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POSITIVE GROUND STATES FOR A CLASS OF SUPERLINEAR (p, q)-LAPLACIAN COUPLED SYSTEMS INVOLVING SCHRÖDINGER EQUATIONS

J. C. DE ALBUQUERQUE, JOÃO MARCOS DO Ó and EDCARLOS D. SILVA ✉

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Abstract

We study the existence of positive ground state solutions for the following class of (p, q)-Laplacian coupled systems

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = f(u) + \alpha\lambda(x)|u|^{\alpha-2}u|v|^{\beta}, & x \in \mathbb{R}^N, \\ -\Delta_q v + b(x)|v|^{q-2}v = g(v) + \beta\lambda(x)|v|^{\beta-2}v|u|^{\alpha}, & x \in \mathbb{R}^N, \end{cases}$$

where $1 < p \le q < N$. Here the coefficient $\lambda(x)$ of the coupling term is related to the potentials by the condition $|\lambda(x)| \le \delta a(x)^{\alpha/p} b(x)^{\beta/q}$, where $\delta \in (0, 1)$ and $\alpha/p + \beta/q = 1$. Using a variational approach based on minimization over the Nehari manifold, we establish the existence of positive ground state solutions for a large class of nonlinear terms and potentials.

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1. Introduction

In this work we study the class of (p, q)-Laplacian coupled systems given by

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = f(u) + \alpha\lambda(x)|u|^{\alpha-2}u|v|^{\beta}, & x \in \mathbb{R}^N, \\ -\Delta_q v + b(x)|v|^{q-2}v = g(v) + \beta\lambda(x)|v|^{\beta-2}v|u|^{\alpha}, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $1 < p \le q < N$. We are concerned with the existence of *ground state solutions*, that is, nontrivial solutions with minimal energy. We study a general class of (p, q)-Laplacian coupled systems, when the potentials $a(x), b(x)$ are nonnegative, bounded and related with the coupling term by the condition $|\lambda(x)| \le \delta a(x)^{\alpha/p} b(x)^{\beta/q}$, for some $\delta \in (0, 1)$ and for all $x \in \mathbb{R}^N$ with $\alpha/p + \beta/q = 1$ and $1 \le \alpha < p$, $1 \le \beta < q$. Notice that this class of systems is a type of '(p, q)-linearly coupled system' due to the presence of the powers α and β in the coupling terms. An important feature of this class of systems is the loss of homogeneity due to the fact that we consider also the case $p \ne q$.

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We study the case where the functions $a(x), b(x), \lambda(x)$ are periodic and asymptotically periodic, that is, $a(x), b(x), \lambda(x)$ are limits of periodic functions when $|x| \to +\infty$. The nonlinearities $f(s)$ and $g(s)$ are p-superlinear and q-superlinear continuous functions respectively, but the classical Ambrosetti–Rabinowitz condition introduced in the seminal work [4] is not imposed. Our main contribution here is to prove the existence of positive ground state solutions for the general class of (p, q)-coupled systems (1.1), which include several classes of nonlinear Schrödinger equations.

1.1. Motivation and related results. In order to introduce the study of the class of (p, q)-Laplacian coupled systems (1.1), we begin by giving a survey on the related problems motivating the present work. If $\lambda = 0$, $f \equiv g$, $a = b$ and $p = q$, then system (1.1) reduces to the following class of quasilinear Schrödinger equations:

$$-\Delta_p u + a(x)|u|^{p-2}u = f(u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Equations involving the p-Laplacian operator arise in various branches of mathematical physics, such as non-Newtonian fluids, elastic mechanics, reaction–diffusion problems, flow through porous media, glaciology, petroleum extraction, nonlinear optics, plasma physics and nonlinear elasticity. For an overview on quasilinear elliptic problems driven by the p-Laplacian we refer the interested reader to the important works [15, 19, 31]. For more physical applications we refer the reader to [14]. When $p = 2$, solutions of (1.2) are related to standing wave solutions of the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \tilde{a}(x)\psi - f(\psi), \quad x \in \mathbb{R}^N,\ t \geq 0, \tag{1.3}$$

where i denotes the imaginary unit and $m, \hbar$ are positive constants. For (1.3), a solution of the form $\psi(x, t) = e^{-iEt/\hbar}u(x)$ is called a *standing wave*. Assuming that $f(t\xi) = f(t)\xi$ for $\xi \in \mathbb{C}$, $|\xi| = 1$, taking $\hbar = \sqrt{2m}$ and denoting $a(x) = \tilde{a}(x) - E$, it is well known that ψ is a solution of (1.3) if and only if u solves Equation (1.2). For more information on the physical background, we refer the reader to [1, 6, 14, 20] and references therein.

The class of Equations (1.2) has been extensively studied by many researchers. In order to overcome the difficulty originating from the lack of compactness, the authors introduced several classes of potentials. For instance, in [34], Rabinowitz studied Schrödinger equations when the potential is coercive and bounded away from zero. In order to improve the behavior of the potential introduced in [34], Bartsch and Wang [7] considered a class of potentials such that the level sets $\{x \in \mathbb{R}^N : a(x) \leq M\}$ have finite Lebesgue measure for all $M > 0$. Here we deal with two classes of nonnegative bounded potentials. For more results concerning nonlinear Schrödinger equations we refer the reader to [2, 12, 13, 18, 26, 27, 32, 41, 42] and references therein.

