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SIGN PROPERTIES OF GREEN'S FUNCTIONS FOR TWO CLASSES OF BOUNDARY VALUE PROBLEMS

BY

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ABSTRACT. Let $G(x,s)$ be the Green's function for the boundary value problem $y^{(n)} = 0$, $Ty = 0$, where $Ty = 0$ represents boundary conditions at two points. The signs of $G(x,s)$ and certain of its partial derivatives with respect to x are determined for two classes of boundary value problems. The results are also carried over to analogous classes of boundary value problems for difference equations.

1. **Introduction**. Let $[a,b] \subseteq \mathbb{R}$. let $n \geq 2$, $2 \leq k \leq n$, $1 \leq r_j$, $1 \leq j \leq k$, be natural numbers. Let $a \leq x_1 < x_2 < \ldots < x_k \leq b$ and consider the boundary value problem (BVP),

$$y^{(n)} = 0, \tag{1.1}$$

$$y^{(i-1)}(x_j) = 0,\ 1 \leq i \leq r_j,\ 1 \leq j \leq k, \tag{1.2}$$

where $\Sigma_{j=1}^{k} r_j = n$. We call a BVP with boundary conditions given by (1.2) a conjugate type BVP. Levin [2], [5] has shown that the Green's function, $G(x,s)$, for the BVP, (1.1), (1.2), satisfies

$$G(x,s)\,(x - x_1)^{r_1}(x - x_2)^{r_2} \ldots (x - x_k)^{r_k} \geq 0, \tag{1.3}$$

on $[a,b] \times [a,b]$. Peterson [6] has obtained (1.3) for a class of two point conjugate type BVP's for more general differential operators; Ridenhour [9] has recently extended Peterson's results to multipoint problems.

Assume $a \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq b$ and permute $\{1, \ldots, n\}$ to $\{r_1, \ldots, r_n\}$. We call a BVP with boundary conditions given by

$$y^{(r_j - 1)}(x_j) = 0,\ 1 \leq j \leq n, \tag{1.4}$$

a focal type BVP, where n is the order of the differential equation. We say the problem is a right focal type BVP if $r_j = j$, $1 \leq j \leq n$. Peterson [7] has shown that the Green's function, $G(x,s)$, for a class of two point right focal type BVP's satisfies

$$\begin{cases} (-1)^{n-k}\, \partial^l/\partial x^l\, G(x,s) \geq 0,\ 0 \leq l \leq k, \\ (-1)^{n-l}\, \partial^l/\partial x^l\, G(x,s) \geq 0,\ k+1 \leq l \leq n-1, \end{cases} \tag{1.5}$$

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on $[a,b] \times [a,b]$, where $a = x_1 = x_2 = \ldots = x_k$, $x_{k+1} = x_{k+2} = \ldots = x_n = b$.

Hartman [4] obtained an analogue of (1.3) for a class of conjugate type BVP's for difference equations and Peterson [8] has recently obtained further results concerning Green's functions for difference equations.

In this paper, we obtain analogues of (1.3) and (1.5) for two classes of two point BVP's. In section 2, we consider two point BVP's for ordinary differential equations; we obtain an analogue of (1.3) for a BVP where the boundary conditions are "between" conjugate type and right focal type conditions, and we obtain an analogue of (1.5) for the two point focal type BVP, (1.1), (1.4). We shall also show that an analogue of (1.3) or (1.5) does not exist, in general, for a multipoint focal type BVP. In section 3, we consider similar questions for difference equations.

2. Boundary value problems for ordinary differential equations. Let $a < b$, $n \geq 1$, $a_0(x), a_1(x), \ldots, a_{n-1}(x) \in C[a,b]$, and let $T: C^{n-1}[a,b] \rightarrow \mathbb{R}^n$ be a continuous, linear map. We begin by characterizing the Green's function, $G(x,s)$, for the BVP,

$$Ly = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0,$$

$$Ty = 0.$$

See [1, p. 190−193], for example.

