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QUANTIFYING THE IMPACT OF PARTIAL INFORMATION ON SHARPE RATIO OPTIMIZATION

Lihong Zhang
School of Economics and Management
Tsinghua University
Beijing 100084, PRC
E-mail: zhanglh2@sem.tsinghua.edu.cn.

Lin Zhao
Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing 100190, PRC
E-mail: zhaolin@iss.ac.cn.

Motivated by the fact that many investors have limited ability to update the expectation regarding future stock returns with the arrival of new information instantly, this paper provides a continuous-time model to study the performance of passive trading strategies. We derive the true Sharp ratio of the passive strategies in terms of the mean and variance of an explicit stochastic process. Based on this expression, we quantify the impact of partial information by performing a thorough comparative static analysis. Such an analysis provides a rationale for why investors with inaccurate information about stock return behave better in the mean-reverting environment than in the i.i.d. environment and why pessimistic investors can achieve better performance than optimistic ones. As a by-product, we propose an analytical approach to compute the "implied" parameters in stock return predictor for both i.i.d. and mean-reverting dynamics, which seems interesting for future research.

1. INTRODUCTION

In financial markets, investors usually have partial information about stock returns. The vast literature studying the implications of partial information generally assumes that investors can infer the information through a perfect learning mechanism. However, due to investors' limited time, limited attention, limited access to information, or limited computing ability, passive strategies which cannot update the expectation regarding future stock returns with the arrival of new information instantly seem more prevalent in real markets. The focus of this paper is on the performance of passive strategies, taking the perfect learner's performance as a benchmark. Since Sharpe ratio is a widely welcomed evaluation method which has a solid theoretical foundation and an easy tractability, we take Sharpe ratio as the performance measure in this paper.[1]

[1] The literature on Sharpe ratio optimization is vast, see Sharpe [31], Sharpe [32], and Modigliani and Modigliani [27], among others.

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There are several recent papers which are also concerned with inaccurate underlying information and non-perfect learning. Peng [29] studies the learning process of a representative investor with a capacity (or attention) constraint and predicts that assets with greater total fundamental volatility will attract more capacity allocation from the investor. Peng and Xiong [30] point out that limited investor attention leads to category-learning behavior, which when combined with investor overconfidence generates important features observed in return comovement. Gomes [18] empirically tests the utility gains from exploiting short-run predictability in the volatility of stock returns and finds that utility gains are quite significant, both ex ante and out-of-sample. We contribute to this literature by investigating non-perfect learning associated with passive trading strategies affects investors' performance in the Sharpe ratio framework.

The new findings of our paper lie in the following three aspects.

First, for an investor who strives to maximize the Sharpe ratio $\mathscr{S}_T$ (hereafter T is a predetermined investment horizon) of his portfolio but follows passive trading strategies without exploiting the dynamics of return predictor, we are able to compute the true Sharp ratio performance of his passive strategies $\mathscr{S}_T^{\text{true}}$ in terms of the mean and variance of an explicit stochastic process, thanks to Proposition 1 below. Since both the pre-specified goal $\mathscr{S}_T$ and the ex post realization $\mathscr{S}_T^{\text{true}}$ are accessible to the investor, Proposition 1 thus provides an analytical approach to compute implied parameters in the actual stock return predictor. For example, given that the true but unobservable drift $\tilde{\mu}$ of the stock price in the market is normally distributed with mean $\bar{v}$ and variance δ_0, independent of the Brownian motion in price fluctuation, if the investor projects his portfolio strategy just based on the passive information $\tilde{\mu} = \bar{v}$, the actual Sharpe ratio produced by the market is doomed to be different from his inferred goal. The investor thus can figure out the variance δ_0 by equating the theoretical prediction with the realized observation. This procedure helps the investor to understand the market trend more thoroughly in an ex-post way, and may serve as a guide to his subsequent investments. An empirical test of the validity of such a procedure seems interesting for future research.

Second, as an application of Proposition 1, we can quantify analytically the gap between the aimed Shape ratio $\mathscr{S}_T$ and the actually realized one $\mathscr{S}_T^{\text{true}}$, denoted by $\mathscr{R}_T^{\text{error}}$, which facilitate the comparative static analysis of the performance loss with respect to fundamental factors in underlying dynamics. Notably, our comparative static analysis provides a rationale for why investors following passive trading strategies behave better in the mean-reverting environment than in the i.i.d. environment and why pessimistic passive investors can achieve better performance than optimistic ones.

Third, the comparative statics shows that investors who follow passive trading strategies without inferring the dynamics of stock return predictor are only competent for short-run investments. For long-run investments, the accumulated potential error of the passive strategies becomes huge, giving rise to the poor reliability. Our numerical results illustrate that for a fixed time horizon, employing multi-period strategies instead of one-period long-run strategies helps passive investors to improve the reliability.

Our paper is related to but different from the following literature. Detemple [11], Dothan and Feldman [12], Gennotte [17], Feldman [16], David [10], Barberis [2], Veronesi [33], Xia [35], Pástor and Veronesi [28], Cvitanić et al. [9] and Leippold, Trojani and Vanini [22] explore the implications of learning in portfolio optimization. However, different from ours, these papers all invoke utility-based maximization to describe the investor's policy. Since utility-based optimization and mean-variance optimization do not agree with each other,[2]

[2] Zhao and Ziemba [37] clarify the non-efficiency of the utility maximization in mean and standard deviation. They also give an intuitive criteria in knowing which approach makes a sound investment decision.

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the conclusions in these papers cannot be applied directly to fund managers who employ Sharpe ratio rule. Xiong and Zhou [36] establish a generalized mathematical solution to mean-variance optimization for Bayesian learners with the same information structure as in our model. However, they do not consider how investors are misled to deviated outcomes by partial information and passive trading strategies, which constitutes the main body of our article. From this point of view, our Proposition 1 is new in the literature in the sense that it firstly gives an analytical expression for the true performance of the portfolio which is constructed on inaccurate stock return predictor. Cvitanić, Lazrakb A and Wang [8] report and quantify some implications of the Sharpe ratio performance for investors who follow up the market information exactly. However, they do not deal with the information processing problem and thus cannot shed light on how inaccurate information misleads investors. One should distinguish the imprecise information in our model from investor's ambiguity or multi-priors. The latter is a popular subject in recent research, see for example Epstein and Wang [15] and Epstein and Schneider [14].

Our Sharpe ratio portfolio model with partial information on stock return predictor is a combination of the Kalman filtering problem and the Sharpe ratio optimization problem. The former problem can be solved by the "separation theorem" (Xiong and Zhou [36]) and the latter problem with perfect market information has already been solved by Cvitanić et al. [8]. The technical contribution of our paper is thus not the solving of an integration of these two problems, but rather the analytical expression for the relation between the pre-specified goal $\mathscr{S}_T$ and the ex post realization $\mathscr{S}_T^{\text{true}}$ and a thorough comparative static analysis of the performance of passive strategies with respect to fundamental factors in underlying dynamics. In this direction, our paper is in the same spirit of Caliendo and Huang [6], who study the implications of maximizing the life-cycle consumption utility based on an unrealistic (optimistic) estimation of asset return.

This paper is organized as follows. Section 2 presents the market information structure to be used to construct the model. In Section 3, we build a quite general model to characterize the way in which investors are misled by partial information. In this section, we compute the aforementioned $\mathscr{S}_T^{\text{true}}$ and $\mathscr{S}_T^{\text{potential}}$, and introduce the reliability measures $\mathscr{R}_T^{\text{error}}$ and $\mathscr{R}_T^{\text{loss}}$. Sections 4 and 5 are devoted to the comparative static studies when the general model is specified in the i.i.d. and the mean-reverting environments respectively. In the i.i.d. environment, a technical contribution is that we can carry through all the above calculations analytically. In the mean-reverting case, we express the desired quantities in a system of ordinary differential equations (ODEs), and such ODEs can be solved numerically in a simple way. In Section 6, we discuss multi-period strategies via numerical illustrations. At the end, Section 7 concludes our work. All proofs and calculations are relegated to the Appendix.

2. ASSET DYNAMICS

The financial market consists of one risk-free bond, paying interest at an exogenously given positive constant rate r, and one risky security, referred to stock, the return of which is unobservable for the investor.[3] To model uncertainty, we adopt a framework where the drift of stock is unobservable. For concreteness, the stock price S_t^u evolves as[4]

$$\frac{dS_t^u}{S_t^u} = (r + \sigma\vartheta_t^u)dt + \sigma dW_t^u. \tag{1}$$

[3] Whether there are more than one risky assets available for the investor is not crucial to our analysis. All the results in this article can be extended straightforwardly to the case with multiple risky assets.

[4] The evolution (1) of price S_t^u is routine in continuous-time financial theory. See Merton [26].

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Here W_t^u is a standard Brownian motion which is defined in a complete probability space $(\Omega, \mathscr{F}, P)$ and $\mathscr{F}_t^u := \sigma\{W_s^u | 0 \le s \le t\}$ denotes the filtration generated by W_t^u. The risk award ϑ_t^u determines the dynamic price of risk in economy, which is a $\mathscr{F}_t^u$-adapted process to be specified later. The volatility σ is assumed to be a known positive constant.[5] Here we use the superscript "u" to identify the scenarios which is unobservable for the investor.

We assume that the investor cannot observe ϑ_t^u or W_t^u directly and has to estimate the current risky return by observing past and present stock prices. Thus the filtration $\mathscr{F}_t^{S^u} := \sigma\{S_s^u | 0 \le s \le t\}$ denotes the information available to the investor up to time t, and the process W_t defined by

$$dW_t := \frac{1}{\sigma}\left(\frac{dS_t^u}{S_t^u} - E[r + \sigma\vartheta_t^u | \mathscr{F}_t^{S^u}]dt\right) \tag{2}$$

signifies the observed randomness in risky stock prices. Notice that the information filtration $\mathscr{F}_t^{S^u}$ is smaller than $\mathscr{F}_t^u$ because of $\mathscr{F}_t^u = \mathscr{F}_t^{S^u} \vee \sigma\{\vartheta_s^u | 0 \le s \le t\}$, and W_t is a standard Brownian motion which is adapted to $\mathscr{F}_t^{S^u}$.[6] The investor infers the posterior risk reward at time t from $\mathscr{F}_t^{S^u}$. Accordingly, from the investor's viewpoint, the evolution of stock price is

$$\frac{dS_t}{S_t} = (r + \sigma\vartheta_t)dt + \sigma dW_t. \tag{3}$$

In the literature, ϑ_t^u is also called predictor of the mean return and ϑ_t is the inferred predictor, see for instance Xia [35] and Wachter [34].

This paper distinguishes two types of investors:

- Type I is called "active investors" or "learners", who attempt to learn about ϑ_t by observing the market price of assets and update their expectation with the arrival of new information. To be specific, we assume these investors take ϑ_t to be $(1+\epsilon)\theta_t$, where

$$\theta_t := E[\vartheta_t^u | \mathscr{F}_t^{S^u}] \tag{4}$$

  is an item arising from Bayesian learning. The parameter ϵ is a constant satisfying $\epsilon > -1$, characterizing the subjective attitude of investor.[7] When $\epsilon > 0$, the investor is optimistic and overestimates the mean return. When $-1 < \epsilon < 0$, the investor is pessimistic and thus underestimates the mean return. When $\epsilon = 0$, the investor estimates the mean return perfectly.
- Type II is labeled as "passive investors".[8] These investors prefer not to implement data mining through learning. In our model, the passive investor thinks of ϑ_t as $(1+\epsilon)\theta_0$. Remember from (4) that θ_0 is the mean of unobservable signal ϑ_t^u at time $t = 0$. The interpretation of factor $(1+\epsilon)$ is exactly the same as that for active investors.

The information inferred by investors is illustrated in Figure 1. Formally, we give the following definition.

[5] It is feasible to obtain perfect estimates of variances but much harder to estimate expected returns (Merton [25]).
[6] By Lévy characterization, Brownian motion is an almost surely continuous martingale with mean zero and quadratic variation t at time t. It is verified apparently that W_t defined in (2) is a martingale with mean zero and quadratic t. The relationship $\mathscr{F}_t^{S^u} \subset \mathscr{F}_t^u$ was already clarified in Xia [35, p. 212.].
[7] The introducing of ϵ resembles the irrational assumption in Kogan et al. [20].
[8] We thank the referee for suggesting this terminology, which makes our statement more precise.

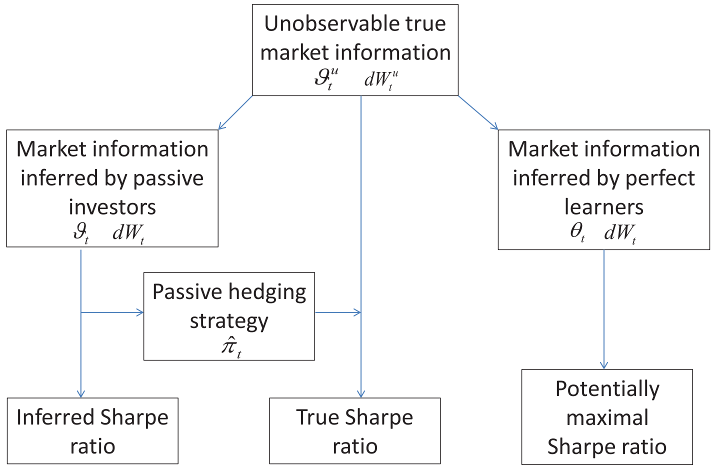

FIGURE 1. (Color online) Performance of "learners" and "passive investors".