Quasilinear elliptic systems have been extensively studied by many researchers. For instance, in [39], Vélin studied the existence of three nontrivial solutions for the following class of (p, q)-gradient elliptic systems with boundary Dirichlet conditions:

$$\begin{cases} -\Delta_p u = \gamma a(x)|u|^{p-2}u + f(x, u, v), & x \in \Omega, \\ -\Delta_q v = \delta b(x)|v|^{q-2}v + g(x, u, v), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

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where $f(x,\cdot,\cdot):\Omega\times\mathbb{R}\times\mathbb{R}\mapsto f(x,u,v)=H_u(x,u,v)$ and $g(x,\cdot,\cdot):\Omega\times\mathbb{R}\times\mathbb{R}\mapsto g(x,u,v)=H_v(x,u,v)$ are weakly lower continuous functionals. In [9], Chen and Fu considered the following class of quasilinear Schrödinger systems:

$$\begin{cases}-\Delta_p u+a(x)|u|^{p-2}u=\kappa d^{-1}F_u(x,u,v)+\lambda|u|^{m-2}u, & x\in\mathbb{R}^N,\\ -\Delta_q v+b(x)|v|^{q-2}v=\kappa d^{-1}F_v(x,u,v)+\mu|v|^{m-2}v, & x\in\mathbb{R}^N,\end{cases}$$

where $1<p\le q\le N$, $\lambda,\mu>0$, $\kappa\in\mathbb{R}$ and $m,d\in(q,p^*)$. The authors proved the existence of infinitely many nonnegative solutions. For more existence results concerning (p,q)-Laplacian elliptic systems we refer the reader to [8, 21, 35, 38, 40, 44] and references therein.

Motivated by the above discussion, we study the class of (p,q)-Laplacian coupled systems (1.1) when $p=q$ or $p\neq q$. In order to establish a variational approach to our problem, throughout the paper we assume that

$$\frac{\alpha}{p}+\frac{\beta}{q}=1\quad\text{and}\quad\begin{cases}p<\alpha+\beta<q, & \text{if } p<q,\\ \alpha+\beta=p=q, & \text{if } p=q.\end{cases}$$

The prototypical example when $p=q=2$ and $\alpha=\beta=1$ is the linearly coupled system

$$\begin{cases}-\Delta u+a(x)u=f(u)+\lambda(x)v, & x\in\mathbb{R}^N,\\ -\Delta v+b(x)v=g(v)+\lambda(x)u, & x\in\mathbb{R}^N.\end{cases}\tag{1.4}$$

In [16, 17], the authors studied the existence of positive ground states for (1.4) when $N=2$ and f,g have exponential growth. For the case $N\ge 2$ we refer the reader to [3, 5, 10–12, 22, 28, 29] and references therein. The class of systems introduced in (1.1) imposes some difficulties. The first is the lack of compactness due to the fact that the system is defined in the whole Euclidean space $\mathbb{R}^N$. We do not assume any coercivity assumption for the potentials $a(x)$ and $b(x)$. Moreover, system (1.1) involves strongly coupled Schrödinger equations because of the coupling terms on the right-hand side. We emphasize that we have different geometry for the energy functional associated to system (1.1) if we consider the coupling term (p,q)-superlinear, (p,q)-asymptotically linear or (p,q)-sublinear. On this subject we refer the reader to [8, 40]. Another difficulty is that the classical Ambrosetti–Rabinowitz condition is not imposed on the nonlinear terms f and g. For the sake of completeness, we recall the classical Ambrosetti–Rabinowitz condition: there exist $\theta_1>p$ and $\theta_2>q$ such that

$$0<F(t)=\theta_1\int_0^t f(\tau)\,d\tau\le tf(t),$$

$$0<G(t)=\theta_2\int_0^t f(\tau)\,d\tau\le tg(t),$$

for all $t\in\mathbb{R}$, where $F(t)=\int_0^t f(\tau)\,d\tau$, $G(t)=\int_0^t g(\tau)\,d\tau, t\in\mathbb{R}$. Instead of the Ambrosetti–Rabinowitz condition, we suppose that f is p-superlinear and g is q-superlinear and we use a variational approach based on minimization over the Nehari manifold to get ground state solutions. Since we are also considering the case $p\neq q$, there is a lack of homogeneity of the energy functional and the standard Nehari manifold is not suitable anymore for the problem studied here. To the best of our knowledge, there are no results concerning the existence of ground state solutions for quasilinear elliptic systems driven by the (p, q)-Laplacian operator with $p \neq q$.

