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A Bound on the Number of Edges in Graphs Without an Even Cycle

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We show that, for each fixed k, an n-vertex graph not containing a cycle of length $2k$ has at most $80\sqrt{k \log k} \cdot n^{1+1/k} + O(n)$ edges.

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1. Introduction

Let $\mathrm{ex}(n, F)$ be the largest number of edges in an n-vertex graph that contains no copy of a fixed graph F. The systematic study of $\mathrm{ex}(n, F)$ was started by Turán [17] over 70 years ago, and it has developed into a central problem in extremal graph theory (see the surveys [8, 10, 15]).

The function $\mathrm{ex}(n, F)$ exhibits a dichotomy: if F is not bipartite, then $\mathrm{ex}(n, F)$ grows quadratically in n, and is fairly well understood. If F is bipartite, $\mathrm{ex}(n, F)$ is subquadratic, and the order of magnitude is known for very few F. The simplest classes of bipartite graphs are trees, complete bipartite graphs, and cycles of even length. Most studies of $\mathrm{ex}(n, F)$ for bipartite F have concentrated on these two classes. In this paper we address the even cycles. For an overview of the status of $\mathrm{ex}(n, F)$ for complete bipartite graphs, see [2]. For a thorough survey of bipartite Turán problems, see [8].

The first bound on the problem is due to Erdős [5], who showed that $\mathrm{ex}(n, C_4) = O(n^{3/2})$. Thanks to the work of Erdős and Rényi [7], Brown [4, Section 3], and Kővari, Sós and Turán [11], it is now known that

$$\mathrm{ex}(n, C_4) = (1/2 + o(1))n^{3/2}.$$

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The current best bounds for $\mathrm{ex}(n, C_6)$ for large values of n are

$$0.5338n^{4/3} < \mathrm{ex}(n, C_6) \leqslant 0.6272n^{4/3},$$

due to Füredi, Naor and Verstraëte [9].

A general bound of $\mathrm{ex}(n, C_{2k}) \leqslant \gamma_k n^{1+1/k}$, for some unspecified constant γ_k, was asserted by Erdős [6, p. 33]. The first proof was by Bondy and Simonovits [3, Lemma 2], who showed that $\mathrm{ex}(n, C_{2k}) \leqslant 20kn^{1+1/k}$ for all sufficiently large n. This was improved by Verstraëte [18] to $8(k-1)n^{1+1/k}$ and by Pikhurko [14] to $(k-1)n^{1+1/k} + O(n)$. The principal result of the present paper is an improvement of these bounds.

Theorem 1.1. *Suppose G is an n-vertex graph that contains no C_{2k}, and $n \geqslant (2k)^{8k^2}$. Then*

$$\mathrm{ex}(n, C_{2k}) \leqslant 80\sqrt{k \log k} \cdot n^{1+1/k} + 10k^2 n.$$

It is our duty to point out that the improvement offered by Theorem 1.1 is of uncertain value, because we still do not know whether $\Theta(n^{1+1/k})$ is the correct order of magnitude for $\mathrm{ex}(n, C_{2k})$. Only for $k = 2, 3, 5$ are constructions of C_{2k}-free graphs with $\Omega(n^{1+1/k})$ edges known [1, 19, 12, 13]. The first author believes it likely that $\mathrm{ex}(n, C_{2k}) = o(n^{1+1/k})$ for all large k. We stress again that the situation is completely different for odd cycles, where the value of $\mathrm{ex}(n, C_{2k+1})$ is known exactly for all large n [16].

Proof method and organization of the paper. Our proof is inspired by that of Pikhurko [14]. Apart from a couple of lemmas that we quote from [14], the present paper is self-contained. However, we advise the reader to at least skim [14] to see the main idea in a simpler setting.

Pikhurko's proof builds a breadth-first search tree, and then argues that a pair of adjacent levels of the tree cannot contain a Θ-graph.[1] He then deduced that each level must be at least $\delta/(k-1)$ times larger than the previous one, where δ is the (minimum) degree. The bound on $\mathrm{ex}(n, C_{2k})$ then follows. The estimate of $\delta/(k-1)$ is sharp when we restrict our attention to a pair of levels.

In our proof we use three adjacent levels. We find a Θ-graph satisfying an extra technical condition that permits an extension of Pikhurko's argument. Annoyingly, this extension requires a bound on the *maximum degree*. To achieve such a bound we use a modification of breadth-first search that avoids the high-degree vertices.

What we really prove in this paper is as follows.

Theorem 1.2. *Suppose $k \geqslant 4$, and suppose G is a bipartite n-vertex graph of minimum degree at least $2d + 5k^2$, where*

$$d \geqslant \max(20\sqrt{k \log k} \cdot n^{1/k}, (2k)^{8k}). \tag{1.1}$$

Then G contains C_{2k}.

[1] We recall the definition of a Θ-graph in Section 3.

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Theorem 1.1 follows from Theorem 1.2 and two well-known facts: every graph contains a bipartite subgraph with half of the edges, and every graph of average degree d_{avg} contains a subgraph of minimum degree at least $d_{\text{avg}}/2$.