DEFINITION 1: *The unobservable true market information is described by* (ϑ_t^u, dW_t^u). *The unbiased learners infer the correct information* (θ_t, dW_t) *and achieve the potentially maximal Sharpe ratio* $\mathscr{S}_T^{\text{potential}}$. *The passive investors infer that the market dynamic is given by* (ϑ_t, dW_t) *and derive the corresponding Sharpe ratio* $\mathscr{S}_T$ *and the passive hedging strategy* $\hat{\pi}_t$ *according to it. The passive investors would actually achieve* $\mathscr{S}_T^{\text{true}}$, *which is the true Sharpe ratio based on the passive strategy* $\hat{\pi}_t$ *and the true market information.*

To deal with partial information, investors prefer to become a learner to follow up the information. The vast literature on learning assumes that the investor can learn the information accurately. However, due to the investor's limited time, limited attention, limited access to information, or limited computing ability, passive strategies seem more prevalent in reality. The focus of this paper is on the performance of passive strategies, taking the learner's performance as a benchmark.

3. SHARPE RATIO OPTIMIZATION

From the viewpoint of investors, the market is complete due to the fact that there is only one stochastic source W_t in price dynamic (3). As it is well known in literature, the process

$$Z_t := \exp\left(-\int_0^t \vartheta_s dW_s - \frac{1}{2}\int_0^t \vartheta_s^2 ds\right) \tag{5}$$

is a martingale with respect to $\mathscr{F}_t^{S^u}$ and the risk-neutral probability measure Q is defined via $dQ = Z_T dP$, where T is a predesigned horizon. Under the risk-neutral measure Q, the discounted price $e^{-rt}S_t$ is a martingale. For an elaboration on martingale and risk-neutral theory, we refer to Duffie [13].

The investor in our paper strives to maximize Sharpe ratio of his portfolio over time span $[0, T]$. This corresponds to the classical Markowitz problem in dynamic context. Let x_0 be initial wealth under management and π_t be the amount invested in risky asset. Without additional influx or deduction of funds, the self-financing wealth process X_t develops

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according to

$$dX_t = [\pi_t \sigma \vartheta_t + rX_t]dt + \pi_t \sigma dW_t, \quad X_0 = x_0. \tag{6}$$

Then the investor's objective is

$$\begin{aligned} \min \quad & \mathrm{Var}[X_T] \\ \text{subject to:} \quad & \text{the budget equation (6) and } E[X_T] = x_0 e^{bT}, \end{aligned} \tag{7}$$

where $b > r$ is an arbitrarily fixed expected return rate. The objective (7) is equivalent to maximizing the observed Sharpe ratio

$$\mathscr{S}_T := \frac{x_0(e^{bT} - e^{rT})}{\sqrt{\mathrm{Var}[X_T]}}.$$

The optimization problem (7) is a multi-period mean-variance problem, which has been studied comprehensively in recent years and closed-form solutions of optimal trading strategies are available in general situations. Among the extensive literature, we cite Korn and Trautmann [21], Bajeux-Besnainou and Portait [1], Li and Ng [23], Xiong and Zhou [36] and Cvitanić et al. [8] as our direct references. The following lemma reviews the main results obtained in Cvitanić et al. [8] for our later uses.

LEMMA 1 [Cvitanić et al. 9]: *Assume that the market is complete such that it possesses an unique risk-neutral kernel Z_t as in (5). Denote*

$$\Lambda_t := \frac{1}{Z_t^2} E[Z_T^2 | \mathscr{F}_t^{S^u}] \quad \text{and} \quad N_t := E[Z_T^2 | \mathscr{F}_t^{S^u}].$$

Define the $\mathscr{F}_t^{S^u}$-adapted process κ_t by

$$\frac{dN_t}{N_t} = \kappa_t dW_t.$$

Then for the minimization problem (7), the following assertions hold true.
(A) The optimal investment to risky asset $\hat{\pi}_t$ is

$$\hat{\pi}_t = -\frac{1}{\sigma}\left[x_0\left(e^{bT} + \frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}\right)e^{-r(T-t)} - \hat{X}_t\right](\vartheta_t + \kappa_t),$$

where $\hat{X}_t$ is the optimal wealth value at time t.
(B) The optimal terminal wealth $\hat{X}_T$ is

$$\hat{X}_T = x_0\left[e^{bT} - \frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}(Z_T - 1)\right].$$

Hence at maturity T, one has

$$E[\hat{X}_T] = x_0 e^{bT}, \quad \mathrm{Var}[\hat{X}_T] = \frac{x_0^2(e^{bT} - e^{rT})^2}{\Lambda_0 - 1}, \quad \mathscr{S}_T = \sqrt{\Lambda_0 - 1}.$$

Lemma 1 is applicable as long as the financial market is complete, whatever the predictor ϑ_t is specialized. The statement (A) describes how the investor manages his portfolio. The statement (B) shows that, following the strategy in (A), the investor takes for granted that the maximal Sharpe ratio is

$$\mathscr{S}_T = \sqrt{\Lambda_0 - 1}.$$

However, when he carries out the inferred strategy $\hat{\pi}_t$ in the market, the true wealth process is not

$$d\hat{X}_t = [\hat{\pi}_t \sigma \vartheta_t + r\hat{X}_t]dt + \hat{\pi}_t \sigma dW_t, \quad \hat{X}_0 = x_0,$$

but

$$d\hat{X}_t = [\hat{\pi}_t \sigma \vartheta_t^u + r\hat{X}_t]dt + \hat{\pi}_t \sigma dW_t^u, \quad \hat{X}_0 = x_0, \tag{8}$$

because true uncertainty faced by the investor is not $\{\vartheta_t, dW_t\}$ but $\{\theta_t^u, dW_t^u\}$. We label the process defined via (8) by $\hat{X}_t^{\text{true}}$.

To solve $\hat{X}_T^{\text{true}}$, it seems necessary to work on the relatively bigger filtration $\mathscr{F}_t^u$. We remark that $\mathscr{F}_t^{S^u} \subseteq \mathscr{F}_t^u$, and W_t^u, W_t are both Brownian motions adapted to $\mathscr{F}_t^u$. The investor in our setup could not observe $\{\mathscr{F}_t^u | t \geq 0\}$. Thanks to the "separation theorem" developed in Detemple [11], Dothan and Feldman [12] and Gennotte [17],[9] we can express $\hat{X}_T^{\text{true}}$ conditional on $\mathscr{F}_t^{S^u}$. In fact, the relationship implied by (2)

$$\theta_t dt + dW_t = \vartheta_t^u dt + dW_t^u \tag{9}$$

clarifies that the dynamics of observed randomness dW_t together with the perfectly inferred drift $\theta_t dt$ coincide exactly with the original dynamics. By virtue of (9), we can reorganize Eq. (8) as

$$d\hat{X}_t = r\hat{X}_t dt + \hat{\pi}\sigma(\vartheta_t^u dt + dW_t^u) = r\hat{X}_t dt + \hat{\pi}\sigma(\theta_t dt + dW_t). \tag{10}$$

All components of the right-hand side of above are observable to Bayesian learners. Eq. (9) also justifies that the unbiased learner follows up the market dynamics precisely.

Now we are ready to calculate the true wealth process which is adapted to $\mathscr{F}_t^u$ explicitly. Define

$$\bar{Z}_t = \exp\left[-\int_0^t \vartheta_s dW_s + \int_0^t \vartheta_s \left(\theta_s - \frac{3}{2}\vartheta_s\right) ds + \int_0^t \kappa_s(\theta_s - \vartheta_s) ds\right]. \tag{11}$$

Proposition 1: *Following the optimal portfolio strategy $\hat{\pi}_t$ in Lemma 1 (A), the actual terminal wealth given by (8) is*

$$\hat{X}_T^{\text{true}} = x_0 \left[e^{bT} - \frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}(\bar{Z}_T - 1)\right].$$

Hence at maturity T, one has

$$E[\hat{X}_T^{\text{true}}] = x_0 e^{bT} - x_0 \left(\frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}\right)(E[\bar{Z}_T] - 1),$$

$$\text{Var}[\hat{X}_T^{\text{true}}] = x_0^2 \left(\frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}\right)^2 \text{Var}[\bar{Z}_T].$$

Moreover, the true Sharpe ratio $\mathscr{S}_T^{\text{true}} := [E[\hat{X}_T^{\text{true}}] - x_0 e^{rT}]/[\sqrt{\text{Var}[\hat{X}_T^{\text{true}}]}]$ equals

$$\mathscr{S}_T^{\text{true}} = \frac{\Lambda_0 - E[\bar{Z}_T]}{\sqrt{\text{Var}[\bar{Z}_T]}}.$$

[9] For a generalized version of "separation theorem" in the learning context, see Xiong and Zhou [36].

Since both the pre-specified mean, variance and $\mathscr{S}_T$ and their ex post realizations are accessible to the investor, Proposition 1 provides an analytical approach to compute the "implied" parameters in the actual stock return predictor θ_t by equating the theoretical mean, variance, or $\mathscr{S}_T$ with the realized observations.[10] Noticing that we can in fact obtain two equations which are concerned with mean and variance respectively, such a procedure can determine at most two independent parameters by solving these two equations together. Because both i.i.d. predictor and mean-reverting predictor usually involve at most two unknown underlying parameters,[11] such a procedure seems ready for determining the parameters in the predictor dynamics. An empirical test of the validity of such a methodology is an interesting topic for future research.

According to Proposition 1, for learners, we have $\vartheta_t = (1+\epsilon)\theta_t$ and

$$\bar{Z}_t = Z_t \exp\left[-\frac{\epsilon}{1+\epsilon}\int_0^t (\vartheta_s + \kappa_s)\vartheta_s ds\right], \tag{12}$$

for passive investors, we have $\vartheta_t = (1+\epsilon)\theta_0$, $\kappa_t = -2\vartheta_t$ and[12]

$$\bar{Z}_t = Z_t \exp\left\{-(1+\epsilon)\theta_0 \int_0^t [\theta_s - (1+\epsilon)\theta_0] ds\right\}. \tag{13}$$

Eqs. (12) and (13) demonstrate that, among active and passive investors, it is only the unbiased learner ($\epsilon = 0$ in (12)) who arrives at $\bar{Z}_t = Z_t$. More precisely we have

THEOREM 1: *In our setup, the following statements hold:*

(A) Among all investors, only the unbiased learner achieves his primitive objective exactly. For other investors, there is always a gap between objective and actual outcome.

(B) Both the inferred and the true maximal Sharpe ratios are determined by underlying dynamics, independent of the exogenous objectives.

As a result of Theorem 1, only being an unbiased learner can he achieve the largest Sharpe ratio which the market can generate potentially, since only an unbiased learning process can follow up market dynamics correctly. All the other investors attain a smaller Sharpe ratio.

When an investor is either passive or error-learning, he could not attain his primitive objective in reality. To quantify the bias, we introduce

$$\mathscr{R}_T^{\text{error}} := \frac{\mathscr{S}_T - \mathscr{S}_T^{\text{true}}}{\mathscr{S}_T}$$

to measure the associated fractional error. Furthermore we define

$$\mathscr{R}_T^{\text{loss}} := \frac{\mathscr{S}_T^{\text{potential}} - \mathscr{S}_T^{\text{true}}}{\mathscr{S}_T^{\text{true}}},$$

which serves as an effective assessment of relative loss of opportunities. Remember that $\mathscr{R}_T^{\text{loss}}$ is always non-negative because of $\mathscr{S}_T^{\text{potential}} \geq \mathscr{S}_T^{\text{true}}$, and the equality holds if and only if the investor is an unbiased learner.

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[10] This procedure for computing the underlying parameters is in the same spirit of computing implied volatility in option pricing.

[11] Usually, in the i.i.d. case, the unknown parameter is the variance, while in the mean-reverting case, the unknown parameters are the mean-reverting intensity and long-run volatility.

[12] When ϑ_t is a constant, it can be computed that $E[Z_T^2|\mathscr{F}_t^{S^u}] = Z_t^2 \exp[\vartheta_t^2(T-t)]$, following which we have $\kappa_t = -2\vartheta_t$.

In the following, we skip the learners with biased inferences ($\epsilon \neq 0$), and focus on differences between the unbiased learner (who serves as a benchmark) and passive investors. The main reason is that learners with inappropriate inferences only bear the losses caused by measurement errors, while, passive investors bear the expenses of being blind to predictor dynamics. Compared to the former, the latter deserves a deeper study.

4. I.I.D. ENVIRONMENT

In this section, we consider the situation in which risky returns are independent and identically distributed. Brennan [4] and Cvitanić et al. [9] have explored the learning problem in the corresponding utility-based optimization.

4.1. Sharpe Ratio Performance

Under the i.i.d. assumption, $\vartheta_t^u = \tilde{v}$. Here $\tilde{v}$ is a normal random variable with mean θ_0 and variance δ_0, which is independent of the Brownian motion W_t^u. $\theta_t = E[\tilde{v}|\mathscr{F}_t^{S^u}]$ evolves as

$$d\theta_t = \left(\frac{\delta_0}{\delta_0 t + 1}\right) dW_t. \tag{14}$$

The unbiased learner believes $\vartheta_t = \theta_t$ whereas the passive investor thinks $\vartheta_t = (1+\epsilon)\theta_0$. To proceed, we need some computational results on relevant quantities, which are presented as Proposition A.1 and Proposition A.2 in Appendix A.