LEMMA 2.1. *Suppose $y \equiv 0$ is the only solution of $Ly = 0$, $Ty = 0$. Then there exists a unique function, $G(x,s)$, defined on $[a,b] \times [a,b]$ having the following properties:*

(i) $\partial^k/\partial x^k\, G(k = 0, 1, \ldots, n-2)$ exist and are continuous on $[a,b] \times [a,b]$. $\partial^k/\partial x^k\, G(k = n-1, n)$ are continuous on triangles $a \leq x < s \leq b$ and $a \leq s < x \leq b$.

(ii) $\lim_{x\rightarrow s^+} \partial^{n-1}/\partial x^{n-1}\, G(x,s) - \lim_{x\rightarrow s^-} \partial^{n-1}/\partial x^{n-1}\, G(x,s) = 1$.

(iii) As a function of x, G satisfies $LG = 0$ if $x \neq s$.

(iv) As a function of x, G satisfies $TG = 0$ for $a \leq s \leq b$. Moreover, for any $f \in C[a,b]$, the unique solution of $Ly = f(x)$, $Ty = 0$ is given by

$$\phi(x) = \int_a^b G(x,s)f(s)ds.$$

Let $n \geq 2$, and $k \in \{1, \ldots, n-1\}$. We consider the BVP,

$$y^{(n)} = 0, \tag{2.1}$$

$$\begin{cases} y^{(r-1)}(a) = 0,\ 1 \leq r \leq k, \\ y^{(s-1)}(b) = 0,\ j+1 \leq s \leq j+n-k, \end{cases} \tag{2.2}$$

where $j \in \{0, \ldots, k\}$. Note that if $j = 0$, (2.2) represents conjugate type boundary conditions, and if $j = k$, (2.2) represents right focal type boundary conditions. Our main result for the BVP, (2.1), (2.2), is that the Green's function, $G(x,s)$, satisfies

$$(-1)^{n-k}\, \partial^l/\partial x^l\, G(x,s) \geq 0,\ 0 \leq l \leq j, \tag{2.3}$$

on $[a,b] \times [a,b]$.

If $j = 0$, (2.3) follows from (1.3), and if $j = k$, (2.3) follows from (1.5). We now assume $j \in \{1, \ldots, k-1\}$ and hence, $k > 1$.

LEMMA 2.2. *Let $G(x,s)$ be the Green's function for the BVP, (2.1), (2.2). Then $\partial/\partial x\, G(x,s)$ is the Green's function for the BVP,*

$$y^{(n^*)} = 0, \tag{2.4}$$

$$\begin{cases} y^{(r-1)}(a) = 0, \ 1 \leq r \leq k^*, \\ y^{(s-1)}(b) = 0, \ j^* + 1 \leq s \leq j^* + n^* - k^*, \end{cases} \tag{2.5}$$

where $n^* = n - 1$, $k^* = k - 1$, $j^* = j - 1$.

PROOF. It is readily verified that $\partial/\partial x\, G(x,s)$ satisfies the four properties listed in Lemma 2.1 which uniquely characterize the Green's function of the BVP, (2.4), (2.5).

THEOREM 2.3. *The Green's function for the BVP, (2.1), (2.2), satisfies (2.3).*

PROOF. As noted above, Theorem 2.3 is true for $j = 0$ and $j = k$. Assume $k > 1$, $j \in \{1, \ldots, k-1\}$. By repeated applications of Lemma 2.2, we note that $\partial^j/\partial x^j\, G(x,s)$ is the Green's function for the conjugate type BVP,

$$y^{(n-j)} = 0,$$

$$\begin{cases} y(a) = \ldots = y^{(k-j-1)}(a) = 0, \\ y(b) = \ldots = y^{(n-k-1)}(b) = 0. \end{cases}$$

By (1.3),

$$(-1)^{n-k}\, \partial^j/\partial x^j\, G(x,s) \geq 0, \tag{2.6}$$

on $[a,b] \times [a,b]$. If $l < j$, use the boundary conditions (2.2) to obtain

$$\partial^l/\partial x^l\, G(x,s) = \int_a^x \partial^{l+1}/\partial t^{l+1}\, G(t,s)dt$$

for all $x \in [a,b]$. Since $\partial^j/\partial x^j\, G(x,s)$ satisfies (2.6), it follows inductively that $G(x,s)$ satisfies (2.3).