1.2. Assumptions and main results. For $s > 1$, let $W^{1,s}(\mathbb{R}^N)$ be the usual Sobolev space with norm

$$\|u\|_{W^{1,s}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |\nabla u|^s \, dx + \int_{\mathbb{R}^N} |u|^s \, dx\Big)^{1/s}.$$

Given a function $c : \mathbb{R}^N \to \mathbb{R}$, we introduce the following space and norm:

$$E_{c,s} = \Big\{u \in W^{1,s}(\mathbb{R}^N) : \int_{\mathbb{R}^N} c(x)|u|^s \, dx < +\infty\Big\},$$

$$\|u\|_{c,s}^s = \int_{\mathbb{R}^N} (|\nabla u|^p + c(x)|u|^s) \, dx.$$

As already mentioned, we are interested in a class of quasilinear elliptic systems with asymptotic periodic potentials. For this purpose, in our argument, it is crucial to analyze the existence of ground states for a class of limit systems with periodic potentials. Specifically, our argument is based on comparison of ground state energy levels among this class of systems. Thus, let us establish the existence of ground states for the following class of systems:

$$\begin{cases} -\Delta_p u + a_o(x)|u|^{p-2}u = f(u) + \alpha\lambda_o(x)|u|^{\alpha-2}u|v|^{\beta}, & x \in \mathbb{R}^N, \\ -\Delta_q v + b_o(x)|v|^{q-2}v = g(v) + \beta\lambda_o(x)|v|^{\beta-2}v|u|^{\alpha}, & x \in \mathbb{R}^N, \end{cases} \tag{S_o}$$

where $1 < p \leq q < N$ and $a_o(x), b_o(x), \lambda_o(x)$ are periodic potentials. In order to establish a variational approach to treat system (S_o), we require suitable assumptions on the potentials. Throughout the paper, these assumptions are as follows.

(V_1) $a_o, b_o, \lambda_o \in C(\mathbb{R}^N)$ are 1-periodic in each $x_1, x_2, \ldots, x_N$.

(V_2) $a_o(x), b_o(x) \geq 0$, for all $x \in \mathbb{R}^N$ and

$$\inf_{u \in E_{a_o,p}} \Big\{\int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} a_o(x)|u|^p \, dx : \int_{\mathbb{R}^N} |u|^p \, dx = 1\Big\} > 0,$$

$$\inf_{v \in E_{b_o,q}} \Big\{\int_{\mathbb{R}^N} |\nabla v|^q \, dx + \int_{\mathbb{R}^N} b_o(x)|v|^q \, dx : \int_{\mathbb{R}^N} |v|^q \, dx = 1\Big\} > 0.$$

(V_3) The inequality $|\lambda_o(x)| \leq \delta a_o(x)^{\alpha/p} b_o(x)^{\beta/q}$ holds for some $\delta \in (0, 1)$ such that

$$\varrho := \frac{1}{q} - \delta \max\Big\{\frac{\alpha}{p}, \frac{\beta}{q}\Big\} > 0.$$

(V_3') Assumption (V_3) holds and there exist $R > 0, \eta_0 > 0$ such that $\lambda_o(x) \geq \eta_0 > 0$, for all $x \in B_R(0)$.

In this work the main interest is to ensure existence of ground states by minimization on the Nehari manifold. For this purpose we assume that $\sup_{t>0} f'(t)t/f(t) < +\infty$, $\sup_{t>0} g'(t)t/g(t) < +\infty$. Furthermore, we make the following assumptions on the nonlinearities.

(F_1) $f, g \in C^1(\mathbb{R})$, $f(t) = o(|t|^{p-2}t)$, $g(t) = o(|t|^{q-2}t)$, as $|t| \to 0$ and

$$\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{p-2}t} = \lim_{|t|\to+\infty} \frac{g(t)}{|t|^{q-2}t} = +\infty.$$

(F_2) There exist $C_1, C_2 > 0$, $r \in (p, p^*)$ and $s \in (q, q^*)$ such that

$$|f(t)| \leq C_1(1 + |t|^{r-1}) \quad \text{and} \quad |g(t)| \leq C_2(1 + |t|^{s-1}) \quad \text{for all } t \in \mathbb{R}.$$

(F_3) $t \mapsto f(t)/|t|^{p-2}t$ and $t \mapsto g(t)/|t|^{q-2}t$ are strictly increasing for any $t > 0$ and decreasing for $t < 0$.

(F_4) $F(t) := \int_0^t f(\tau)\, d\tau \leq F(|t|)$ and $G(t) := \int_0^t g(\tau)\, d\tau \leq G(|t|)$, for all $t \in \mathbb{R}$.

Under our assumptions $E_{a_o,p}$ and $E_{b_o,q}$ are reflexive Banach spaces, and consequently the product space $E_o = E_{a_o,p} \times E_{b_o,q}$, when endowed with the norm $\|(u, v)\|_o = \|u\|_{a_o,p} + \|v\|_{b_o,q}$, is a reflexive Banach space. We shall consider the energy functional of C^1 class $I_o : E_o \to \mathbb{R}$ given by

$$I_o(u, v) = \frac{1}{p}\|u\|^p_{a_o,p} + \frac{1}{q}\|v\|^q_{b_o,q} - \int_{\mathbb{R}^N} (F(u) + G(v) + \lambda_o(x)|u|^\alpha|v|^\beta)\, dx.$$

From a standard mathematical point of view, finding weak solutions to the elliptic problem (S_o) is equivalent to finding critical points for the energy functional I_o. In order to get ground state solutions it is usual to consider the Nehari method. The standard Nehari manifold for system (S_o) is defined by

$$\mathcal{M}_o = \{(u, v) \in E_o\backslash\{(0, 0)\} : \langle I_o'(u, v), (u, v)\rangle = 0\}.$$