The rest of the paper is organized as follows. We present our modification of breadth-first search in Section 2. In Section 3, which is the heart of the paper, we explain how to find Θ-graphs in triples of consecutive levels. Finally, in Section 4 we assemble the pieces of the proof.

2. Graph exploration

Our aim is to have vertices of degree at most Δd for some $k \ll \Delta \ll d^{1/k}$. The particular choice is fairly flexible; we choose to use

$$\Delta \stackrel{\text{def}}{=} k^3.$$

Let G be a graph, and let x be any vertex of G. We start our exploration with the set $V_0 = \{x\}$, and mark the vertex x as explored. Suppose $V_0, V_1, \ldots, V_{i-1}$ are the sets explored in the 0th, 1st, $\ldots, (i-1)$st steps respectively. We then define V_i as follows.

(1) Let V_i' consist of those neighbours of V_{i-1} that have not yet been explored. Let Bg_i be the set of those vertices in V_i' that have more than Δd unexplored neighbours, and let $\mathrm{Sm}_i = V_i' \setminus \mathrm{Bg}_i$.

(2) Define

$$V_i = \begin{cases} V_i' & \text{if } |\mathrm{Bg}_i| > \dfrac{1}{2k}|V_i'|, \\ \mathrm{Sm}_i & \text{if } |\mathrm{Bg}_i| \leqslant \dfrac{1}{2k}|V_i'|. \end{cases}$$

The vertices of V_i are then marked as explored.

We call the sets $V_0, V_1, \ldots$ *levels* of G. A level V_i is *big* if

$$|\mathrm{Bg}_i| > \frac{1}{2k}|V_i'|,$$

and it is *normal* otherwise.

Lemma 2.1. *If $\delta \leqslant \Delta d$, and G is a bipartite graph of minimum degree at least δ, then each $v \in V_{i+1}$ has at least δ neighbours in $V_i \cup V_{i+2}'$.*

Proof. Fix a vertex $v \in V(G)$. We will show, by induction on i, that if $v \notin V_1 \cup \cdots \cup V_i$, then v has at least δ neighbours in $V(G) \setminus (V_1 \cup \cdots \cup V_{i-1})$. The base case $i = 1$ is clear. Suppose $i > 1$. If $v \in \mathrm{Bg}_i$, then v has $\Delta d \geqslant \delta$ neighbours in the required set. Otherwise, v is not in V_i' and hence has no neighbours in V_{i-1}. Hence, v has as many neighbours in $V(G) \setminus (V_1 \cup \cdots \cup V_{i-1})$ as in $V(G) \setminus (V_1 \cup \cdots \cup V_{i-2})$, and our claim follows from the induction hypothesis.

If $v \in V_{i+1}$, then the neighbours of v are a subset of $V_1 \cup \cdots \cup V_i \cup V_{i+2}'$. Hence, at least δ of these neighbours lie in $V_i \cup V_{i+2}'$. □

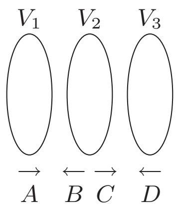

Figure 1. A trilayered graph

Trilayered graphs. We abstract out the properties of a triple of consecutive levels into the following definition. A *trilayered graph* with layers V_1, V_2, V_3 is a graph G on a vertex set $V_1 \cup V_2 \cup V_3$ such that the only edges in G are between V_1 and V_2, and between V_2 and V_3. If $V_1' \subset V_1$, $V_2' \subset V_2$ and $V_3' \subset V_3$, then we let $G[V_1', V_2', V_3']$ denote the trilayered subgraph induced by three sets V_1', V_2', V_3'. Because the graph G that has been explored is bipartite, there are no edges inside each level. Therefore any three sets V_{i-1}, V_i, V_{i+1}' from the exploration process naturally form a trilayered graph; these graphs and their subgraphs are the only trilayered graphs that appear in this paper.

We say that a trilayered graph has *minimum degree* at least $[A : B, C : D]$ if each vertex in V_1 has at least A neighbours in V_2, each vertex in V_2 has at least B neighbours in V_1, each vertex in V_2 has at least C neighbours in V_3, and each vertex in V_3 has at least D neighbours in V_2. A schematic drawing of such a graph is shown in Figure 1.

3. Θ-graphs

A Θ-*graph* is a cycle of length at least $2k$ with a chord. We shall use several lemmas from previous work.

Lemma 3.1 (Lemma 2.1 in [14], also Lemma 2 in [18]). *Let F be a Θ-graph and $1 \leqslant l \leqslant |V(F)| - 1$. Let $V(F) = W \cup Z$ be an arbitrary partition of its vertex set into two non-empty parts such that every path in F of length l that begins in W necessarily ends in W. Then F is bipartite with parts W and Z.* □

Lemma 3.2 (Lemma 2.2 in [14]). *Let $k \geqslant 3$. Any bipartite graph H of minimum degree at least k contains a Θ-graph.* □

Corollary 3.3. *Let $k \geqslant 3$. Any bipartite graph H of average degree at least $2k$ contains a Θ-graph.* □

For a graph G and a set $Y \subset V(G)$, let $G[Y]$ denote the graph induced on Y. For disjoint $Y, Z \subset V(G)$, let $G[Y, Z]$ denote the bipartite subgraph of G that is induced by the bipartition $Y \cup Z$.