PROPOSITION 2: *Under the i.i.d. assumption, the largest Sharpe ratio that the market can potentially yield is*

$$\mathscr{S}_T^{\text{potential}} = \left(\frac{1+\delta_0 T}{\sqrt{1+2\delta_0 T}} e^{(T\theta_0^2/(1+2\delta_0 T)} - 1\right)^{1/2}.$$

The associated efficient strategy is

$$\hat{\pi}_t = \frac{1}{\sigma}\left[x_0\left(e^{bT} + \frac{e^{bT} - e^{rT}}{\frac{1+\delta_0 T}{\sqrt{1+2\delta_0 T}} e^{(T\theta_0^2/1+2\delta_0 T)} - 1}\right) e^{-r(T-t)} - \hat{X}_t\right]\left[1 - \frac{2\delta_0(T-t)}{1+\delta_0(2T-t)}\right]\theta_t,$$

where $\hat{X}_t$ is the optimal wealth value at time t.

The optimal stock holding $\hat{\pi}_t$ is a function of $\hat{X}_t$, θ_t, and δ_0. In the long run, when $\hat{X}_t$ is large, the optimal risky holding $\hat{\pi}_t$ tends to be reduced. This is the so-called "reversal" feature for investors in the traditional i.i.d. environment.[13] In the short run, an additional force driving $\hat{\pi}_t$ is θ_t. Because stocks are allowed to be traded instantaneously without frictions, the motivation to earn the temporary profit makes $\hat{\pi}_t$ increase when θ_t is high, and vice versa.

A noteworthy phenomenon implied by Proposition 2 is that, there is a positive constant δ^* such that $\mathscr{S}_T^{\text{potential}}$ is monotonously decreasing in $\delta_0 \in (0, \delta^*)$, while monotonously

[13] See Cvitanić et al. [8] for a detailed discussion on the reversal feature in the i.i.d. case.

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increasing in $\delta_0 \in (\delta^*, +\infty)$ with $\lim_{\delta_0 \to +\infty} \mathscr{S}_T^{\text{potential}} = +\infty$. It can be verified that

$$\delta^* = \frac{4\theta_0^2}{(1 - 2T\theta_0^2) + \sqrt{(1 - 2T\theta_0^2)^2 + 16T\theta_0^2}}. \quad (15)$$

There are two conflicting influences of δ_0 in determining the maximal Sharpe ratio. It increases the underlying risk as well as brings opportunities of high returns. When $\delta_0 < \delta^*$, the former impact dominates while when $\delta_0 > \delta^*$, the latter impact dominates.

Remark 1: We list some detailed analysis on Eq. (15).

- Observe that δ^* is fully determined by θ_0 and T. The fact $\lim_{\theta_0 \to +\infty} \delta^* = +\infty$ tells us that when θ_0 is high, δ_0 tends to depress the Sharpe ratio, while $\lim_{\theta_0 \to 0} \delta^* = 0$ says that when θ_0 is low, δ_0 tends to raise the Sharpe ratio. In the extreme case of $\theta_0 = 0$, $\mathscr{S}_T^{\text{potential}}$ is always increasing in δ_0, because in this case learners benefit merely from the fluctuation of returns.
- The limit points $\lim_{T \to 0} \delta^* = 2\theta_0^2$ and $\lim_{T \to +\infty} \delta^* = \theta_0^2$ are notable. If a learner has access to a variety of stocks with the same mean return θ_0, he is then advised by (15) to invest in the stock whose signal variance δ_0 is distant from δ^*.
- The main reason why the learner seems risk-seeking is that he is capable of uncovering the market information completely in our model. Note that in reality, such perfect learners are rare in markets. One worthy research subject is to explore the behavioral mechanism of passive investors.

For passive investors, by virtue of Proposition 1, Proposition 2, Proposition A.1 and Proposition A.2, we have

$$E[\hat{X}_T] = x_0 e^{bT}, \quad (16)$$

$$\text{Var}[\hat{X}_T] = \frac{x_0^2 (e^{bT} - e^{rT})^2}{e^{(1+\epsilon)^2 \theta_0^2 T} - 1}, \quad (17)$$

$$\mathscr{S}_T = \left[e^{(1+\epsilon)^2 \theta_0^2 T} - 1\right]^{1/2}, \quad (18)$$

and

$$E[\hat{X}_T^{\text{true}}] = x_0 e^{bT} - x_0 \left[\frac{e^{bT} - e^{rT}}{e^{(1+\epsilon)^2 \theta_0^2 T} - 1}\right] \left[e^{\frac{1}{2}(1+\epsilon)^2 \theta_0^2 T([2\epsilon/(1+\epsilon)] + \delta_0 T)} - 1\right], \quad (19)$$

$$\text{Var}[\hat{X}_T^{\text{true}}] = x_0^2 \left[\frac{e^{bT} - e^{rT}}{e^{(1+\epsilon)^2 \theta_0^2 T} - 1}\right]^2 e^{(1+\epsilon)^2 \theta_0^2 T([2\epsilon/(1+\epsilon)] + \delta_0 T)} [e^{(1+\epsilon)^2 \theta_0^2 T(1+\delta_0 T)} - 1], \quad (20)$$

$$\mathscr{S}_T^{\text{true}} = \frac{e^{\frac{1}{2}(1+\epsilon)^2 \theta_0^2 T([2/(1+\epsilon)] - \delta_0 T)} - 1}{\left[e^{(1+\epsilon)^2 \theta_0^2 T(1+\delta_0 T)} - 1\right]^{1/2}}. \quad (21)$$

To compare the actual and the inferred mean, variance and the Sharpe ratio respectively, we deduce from (16) and (19) to obtain

$$E[\hat{X}_T] - E[\hat{X}_T^{\text{true}}] = x_0 \left[\frac{e^{bT} - e^{rT}}{e^{(1+\epsilon)^2 \theta_0^2 T} - 1}\right] [e^{\frac{1}{2}(1+\epsilon)^2 \theta_0^2 T([2\epsilon/(1+\epsilon)] + \delta_0 T)} - 1],$$

and from (17) and (20) to get

$$\frac{\mathrm{Var}[\hat{X}_T^{\mathrm{true}}]}{\mathrm{Var}[\hat{X}_T]} = e^{(1+\epsilon)^2\theta_0^2 T([2\epsilon/(1+\epsilon)]+\delta_0 T)}\left[\frac{e^{(1+\epsilon)^2\theta_0^2 T(1+\delta_0 T)}-1}{e^{(1+\epsilon)^2\theta_0^2 T}-1}\right].$$

It is easy to see that $E[\hat{X}_T] > E[\hat{X}_T^{\mathrm{true}}]$ if and only if $\epsilon > -[\delta_0 T/(2+\delta_0 T)]$. However, for variance, we only have a partial assertion that $\epsilon > -[\delta_0 T/(2+\delta_0 T)]$ implies $\mathrm{Var}[\hat{X}_T^{\mathrm{true}}] > \mathrm{Var}[\hat{X}_T]$. When $\epsilon < -(\delta_0 T)/(2+\delta_0 T)$, the relative size of $\mathrm{Var}[\hat{X}_T^{\mathrm{true}}]$ and $\mathrm{Var}[\hat{X}_T]$ is uncertain. A numerical approach can be applied for detailed analysis. Implications of the above comparisons are listed below.

- When $\epsilon \geq 0$, it is definite that $E[\hat{X}_T^{\mathrm{true}}] < E[\hat{X}_T]$, $\mathrm{Var}[\hat{X}_T^{\mathrm{true}}] > \mathrm{Var}[\hat{X}_T]$ and in turn $\mathscr{S}_T^{\mathrm{true}} < \mathscr{S}_T$. It indicates that for optimistic investors, passive strategies lead them to a lower portfolio mean and a higher portfolio variance.
- The situation $\epsilon < 0$ is a little complicated.
  - When $-[\delta_0 T/(2+\delta_0 T)] < \epsilon < 0$, it remains that $E[\hat{X}_T^{\mathrm{true}}] < E[\hat{X}_T]$, $\mathrm{Var}[\hat{X}_T^{\mathrm{true}}] > \mathrm{Var}[\hat{X}_T]$ and $\mathscr{S}_T^{\mathrm{true}} < \mathscr{S}_T$. The intuition is that the slight pessimism could not offset the effects caused by ignoring predictor dynamics.
  - When $\epsilon < -[\delta_0 T/(2+\delta_0 T)]$, definitely $E[\hat{X}_T^{\mathrm{true}}] > E[\hat{X}_T]$, but the relative size of $\mathrm{Var}[\hat{X}_T^{\mathrm{true}}]$ and $\mathrm{Var}[\hat{X}_T]$ is uncertain. However, the fact $\lim_{\epsilon\to -1}(\mathscr{S}_T^{\mathrm{true}}/(\mathscr{S}_T) = +\infty$ tells us that the passive investor with more severe pessimism could earn a higher Sharpe ratio.

Intuitively, the optimistic investor overestimates the portfolio mean, underestimates the portfolio risk and in turn obtain a lower Sharpe ratio. On the contrary, the pessimistic underestimates the portfolio mean, overestimates the portfolio risk, and in turn obtain a higher Sharpe ratio.[14] Notice that the passive investor with slight pessimism $(-[\delta_0 T/(2+\delta_0 T)] < \epsilon < 0)$ can nevertheless obtain a lower Sharpe ratio. Eq. (27) below indicates that when θ_0 is relatively high enough, the threshold between $\mathscr{S}_T^{\mathrm{true}} > \mathscr{S}_T$ and $\mathscr{S}_T^{\mathrm{true}} < \mathscr{S}_T$ is $\epsilon = -[\delta_0 T/(1+\delta_0 T)]$.

To visualize the above analysis, we calibrate the model to a set of empirical parameters. Without loss of generality, we set initial wealth $x_0 = 1$ throughout this paper. We take the yearly data $r = 3\%$, $b = 10\%$, $\theta_0 = 0.35$, and $\delta_0 = 0.025$ as given.[15] For passive investors, panels A, B, and C in Figure 2 compare the actual and the inferred mean, variance and the Sharpe ratio, respectively, with ϵ ranging from -0.5 to 0.5. Notice that in panel B, all the three possibilities $\mathrm{Var}[\hat{X}_T^{\mathrm{true}}] < \mathrm{Var}[\hat{X}_T]$, $\mathrm{Var}[\hat{X}_T^{\mathrm{true}}] = \mathrm{Var}[\hat{X}_T]$ and $\mathrm{Var}[\hat{X}_T^{\mathrm{true}}] > \mathrm{Var}[\hat{X}_T]$ can happen when $\epsilon < -[\delta_0 T/(2+\delta_0 T)]$. In panel C, it is obvious that when ϵ is close to -1, there holds $\mathscr{S}_T^{\mathrm{true}} > \mathscr{S}_T$. Panel D in Figure 2 compares the true and the potentially maximal Sharpe ratio. The next subsection discusses these gaps in detail.

4.2. $\mathscr{R}_T^{\mathbf{Error}}$ and $\mathscr{R}_T^{\mathbf{Loss}}$

In order to see how the Sharpe ratio performance is influenced by fundamental parameters, we study the comparative statics of $\mathscr{R}_T^{\mathrm{error}}$ and $\mathscr{R}_T^{\mathrm{loss}}$. Recall Proposition 2, (18) and (21).

[14] Eq. (25) demonstrates that when predictor information is completely exposed to investors, $\epsilon > 0$ always results in a lower $\mathscr{S}_T^{\mathrm{true}}$ while $\epsilon < 0$ always results in a higher $\mathscr{S}_T^{\mathrm{true}}$, relative to $\mathscr{S}_T$.

[15] When we consult a relevant empirical study in Brennan [4], Table 1 therein, the four possible values of δ_0 are $\delta_0 = v_0/\sigma^2 = (0.0243/0.202)^2 = 0.014$, $\delta_0 = v_0/\sigma^2 = (0.0243/0.140)^2 = 0.030$, $\delta_0 = v_0/\sigma^2 = (0.0452/0.202)^2 = 0.050$, $\delta_0 = v_0/\sigma^2 = (0.0452/0.140)^2 = 0.104$, for slightly different databases. The notations v_0 and σ are also from Table 1 there.