In the present work we are interested in ensuring the existence of ground state solutions for the elliptic problem (S_o) with $1 < p \leq q < N$. When $p \neq q$ the principal part in the energy functional is not homogeneous. As a consequence the Nehari manifold $\mathcal{M}_o$ is not suitable for our work. The main problem is to guarantee that any Palais–Smale sequence in $\mathcal{M}_o$ is bounded. Another difficulty is to ensure that any nonzero pair $(u, v) \in E_o$ admits a unique projection in the standard Nehari manifold $\mathcal{M}_o$. Furthermore, assuming that $p \neq q$, it is not clear whether $\mathcal{M}_o$ is a C^1 manifold, which is crucial in our arguments. In order to overcome these difficulties we shall introduce the Nehari manifold

$$\mathcal{N}_o = \left\{(u, v) \in E_o\backslash\{(0, 0)\} : \left\langle I_o'(u, v), \left(\frac{1}{p}u, \frac{1}{q}v\right)\right\rangle = 0\right\}.$$

Here we mention that $\mathcal{N}_o$ is a C^1 manifold and any Palais–Smale sequence over $\mathcal{N}_o$ is bounded away from zero; see Lemma 4.1. Moreover, we have that I_o is coercive over

$\mathcal{N}_o$. Related to the Nehari manifold, we also need to consider fibering maps, which are a powerful tool in the Nehari method. Due to the loss of homogeneity, we introduce the fibering maps $t \to I_o(t^{1/p}u, t^{1/q}v)$ which coincide with the usual one when $p = q$. Thanks to the fibering maps, we can prove that any nonzero pair $(u, v) \in E_o$ admits a unique projection in the Nehari manifold $\mathcal{N}_o$; see Lemma 4.2. We are now in a position to state our first result.

Theorem 1.1. *If* (V_1)–(V_3) *and* (F_1)–(F_4) *hold, then there exists a ground state for system* (S_o). *Moreover, we have the following statements.*

(i) *Assume also that* $\lambda_o(x) \geq 0$ *for all* $x \in \mathbb{R}^N$. *Then there exists a nonnegative ground state for system* (S_o).
(ii) *Assume also that* (V_3') *holds and* $\lambda_o(x) \geq 0$ *for all* $x \in \mathbb{R}^N$. *Then there exists a positive ground state for system* (S_o), *for all* $\eta_0 > 0$ *large enough.*

Based on this framework, we are able to consider the asymptotically periodic case. More precisely, the asymptotically periodic case says that the periodic functions $a_o(x)$, $b_o(x)$ and $\lambda_o(x)$ are the limit as x goes to $\pm\infty$ of the potentials $a(x), b(x)$ and $\lambda(x)$, respectively. In other words, the potentials $a(x)$, $b(x)$ and $\lambda(x)$ satisfy the assumptions

$$\lim_{|x|\to+\infty} |a_o(x) - a(x)| = \lim_{|x|\to+\infty} |b_o(x) - b(x)| = \lim_{|x|\to+\infty} |\lambda(x) - \lambda_o(x)| = 0. \tag{1.5}$$

For our purpose we shall consider the following hypothesis:

(V_4) The limits given in (1.5) hold. Assume also that $a(x) < a_o(x)$, $b(x) < b_o(x)$, $\lambda_o(x) < \lambda(x)$, for all $x \in \mathbb{R}^N$.

Under these conditions we shall consider the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = f(u) + \alpha\lambda(x)|u|^{\alpha-2}u|v|^{\beta}, & x \in \mathbb{R}^N, \\ -\Delta_q v + b(x)|v|^{q-2}v = g(v) + \beta\lambda(x)|v|^{\beta-2}v|u|^{\alpha}, & x \in \mathbb{R}^N. \end{cases} \tag{S}$$

Furthermore, we consider the following hypotheses.

(V_5) $a(x), b(x) \geq 0$, for all $x \in \mathbb{R}^N$ and

$$\inf_{u\in E_{a,p}} \left\{ \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} a(x)|u|^p \, dx : \int_{\mathbb{R}^N} |u|^p \, dx = 1 \right\} > 0,$$

$$\inf_{v\in E_{b,q}} \left\{ \int_{\mathbb{R}^N} |\nabla v|^q \, dx + \int_{\mathbb{R}^N} b(x)|v|^q \, dx : \int_{\mathbb{R}^N} |v|^q \, dx = 1 \right\} > 0.$$

(V_6) We assume $|\lambda(x)| \leq \delta a(x)^{\alpha/p} b(x)^{\beta/q}$, for some $\delta \in (0, 1)$, such that $\varrho > 0$, where ϱ was defined in (V_3).

(V_6') We suppose (V_6) holds and there exist $R > 0, \eta > 0$ such that $\lambda(x) \geq \eta > 0$, for all $x \in B_R(0)$.

We set the product space $E = E_{a,p} \times E_{b,q}$ endowed with the norm $\|(u, v)\| = \|u\|_{a,p} + \|v\|_{b,q}$. Under these assumptions we are able to state the following result.