Well-placed Θ-graphs. Suppose G is a trilayered graph with layers V_1, V_2, V_3. We say that a Θ-graph $F \subset G$ is *well-placed* if each vertex of $V(F) \cap V_2$ is adjacent to some vertex in

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$V_1 \setminus V(F)$. The condition ensures that, for each vertex v of F in V_2, there exists a path from the root to v that avoids F.

Lemma 3.4. *Suppose G is a trilayered graph with layers V_1, V_2, V_3 such that the degree of every vertex in V_2 is at least $2d + 5k^2$, and no vertex in V_2 has more than Δd neighbours in V_3. Suppose t is a non-negative integer, and let*

$$F = \frac{d \cdot e(V_1, V_2)}{8k|V_3|}.$$

Assume that

$$F \geqslant 2, \tag{3.1a}$$

$$e(V_1, V_2) \geqslant 2kF|V_1|, \tag{3.1b}$$

$$e(V_1, V_2) \geqslant 8k(t+1)^2(2\Delta k)^{2k-1}|V_1|, \tag{3.1c}$$

$$e(V_1, V_2) \geqslant 8(et/F)^t k|V_2|, \tag{3.1d}$$

$$e(V_1, V_2) \geqslant 20(t+1)^2|V_2|. \tag{3.1e}$$

Then at least one of the following holds.

(I) *There is a Θ-graph in $G[V_1, V_2]$.*
(II) *There is a well-placed Θ-graph in $G[V_1, V_2, V_3]$.*

The proof of Lemma 3.4 is in two parts: finding a trilayered subgraph of large minimum degree (Lemmas 3.5 and 3.6), and finding a well-placed Θ-graph inside that trilayered graph (Lemma 3.7).

Finding a trilayered subgraph of large minimum degree. The disjoint union of two bipartite graphs shows that a trilayered graph with many edges need not contain a trilayered subgraph of large minimum degree. We show that, in contrast, if a trilayered graph contains no Θ-graph between two of its levels, then it must contain a subgraph of large minimum degree. The next lemma demonstrates a weaker version of this claim: it leaves open the possibility that the graph contains a denser trilayered subgraph. In that case, we can iterate inside that subgraph, which is done in Lemma 3.6.

Lemma 3.5. *Let a, A, B, C, D be positive real numbers. Suppose G is a trilayered graph with layers V_1, V_2, V_3 and the degree of every vertex in V_2 is at least $d + 4k^2 + C$. Assume also that*

$$a \cdot e(V_1, V_2) \geqslant (A + k + 1)|V_1| + B|V_2|. \tag{3.2}$$

Then one of the following holds.

(I) *There is a Θ-graph in $G[V_1, V_2]$.*
(II) *There exist non-empty subsets $V_1' \subset V_1$, $V_2' \subset V_2$, $V_3' \subset V_3$ such that the induced trilayered subgraph $G[V_1', V_2', V_3']$ has minimum degree at least $[A : B, C : D]$.*
(III) *There is a subset $\widetilde{V}_2 \subset V_2$ such that $e(V_1, \widetilde{V}_2) \geqslant (1-a)e(V_1, V_2)$, and $|\widetilde{V}_2| \leqslant D|V_3|/d$.*

Proof. We suppose that alternative (I) does not hold. Then, by Corollary 3.3, the average degree of every subgraph of $G[V_1, V_2]$ is at most $2k$.

Consider the process that aims to construct a subgraph satisfying (II). The process starts with $V_1' = V_1$, $V_2' = V_2$ and $V_3' = V_3$, and at each step removes one of the vertices that violate the minimum degree condition on $G[V_1', V_2', V_3']$. The process stops when either no vertices are left or the minimum degree of $G[V_1', V_2', V_3']$ is at least $[A : B, C : D]$. Since in the latter case we are done, we assume that this process eventually removes every vertex of G.

Let R be the vertices of V_2 that were removed because at the time of removal they had fewer than C neighbours in V_3'. Put

$$E' \stackrel{\text{def}}{=} \{uv \in E(G) : u \in V_2,\ v \in V_3,\ \text{and } v \text{ was removed before } u\},$$
$$S \stackrel{\text{def}}{=} \{v \in V_2 : v \text{ has at least } 4k^2 \text{ neighbours in } V_1\}.$$

Note that $|E'| \leqslant D|V_3|$. We cannot have $|S| \geqslant |V_1|/k$, for otherwise the average degree of the bipartite graph $G[V_1, S]$ would be at least

$$\frac{4k}{1+1/k} \geqslant 2k.$$

So $|S| \leqslant |V_1|/k$.