We now consider a focal type BVP. Let $a < b$, $n \geq 2$, $k \in \{1, \ldots, n-1\}$. Let $\{r_1, \ldots, r_k\}$, $\{s_1, \ldots, s_{n-k}\}$ be a partition of $\{1, \ldots, n\}$ such that $r_1 < r_2 < \ldots < r_k$ and $s_1 < s_2 < \ldots < s_{n-k}$. Consider the two point focal type BVP,

$$y^{(n)} = 0, \tag{2.7}$$

$$\begin{cases} y^{(r_i-1)}(a) = 0, \ 1 \leq i \leq k, \\ y^{(s_j-1)}(b) = 0, \ 1 \leq j \leq n-k. \end{cases} \tag{2.8}$$

Our main result concerning the BVP, (2.7, (2.8), is that the Green's function, $G(x,s)$, satisfies

$$(-1)^{\sigma_l}\,\partial^l/\partial x^l\, G(x,s) \geq 0, \tag{2.9}$$

on $[a,b] \times [a,b]$, for $0 \leq l \leq n-1$, where $\sigma_l = \text{card}\,\{j: s_j > l\}$.

Note that if $G(x,s)$ is the Green's function for (2.7), (2.8), then $\partial/\partial x\, G(x,s)$ is the Green's function for either the BVP

$$y^{(n^*)} = 0,$$

$$\begin{cases} y^{(r_i^*-1)}(a) = 0, \ 2 \leq i \leq k, \\ y^{(s_j^*-1)}(b) = 0, \ 1 \leq j \leq n-k, \end{cases}$$

when $r_1 = 1$, or the BVP

$$y^{(n^*)} = 0,$$

$$\begin{cases} y^{(r_j^*-1)}(a) = 0, \ 1 \leq i \leq k, \\ y^{(s_j^*-1)}(b) = 0, \ 2 \leq j \leq n-k, \end{cases}$$

when $s_1 = 1$, where $n^* = n-1$, $r_i^* = r_i - 1$, $s_j^* = s_j - 1$. If $k = 1$ or $n-1$, then we consider initial value problems at b or a, respectively. We again use Lemma 2.1 to see that $\partial/\partial x\, G(x,s)$ is the Green's function for a focal type BVP. Proceed inductively and note that $\partial^{n-1}/\partial x^{n-1}\, G(x,s)$ is the Green's function for the first order problem

$$y' = 0, \ y(a) = 0, \text{ when } r_k = n,$$

or the problem

$$y' = 0, \ y(b) = 0, \text{ when } s_{n-k} = n.$$

THEOREM 2.4. *The Green's function, $G(x,s)$, for the focal type BVP, (2.7), (2.8), satisfies (2.9).*

PROOF. First assume $r_k = n$. Then, since $\partial^{n-1}/\partial x^{n-1}\, G(x,s)$ is the Green's function for the problem, $y' = 0$, $y(a) = 0$,

$$\partial^{n-1}/\partial x^{n-1}\, G(x,s) = \begin{cases} 1, \ a \leq s < x \leq b, \\ 0, \ a \leq x < s \leq b. \end{cases}$$

Let $l \in \{r_1, \ldots, r_k\}$. Using the boundary conditions, (2.8),

$$\partial^{l-1}/\partial x^{l-1}\, G(x,s) = \int_a^x \partial^l/\partial t^l\, G(t,s)dt.$$

If $\partial^l/\partial x^l\, G(x,s)$ is of constant sign on $[a,b] \times [a,b]$, then $\partial^{l-1}/\partial x^{l-1}\, G(x,s)$ is of constant sign and has the same sign.