THEOREM 1.2. *If* (V_1)–(V_6) *and* (F_1)–(F_4) *hold, then there exists a ground state for system* (S)*. Moreover, we have the following statements:*

(i) *assume also that* $\lambda(x) \geq 0$ *for all* $x \in \mathbb{R}^N$*. Then there exists a nonnegative ground state for system* (S)*;*
(ii) *assume also that* (V_6') *holds and* $\lambda(x) \geq 0$ *for all* $x \in \mathbb{R}^N$*. Then there exists a positive ground state for system* (S)*, for all* $\eta > 0$ *large enough.*

REMARK 1.3. We point out that in the coercive case, that is, when $a(x) \to +\infty$ and $b(x) \to +\infty$ as $|x| \to +\infty$, the embedding $E = E_{a,p} \times E_{b,q} \hookrightarrow L^{s_1}(\mathbb{R}^N) \times L^{s_2}(\mathbb{R}^N)$ is compact for each $s_1 \in [p, p^*)$ and $s_2 \in [q, q^*)$; see [30, Lemma 2.2]. For related results involving the Laplacian operator we refer the reader to [12]. We also can recover the compactness by requiring that for any $M > 0$ the set $\{x \in \mathbb{R}^N : a(x) \leq M, b(x) \leq M\}$ has finite Lebesgue measure. In fact, any hypothesis on the potentials $a(x)$ and $b(x)$ which ensures the compact embedding, implies that System (1.1) admits at least one ground state solution via minimization over the Nehari method. In this direction, we refer the reader to [7, 12, 34]. Here we do not require any kind of conditions on the potentials which ensure some kind of compactness of the associated energy functional.

REMARK 1.4. Typical examples of nonlinearities satisfying (F_1)–(F_4) are given by $f(t) = |t|^{p-2}t\ln(1+|t|)$ and $g(t) = |t|^{q-2}t\ln(1+|t|)$. More generally, we can consider also $f(t) = |t|^{p-2}t\ln^{\gamma}(1+|t|)$ and $g(t) = |t|^{q-2}t\ln^{\gamma}(1+|t|)$, where $\gamma \geq 1$ is a parameter and $p, q > 1$. In these examples the functions satisfy assumptions (F_1)–(F_4). However, these functions do not satisfy the Ambrosetti–Rabinowitz condition.

The remainder of this paper is organized as follows. In Section 2 we introduce the variational framework for our problem. In Section 3 we prove some useful facts which we will use throughout the paper. In Section 4 we introduce and give some properties of the Nehari manifold associated with the energy functional. In Section 5 we use a minimization technique over the Nehari manifold in order to get a nontrivial ground state solution for system (S_o). In this case, we make use of Lions's lemma [25, Lemma I.1] in the following form.

LEMMA 1.5. *Let* $1 < p < \infty$ *and* $r \in (p, p^*)$*, where* $p^* = Np/(N-p)$*. Consider a bounded sequence* $(u_n)_n \in W^{1,p}(\mathbb{R}^N)$ *such that*

$$\liminf_{n\to\infty}\left(\sup_{y\in\mathbb{R}^N}\int_{B_R(y_n)}|u_n|^p\,dx\right) = 0.$$

Then $u_n \to 0$ *in* $L^r(\mathbb{R}^N)$.

Using Lions's lemma together with the invariance of the energy functional, we obtain the existence of nontrivial critical points. Then we use the known ground state to get another one which will be nonnegative. By using the strong maximum principle we conclude that this ground state will be strictly positive. In Section 6 we study the case where the potentials are asymptotically periodic. For this purpose, we establish a relation between the energy levels for ground state solutions associated to systems (S_o) and (S).

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2. The variational framework

The goal of this section is to provide the framework in which the existence of solutions to systems (S_o) and (S) may be established by a variational approach. Associated to system (S_o) we have the energy functional $I_o : E_o \to \mathbb{R}$ given by

$$I_o(u, v) = \frac{1}{p}\|u\|^p_{a_o,p} + \frac{1}{q}\|v\|^q_{b_o,q} - \int_{\mathbb{R}^N} (F(u) + G(v) + \lambda_o(x)|u|^\alpha |v|^\beta)\, dx.$$

It follows from assumptions (F_1) and (F_2) that for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$|f(t)| \le \varepsilon|t|^{p-1} + C_\varepsilon|t|^{r-1} \quad \text{and} \quad |g(t)| \le \varepsilon|t|^{q-1} + C_\varepsilon|t|^{s-1} \quad \text{for all } t \in \mathbb{R}, \tag{2.1}$$

which implies that

$$|F(t)| \le \varepsilon|t|^p + C_\varepsilon|t|^r \quad \text{and} \quad |g(t)| \le \varepsilon|t|^q + C_\varepsilon|t|^s \quad \text{for all } t \in \mathbb{R}. \tag{2.2}$$

Assumptions (V_2) and (V_5) imply that the spaces $E_{a_{o,p}}, E_{a,p}$ are continuously embedded into $L^r(\mathbb{R}^N)$ for all $r \in [p, p^*]$ and the spaces $E_{b_{o,q}}, E_{b,q}$ are continuously embedded into $L^s(\mathbb{R}^N)$ for all $s \in [q, q^*]$; see [18, Lemma 2.1]. By using (2.2) one sees that I_o is well defined. Moreover, $I_o \in C^1(E, \mathbb{R})$ and its derivative is given by

$$\begin{aligned}\langle I_o'(u, v), (\phi, \psi)\rangle &= \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla\phi + a_o(x)|u|^{p-2}u\phi)\, dx \\ &+ \int_{\mathbb{R}^N} (|\nabla v|^{q-2}\nabla v\nabla\psi + b_o(x)|v|^{q-2}v\psi)\, dx - \int_{\mathbb{R}^N} (f(u)\phi + g(v)\psi)\, dx \\ &- \int_{\mathbb{R}^N} \lambda_o(x)(\alpha|u|^{\alpha-2}u|v|^\beta\phi + \beta|u|^\alpha|v|^{\beta-2}v\psi)\, dx.\end{aligned}$$

Hence, critical points of I_o are precisely the weak solutions of system (S_o).

