho} \frac{x^\rho g(1-x/k)}{k^\rho L(1/k)}\,\mathrm{d}x + \int_N^k \left(1 - \frac{x}{k}\right)^k x^{-\rho} \frac{x^\rho g(1-x/k)}{k^\rho L(1/k)}\,\mathrm{d}x \\
&\quad = \int_0^N \left(1 - \frac{x}{k}\right)^k x^{-\rho}\left(\left(\frac{x}{k}\right)^\rho \frac{g(1-x/k)}{(L(x/k) - o(1)L(1/k))} - 1\right)\mathrm{d}x \\
&\qquad + \int_0^N \left(1 - \frac{x}{k}\right)^k x^{-1-\rho}\,\mathrm{d}x + \int_N^k \left(1 - \frac{x}{k}\right)^k x^{-\rho} \frac{x^\rho g(1-x/k)}{k^\rho L(1/k)}\,\mathrm{d}x \\
&\quad \to \frac{\Gamma(1-\rho)}{\rho}.
\end{aligned}$$

Since the second integral on the right-hand side converges to $\Gamma(1-\rho)$ and the third integral on the right-hand side is nonnegative, the first integral on the right-hand side must converge to 0, as first $k \to \infty$ and then $N \to \infty$; hence,

$$\frac{(x/k)^\rho g(1-x/k)}{L(x/k) - o(1)L(1/k)} \to 1.$$

Setting $s = x/k \to 0$ and using $L(1/k) \sim L(x/k)$ as before, it follows that $g(1-s) \sim s^{-\rho} L(s)$ as required.

Proof of Lemma 2. Consider the lower and upper sums of $(1-s^k)/(f(s)-s)$ on the subdivision of the unit interval by iterates of f: $0<f(0)<f_{(2)}(0)<\cdots$, where $f_{(i)}(0)$ is the ith functional iterate of f, with $f_{(0)}(0):=0$. We have

$$\sum_{i=0}^{\infty}(1-f_{(i)}(0)^k)<\int_0^1\frac{1-s^k}{f(s)-s}\,\mathrm{d}s<\sum_{i=0}^{\infty}\frac{1-f_{(i)}(0)}{f_{(i+1)}(0)-f_{(i)}(0)}(f_{(i)}(0)-f_{(i-1)}(0)).$$

The lower sum is precisely $\mathrm{E}\,T^k$. Denote the lower sums on the right-hand side by $S_{\mathrm{l}}(k)$ and the upper sums on the left-hand side by $S_{\mathrm{u}}(k)$, and let $\varepsilon\in(0,1)$. Note that

$$\frac{f_{(i)}(0)-f_{(i-1)}(0)}{f_{(i+1)}(0)-f_{(i)}(0)}\downarrow 1,$$

since these are reciprocals of the gradients of disjoint chords on $f(s)$ (monotonicity follows from that of f using the mean value theorem) and we can choose $i^*=i^*(\varepsilon)$ such that, for all $j\ge i^*$, $(f_{(j)}(0)-f_{(j-1)}(0))/(f_{(j+1)}(0)-f_{(j)}(0))-1<\varepsilon/2$. Observe that $S_{\mathrm{l}}(k)$ grows unboundedly with k. Choose large enough k so that $i^*f(0)/(S_{\mathrm{l}}(k)(f_{(2)}(0)-f(0)))<\varepsilon/2$. Then we obtain

$$\begin{aligned}\frac{S_{\mathrm{u}}}{S_{\mathrm{l}}}&=\frac{\sum_{i=0}^{i^*-1}(1-f_{(i)}(0)^k)(f_{(i)}(0)-f_{(i-1)}(0))/(f_{(i+1)}(0)-f_{(i)}(0))}{\sum_{i=0}^{\infty}(1-f_{(i)}(0)^k)}\\&\quad+\frac{\sum_{i^*}^{\infty}(1-f_{(i)}(0)^k)(f_{(i)}(0)-f_{(i-1)}(0))/(f_{(i+1)}(0)-f_{(i)}(0))}{\sum_{i=0}^{\infty}(1-f_{(i)}(0)^k)}\\&\le\frac{i^*f(0)}{S_{\mathrm{l}}(k)(f_{(2)}(0)-f(0))}+\left(1+\frac{\varepsilon}{2}\right)\\&<1+\varepsilon.\end{aligned}$$

Since ε can be chosen arbitrarily small, we see that $S_{\mathrm{u}}\sim S_{\mathrm{l}}$. This completes the proof.

4. Mean time to extinction when extinction has occurred by time *n*—the case of finite variance

Suppose we know that a critical Galton–Watson individual with $\mathrm{E}\,Z^2<\infty$ dies by time n (‘large’ n). What is our best guess on the actual time of death? Let $T:=T(1)$, the time to extinction of one particle, and let $T_n:=(T\mid T\le n)$.

Theorem 3. *For a critical Galton–Watson process with $\sigma^2<\infty$,*

$$\mathrm{E}\,T_n\sim\frac{\sigma^2}{2}\log n.$$

Proof. We condition on extinction by time n and use $J(k)$ to approximate this conditional mean time to extinction. We have

$$\mathrm{E}\,T_n=\frac{1}{f_{(n)}(0)}\sum_{i=1}^{n-1}(nf_{(n)}(0)-f_{(i)}(0))\sim\sum_{i=0}^{n}(1-f_{(i)}(0)).$$

As before, we have

$$\mathrm{E}\,T_n\sim\int_0^{f_{(n)}(0)}\frac{1-s^k}{f(s)-s}\,\mathrm{d}s.$$

Let f be a PGF of a critical Galton–Watson process with $\sigma^2 < \infty$. Then $1 - f_{(n)}(0) \sim 2/\sigma^2 n$, and we can write $f(s) - s = (1-s)^2 f''(\xi_s)$, where $s \le \xi_s \le 1$. Then, for any $0 < \alpha < 1$,

$$\mathrm{E}\, T_n \sim \int_0^{f_{(n)}(0)} \frac{\mathrm{d}s}{(1-s)f''(\xi_s)} \sim \int_\alpha^{1-2/\sigma^2 n} \frac{\mathrm{d}s}{(1-s)f''(\xi_s)} \sim \frac{\sigma^2}{2}\log n.$$

Unfortunately, an analogue of $J(k)$ for noncritical processes is not yet available and a similar analysis to the foregoing cannot be performed. We conjecture, however, that the conditional mean time to extinction for supercritical processes with $\mathrm{E}\, Z \log Z < \infty$ grows linearly in n.

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