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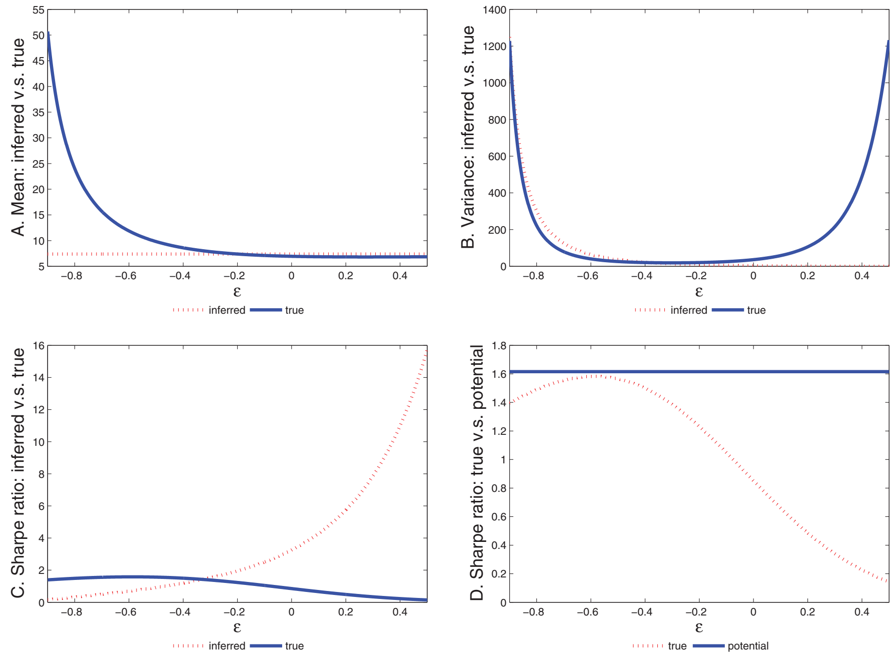

FIGURE 2. (Color online) Relationship between passive investors' performances and ϵ for the i.i.d. case. In this figure, we compare $E[\hat{X}_T]$ and $E[\hat{X}_T^{\text{true}}]$, $\text{Var}[\hat{X}_T]$ and $\text{Var}[\hat{X}_T^{\text{true}}]$, $\mathscr{S}_T$ and $\mathscr{S}_T^{\text{true}}$, $\mathscr{S}_T^{\text{true}}$, and $\mathscr{S}_T^{\text{potential}}$, respectively. Here we set $x_0 = 1$, $r = 3\%$, $b = 10\%$, $\theta_0 = 0.35$, $\delta_0 = 0.025$ and $T = 20$.

Comparative statics with respect to time horizon T.

- When T goes to infinity, one has

$$\lim_{T\to+\infty} \mathscr{S}_T = +\infty, \quad \lim_{T\to+\infty} \mathscr{S}_T^{\text{true}} = 0, \quad \lim_{T\to+\infty} \mathscr{S}_T^{\text{potential}} = +\infty,$$

which yield

$$\lim_{T\to+\infty} \mathscr{R}_T^{\text{error}} = 1 \quad \text{and} \quad \lim_{T\to+\infty} \mathscr{R}_T^{\text{loss}} = +\infty. \tag{22}$$

As T goes to infinity, both $\mathscr{S}_T$ and $\mathscr{S}_T^{\text{potential}}$ rise to infinity. However, the true outcome $\mathscr{S}_T^{\text{true}}$ approaches zero. This indicates that the passive investor's belief deviates from the truth seriously, especially for long-run investments. Moreover, the passive investor's portfolio strategy is doomed to behave badly in reality.

- When T goes to zero, one has

$$\lim_{T\to 0} \mathscr{R}_T^{\text{error}} = \frac{\epsilon}{1+\epsilon} \quad \text{and} \quad \lim_{T\to 0} \mathscr{R}_T^{\text{loss}} = 0. \tag{23}$$

In short-time horizon, the loss caused by ignorance of predictor dynamics is small enough such that passive investors are competent. The fact $\lim_{T\to 0} \mathscr{R}_T^{\text{error}} = \epsilon/(1+\epsilon)$ suggests that the mis-perception on Sharpe ratio caused by subjective attitude could not be removed thoroughly.

The tendencies of (22) and (23) are illustrated in panels A and B of Figure 3.

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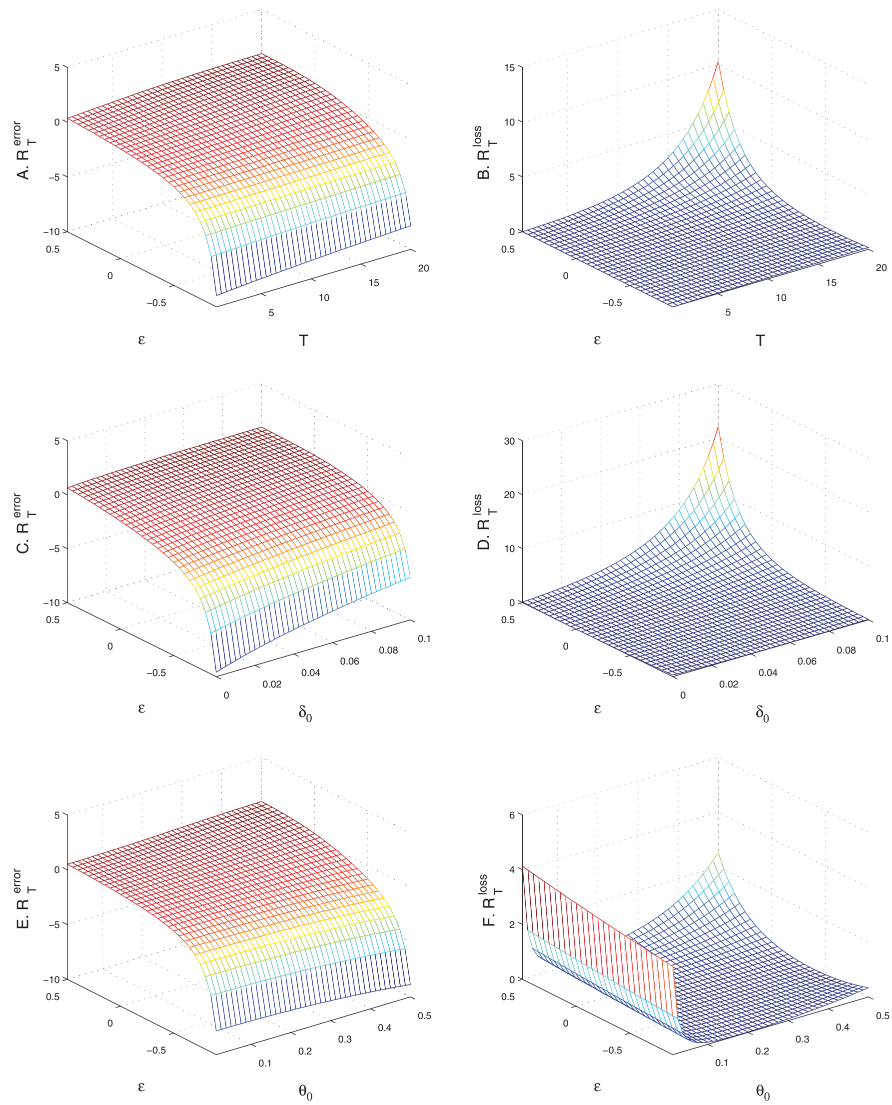

FIGURE 3. (Color online) Relationship between passive investors' $\mathscr{R}$ values and $(T, \delta_0, \theta_0, \epsilon)$ for the i.i.d. case. In this figure, the benchmark is given by $x_0 = 1$, $r = 3\%$, $b = 10\%$, $\theta_0 = 0.35$, $\delta_0 = 0.025$, and $T = 10$. Panels A, B, C, D, E, F plot the dependencies of $\mathscr{R}_T^{\text{error}}$ and $\mathscr{R}_T^{\text{loss}}$ on (T, ϵ), (δ_0, ϵ), (θ_0, ϵ), respectively. When we picturize the dependencies of $\mathscr{R}_T^{\text{error}}$ and $\mathscr{R}_T^{\text{loss}}$ on ϵ and one of these parameters, we keep other parameters equal to the benchmark.

Comparative statics with respect to δ_0.

- When δ_0 goes to infinity, one has

$$\lim_{\delta_0 \to +\infty} \mathscr{S}_T^{\text{true}} = 0 \quad \text{and} \quad \lim_{\delta_0 \to +\infty} \mathscr{S}_T^{\text{potential}} = +\infty,$$

which yield

$$\lim_{\delta_0 \to +\infty} \mathscr{R}_T^{\text{error}} = 1 \quad \text{and} \quad \lim_{\delta_0 \to +\infty} \mathscr{R}_T^{\text{loss}} = +\infty. \tag{24}$$

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These inequalities confirm again the fact that judgement error and potential loss are mainly caused by the ignorance of δ_0.

- When δ_0 goes to zero, one has

$$\lim_{\delta_0\to 0}\mathscr{S}_T^{\text{true}} = \frac{e^{(1+\epsilon)\theta_0^2 T}-1}{\left[e^{(1+\epsilon)^2\theta_0^2 T}-1\right]^{1/2}} \quad \text{and} \quad \lim_{\delta_0\to 0}\mathscr{S}_T^{\text{potential}} = \left(e^{\theta_0^2 T}-1\right)^{1/2},$$

which yield

$$\lim_{\delta_0\to 0}\mathscr{R}_T^{\text{error}} = \frac{e^{(1+\epsilon)^2\theta_0^2 T}-e^{(1+\epsilon)\theta_0^2 T}}{e^{(1+\epsilon)^2\theta_0^2 T}-1}, \tag{25}$$

$$\lim_{\delta_0\to 0}\mathscr{R}_T^{\text{loss}} = \frac{\sqrt{(e^{\theta_0^2 T}-1)[e^{(1+\epsilon)^2\theta_0^2 T}-1]}}{e^{(1+\epsilon)\theta_0^2 T}-1}-1. \tag{26}$$

When $\delta_0 = 0$, our model reduces to be the one in which the predictor information is completely exposed to investors. In this case, Eq. (25) shows that $\epsilon > 0$ always results in a lower $\mathscr{S}_T^{\text{true}}$ while $\epsilon < 0$ always results in a higher $\mathscr{S}_T^{\text{true}}$, relative to $\mathscr{S}_T$. Eq. (26) shows that potential loss is always positive unless $\epsilon = 0$, which implies that only the investor who estimates the predictor precisely can prevent himself from suffering potential Sharpe ratio loss.

The tendencies of (24), (25), and (26) are illustrated in panels C and D of Figure 3.

Comparative statics with respect to θ_0.

- When θ_0 goes to infinity, one has

$$\lim_{\theta_0\to+\infty}\mathscr{S}_T = +\infty, \quad \lim_{\theta_0\to+\infty}\mathscr{S}_T^{\text{potential}} = +\infty,$$

and

$$\lim_{\theta_0\to+\infty}\mathscr{S}_T^{\text{true}} = \begin{cases} +\infty, & \text{if} \quad \epsilon < \dfrac{1-2\delta_0 T}{1+2\delta_0 T}, \\ 1, & \text{if} \quad \epsilon = \dfrac{1-2\delta_0 T}{1+2\delta_0 T}, \\ 0, & \text{if} \quad \epsilon > \dfrac{1-2\delta_0 T}{1+2\delta_0 T}. \end{cases}$$

Moreover,

$$\lim_{\theta_0\to+\infty}\mathscr{R}_T^{\text{error}} = \begin{cases} -\infty, & \epsilon < -\dfrac{\delta_0 T}{1+\delta_0 T}, \\ 0, & \epsilon = -\dfrac{\delta_0 T}{1+\delta_0 T}, \\ 1, & \epsilon > -\dfrac{\delta_0 T}{1+\delta_0 T}, \end{cases} \tag{27}$$

and

$$\lim_{\theta_0\to+\infty}\mathscr{R}_T^{\text{loss}} = \begin{cases} +\infty, & \epsilon \neq -\dfrac{2\delta_0 T}{1+2\delta_0 T}, \\ 0, & \epsilon = -\dfrac{2\delta_0 T}{1+2\delta_0 T}. \end{cases} \tag{28}$$

Eq. (27) suggests that when θ_0 is high enough, the judgement of passive investor with $\epsilon = [-\delta_0 T/(1+\delta_0 T)]$ approximates reality. For $\epsilon < [-\delta_0 T/(1+\delta_0 T)]$, the passive investor underestimates the true Sharpe ratio while for $\epsilon > [-\delta_0 T/(1+\delta_0 T)]$, the passive investor overestimates the true Sharpe ratio. To understand Eq. (28) well, let us consider the special case of $\delta_0 = 0$. In this case, the gap between $\mathscr{S}_T^{\text{true}}$ and $\mathscr{S}_T^{\text{potential}}$ is caused by ϵ only and $\mathscr{R}_T^{\text{loss}} = 0$ holds only for $\epsilon = 0$. All the other investors with $\epsilon \neq 0$ tend to lose in Sharpe ratio substantially. For general cases, Eq. (28) indicates that performance of the passive investor with $\epsilon = [-2\delta_0 T/(1+2\delta_0 T)]$ is close to optimality when θ_0 is high.

- When θ_0 goes to zero, one has

$$\lim_{\theta_0 \to 0} \mathscr{S}_T = 0, \quad \lim_{\theta_0 \to 0} \mathscr{S}_T^{\text{true}} = 0, \quad \lim_{\theta_0 \to 0} \mathscr{S}_T^{\text{potential}} = \left(\frac{1+\delta_0 T}{\sqrt{1+2\delta_0 T}} - 1 \right)^{1/2}.$$

Moreover,

$$\lim_{\theta_0 \to 0} \mathscr{R}_T^{\text{error}} = 1 - \frac{[1/(1+\epsilon)] - \frac{1}{2}\delta_0 T}{\sqrt{1+\delta_0 T}} \quad \text{and} \quad \lim_{\theta_0 \to 0} \mathscr{R}_T^{\text{loss}} = +\infty, \tag{29}$$

which confirm the fact that passive investors behave badly when θ_0 is not significant.