The average degree condition on $G[V_1, S]$ implies that

$$e(V_1, S) \leqslant k(|V_1| + |S|) \leqslant (k+1)|V_1|.$$

Let u be any vertex in $R \setminus S$. Since it is connected to at least $(d + 4k^2 + C) - 4k^2 = d + C$ vertices of V_3, it must be adjacent to at least d edges of E'. Thus,

$$|R \setminus S| \leqslant |E'|/d \leqslant D|V_3|/d.$$

Assume that the conclusion (III) does not hold with $\widetilde{V}_2 = R \setminus S$. Then

$$e(V_1, R \setminus S) < (1-a)e(V_1, V_2).$$

Since the total number of edges between V_1 and V_2 that were removed due to the minimal degree conditions on V_1 and V_2 is at most $A|V_1|$ and $B|V_2|$ respectively, we conclude that

$$\begin{aligned} e(V_1, V_2) &\leqslant e(V_1, S) + e(V_1, R \setminus S) + A|V_1| + B|V_2| \\ &< (k+1)|V_1| + (1-a)e(V_1, V_2) + A|V_1| + B|V_2|, \end{aligned}$$

implying that

$$a \cdot e(V_1, V_2) < (A + k + 1)|V_1| + B|V_2|.$$

The contradiction with (3.2) completes the proof. □

Remark. The next lemma can be eliminated at the cost of obtaining the bound

$$\operatorname{ex}(n, C_{2k}) = O(k^{2/3} n^{1+1/k})$$

in place of

$$\operatorname{ex}(n, C_{2k}) = O(\sqrt{k \log k} \cdot n^{1+1/k}).$$

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To do that, we can set $B \approx k^{2/3}$, $D \approx k^{1/3}$ and $a = 1/2$. One can then show that when applied to trilayered graphs arising from the exploration process, the alternative (III) leads to a subgraph of average degree $2k$. The two remaining alternatives are dealt by Corollary 3.3 and Lemma 3.7. However, it is possible to obtain a better bound by iterating the preceding lemma.

Lemma 3.6. *Let C be a positive real number. Suppose G is a trilayered graph with layers V_1, V_2, V_3, and the degree of every vertex in V_2 is at least $d + 4k^2 + C$. Let*

$$F = \frac{d \cdot e(V_1, V_2)}{8k|V_3|},$$

and assume that F and $e(V_1, V_2)$ satisfy (3.1) *for some integer $t \geqslant 1$. Then one of the following holds.*

(I) *There is a Θ-graph in $G[V_1, V_2]$.*

(II) *There exist numbers A, B, D and non-empty subsets $V_1' \subset V_1$, $V_2' \subset V_2$, $V_3' \subset V_3$ such that the induced trilayered subgraph $G[V_1', V_2', V_3']$ has minimum degree at least $[A : B, C : D]$, with the following inequalities that bind A, B, and D:*

$$\begin{aligned} &B \geqslant 5, \qquad (B-4)D \geqslant 2k, \\ &A \geqslant 2k(\Delta D)^{D-1}. \end{aligned} \tag{3.3}$$

Proof. Assume, for the sake of contradiction, that neither (I) nor (II) holds. Set

$$a_j = \frac{1}{t-j+1} \quad \text{for } j = 0, \ldots, t-1.$$

We shall define a sequence of sets $V_2 = V_2^{(0)} \supseteq V_2^{(1)} \supseteq \cdots \supseteq V_2^{(t)}$ inductively. We let

$$d_i \stackrel{\text{def}}{=} e(V_1, V_2^{(i)})/|V_2^{(i)}|$$

denote the average degree from $V_2^{(i)}$ into V_1. The sequence $V_2^{(0)}, V_2^{(1)}, \ldots, V_2^{(t)}$ will be constructed so as to satisfy

$$e(V_1, V_2^{(i+1)}) \geqslant (1 - a_i)e(V_1, V_2^{(i)}), \tag{3.4}$$

$$d_{i+1} \geqslant d_i \cdot Fa_i \prod_{j=0}^{i}(1 - a_j). \tag{3.5}$$

Note that (3.4) and the choice of $a_0, \ldots, a_i$ imply that

$$e(V_1, V_2^{(i)}) \geqslant \frac{1}{t+1} e(V_1, V_2). \tag{3.6}$$

The sequence starts with $V_2^{(0)} = V_2$. Assume $V_2^{(i)}$ has been defined. We proceed to define $V_2^{(i+1)}$. Put

$$\begin{aligned} A &= a_i e(V_1, V_2^{(i)})/2|V_1| - k - 1, \\ B &= a_i d_i/4 + 5, \\ D &= \min(2k, 8k/a_i d_i). \end{aligned}$$

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With the help of (3.6) and (3.1c) it is easy to check that the inequalities (3.3) hold for this choice of constants.

In addition,

$$\begin{aligned}(A+k+1)|V_1| + B|V_2^{(i)}| &= \frac{3}{4}a_i e(V_1, V_2^{(i)}) + 5|V_2^{(i)}| \\ &\overset{(3.1e)}{\leqslant} \frac{3}{4}a_i e(V_1, V_2^{(i)}) + \frac{1}{4(t+1)^2} e(V_1, V_2) \\ &\overset{(3.6)}{\leqslant} a_i e(V_1, V_2^{(i)}).\end{aligned}$$