Similarly, let $l \in \{s_1, \ldots, s_{m-k}\}$ and use (2.8) to obtain

$$\partial^{l-1}/\partial x^{l-1}\, G(x,s) = \int_b^x \partial^l/\partial t^l\, G(t,s)dt.$$

If $\partial^l/\partial x^l\, G(x,s)$ has constant sign, then $\partial^{l-1}/\partial x^{l-1}\, G(x,s)$ has constant sign and has opposite sign. Since $\partial^{n-1}/\partial x^{n-1}\, G(x,s) \geq 0$ on $[a,b] \times [a,b]$, it follows inductively that $G(x,s)$ satisfies (2.9).

Assume $s_{n-k} = n$. Then $\partial^{n-1}/\partial x^{n-1} G(x,s)$ is the Green's function for the problem, $y' = 0$, $y(b) = 0$. That is,

$$\partial^{n-1}/\partial x^{n-1}\, G(x,s) = \begin{cases} 0,\ a \leq s < x \leq b, \\ -1,\ a \leq x < s \leq b. \end{cases}$$

It is shown similarly that $G(x,s)$ satisfies (2.9).

REMARKS: i) The proofs of Theorems 2.3 and 2.4 can be used to obtain the Green's function, $G(x,s)$, for the BVP, (2.1), (2.2) and the focal type BVP, (2.7), (2.8), in closed form. For example,

$$H(x,s) = \begin{cases} -x\ ,\ 0 \leq x < s \leq 1, \\ -s\ ,\ 0 \leq s < x \leq 1, \end{cases}$$

is the Green's function for the focal type BVP $y'' = 0$, $y(0) = y'(1) = 0$. Thus,

$$G_1(x,s) = \int_0^x H(t,s)dt = \begin{cases} -x^2/2,\ 0 \leq x < s \leq 1, \\ s^2/2 - sx,\ 0 \leq s < x \leq 1, \end{cases}$$

is the Green's function for the focal type BVP $y''' = 0$, $y(0) = y'(0) = y''(1) = 0$; similarly,

$$G_2(x,s) = \int_1^x H(t,s)dt = \begin{cases} s - \dfrac{s^2}{2} - \dfrac{x^2}{2},\ 0 \leq x < s \leq 1, \\ s(1-x),\ 0 \leq s < x \leq 1, \end{cases}$$

is the Green's function for the focal type BVP $y''' = 0$, $y(1) = y'(0) = y''(1) = 0$.

ii) Let $\rho_i(x)$, $1 \leq i \leq n+1$ be positive, $n+1-i$ times continuously differentiable functions on $[a, b]$ and define quasi-derivatives D_i, $0 \leq i \leq n$, by

$$D_0 y = \rho_1(x)y,$$

$$D_i y = \rho_{i+1}(x)(D_{i-1}y)',\ 1 \leq i \leq n.$$

See [7], for example. Theorem 2.4 remains valid if we replace d^i/dx^i with corresponding quasi-derivatives D_i, $0 \leq i \leq n$, where $D_0 G = \rho_1(x)G(x,s)$, $D_i G(x,s) = \rho_{i+1}(x)$ $\partial/\partial x\, D_{i-1}G(x,s)$, $1 \leq i \leq n$. The proof is similar if we note that if $G(x,s)$ is the Green's function for the BVP

$$D_n y = 0$$

$$\begin{cases} D_{r_i - 1} y(a) = 0, \ 1 \leq i \leq k, \\ D_{s_j - 1} y(b) = 0, \ 1 \leq j \leq n - k, \end{cases}$$