The tendencies of (27), (28), and (29) are illustrated in panels E and F of Figure 3.

5. MEAN-REVERTING ENVIRONMENT

In this section, we study the mean-reverting case. Different from the i.i.d. case, the main purpose in this section is to investigate the effects of mean-reverting intensity and long-run volatility.

As in Wachter [34], we assume ϑ_t^u follows an Ornstein–Uhlenbeck process

$$d\vartheta_t^u = \lambda(\bar{\theta} - \vartheta_t^u)dt - \sigma_\theta dW_t^u, \quad \vartheta_0^u = \tilde{v}, \tag{30}$$

where λ, $\bar{\theta}$, σ_θ are non-negative constants and $\tilde{v}$ is a random variable independent of W_t^u. The stock price S_t^u and the state variable ϑ_t^u are perfectly negatively correlated, which ensures the completeness of the market and subsequently enables us to use the martingale approach.[16] By Kalman filtering lemma, the dynamic of $\theta_t = E[\vartheta_t^u | \mathscr{F}_t^{S^u}]$ is[17]

$$d\theta_t = \lambda(\bar{\theta} - \theta_t)dt - (\sigma_\theta - \delta_t)dW_t, \quad \theta_0 = E[\tilde{v}], \tag{31}$$

where $\delta_t := \text{Var}[\vartheta_t^u] - \text{Var}[\theta_t]$ solves the Riccati ODE

$$\frac{d}{dt}\delta_t = -2\lambda\delta_t + \sigma_\theta^2 - (\delta_t - \sigma_\theta)^2, \quad \delta_0 = \text{Var}[\tilde{v}].$$

The solution of δ_t is

$$\delta_t = \begin{cases} \dfrac{\delta_0}{\delta_0 t + 1}, & \text{if} \quad w = 0, \\ \dfrac{2\delta_0 w e^{-2wt}}{\delta_0(1 - e^{-2wt}) + 2w}, & \text{if} \quad w \neq 0, \end{cases}$$

[16] For assumption (30), see for instance Kim and Omberg [19], Brennan, Schwartz and Lagnado [5], Campbell and Viceira [7], Wachter [34], etc. Barberis [2] finds an empirical justification of such correlation to be −0.93.

[17] Eq. (31) follows immediately from Theorem 10.3 in Liptser and Shiryayev [24].

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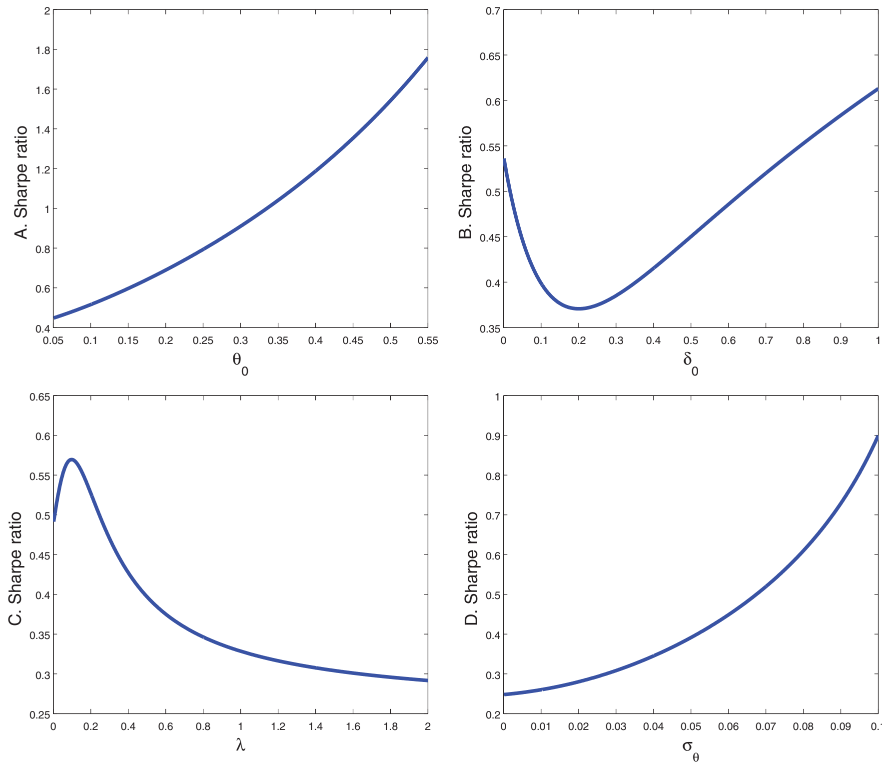

FIGURE 4. (Color online) Relationship between $\mathscr{S}_T^{\text{potential}}$ and $(\theta_0, \delta_0, \lambda, \sigma_\theta)$ for the mean-reverting case. In this figure, panels A, B, C, and D plot the dependencies of $\mathscr{S}_T^{\text{potential}}$ on θ_0, δ_0, λ, and σ_θ, respectively. The benchmark is given by $\theta_0 = \bar{\theta} = 0.0788$, $\lambda = 0.2712$, $\sigma_\theta = 0.0655$, $\delta_0 = 0.025$, and $T = 10$. When we picturize the dependency of $\mathscr{S}_T^{\text{potential}}$ on one of these parameters, we keep other parameters equal to the benchmark.

with $w := \lambda - \sigma_\theta$.[18] The positivity of δ_t indicates that the investor underestimates the predictor volatility in general. When $\sigma_\theta \le \lambda$, such bias diminishes as t goes to infinity. When $\sigma_\theta > \lambda$, such bias approaches a fixed positive level $2(\sigma_\theta - \lambda)$.

5.1. Sharpe Ratio Performance

Keep in mind that the unbiased learner regards $\vartheta_t = \theta_t$, while passive investors believe $\vartheta_t = (1+\epsilon)\theta_0$. The required calculations are in Proposition A.3 and Proposition A.4 in Appendix A. Due to the lack of explicit solutions, we resort to numerical results to illustrate the main points. To make sense, we have tried many plausible parameters in related numerical experiments. The viewpoints to be given below present their common features as well as can be interpreted intuitively.

Based on the monthly data in Wachter [34],[19] matching the moments generates the yearly data: $\bar{\theta} = 0.0788$, $\lambda = 0.0226 \times 12 = 0.2712$, and $\sigma_\theta = 0.0189 \times \sqrt{12} = 0.0655$. We picturize $\mathscr{S}_T^{\text{potential}}$, with θ_0, δ_0, λ and σ_θ varying in a reasonable range around the above benchmark respectively in Figure 4.

[18] Similar calculations can be found in Gennotte [17] and David [10].
[19] See Table 1 therein.

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- Panel A shows that $\mathscr{S}_T^{\text{potential}}$ is increasing in θ_0. This is intuitive since a high initial mean return is always profitable for investors.
- Panel B shows that $\mathscr{S}_T^{\text{potential}}$ does rely on δ_0 in a pattern similar to the i.i.d. case. Namely, $\mathscr{S}_T^{\text{potential}}$ is decreasing in δ_0 if δ_0 is small while increasing in δ_0 if δ_0 is big enough. However, the critical point is different from the i.i.d. case. To illustrate it more clearly, we choose $\theta_0 = \bar{\theta} = 0.0788$ and see that $\mathscr{S}_T^{\text{potential}}$ is increasing in δ_0 when $\delta_0 > \delta_1 = 0.2$ approximately in panel B, while in the i.i.d situation, we find that for the same value of θ_0, $\mathscr{S}_T^{\text{potential}}$ is increasing in δ_0 just when $\delta_0 > \delta_2 = 0.0113$ from Eq. (15). Notice that δ_1 is almost 18 times of δ_2. Nevertheless, the insights in Remark 1 remain valid.
- Panel C describes that $\mathscr{S}_T^{\text{potential}}$ is at the beginning increasing and then decreasing in λ. The interpretation is that as λ becomes positive from zero, the mean-reverting intensity of θ_t is increased, which drives $\mathscr{S}_T^{\text{potential}}$ upwards. As λ becomes larger, the fluctuation of θ_t decays further such that θ_t converges to a stable level $\bar{\theta}$. The decline of predictor risk also reduces the potential profits from risk, and therefore leads to a smaller $\mathscr{S}_T^{\text{potential}}$. The influence of λ on $\mathscr{S}_T^{\text{potential}}$ is substantial, supported by the evidence that, when λ is strengthened to a slight extent, $\lambda \geq 0.2$, for instance, $\mathscr{S}_T^{\text{potential}}$ is sharply reduced.
- Panel D shows that $\mathscr{S}_T^{\text{potential}}$ increases rapidly in σ_θ. This indicates that the long-run volatility in predictor contributes substantially to $\mathscr{S}_T^{\text{potential}}$.

For passive investors, to make the results comparable to the i.i.d. situation, we use an artificial benchmark: $r = 3\%$, $b = 10\%$, $\bar{\theta} = 0.35$, and $\delta_0 = 0.025$. We consider two special cases of the mean-reverting model to emphasize the effect of each factor. In the first case, we set $\sigma_\theta = 0$, to stress the mean-reverting feature. In the second case, we set $\lambda = 0$, to isolate the role of long-run volatility.

- Figure 5 depicts the Sharpe ratio performance in the first case. Compared with Figure 2, the mean-reverting trend does not change the pattern of passive errors, but mitigate the error magnitude. Intuitively, the mean-reverting tendency reduces the risk compared with the i.i.d. situation.
- Figure 6 shows the Sharpe ratio performance in the second case. It is notable that even an optimistic investor can underestimate his Sharpe ratio due to the loss of information on long-run volatility when we compare the corresponding panel C's between Figure 2 and Figure 6.

5.2. $\mathscr{R}_T^{\text{Error}}$ and $\mathscr{R}_T^{\text{Loss}}$

In this subsection, we study the comparative statics on $\mathscr{R}_T^{\text{error}}$ and $\mathscr{R}_T^{\text{loss}}$. It is not surprising to find that $\mathscr{R}_T^{\text{error}}$ and $\mathscr{R}_T^{\text{loss}}$ diverge as T, δ_0 or θ_0 goes to infinity through our numerical studies, which we prefer not to present here. To focus on the long-run feature, we draw the pictures of $\mathscr{R}_T^{\text{error}}$ and $\mathscr{R}_T^{\text{loss}}$, taking (λ, ϵ) and $(\sigma_\theta, \epsilon)$ as variables in Figure 7. To isolate the influences of λ from σ_θ, we consider the above two subcases respectively.

- Panels A and B show that when mean-reverting intensity λ is big (say $\lambda > 0.2$ in our figure), judgement error and potential loss mainly depend on ϵ, since the information $\theta_t = \theta_0$ is close to reality. Panel B also shows that $\mathscr{R}_T^{\text{loss}}$ is reduced substantially as $\lambda > 0.2$ in our figure.

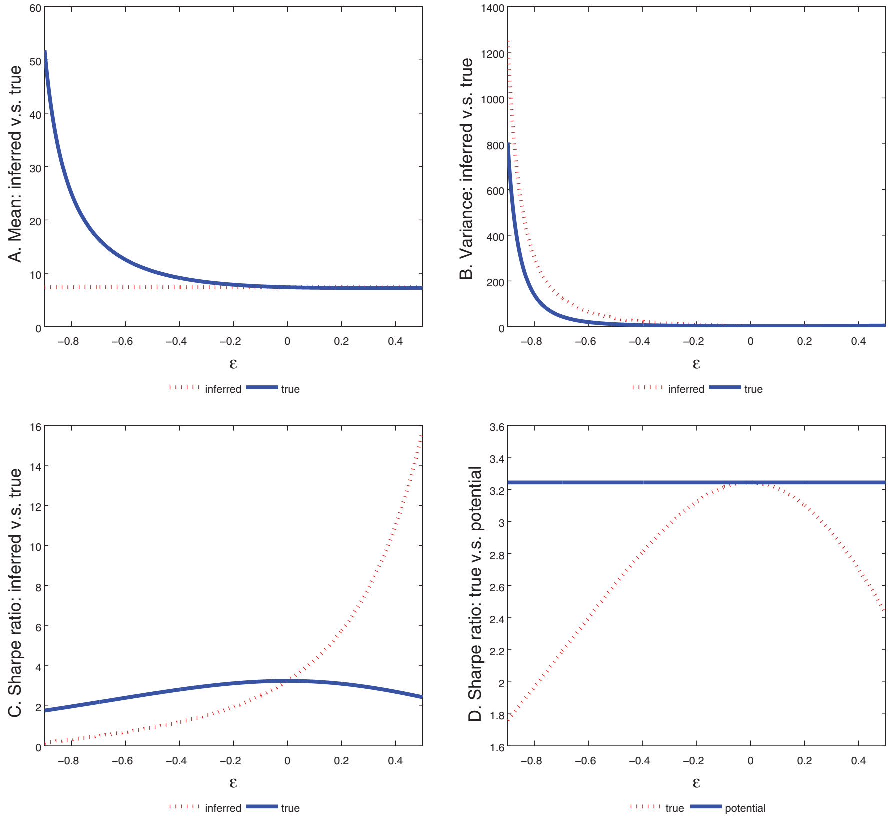

FIGURE 5. (Color online) Relationship between passive investors' performances and ϵ for the mean-reverting case with $\sigma_\theta = 0$. In this figure, we compare $E[\hat{X}_T]$ and $E[\hat{X}_T^{\text{true}}]$, $\text{Var}[\hat{X}_T]$ and $\text{Var}[\hat{X}_T^{\text{true}}]$, $\mathscr{S}_T$ and $\mathscr{S}_T^{\text{true}}$, $\mathscr{S}_T^{\text{true}}$, and $\mathscr{S}_T^{\text{potential}}$, respectively. Here we set $x_0 = 1$, $r = 3\%$, $b = 10\%$, $\theta_0 = \bar{\theta} = 0.35$, $\delta_0 = 0.025$, $\lambda = 1.0$, $\sigma_\theta = 0$, and $T = 20$.