So, the condition (3.2) of Lemma 3.5 is satisfied for the graph $G[V_1, V_2^{(i)}, V_3]$. By Lemma 3.5 there is a subset $V_2^{(i+1)} \subset V_2^{(i)}$ satisfying (3.4) and

$$|V_2^{(i+1)}| \leqslant D|V_3|/d.$$

Next we show that the set $V_2^{(i+1)}$ satisfies inequality (3.5). Indeed, we have

$$\begin{aligned} d_{i+1} &= \frac{e(V_1, V_2^{(i+1)})}{|V_2^{(i+1)}|} \geqslant \frac{(1-a_i)e(V_1, V_2^{(i)})}{D|V_3|/d} \geqslant (1-a_i)a_i d_i \frac{d}{8k|V_3|} e(V_1, V_2^{(i)}) \\ &\overset{(3.4)}{\geqslant} (1-a_i)a_i d_i \frac{d \cdot e(V_1, V_2)}{8k|V_3|} \prod_{j=0}^{i-1}(1-a_j) = d_i \cdot F a_i \prod_{j=0}^{i}(1-a_j).\end{aligned}$$

Iterative application of (3.5) implies

$$d_t \geqslant d_0 F^t \prod_{j=0}^{t-1} a_j (1-a_j)^{t-j} \geqslant d_0 F^t \prod_{j=0}^{t-1} \frac{e^{-1}}{t-j+1} = d_0 \frac{(F/e)^t}{(t+1)!}. \tag{3.7}$$

If we have $|V_2^{(t)}| < |V_1|$, then the average degree of induced subgraph $G[V_1, V_2^{(t)}]$ is greater than

$$e(V_1, V_2^{(t)})/|V_1| \overset{(3.6)}{\geqslant} e(V_1, V_2)/(t+1)|V_1| \overset{(3.1c)}{\geqslant} 2k,$$

which by Corollary 3.3 leads to outcome (I).

If $|V_2^{(t)}| \geqslant |V_1|$ and $d_t \geqslant 4k$, then the average degree of $G[V_1, V_2^{(t)}]$ is at least $d_t/2 \geqslant 2k$ because d_t is the average degree of $V_2^{(t)}$ into V_1, again leading to outcome (I). So, we may assume that $d_t < 4k$. Since $(t+1)! \leqslant 2t^t$ we deduce from (3.7) that

$$d_0 < 4k(t+1)!(e/F)^t \leqslant 8k(et/F)^t.$$

This contradicts (3.1d), and so the proof is complete. □

Locating well-placed Θ-graphs in trilayered graphs. We come to the central argument of the paper. It shows how to embed well-placed Θ-graphs into trilayered graphs of large minimum degree. Or rather, it shows how to embed well-placed Θ-graphs into regular trilayered graphs; the contortions of the previous two lemmas, and the factor of $\sqrt{\log k}$ in the final bound, come from the authors' inability to deal with irregular graphs.

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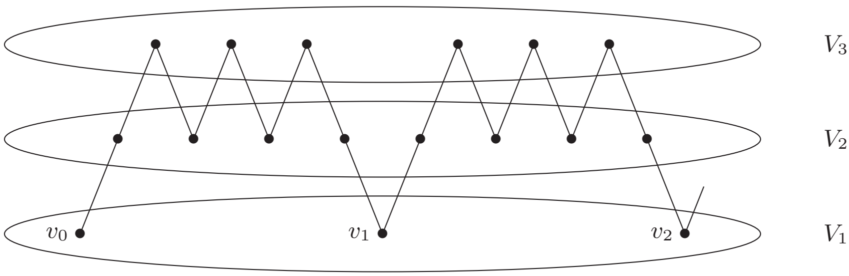

Figure 2. A path in a trilayered graph

Lemma 3.7. *Let A, B, D be positive real numbers. Let G be a trilayered graph with layers V_1, V_2, V_3 of minimum degree at least $[A : B, d+k : D]$. Suppose that no vertex in V_2 has more than Δd neighbours in V_3. Assume also that*

$$B \geqslant 5, \tag{3.8}$$

$$(B-4)D \geqslant 2k-2, \tag{3.9}$$

$$A \geqslant 2k(\Delta D)^{D-1}. \tag{3.10}$$

Then G contains a well-placed Θ-graph.

Proof. Assume, for the sake of contradiction, that G contains no well-placed Θ-graphs. Leaning on this assumption, we shall build an arbitrary long path P of the form shown in Figure 2, where, for each i, vertices v_i and v_{i+1} are joined by a path of length $2D$ that alternates between V_2 and V_3. Since the graph is finite, this would be a contradiction.

While building the path we maintain the following property:

$$\text{Every } v \in P \cap V_2 \text{ has at least one neighbour in } V_1 \setminus P. \tag{$\star$}$$

We call a path satisfying ($\star$) *good*.

We construct the path inductively. We begin by picking v_0 arbitrarily from V_1. Suppose a good path $P = v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1}$ has been constructed, and we wish to find a path extension $v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow v_l$.

There are at least about A ways to extend the path by a single vertex. The idea of the following argument shows that many of these extensions can be extended to another vertex, and then another, and so on.

For each $i = 1, 2, \ldots, 2D-1$, we shall define a family $\mathcal{Q}_i$ of good paths that satisfy the following.

(1) Each path in $\mathcal{Q}_i$ is of the form $v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u$, where $v_{l-1} \leftrightsquigarrow u$ is a path of length i that alternates between V_2 and V_3. The vertex u is called a *terminal* of the path. The set of terminals of the paths in $\mathcal{Q}_i$ is denoted by $T(\mathcal{Q}_i)$. Note that $T(\mathcal{Q}_i) \subset V_2$ for odd i and $T(\mathcal{Q}_i) \subset V_3$ for even i.