then $1/\rho_2(x) D_1 G(x, s)$ is the Green's function for the BVP

$$D_{n^*}^* y = 0,$$

$$D_{r_j^* - l}^* y(a) = 0, \ 2 \leq i \leq k,$$

$$D_{s_j^* - 1}^* y(b) = 0, \ 1 \leq j \leq n - k,$$

when $r_1 = 1$, and the BVP

$$D_{n^*}^* y = 0,$$

$$\begin{cases} D_{r_i^* - 1}^* y(a) = 0, \ 1 \leq i \leq k, \\ D_{s_j^* - 1}^* y(b) = 0, \ 2 \leq j \leq n - k, \end{cases}$$

when $s_1 = 1$, where $n^* = n - 1$, $r_i^* = r_i - 1$, $s_j^* = s_j - 1$, $D_0^* y = \rho_2 y$, $D_i^* y = \rho_{i+2}(x)$ $(D_{i-1}^* y)'$, $1 \leq i \leq n^*$. It follows inductively that $1/\rho_n(x) D_{n-1} G(x, s)$ is the Green's function for the first order problem,

$$\rho_{n+1}(x)(\rho_n y)' = 0, \ \rho_n y(a) = 0, \text{ if } r_k = n,$$

$$\rho_{n+1}(x)(\rho_n y)' = 0, \ \rho_n y(b) = 0, \text{ if } s_{n-k} = n.$$

Theorem 2.3 also remains valid if we replace d^i/dx^i with corresponding quasi-derivatives D_i, $0 \leq i \leq n$. To see this, note that $D_n y = 0$ is disconjugate on $[a,b]$ See [2]. Also, conjugate type boundary conditions

$$\begin{cases} D_0 y(a) = \ldots = D_{k-1} y(a) = 0, \\ D_0 y(b) = \ldots = D_{n-k-1} y(b) = 0, \end{cases}$$

imply conjugate type boundary conditions,

$$\begin{cases} y(a) = \ldots = y^{(k-1)}(a) = 0, \\ y(b) = \ldots = y^{(n-k-1)}(b) = 0. \end{cases}$$

Thus, Levin's inequality (1.3) can be used precisely as it was used in the proof of Theorem 2.3 to obtain an analogue of Theorem 2.3.

iii) An analogue of (1.3) or (1.5) does not exist, in general, for a multipoint focal type BVP, (1.1), (1.4). For example, consider the BVP,

$$y''' = 0,$$

$$y(0) = y'(1) = y''(2 + \delta) = 0,$$

$\delta > 0$. The Green's function, $G(x,s)$, has the following representation. For $s \in [0,1]$,

$$G(x,s) = \begin{cases} s^2/2,\ 0 \le s < x \le 2 + \delta, \\ x(2s - x)/2,\ 0 \le x < s \le 1. \end{cases}$$

For $s \in [1, 2 + \delta]$,

$$G(x,s) = \begin{cases} s^2/2 - x(s-1),\ 1 \le s < x \le 2 + \delta, \\ x(2 - x)/2,\ 0 \le x < s \le 2 + \delta. \end{cases}$$

Although G_{xx} is the Green's function for $y' = 0$, $y(2 + \delta) = 0$, and

$$G_{xx}(x,s) = \begin{cases} 0,\ 0 \le s < x \le 2 + \delta, \\ -1,\ 0 \le x < s \le 2 + \delta, \end{cases}$$

the above techniques do not apply. Note that $G(x,s)$ changes sign for $x \in [1, 2 + \delta]$ at $x = 2$ if $s > x$.