Analogously, to analyze system (S) variationally, we introduce the C^1 energy functional $I : E \to \mathbb{R}$ related to the functions $a(x), b(x)$ and $\lambda(x)$. Under our assumptions the energy functional I is well defined and the critical points correspond to solutions of system (S).

3. Preliminary results

In this section we provide some basic lemmas which will be used throughout the paper.

LEMMA 3.1. *Suppose (V_2), (V_3). Then*

$$\int_{\mathbb{R}^N} \lambda_o(x)|u|^\alpha|v|^\beta\, dx \le \delta \max\left\{\frac{\alpha}{p}, \frac{\beta}{q}\right\}(\|u\|^p_{a_o,p} + \|v\|^q_{b_o,q}), \tag{3.1}$$

for all $(u, v) \in E_o$.

PROOF. In fact, it follows from assumption (V_3) that

$$\int_{\mathbb{R}^N} \lambda_o(x)|u|^\alpha |v|^\beta \, dx \le \delta \int_{\mathbb{R}^N} a_o(x)^{\alpha/p}|u|^\alpha b_o(x)^{\beta/q}|v|^\beta \, dx.$$

Since $\alpha/p + \beta/q = 1$, we can use Young's inequality to conclude that

$$\int_{\mathbb{R}^N} \lambda_o(x)|u|^\alpha |v|^\beta \, dx \le \delta \max\left\{\frac{\alpha}{p}, \frac{\beta}{q}\right\} \int_{\mathbb{R}^N} (a_o(x)|u|^p + b_o(x)|v|^q) \, dx,$$

which implies (3.1). This concludes the proof. □

REMARK 3.2. It is important to point out that (3.1) remains true for the asymptotically periodic case.

LEMMA 3.3. *Suppose (F_3) holds. Then the functions*

$$f(t)t - pF(t) \quad \text{and} \quad g(t)t - qG(t) \tag{3.2}$$

are increasing for $t > 0$ and decreasing for $t < 0$. Furthermore, we have

$$f'(t)t^2 - (p-1)f(t)t > 0 \quad \text{and} \quad g'(t)t^2 - (q-1)g(t)t > 0 \tag{3.3}$$

for all $t \neq 0$.

PROOF. Let $0 < t_1 < t_2$ be fixed. By using (F_3) we deduce that

$$f(t_1)t_1 - pF(t_1) < \frac{f(t_2)}{t_2^{p-1}} t_1^p - pF(t_2) + p\int_{t_1}^{t_2} f(\tau)\, d\tau. \tag{3.4}$$

Moreover, we have

$$p\int_{t_1}^{t_2} f(\tau)\, d\tau < p\frac{f(t_2)}{t_2^{p-1}} \int_{t_1}^{t_2} \tau^{p-1}\, d\tau = \frac{f(t_2)}{t_2^{p-1}}(t_2^p - t_1^p). \tag{3.5}$$

Combining (3.4) and (3.5), we conclude that

$$f(t_1)t_1 - pF(t_1) < f(t_2)t_2 - pF(t_2).$$

The same argument can be used to get the result when $t < 0$. Analogously, the arguments can be applied for the function $g(t)t - qG(t)$.

It follows from (F_3) that for $t \in (0, +\infty)$ we have

$$\frac{d}{dt}\left(\frac{f(t)}{t^{p-1}}\right) > 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{g(t)}{t^{q-1}}\right) > 0,$$

which implies (3.3). Analogously, we get the same result when $t \in (-\infty, 0)$. This concludes the proof. □

REMARK 3.4. It is important to mention that in view of the preceding lemma, the functions $f(t)t - pF(t)$ and $g(t)t - qG(t)$ are nonnegative for all $t \in \mathbb{R}$.

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4. The Nehari manifold

We begin this section considering the Nehari manifold associated to system (S_o) defined by

$$\mathcal{N}_o := \Big\{(u,v) \in E_o\backslash\{(0,0)\} : \Big\langle I_o'(u,v), \Big(\frac{1}{p}u, \frac{1}{q}v\Big)\Big\rangle = 0\Big\}.$$

Hence, $(u,v) \in \mathcal{N}_o$ if and only if it satisfies

$$\frac{1}{p}\|u\|_{a_o,p}^p + \frac{1}{q}\|v\|_{b_o,q}^q - \int_{\mathbb{R}^N} \lambda_o(x)|u|^\alpha|v|^\beta\,dx = \int_{\mathbb{R}^N}\Big(\frac{1}{p}f(u)u + \frac{1}{q}g(v)v\Big)\,dx. \tag{4.1}$$

We also provide some properties for the Nehari manifold $\mathcal{N}_o$.