(2) For each i, the paths in $\mathcal{Q}_i$ have distinct terminals.

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(3) For odd-numbered indices, we have the inequality

$$|\mathcal{Q}_{2i+1}| \geqslant -3k + A\left(\frac{1}{\Delta}\right)^i \prod_{j\leqslant i}\left(1-\frac{j}{D}\right). \tag{3.11}$$

(4) For even-numbered indices, we have the inequality

$$e(T(\mathcal{Q}_{2i}), V_2) \geqslant d|\mathcal{Q}_{2i-1}|. \tag{3.12}$$

Let

$$t \stackrel{\text{def}}{=} \lceil B/2 \rceil.$$

We will repeatedly use the following straightforward fact, which we call the *small-degree argument*: whenever Q is a good path and $u \in V_2 \setminus Q$ is adjacent to the terminal of Q, then u is adjacent to fewer than t vertices in $V_1 \cap Q$. Indeed, if vertex u were adjacent to $v_{j_1}, v_{j_2}, \dots, v_{j_t} \in V_1 \cap Q$ with $j_1 < j_2 < \dots < j_k$, then $v_{j_2} \leftrightsquigarrow u$ (along path Q) and the edge uv_{j_2} would form a cycle of total length at least

$$2D(t-2)+2 \geqslant 2D(B/2-2)+2 \overset{(3.9)}{\geqslant} 2k.$$

As uv_{j_3} is a chord of the cycle, and u is adjacent to v_{j_1} that is not on the cycle, that would contradict the assumption that G contains no well-placed Θ-graph.

The set $\mathcal{Q}_1$ consists of all paths of the form Pu for $u \in V_2 \setminus P$. Let us check that the preceding conditions hold for $\mathcal{Q}_1$. Vertex v_{l-1} cannot be adjacent to k or more vertices in $P \cap V_2$, for otherwise G would contain a well-placed Θ-graph with a chord through v_{l-1}. So, $|\mathcal{Q}_1| \geqslant A-k$. Next, consider any $u \in V_2 \setminus P$ that is a neighbour of v_{l-1}. By the small-degree argument vertex u cannot be adjacent to t or more vertices of $P \cap V_1$, and Pu is good.

Suppose $\mathcal{Q}_{2i-1}$ has been defined, and we wish to define $\mathcal{Q}_{2i}$. Consider an arbitrary path $Q = v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u \in \mathcal{Q}_{2i-1}$. Vertex u cannot have k or more neighbours in $Q \cap V_3$, for otherwise G would contain a well-placed Θ-graph with a chord through u. Hence, there are at least d edges of the form uw, where $w \in V_3 \setminus Q$. As we vary u we obtain a family of at least $d|\mathcal{Q}_{2i-1}|$ paths. We let $\mathcal{Q}_{2i}$ consist of any maximal subfamily of such paths with distinct terminals. The condition (3.12) follows automatically as each vertex of $T(\mathcal{Q}_{2i-1})$ has at least d neighbours in $T(\mathcal{Q}_{2i})$.

Suppose $\mathcal{Q}_{2i}$ has been defined, and we wish to define $\mathcal{Q}_{2i+1}$. Consider an arbitrary path $Q = v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u \in \mathcal{Q}_{2i}$. An edge uw is called *long* if $w \in P$, and w is at a distance exceeding $2k$ from u along path Q. If uw is a long edge, then from u to Q there is only one edge, namely the edge to the predecessor of u on Q, for otherwise there is a well-placed Θ-graph. Also, at most i neighbours of u lie on the path $v_{l-1} \leftrightsquigarrow u$. Since $\deg u \geqslant D$, it follows that there are at least $(1-i/D)\deg u$ short edges from u that miss $v_{l-1} \leftrightsquigarrow u$. Thus there is a set $\mathcal{W}$ of at least $(1-i/D)e(T(\mathcal{Q}_{2i}), V_2)$ walks (not necessarily paths!) of the form $v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow uw$ such that $v_{l-1} \leftrightsquigarrow uw$ is a path and w occurs only among the last $2k$ vertices of the walk.

From the maximum degree condition on V_2 it follows that walks in $\mathcal{W}$ have at least

$$(1-i/D)e(T(\mathcal{Q}_{2i}), V_2)/\Delta d$$

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distinct terminals. A walk fails to be a path only if the terminal vertex lies on P. However, since the edge uw is short, this can happen for at most $2k$ possible terminals. Hence, there is a $\mathcal{Q}_{2i+1} \subset \mathcal{W}$ of size

$$|\mathcal{Q}_{2i+1}| \geqslant (1 - i/D)e(T(\mathcal{Q}_{2i}), V_2)/\Delta d - 2k \tag{3.13}$$

that consists of paths with distinct terminals. It remains to check that every path in $\mathcal{Q}_{2i+1}$ is good. The only way that $Q = v_0 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow uw \in \mathcal{Q}_{2i+1}$ may fail to be good is if w has no neighbours in $V_1 \setminus Q$. By the small-degree argument w has fewer than t neighbours in V_1. Since w has at least B neighbours in V_1 and $B \geqslant t + 2$, we conclude that w has at least *two* neighbours in V_1 outside the path. Of course, the same is true for every terminal of a path in $\mathcal{Q}_{2i+1}$. The condition (3.11) for $\mathcal{Q}_{2i+1}$ follows from (3.13), (3.12) and from the validity of (3.11) for $\mathcal{Q}_{2i-1}$.