- Panels C and D show that the ignorance of σ_θ contribute substantially to judgement error and potential loss. Moreover, $\mathscr{R}_T^{\text{loss}}$ tends to increase no matter what the sign of ϵ is. This confirms that for investors, no matter optimistic or pessimistic, being blind to the long-run volatility eventually result in a positive loss in Sharpe ratio.

Remark 2: Panel B in Figure 7 indicates that passive investors behave much better in the mean-reverting environment ($\lambda > 0.2$) than in the i.i.d. environment ($\lambda = 0$). Since the realistic market trend is influenced by periodic shocks (see Braun and Larrain [3] among others) and exhibits the mean-reverting character, passive investors also have many opportunities to perform closely to optimality, especially when the reverting intensity λ is large.

6. MULTI-PERIOD STRATEGIES

Cvitanić et al. [8] demonstrate that the manager's focus on the short-run is detrimental to the long-run investor with the same time horizon. Their insights are applicable for unbiased learners, who follow up the underlying information exactly. In our paper, we

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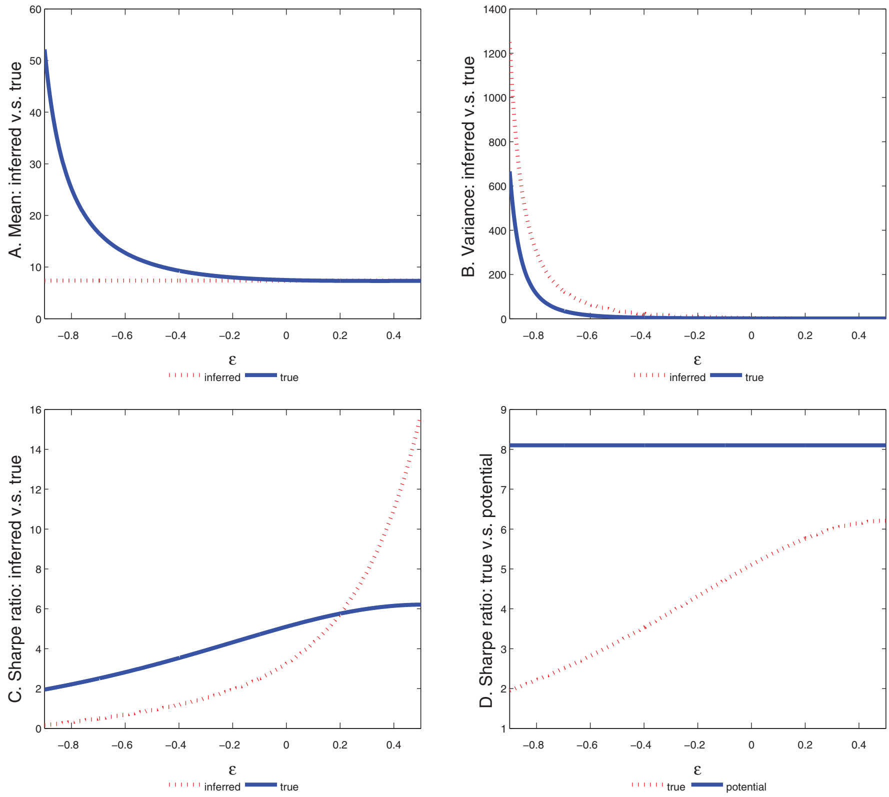

FIGURE 6. (Color online) Relationship between passive investors' performances and ϵ for the mean-reverting case with $\lambda = 0$. In this figure, we compare $E[\hat{X}_T]$ and $E[\hat{X}_T^{\text{true}}]$, $\text{Var}[\hat{X}_T]$ and $\text{Var}[\hat{X}_T^{\text{true}}]$, $\mathscr{S}_T$ and $\mathscr{S}_T^{\text{true}}$, $\mathscr{S}_T^{\text{true}}$ and $\mathscr{S}_T^{\text{potential}}$, respectively. Here we set $x_0 = 1$, $r = 3\%$, $b = 10\%$, $\theta_0 = \bar{\theta} = 0.35$, $\delta_0 = 0.025$, $\lambda = 0$, $\sigma_\theta = 0.05$ and $T = 20$.

mainly focus on passive investors, taking learners as a benchmark. The following questions arise naturally:

- Is the result of Cvitanić et al. [8] valid for passive investors?
- Is there anything new for passive investors when invoking a multi-period strategy?

In this section, we discuss these questions through numerical examples.

Let N be the number of time periods and $\tau = T/N$ be the length of each period. At the beginning of each quarter $T_n = n \times \tau$, the passive investor restarts his policy according to

$$\begin{aligned} \min: \quad & \text{Var}[X_{T_{n+1}} | \mathscr{F}_{T_n}^{S^u}] := E[(X_{T_{n+1}} - E[X_{T_{n+1}} | \mathscr{F}_{T_n}^{S^u}])^2 | \mathscr{F}_{T_n}^{S^u}] \\ \text{subject to}: \quad & \text{the budget equation (6) and } E[X_{T_{n+1}} | \mathscr{F}_{T_n}^{S^u}] = X_{T_n} e^{b\tau}. \end{aligned}$$

The equation $E[X_{T_{n+1}} | \mathscr{F}_{T_n}^{S^u}] = X_{T_n} e^{b\tau}$ reflects the dynamic adjustments. Applying Lemma 1 recursively, we get the optimal wealth process

$$\hat{X}_T = x_0 \prod_{n=0}^{N-1} \left[e^{b\tau} - \frac{e^{b\tau} - e^{r\tau}}{\text{Var}[Z_\tau]} (Z_\tau - 1) \right]. \tag{32}$$

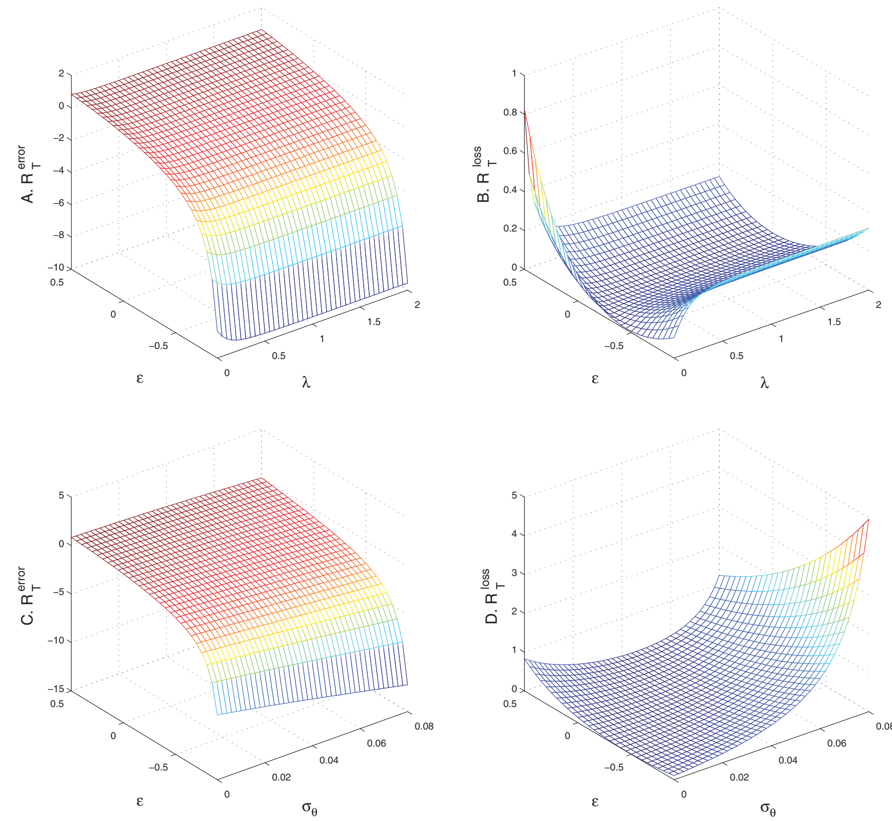

FIGURE 7. (Color online) Relationship between passive investors' $\mathscr{R}$ values and $(\lambda, \sigma_\theta, \epsilon)$ for the mean-reverting case. In this figure, the benchmark is given by $x_0 = 1$, $r = 3\%$, $b = 10\%$, $\theta_0 = \bar{\theta} = 0.35$, $\delta_0 = 0.025$, $\lambda = 0$, $\sigma_\theta = 0$, and $T = 10$. Panels A, B, C, D plot the dependencies of $\mathscr{R}_T^{\text{error}}$ and $\mathscr{R}_T^{\text{loss}}$ on (λ, ϵ), $(\sigma_\theta, \epsilon)$, respectively. When we picturize the dependencies of $\mathscr{R}_T^{\text{error}}$ and $\mathscr{R}_T^{\text{loss}}$ on ϵ and one of these parameters, we keep other parameters equal to the benchmark.

Parallel to Proposition 1, the true wealth of this portfolio is

$$\hat{X}_T^{\text{true}} = x_0 \prod_{n=0}^{N-1} \left[e^{b\tau} - \frac{e^{b\tau} - e^{r\tau}}{\text{Var}[Z_\tau]} (\bar{Z}_\tau - 1) \right]. \tag{33}$$

Choose $r = 3\%$, $b = 10\%$, $\theta_0 = \bar{\theta} = 0.35$, $\delta_0 = 0.025$, and $T = 10$ as before. We calculate the inferred Sharpe ratio $\mathscr{S}_T$, the true Sharpe ratio $\mathscr{S}_T^{\text{true}}$, the potentially maximal Sharpe ratio $\mathscr{S}_T^{\text{potential}}$, the reliability evaluations $\mathscr{R}_T^{\text{error}}$, and $\mathscr{R}_T^{\text{loss}}$ when N takes different values.[20] Our examples contain four typical structures: i.i.d. ($\lambda = \sigma_\theta = 0$), mean-reverting only ($\lambda = 1.0$,

[20] It follows from simple computations that Sharpe ratios associated to (32) and (33) are

$$\mathscr{S}_T = \frac{e^{bT} - e^{rT}}{\sqrt{\left[e^{2b\tau} + \frac{(e^{b\tau} - e^{r\tau})^2}{\text{Var}[Z_\tau]} \right]^N - e^{2bT}}},$$

$$\mathscr{S}_T^{\text{true}} = \frac{(\alpha_\tau - \beta_\tau E[\bar{Z}_\tau])^N - e^{rT}}{\sqrt{(\alpha_\tau^2 - 2\alpha_\tau\beta_\tau E[\bar{Z}_\tau] + \beta_\tau^2 E[\bar{Z}_\tau^2])^N - (\alpha_\tau - \beta_\tau E[\bar{Z}_\tau])^{2N}}},$$

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TABLE 1. Values of $\mathscr{S}_T$, $\mathscr{S}_T^{\text{true}}$, $\mathscr{S}_T^{\text{potential}}$, $\mathscr{R}_T^{\text{error}}$, and $\mathscr{R}_T^{\text{loss}}$

	(λ, σ_θ)	$\mathscr{S}_T$	$\mathscr{S}_T^{\text{true}}$	$\mathscr{S}_T^{\text{potential}}$	$\mathscr{R}_T^{\text{error}}$	$\mathscr{R}_T^{\text{loss}}$
$N = 1$	$(0, 0)$	1.5505	1.0090	1.1444	0.3492	0.1341
	$(1.0, 0)$	1.5505	1.5438	1.5441	0.0043	0.0002
	$(0, 0.08)$	1.5505	2.8434	8.7639	−0.8338	2.0822
	$(1.0, 0.08)$	1.5505	1.9684	2.1448	−0.2695	0.0896
$N = 2$	$(0, 0)$	1.0805	0.9141	0.9423	0.1541	0.0309
	$(1.0, 0)$	1.0805	1.0734	1.0736	0.0066	0.0002
	$(0, 0.08)$	1.0805	1.4667	1.7330	−0.3573	0.1816
	$(1.0, 0.08)$	1.0805	1.2700	1.3291	−0.1753	0.0465
$N = 5$	$(0, 0)$	0.8539	0.8076	0.8126	0.0542	0.0062
	$(1.0, 0)$	0.8539	0.8450	0.8453	0.0104	0.0004
	$(0, 0.08)$	0.8539	0.9578	0.9856	−0.1217	0.0290
	$(1.0, 0.08)$	0.8539	0.9336	0.9513	−0.0933	0.0190
$N = 10$	$(0, 0)$	0.7847	0.7646	0.7664	0.0256	0.0023
	$(1.0, 0)$	0.7847	0.7766	0.7769	0.0103	0.0004
	$(0, 0.08)$	0.7847	0.8293	0.8380	−0.0568	0.0105
	$(1.0, 0.08)$	0.7847	0.8249	0.8323	−0.0512	0.0090
$N = 20$	$(0, 0)$	0.7510	0.7417	0.7424	0.0124	0.0009
	$(1.0, 0)$	0.7510	0.7452	0.7455	0.0077	0.0004
	$(0, 0.08)$	0.7510	0.7715	0.7750	−0.0273	0.0044
	$(1.0, 0.08)$	0.7510	0.7708	0.7740	−0.0263	0.0042

In this table, we use the parameter setting $r = 3\%$, $b = 10\%$, $\theta_0 = \bar{\theta} = 0.35$, $\delta_0 = 0.025$, $T = 10$, and $\epsilon = 0$.