Lemma 4.1. *Suppose (V_2), (V_3) and (F_1)–(F_3). Then the following assertions hold.*

(i) *$\mathcal{N}_o$ is a C^1-manifold.*
(ii) *There exists $\gamma > 0$ such that $\|(u,v)\|_o \geq \gamma$, for all $(u,v) \in \mathcal{N}_o$.*

Proof. Let $\varphi : E_o\backslash\{(0,0)\} \to \mathbb{R}$ be defined by $\varphi(u,v) = \langle I_o'(u,v), ((1/p)u, (1/q)v)\rangle$. It is not hard to verify that φ belongs to class C^1. Notice that $\mathcal{N}_o = \varphi^{-1}(0)$. Thus, it is enough to verify that 0 is a regular value for the functional φ. Using (3.3) and (4.1), we can deduce that

$$\Big\langle \varphi'(u,v), \Big(\frac{1}{p}u, \frac{1}{q}v\Big)\Big\rangle \leq -\frac{1}{p^2}\int_{\mathbb{R}^N}(f'(u)u^2 - (p-1)f(u)u) - \frac{1}{q^2}\int_{\mathbb{R}^N}(g'(v)v^2 - (q-1)g(v)v) < 0,$$

which implies that 0 is a regular value of φ. Therefore, $\mathcal{N}_o$ is a C^1-manifold.

In order to prove (ii), we note by Lemma 3.1 that

$$\varrho(\|u\|_{a_o,p}^p + \|v\|_{b_o,q}^q) \leq \frac{1}{p}\|u\|_{a_o,p}^p + \frac{1}{q}\|v\|_{b_o,q}^q - \int_{\mathbb{R}^N}\lambda_o(x)|u|^\alpha|v|^\beta\,dx,$$

where $\varrho > 0$ was defined in (V_3). Hence, by using (2.1) and (4.1) we can deduce that

$$\varrho(\|u\|_{a_o,p}^p + \|v\|_{b_o,q}^q) \leq \varepsilon(\|u\|_{a_o,p}^p + \|v\|_{b_o,q}^q) + \tilde{C}_\varepsilon(\|u\|_{a_o,p}^r + \|v\|_{b_o,q}^s).$$

Taking $\varepsilon > 0$ sufficiently small such that $\varrho - \varepsilon > 0$, we conclude that

$$0 < \frac{1}{\tilde{C}_\varepsilon}(\varrho - \varepsilon) \leq \|u\|_{a_o,p}^{r-p} + \|v\|_{b_o,q}^{s-q},$$

which implies (ii). This completes the proof. □

Now, by studying the fiber mapping, we prove that any nontrivial element of E_o can be projected over the Nehari manifold $\mathcal{N}_o$.

Lemma 4.2. *Suppose (V_2), (V_3) and (F_1)–(F_3). For any $(u,v) \in E_o\backslash\{(0,0)\}$ there exists a unique $t_0 > 0$, depending on (u,v), such that*

$$(t_0^{1/p}u, t_0^{1/q}v) \in \mathcal{N}_o \quad \text{and} \quad I_o(t_0^{1/p}u, t_0^{1/q}v) = \max_{t\geq 0} I_o(t^{1/p}u, t^{1/q}v).$$

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PROOF. Let $(u,v) \in E_o \backslash \{(0,0)\}$ be fixed. We consider the fiber mapping $h : [0,+\infty) \to \mathbb{R}$ defined by $h(t) = I_o(t^{1/p}u, t^{1/q}v)$. Note that

$$h'(t)t = \Big\langle I_o'(t^{1/p}u, t^{1/q}v), \Big(\frac{1}{p}t^{1/p}u, \frac{1}{q}t^{1/q}v\Big)\Big\rangle.$$

Thus, t_0 is a positive critical point of $h(t)$ if and only if $(t_0^{1/p}u, t_0^{1/q}v) \in \mathcal{N}_o$. Using Lemma 3.1, the growth conditions of the nonlinearities and Sobolev embedding, we can deduce that

$$h(t) \geq t[(\varrho - C\varepsilon)(\|u\|^p_{a_o,p} + \|v\|^q_{b_o,q}) - C_\varepsilon t^{(r-p/p)}\|u\|^r_{a_o,p} - C_\varepsilon t^{(s-q/q)}\|v\|^s_{b_o,q}].$$

Taking ε sufficiently small, we conclude that $h(t) \geq 0$ provided that $t > 0$ is small. On the other hand, we can deduce that

$$\begin{aligned}\frac{h(t)}{t} \leq \frac{1}{p}\|u\|^p_{a_o,p} + \frac{1}{q}\|v\|^q_{b_o,q} - \int_{\{u\neq 0\}} \frac{F(t^{1/p}u)}{(t^{1/p}|u|)^p}|u|^p\,dx \\ - \int_{\{v\neq 0\}} \frac{G(t^{1/q}v)}{(t^{1/q}|v|)^q}|v|^q\,dx - \int_{\mathbb{R}^N} \lambda_o(x)|u|^\alpha|v|^\beta\,dx,\end{aligned}$$

which together with (F_1) implies that $h(t) \leq 0$ for $t > 0$ large. Thus, $h(t)$ has maximum points in $(0,+\infty)$. Now, note that every critical point $t \in (0,+\infty)$ of $h(t)$ satisfies

$$\begin{aligned}\frac{1}{p}\|u\|^p_{a_o,p} + \frac{1}{q}\|v\|^q_{b_o,q} - \int_{\mathbb{R}^N} \lambda_o(x)|u|^\alpha|v|^\beta\,dx =& \frac{1}{p}\int_{\mathbb{R}^N} \frac{f(t^{1/p}u)u}{t^{1-1/p}}\,dx \\ &+ \frac{1}{q}\int_{\mathbb{R}^N} \frac{g(t^{1/q}v)v}{t^{1-1/q}}\,dx.\end{aligned} \tag{4.2}$$