Note that $\mathcal{Q}_{2D-1}$ is non-empty. Let $Q = v_0 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u \in \mathcal{Q}_{2D-1}$ be an arbitrary path. Note that since $2D - 1$ is odd, $u \in V_2$. By the property of terminals of V_i (odd i) that we noted in the previous paragraph, there are two vertices in $V_1 \setminus Q$ that are neighbours of u. Let v_l be any of them, and let the new path be $Qv_l = v_0 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow uv_l$. This path can fail to be good if there is a vertex w on the path Q that is good in Q, but is bad in Qv_l. By the small-degree argument, w is adjacent to fewer than t vertices in $Q \cap V_1$ that precede w in Q. The same argument applied to the reversal of the path Qv_l shows that w is adjacent to fewer than t vertices in $Q \cap V_1$ that succeeds w in Q. Since $2t - 2 < B$, the path Qv_l is good.

Hence, it is possible to build an arbitrarily long path in G. This contradicts the finiteness of G. □

Lemma 3.4 follows from Lemmas 3.6 and 3.7 by setting $C = d + k$, in view of inequality $4k^2 + k \leqslant 5k^2$. We lose $k^2 - k$ here for cosmetic reasons: $5k^2$ is tidier than $4k^2 + k$.

4. Proof of Theorem 1.2

Suppose that G is a bipartite graph of minimum degree at least $2d + 5k^2$ and contains no C_{2k}. Pick a root vertex x arbitrarily, and let $V_0, V_1, \ldots, V_{k-1}$ be the levels obtained from the exploration process in Section 2.

Lemma 4.1. *For $1 \leqslant i \leqslant k - 1$, the graph $G[V_{i-1}, V_i, V_{i+1}]$ contains no well-placed Θ-graph.*

Proof. The following proof is almost an exact repetition of the proof of Claim 3.1 from [14] (which is also reproduced as Lemma 4.2 below).

Suppose, for the sake of contradiction, that a well-placed Θ-graph $F \subset G[V_{i-1}, V_i, V_{i+1}]$ exists. Let $Y = V_i \cap V(F)$. Since F is well-placed, for every vertex of Y there is a path avoiding $V(F)$ of length i to the vertex x. The union of these paths forms a tree T with x as a root. Let y be the vertex farthest from x such that every vertex of Y is a T-descendant of y. Paths that connect x to Y branch at y. Pick one such branch, and let $W \subset Y$ be the set of all the T-descendants of that branch. Let $Z = V(F) \setminus W$. From $W \neq V_i \cap V(F)$ it follows that Z is not an independent set of F, and so $W \cup Z$ is not a bipartition of F.

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Let ℓ be the distance between x and y. We have $\ell < i$ and $2k - 2i + 2\ell < 2k \leqslant |V(F)|$. By Lemma 3.1 in F there is a path P of length $2k - 2i + 2\ell$ that starts at some $w \in W$ and ends in $z \in Z$. Since the length of P is even, $z \in Y$. Let P_w and P_z be unique paths in T that connect y to w and z, respectively. They intersect only at y. Each of P_w and P_z has length $i - \ell$. The union of paths P, P_w, P_z forms a $2k$-cycle in G. □

The same argument (with a different Y) also proves the next lemma.

Lemma 4.2 (Claim 3.1 in [14]). *For $1 \leqslant i \leqslant k-1$, neither of $G[V_i]$ and $G[V_i, V_{i+1}]$ contains a bipartite Θ-graph.* □

The next step is to show that the levels $V_0, V_1, V_2, \ldots$ increase in size. We shall show by induction on i that

$$e(V_i, V_{i+1}) \geqslant d|V_i|, \tag{4.1}$$

$$e(V_i, V_{i+1}) \leqslant 2k|V_{i+1}|, \tag{4.2}$$

$$e(V_i, V'_{i+1}) \leqslant 2k|V'_{i+1}|, \tag{4.3}$$

$$|V_{i+1}| \geqslant (2k)^{-1} d|V_i|, \tag{4.4}$$

$$|V_{i+1}| \geqslant \frac{d^2}{400k \log k}|V_{i-1}|. \tag{4.5}$$

To prove Theorem 1.2, we only need (4.5); the remaining inequalities play auxiliary roles in the derivation of (4.5).

Clearly, these inequalities hold for $i = 0$ since each vertex of V_1 sends only one edge to V_0.