$\sigma_\theta = 0$), diffusion only ($\lambda = 0$, $\sigma_\theta = 0.08$), and mixed mean-reverting ($\lambda = 1.0$, $\sigma_\theta = 0.08$). We caution readers that in contrast to the single-period case, the Sharpe ratio relies on the expected return rate b in a multi-period context. Therefore we redefine $\mathscr{S}_T^{\text{potential}}$ to be the Sharpe ratio acquired by the unbiased learner who aims at the expected return $e^{bT} := E[\hat{X}_T^{\text{true}}]$ in advance.

The results are listed in Table 1. In agreement with what Cvitanić et al. [8] documented, $\mathscr{S}_T$, $\mathscr{S}_T^{\text{true}}$, and $\mathscr{S}_T^{\text{potential}}$ all become smaller as the number N increases. This illustrates that the strategy of maximizing the short-run Sharpe ratio results in a significant loss of performance for both active and passive investors who care about the long-run Sharpe ratio. For all the four subcases and all N, the performance of learners is always better than that of passive investors. However, as N increases, $\mathscr{S}_T^{\text{potential}}$ approximates $\mathscr{S}_T$ more and more closely, which implies that the effect of learning becomes less significant as τ becomes smaller. Intuitively, when τ is infinitesimal, the effect of learning in such small time interval diminishes and hence $\mathscr{S}_T^{\text{potential}}$ converges monotonously to $\mathscr{S}_T$. As a result, $\mathscr{R}_T^{\text{loss}}$ and $\mathscr{R}_T^{\text{error}}$ go to zero as N becomes infinite. Relative to the one-period strategy, multi-period strategies help passive investors to improve the reliability.

7. CONCLUSION

This paper investigates how inaccurate information about the stock return influences the Sharpe ratio maximizer's performance in a continuous-time model. The technical contribution of our paper is not the solving of an integration of Kalman filtering and Sharpe

where $\alpha_\tau = e^{b\tau} + \{(e^{b\tau} - e^{r\tau})/\text{Var}[Z_\tau]\}$ and $\beta_\tau = (e^{b\tau} - e^{r\tau})/\text{Var}[Z_\tau]$. The details are in line with Proposition 5 in Cvitanić et al. [8] and thus are omitted.

ratio optimization, but rather an analytical expression for the relation between the prespecified goal $\mathscr{S}_T$ and the ex post realization $\mathscr{S}_T^{\text{true}}$ and a thorough comparative static analysis of the performance of passive strategies with respect to fundamental factors in underlying dynamics. Our framework also suggests an approach to compute the implied parameters in stock return predictor by equating the theoretical prediction with the realized observation.

All investors in this paper could not observe the fundamental Brownian motion W_t^u or the risk award ϑ_t^u directly. Both active and passive investors infer the Brownian motion W_t by observing market prices. The unique distinction between active and passive investors lies in that active investors conduct a Bayesian procedure to complete their information on ϑ_t, while passive investors give up learning and just take for granted ϑ_t to be constant. For learners, matching the observed randomness W_t with the unbiased predictor θ_t allows them to follow up the underlying dynamic precisely, and in turn to achieve the potentially maximal Sharpe ratio that the market can provide. For passive investors, due to the partial information, they are certainly confronted with a gap between their expected goals and actual realizations. This gap is measured by $\mathscr{R}_T^{\text{error}}$. Aside from ill-judged errors, passive investors also incur heavy losses in profit opportunities. The losses are quantified by $\mathscr{R}_T^{\text{loss}}$.

We consider four typical dynamic structures: i.i.d., mean-reverting only ($\lambda \neq 0, \sigma_\theta = 0$), diffusion only ($\lambda = 0, \sigma_\theta \neq 0$), and the mixture of mean-reverting and diffusion ($\lambda \neq 0, \sigma_\theta \neq 0$). Overall, the uncertainty in predictor has two conflicting influences on the Sharpe ratio. It gives rise to a higher underlying risk whereas it offers opportunities of higher returns. As a benchmark, the unbiased learner captures the dynamic of θ_t accurately, and thus can hedge its risk in an efficient way as well as benefit from its potentially high realizations.

For passive investors, being blind to predictor uncertainty makes them lose profit opportunities. In both i.i.d. and mean-reverting environments, $\mathscr{R}_T^{\text{error}}$ and $\mathscr{R}_T^{\text{loss}}$ diverge as δ_0 increases, while the reliability of passive investment is improved greatly when the mean-reverting intensity λ is significant. Both $\mathscr{R}_T^{\text{error}}$ and $\mathscr{R}_T^{\text{loss}}$ depart steeply from zero as σ_θ grows. The above yields three implications. First, in the most common mean-reverting situation, passive investors behave much better than in the i.i.d. case. This partially explains the reasonableness of the existence of passive investments. Second, the numerical result illustrates that the factor dominating the passive investor's performance in the mean-reverting environment is the long-run volatility σ_θ. Third, without exact information on the stock return, pessimistic passive investors can achieve better performance than optimistic passive ones.

For long-run investments, the accumulated potential error of passive strategies becomes huge, giving rise to the poor reliability. Passive investors would face a lower Sharpe ratio in the multi-period context relative to the one-period strategy with the same time horizon, which is consistent with the result of Cvitanić et al. [8] for learners. However, our numerical results illustrate that, multi-period strategies help passive investors to reduce $\mathscr{R}_T^{\text{error}}$ and $\mathscr{R}_T^{\text{loss}}$.

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APPENDIX A

A.1. Technical Results

Proposition A.1: *Under the i.i.d. assumption, solutions of minimization problem (7) are given below.*

(A) For learners, $\kappa_t = [\delta_t A_t - 2(1+\epsilon)]\theta_t$ *and* $\Lambda_0 = \exp\left([A_0/2]\theta_0^2 + B_0\right)$, *where* A_t *and* B_t *are uniquely determined by (B.9) and (B.10) in Appendix B.*

(B) For passive investors, $\kappa_t = -2(1+\epsilon)\theta_0$ *and* $\Lambda_0 = \exp[(1+\epsilon)^2\theta_0^2 T]$.

Proposition A.2: *In the i.i.d. environment, moments of* $\bar{Z}_T$ *defined in (11) for passive investors are* $E[\bar{Z}_T] = \exp\left[\epsilon(1+\epsilon)\theta_0^2 T + \frac{1}{2}\delta_0(1+\epsilon)^2\theta_0^2 T^2\right]$ *and* $E[\bar{Z}_T^2] = \exp[(1+\epsilon)(1+3\epsilon)\theta_0^2 T + 2\delta_0(1+\epsilon)^2\theta_0^2 T^2]$.

Proposition A.3: *Under the mean-reverting assumption, solutions of minimization problem (7) are given below.*

(A) For learners, $\kappa_t = -2(1+\epsilon)\theta_t - (\sigma_\theta - \delta_t)(A_t\theta_t + \bar{\theta}B_t)$ *and* $\Lambda_0 = \exp\left(\frac{1}{2}A_0\theta_0^2 + \bar{\theta}B_0\theta_0 + \bar{\theta}^2 C_0 + D_0\right)$, *where* A_t, B_t, C_t, *and* D_t *are uniquely solved by (B.12)–(B.15) in Appendix B.*

(B) For passive investors, $\kappa_t = -2(1+\epsilon)\theta_0$ *and* $\Lambda_0 = \exp[(1+\epsilon)^2\theta_0^2 T]$.

Proposition A.4: *In the mean-reverting environment, moments of* $\bar{Z}_T$ *defined in (11) for passive investors are* $E[\bar{Z}_T] = e^{A+(B/2)}$ *and* $E[\bar{Z}_T^2] = e^{2A+2B}$, *where* $A = \frac{1}{2}(1+\epsilon)^2\theta_0^2 T - (1+\epsilon)\theta_0\bar{\theta}T - \frac{1}{\lambda}(1+\epsilon)\theta_0(\theta_0 - \bar{\theta})(1 - e^{-\lambda T})$ *and* $B = (1+\epsilon)^2\theta_0^2\int_0^T \left\{1 - \frac{1}{\lambda}\left[1 - e^{-\lambda(T-t)}\right](\sigma_\theta - \delta_t)\right\}^2 dt$.

APPENDIX B

B.1. Detailed Proofs

Proof of Proposition 1: Recall from Proposition 1 (A) that the optimal portfolio $\hat{\pi}_t$ is

$$\hat{\pi}_t = -\frac{1}{\sigma}\left[x_0\left(e^{bT} + \frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}\right)e^{-r(T-t)} - \hat{X}_t\right](\vartheta_t + \kappa_t).$$

Let $c = x_0\left(e^{bT} + [e^{bT} - e^{rT}/(\Lambda_0 - 1)\right)]e^{-rT}$, thus the self-financing equation (8) can be written in the form

$$\begin{aligned} &d\hat{X}_t - \hat{X}_t\{(\vartheta_t + \kappa_t)dW_t^u + [(\vartheta_t + \kappa_t)\vartheta_t^u + r]dt\} \\ &\quad = -ce^{rt}[(\vartheta_t + \kappa_t)dW_t^u + (\vartheta_t + \kappa_t)\vartheta_t^u dt]. \end{aligned} \tag{B.1}$$

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Choose a stochastic multiplier $G_t = \exp\left(\int_0^t a_s dW_s^u + \int_0^t b_s ds\right)$, where a_t and b_t are $\mathscr{F}_t^u$-adapted stochastic processes to be determined later. Applying the Itô formula to the process $\hat{X}_t G_t$ yields

$$\begin{aligned} d(\hat{X}_t G_t) &= G_t d\hat{X}_t + \hat{X}_t dG_t + d\hat{X}_t dG_t \\ &= G_t d\hat{X}_t + \hat{X}_t G_t \left[a_t dW_t^u + \left(\frac{a_t^2}{2} + b_t\right) dt\right] + (\hat{X}_t - ce^{rt})(\vartheta_t + \kappa_t) G_t a_t dt \\ &= G_t d\hat{X}_t + \hat{X}_t G_t \left\{a_t dW_t^u + \left[\frac{a_t^2}{2} + b_t + (\vartheta_t + \kappa_t) a_t\right] dt\right\} \\ &\quad - ce^{rt} G_t (\vartheta_t + \kappa_t) a_t dt. \end{aligned} \tag{B.2}$$

After taking $a_t = -(\vartheta_t + \kappa_t)$ and $b_t = -(\vartheta_t + \kappa_t)\vartheta_t^u - r + \frac{1}{2}(\vartheta_t + \kappa_t)^2$, we insert (B.1) into (B.2) and find

$$\begin{aligned} d(\hat{X}_t G_t) &= -ce^{rt} G_t [(\vartheta_t + \kappa_t) dW_t^u + (\vartheta_t + \kappa_t)(\vartheta_t^u + a_t) dt] \\ &= ce^{rt}(dG_t + G_t r dt). \end{aligned}$$

Integrating the above, we get

$$\begin{aligned} \hat{X}_t &= \frac{x_0}{G_t} + \frac{c}{G_t} \int_0^t e^{rs}(dG_s + rG_s ds) \\ &= \frac{x_0}{G_t} + \frac{c}{G_t}(e^{rt} G_t - 1) \\ &= x_0 e^{-r(T-t)} \left(e^{bT} + \frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}\right) + \frac{x_0}{G_t}\left[1 - e^{-rT}\left(e^{bT} + \frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}\right)\right]. \end{aligned}$$

Notice that $1/G_t = e^{rt} Y_t$, where

$$Y_t := \exp\left\{\int_0^t (\vartheta_s + \kappa_s) dW_s^u + \int_0^t \left[(\vartheta_s + \kappa_s)\vartheta_s^u - \frac{1}{2}(\vartheta_s + \kappa_s)^2\right] ds\right\}, \tag{B.3}$$

we can rewrite $\hat{X}_t$ as

$$\hat{X}_t = x_0 e^{-r(T-t)} \left(e^{bT} + \frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}\right) + x_0 e^{rt}\left[1 - e^{-rT}\left(e^{bT} + \frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}\right)\right] Y_t.$$

Especially, when $t = T$,

$$\hat{X}_T = x_0 \left(e^{bT} + \frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}\right) + x_0 \left[e^{rT} - \left(e^{bT} + \frac{e^{bT} - e^{rT}}{\Lambda_0 - 1}\right)\right] Y_T. \tag{B.4}$$