3. Boundary value problems for difference equations. Let $I = \{a, a + 1, \ldots, b\} \subset \mathbb{R}$, $n \ge 2$, and $I^j = \{a, a + 1, \ldots, b + j\}$, $0 \le j \le n$. Define $\Delta^0 u(m) \equiv u(m)$, $\Delta^1 u(m) = u(m + 1) - u(m)$, and $\Delta^j u(m) = \Delta(\Delta^{j-1}u)(m)$, $1 < j \le n$. Let $k \in \{1, \ldots, n - 1\}$ and let $\{r_1, \ldots, r_k\}$, $\{s_1, \ldots, s_{n-k}\}$ be a partition of $\{1, \ldots, n\}$ such that $r_1 < r_2 < \ldots < r_k$ and $s_1 < s_2 < \ldots < s_{n-k}$. Consider the two point focal type BVP

$$\Delta^n u(m) = 0,\ m \in I, \tag{3.1}$$

$$\begin{cases} \Delta^{r_i - 1} u(a) = 0,\ 1 \le i \le k, \\ \Delta^{s_j - 1} u(b + n - (s_j - 1)) = 0,\ 1 \le j \le n - k. \end{cases} \tag{3.2}$$

We shall show that the Green's function, $G(m,s)$, for the BVP, (3.1), (3.2), satisfies

$$(-1)^{\sigma_l} \Delta_m^l\, G(m,s) \ge 0 \tag{3.3}$$

on $I^{n-l} \times I$, for $0 \le l \le n - 1$, where $\sigma_l = \mathrm{card}\{j: s_j > l\}$ and $\Delta_m^0\, G(m,s) \equiv G(m,s)$, $\Delta_m^1\, G(m,s) = G(m + 1, s) - G(m,s)$, $\Delta_m^j\, G(m,s) = \Delta(\Delta_m^{j-1}\, G(m,s))$, $1 < j \le n$.

THEOREM 3.1. Suppose $u: I^n \to \mathbb{R}$ satisfies the difference inequality, $\Delta^n u \ge 0$ on I, and the boundary conditions (3.2). Then

$$(-1)^{\sigma_l}\, \Delta^l u \ge 0, \tag{3.4}$$

on I^{n-l}, $0 \le l \le n - 1$, where $\sigma_l = \mathrm{card}\{j: s_j > l\}$.

PROOF. Let $l \in \{r_1, \ldots, r_k\}$. Using the boundary conditions (3.2),

$$\Delta^{l-1} u(m) = \sum_{t=a}^{m-1} \Delta^l u(t),\ m \in I^{n-(l-1)}.$$

If $\Delta^l u(m)$ has constant sign on I^{n-l}, then $\Delta^{l-1} u(m)$ has constant sign and has the same sign on $I^{n-(l-1)}$. If $l \in \{s_1, \ldots, s_{n-k}\}$,

$$\Delta^{l-1}u(m) = -\sum_{t=m}^{b+n-l} \Delta^l u(t),\ m \in I^{n-(l-1)}.$$

If $\Delta^l u(m)$ has constant sign on I^{n-l}, then $\Delta^{l-1}u(m)$ has constant sign and has opposite sign on $I^{n-(l-1)}$. Since $\Delta^n u \geq 0$ on I, it follows inductively that u satisfies (3.4).

COROLLARY 3.2. The Green's function, $G(m, s)$, for the BVP, (3.1), (3.2), satisfies (3.3).

PROOF. The Green's function, $G(m,s)$, has the following characterization: for each $s \in I$, $v(m) = G(m, s)$ is the unique solution of the BVP, $\Delta^n v(m) = \delta_{m,s}$, (the Kronecker delta), (3.2). See [3, p. 144] or [4, p. 20], for example, Since $\delta_{m,s} \geq 0$ on $I^n \times I$, the result follows immediately from Theorem 3.1.

REMARK. In [4], Hartman obtained an analogue of the Levin inequality (1.3) for a conjugate type BVP for difference equations. Analogues of Lemma 2.2 and Theorem 2.3 can be proved for a BVP,

$$\Delta^n u(m) = 0,\ m \in I,$$

$$\Delta^{r-1}u(\mathrm{a}) = 0,\ 1 \leq r \leq k,$$

$$\Delta^{s-1}u(b + n - (s - 1)) = 0, j + l \leq s \leq j + n - k,$$

where $k \in \{1, \ldots, n - 1\}$ and $j \in \{0, \ldots, k\}$.

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