Proof of (4.1). By Lemma 2.1 the degree of every vertex in V_i is at least $2d + 4k$, and so

$$e(V_i, V'_{i+1}) \geqslant (2d + 4k)|V_i| - e(V_{i-1}, V_i) \overset{\text{induc.}}{\geqslant} (2d + 2k)|V_i|.$$

We next distinguish two cases depending on whether V_{i+1} is big (in the sense of the definition from Section 2). If V_{i+1} is big, then $e(V_i, V_{i+1}) = e(V_i, V'_{i+1})$, and (4.1) follows. If V_{i+1} is normal, then Corollary 3.3 and Lemma 4.2 imply that

$$e(V_i, \mathrm{Bg}_{i+1}) \leqslant k(|V_i| + |\mathrm{Bg}_{i+1}|) \leqslant k\left(|V_i| + \frac{1}{2k}|V'_{i+1}|\right) \leqslant k|V_i| + \frac{1}{2}e(V_i, V'_{i+1})$$

and so

$$e(V_i, V_{i+1}) = e(V_i, V'_{i+1}) - e(V_i, \mathrm{Bg}_{i+1}) \geqslant \frac{1}{2}e(V_i, V'_{i+1}) - k|V_i| \geqslant d|V_i|$$

implying (4.1). □

Proof of (4.2). Consider the graph $G[V_i, V_{i+1}]$. Inequality (4.1) asserts that the average degree of V_i is at least $d \geqslant 2k$. If (4.2) does not hold, then the average degree of V_{i+1} is at least $2k$ as well, contradicting Corollary 3.3 and Lemma 4.2. □

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Proof of (4.3). The argument is the same as for (4.2) with $G[V_i, V'_{i+1}]$ in place of $G[V_i, V_{i+1}]$. □

Proof of (4.4). This follows from (4.2) and (4.1). □

Proof of (4.5) in the case V_i is a normal level. We assume that (4.5) does not hold and will derive a contradiction. Consider the trilayered graph $G[V_{i-1}, V_i, V'_{i+1}]$. Let $t = 2\log k$. Suppose for the moment that the inequalities (3.1) in Lemma 3.4 hold. Then, since V_i is normal, each vertex in V_i has at most Δd neighbours in V'_{i+1}, and so Lemma 3.4 applies. However, the lemma's conclusion contradicts Lemmas 4.1 and 4.2. Hence, to prove (4.5) it suffices to verify inequalities (3.1a)–(3.1d) with $F = d \cdot e(V_{i-1}, V_i)/8k|V'_{i+1}|$.

We may assume that

$$F \geqslant 2e^2 \log k, \tag{4.6}$$

and in particular that (3.1a) holds. Indeed, if (4.6) were not true, then inequality (4.1) would imply

$$|V'_{i+1}| \geqslant (d^2/16e^2 k \log k)|V_{i-1}|,$$

and thus

$$|V_{i+1}| \geqslant \left(1 - \frac{1}{k}\right)|V'_{i+1}| \geqslant (d^2/32e^2 k \log k)|V_{i-1}|,$$

and so (4.5) would follow in view of $32e^2 \leqslant 400$.

Inequality (3.1b) is implied by (4.4). Indeed,

$$e(V_{i-1}, V_i) = 8k|V'_{i+1}|F/d \geqslant 8k|V_{i+1}|F/d \overset{(4.4)}{\geqslant} 4F|V_i| \overset{(4.4)}{\geqslant} 2k^{-1}dF|V_{i-1}|,$$

and $d \geqslant k^2$ by the definition of d from (1.1).

Inequality (3.1c) is implied by (1.1) and (4.1).

Next, suppose (3.1d) were not true. Since $F/t \geqslant e^2$ by (4.6), we would then conclude

$$\begin{aligned} |V_{i+1}| &\overset{(4.4)}{\geqslant} (2k)^{-1}d|V_i| \geqslant d(16k^2)^{-1}(F/et)^t e(V_{i-1}, V_i) \\ &\geqslant d(16k^2)^{-1}e^{2\log k}e(V_{i-1}, V_i) \overset{(4.1)}{\geqslant} \frac{1}{16}d^2|V_{i-1}|, \end{aligned}$$

and so (4.5) would follow.

Finally, (3.1e) is a consequence of (4.1). □

Proof of (4.5) in the case V_i is a big level. We have

$$\begin{aligned} |V_{i+1}| &\geqslant \frac{1}{2}|V'_{i+1}| \overset{(4.3)}{\geqslant} (4k)^{-1}e(V_i, V'_{i+1}) \geqslant (4k)^{-1}e(\mathrm{Bg}_i, V'_{i+1}) \geqslant (4k)^{-1}\Delta d|\mathrm{Bg}_i| \\ &\geqslant (8k^2)^{-1}\Delta d|V_i| \overset{(4.4)}{\geqslant} (16k^3)^{-1}\Delta d^2|V_{i-1}| = \frac{1}{16}d^2|V_{i-1}|, \end{aligned}$$

and so (4.5) holds. □

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We are ready to complete the proof of Theorem 1.2. If k is even, then $k/2$ applications of (4.5) yield

$$|V_k| \geqslant \frac{d^k}{(400k\log k)^{k/2}}.$$

If k is odd, then $(k-1)/2$ applications of (4.5) yield

$$|V_k| \geqslant \frac{d^{k-1}}{(400k\log k)^{(k-1)/2}}|V_1| \geqslant \frac{d^k}{(400k\log k)^{(k-1)/2}}.$$

Either way, since $|V_k| < n$ we conclude that $d < 20\sqrt{k\log k}\cdot n^{1/k}$.

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