By definition of N_t, one gets $N_t = \Lambda_0 \exp\left(\int_0^t k_s dW_s - \frac{1}{2}\int_0^t k_s^2 ds\right)$. Because of $N_T = Z_T^2 = \exp\left(-2\int_0^T \vartheta_s dW_s - \int_0^T \vartheta_s^2 ds\right)$, one obtains

$$\exp\left[\int_0^T (2\vartheta_s + \kappa_s) dW_s + \int_0^T \left(\vartheta_s^2 - \frac{1}{2}\kappa_s^2\right) ds\right] = \frac{1}{\Lambda_0}. \tag{B.5}$$

From (2), we have

$$dW_t = dW_t^u + (\vartheta_t^u - \theta_t) dt. \tag{B.6}$$

Inserting (B.5) and (B.6) into (B.3) yields

$$Y_T = \frac{1}{\Lambda_0} \exp\left[-\int_0^T \vartheta_s dW_s + \int_0^T \vartheta_s \left(\theta_s - \frac{3}{2}\vartheta_s\right) ds + \int_0^T \kappa_s(\theta_s - \vartheta_s) ds\right]. \tag{B.7}$$

The combination of (B.4) with (B.7) completes the proof. ■

Proof of Proposition A.1: To prove (A), recall that

$$Z_t = \exp\left[-(1+\epsilon)\int_0^t \theta_s dW_s - \frac{1}{2}(1+\epsilon)^2 \int_0^t \theta_s^2 ds\right].$$

We calculate $\Lambda_t = \Lambda_t(t, \theta_t)$ which is defined in Proposition 1. Note that Λ_t is a function of t and θ_t, where θ_t denotes the realization of the corresponding stochastic process. Let $\delta_t = \delta_0/(\delta_0 t + 1)$. Applying the Itô formula to the process $Z_t^2 \Lambda(t, \theta_t)$ yields

$$\begin{aligned} d[Z_t^2 \Lambda(t,\theta_t)] &= Z_t^2 d\Lambda(t,\theta_t) + \Lambda(t,\theta_t) dZ_t^2 + dZ_t^2 d\Lambda(t,\theta_t) \\ &= Z_t^2 \mathcal{Q}\Lambda(t,\theta_t)dt + Z_t^2 \left[\delta_t \frac{\partial}{\partial \theta_t} - 2(1+\epsilon)\theta_t\right] \Lambda(t,\theta_t) dW_t \end{aligned} \tag{B.8}$$

with

$$\mathcal{Q} = \frac{\partial}{\partial t} - 2(1+\epsilon)\theta_t \delta_t \frac{\partial}{\partial \theta_t} + \frac{1}{2}\delta_t^2 \frac{\partial^2}{\partial \theta_t^2} + (1+\epsilon)^2 \theta_t^2.$$

Since $Z_t^2 \Lambda(t,\theta_t)$ is a martingale under P, the drift in the right-hand side of (B.8) should vanish. Therefore we obtain the partial differential equation (PDE) governing $\Lambda(t,\theta_t)$ from $\mathcal{Q}\Lambda(t,\theta_t) = 0$ with the terminal condition $\Lambda(T,\theta) = 1$ for all $\theta \in \Re$. We fit the solution in the form $\Lambda(t,\theta_t) = \exp\left(\frac{A_t}{2}\theta_t^2 + B_t\right)$ and separate the polynomial coefficients of θ_t. Then we obtain that

$$\frac{1}{2}\frac{d}{dt}A_t - 2(1+\epsilon)\delta_t A_t + \frac{1}{2}\delta_t^2 A_t^2 + (1+\epsilon)^2 = 0, \quad A_T = 0, \tag{B.9}$$

$$\frac{d}{dt}B_t + \frac{1}{2}\delta_t^2 A_t = 0, \quad B_T = 0. \tag{B.10}$$

The ODE (B.9) is of Riccati type with variable coefficients and $B_t = \frac{1}{2}\int_t^T \delta_s^2 A_s ds$. Having obtained the expression of $\Lambda(t,\theta_t)$, one can calculate easily the process κ_t from (B.8) by

$$\kappa_t = \frac{1}{\Lambda(t,\theta_t)}\left[\delta_t \frac{\partial}{\partial \theta_t} - 2(1+\epsilon)\theta_t\right]\Lambda(t,\theta_t) = [\delta_t A_t - 2(1+\epsilon)]\theta_t.$$

For (B), $\vartheta_t = (1+\epsilon)\theta_0$ is a constant. In this case, $E[Z_T^2 | \mathscr{F}_t^{S^u}] = Z_t^2 \exp[(1+\epsilon)^2\theta_0^2(T-t)]$, $\kappa_t = -2(1+\epsilon)\theta_0$ and $\Lambda_0 = \exp[(1+\epsilon)^2\theta_0^2 T]$. ■

Proof of Proposition A.2: When $\vartheta_t = (1+\epsilon)\theta_0$, one has $\bar{Z}_T = e^{Y_T}$, with

$$\begin{aligned} Y_T &= \frac{1}{2}(1+\epsilon)^2\theta_0^2 T - (1+\epsilon)\theta_0\left(\int_0^T dW_t + \int_0^T \theta_t dt\right) \\ &= \frac{1}{2}(1+\epsilon)^2\theta_0^2 T - (1+\epsilon)\theta_0\left(\int_0^T dW_t + T\theta_T - \int_0^T t d\theta_t\right) \\ &= \frac{1}{2}(\epsilon^2 - 1)\theta_0^2 T - (1+\epsilon)(1+\delta_0 T)\theta_0 \int_0^T \frac{1}{\delta_0 t + 1} dW_t, \end{aligned}$$

which is normally distributed with mean $\frac{1}{2}(\epsilon^2-1)\theta_0^2 T$ and variance $(1+\epsilon)^2(1+\delta_0 T)\theta_0^2 T$. Thanks to the familiar formula $E[e^{N(\mu,\sigma^2)}] = e^{\mu+\frac{\sigma^2}{2}}$, one gets the desired easily. ■

Proof of Proposition 2: The Riccati ODE with variable coefficients usually admits no closed-form solutions. Even if it is solved analytically, the solution is always complicated. However, in the case $\epsilon = 0$, the ODE (B.9) is easily solved. In detail, (B.9) can be transformed as

$$\frac{d(A_t\delta_t)}{(A_t\delta_t - 1)(A_t\delta_t - 2)} = -\delta_t dt,$$

which can be solved completely by $A_t = [2(1+\delta_0 t)(T-t)]/[1+\delta_0(2T-t)]$. Next, (B.10) gives us

$$B_t = \frac{1}{2}\ln\left\{\frac{(1+\delta_0 T)^2}{(1+\delta_0 t)[1+\delta_0(2T-t)]}\right\}.$$

Thus Proposition 2 follows straightforwardly from Proposition 1 and Proposition A.1. ∎

Proof of Proposition A.3: For (A), denote $\Lambda_t = \Lambda_t(t,\theta_t)$, where the variable θ_t is involved in the information $\mathscr{F}_t^{S^u}$. Applying Itô formula to the process $Z_t^2\Lambda(t,\theta_t)$, we get

$$\begin{aligned} d[Z_t^2\Lambda(t,\theta_t)] &= Z_t^2 d\Lambda(t,\theta_t) + \Lambda(t,\theta_t)dZ_t^2 + dZ_t^2 d\Lambda_t(t,\theta_t) \\ &= Z_t^2\mathcal{Q}\Lambda(t,\theta_t)dt - Z_t^2\left[(\sigma_\theta - \delta_t)\frac{\partial}{\partial\theta_t} + 2(1+\epsilon)\theta_t\right]\Lambda(t,\theta_t)dW_t \end{aligned} \quad \textbf{(B.11)}$$

with

$$\mathcal{Q} = \frac{\partial}{\partial t} + [\lambda(\bar\theta - \theta_t) + 2(1+\epsilon)\theta_t(\sigma_\theta - \delta_t)]\frac{\partial}{\partial\theta_t} + \frac{1}{2}(\sigma_\theta - \delta_t)^2\frac{\partial^2}{\partial\theta_t^2} + (1+\epsilon)^2\theta_t^2.$$

Since $Z_t^2\Lambda(t,\theta_t)$ is a martingale under P, the drift in the right-hand side of (B.11) should be zero. Hence the PDE of $\Lambda(t,\theta_t)$ is $\mathcal{Q}\Lambda(t,\theta_t) = 0$, with the terminal condition $\Lambda(T,\theta) = 1$ for all $\theta \in \Re$. We fit the solution in the form $\Lambda(t,\theta_t) = \exp\left([A_t/2]\theta_t^2 + \bar\theta B_t\theta_t + \bar\theta^2 C_t + D_t\right)$ and separate the polynomial coefficients of θ_t to obtain that

$$\frac{1}{2}\frac{d}{dt}A_t + [2(1+\epsilon)(\sigma_\theta - \delta_t) - \lambda]A_t + \frac{1}{2}(\sigma_\theta - \delta_t)^2 A_t^2 + (1+\epsilon)^2 = 0, \quad A_T = 0, \quad \textbf{(B.12)}$$

$$\frac{d}{dt}B_t + [(\sigma_\theta - \delta_t)^2 A + 2(1+\epsilon)(\sigma_\theta - \delta_t) - \lambda]B_t + \lambda A_t = 0, \quad B_T = 0, \quad \textbf{(B.13)}$$

$$\frac{d}{dt}C_t + \lambda B_t + \frac{1}{2}(\sigma_\theta - \delta_t)^2 B_t^2 = 0, \quad C_T = 0. \quad \textbf{(B.14)}$$

$$\frac{d}{dt}D_t + \frac{1}{2}(\sigma_\theta - \delta_t)^2 A_t = 0, \quad D_T = 0. \quad \textbf{(B.15)}$$

The ODE (B.12) is of Riccati type with variable coefficients and it admits a unique positive solution theoretically. We solve (B.13)–(B.15) by

$$\begin{aligned} B_t &= \lambda\int_t^T \exp\left\{\int_t^s [(\sigma_\theta - \delta_\tau)^2 A_\tau + 2(1+\epsilon)(\sigma_\theta - \delta_\tau) - \lambda]d\tau\right\} A_s ds, \\ C_t &= \int_t^T \left[\lambda B_s + \frac{1}{2}(\sigma_\theta - \delta_s)^2 B_s^2\right] ds \quad \text{and} \quad D_t = \frac{1}{2}\int_t^T (\sigma_\theta - \delta_s)^2 A_s ds. \end{aligned}$$

Having obtained the expression of $\Lambda(t,\theta_t)$, the process κ_t follows directly from (B.11) by

$$\begin{aligned} \kappa_t &= -\frac{1}{\Lambda(t,\theta_t)}\left[(\sigma_\theta - \delta_t)\frac{\partial}{\partial\theta_t} + 2(1+\epsilon)\theta_t\right]\Lambda(t,\theta_t) \\ &= -2(1+\epsilon)\theta_t - (\sigma_\theta - \delta_t)(A_t\theta_t + \bar\theta B_t). \end{aligned}$$

The statement (B) is exactly the same as that in Proposition A.1. ∎

Proof of Proposition A.4: When $\vartheta_t = (1+\epsilon)\theta_0$, one has $\bar{Z}_T = e^{Y_T}$, with

$$Y_T = \frac{1}{2}(1+\epsilon)^2\theta_0^2 T - (1+\epsilon)\theta_0\left(\int_0^T dW_t + \int_0^T \theta_t dt\right), \tag{B.16}$$

where

$$\theta_t = e^{-\lambda t}\theta_0 + (1-e^{-\lambda t})\bar{\theta} - \int_0^t e^{-\lambda(t-s)}(\sigma_\theta - \delta_s)dW_s.$$

Thus

$$\begin{aligned}\int_0^T \theta_t dt &= T\theta_T - \int_0^T t d\theta_t \\ &= \bar{\theta}T + \frac{1}{\lambda}(1-e^{-\lambda T})(\theta_0 - \bar{\theta}) - \frac{1}{\lambda}\int_0^T \left[1 - e^{-\lambda(T-t)}\right](\sigma_\theta - \delta_t)dW_t. \end{aligned}\tag{B.17}$$

Inserting (B.17) into (B.16), we find

$$\begin{aligned} Y_T = \frac{1}{2}(1+\epsilon)^2\theta_0^2 T - (1+\epsilon)\theta_0\bar{\theta}T - \frac{1}{\lambda}(1+\epsilon)\theta_0(\theta_0 - \bar{\theta})(1-e^{-\lambda T}) \\ - (1+\epsilon)\theta_0 \int_0^T \left\{1 - \frac{1}{\lambda}\left[1 - e^{-\lambda(T-t)}\right](\sigma_\theta - \delta_t)\right\} dW_t. \end{aligned}$$

Notice that Y_T is normally distributed with mean $\frac{1}{2}(1+\epsilon)^2\theta_0^2 T - (1+\epsilon)\theta_0\bar{\theta}T - (1/\lambda)(1+\epsilon)\theta_0(\theta_0-\bar{\theta})(1-e^{-\lambda T})$ and variance $(1+\epsilon)^2\theta_0^2\int_0^T\{1-(1/\lambda)[1-e^{-\lambda(T-t)}](\sigma_\theta-\delta_t)\}^2 dt$. By virtue of $E[e^{N(\mu,\sigma^2)}] = e^{[\mu+(\sigma^2/2)]}$, one gets the desired easily. ■