By using (3.3), we have

$$\frac{d}{dt}\Big(\frac{f(t^{1/p}u)u}{t^{1-1/p}}\Big) > 0 \quad \text{and} \quad \frac{d}{dt}\Big(\frac{g(t^{1/q}v)v}{t^{1-1/q}}\Big) > 0. \tag{4.3}$$

Therefore, the right-hand side of (4.2) is increasing on $t > 0$ which implies that the critical point is unique. This concludes the proof. □

5. Proof of Theorem 1.1

In order to prove Theorem 1.1, we introduce the Nehari energy level associated with system (S_o) defined by

$$c_{\mathcal{N}_o} = \inf_{(u,v)\in\mathcal{N}_o} I_o(u,v).$$

Let $(u_n, v_n)_n \subset \mathcal{N}_o$ be a Palais–Smale sequence to $c_{\mathcal{N}_o}$, that is,

$$I_o(u_n,v_n) \to c_{\mathcal{N}_o} \quad \text{and} \quad I_o'(u_n,v_n) \to 0 \quad \text{as } n \to +\infty. \tag{5.1}$$

PROPOSITION 5.1. *Suppose (V_1)–(V_3) and (F_1)–(F_3). Then any sequence satisfying (5.1) is bounded in E_o.*

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Proof. Arguing by contradiction, we suppose that $\|(u_n, v_n)\|_o = \|u_n\|_{a_o,p} + \|v_n\|_{b_o,q} \to +\infty$, as $n \to +\infty$. We define $w_n = u_n/K_n^{1/p}$ and $z_n = v_n/K_n^{1/q}$, where $K_n := \|u_n\|_{a_o,p}^p + \|v_n\|_{b_o,q}^q$. Thus,

$$\|w_n\|_{a_o,p}^p + \|z_n\|_{b_o,q}^q = 1 \quad \text{and} \quad K_n \to +\infty \quad \text{as } n \to +\infty.$$

Hence, $(w_n, z_n)_n$ is bounded in E_o. Thus, we may assume up to a subsequence that:

- $(w_n, z_n) \rightharpoonup (w_0, z_0)$ weakly in E_o;
- $w_n \to w_0$ strongly in $L^r_{\rm loc}(\mathbb{R}^N)$, for all $p \le r < p^*$;
- $z_n \to z_0$ strongly in $L^s_{\rm loc}(\mathbb{R}^N)$, for all $q \le s < q^*$.
- $w_n(x) \to w_0(x)$ and $z_n(x) \to z_0(x)$, almost everywhere in $\mathbb{R}^N$.

We split the argument into two cases.

Case 1. $(w_0, z_0) \neq (0, 0)$.

Let us assume without loss of generality that $w_0 \neq 0$. By using Lemma 3.1 and (5.1) we can deduce that

$$o_n(1) = \frac{I_o(u_n, v_n)}{K_n} \le \frac{1}{p} + \delta \max\left\{\frac{\alpha}{p}, \frac{\beta}{q}\right\} - \int_{\{u_n \neq 0\}} \frac{F(u_n)}{K_n}\, dx.$$

The last inequality jointly with (F_1) and Fatou's lemma leads to

$$\frac{1}{p} + \delta \max\left\{\frac{\alpha}{p}, \frac{\beta}{q}\right\} \ge \int_{\{u_n \neq 0\}} \liminf_{n\to+\infty} \frac{F(u_n)}{|u_n|^p} |w_n|^p\, dx = +\infty,$$

which is a contradiction.

Case 2. $(w_0, z_0) = (0, 0)$.

First, we claim that for any $R > 0$ we have

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} (|w_n|^p + |z_n|^q)\, dx = 0. \tag{5.2}$$

In fact, if (5.2) does not hold, then there exist $R, \eta > 0$ such that

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} (|w_n|^p + |z_n|^q)\, dx \ge \eta > 0.$$

Hence, we can consider a sequence $(y_n)_n \subset \mathbb{Z}^N$ such that

$$\lim_{n\to+\infty} \int_{B_R(y_n)} (|w_n|^p + |z_n|^q)\, dx \ge \frac{\eta}{2} > 0.$$

We define the shift sequence $(\tilde{w}_n(x), \tilde{z}_n(x)) = (w_n(x + y_n), z_n(x + y_n))$. Since $a_o(\cdot)$ and $b_o(\cdot)$ are periodic, we have $\|(w_n, z_n)\|_o = \|(\tilde{w}_n, \tilde{z}_n)\|_o$. Thus, up to a subsequence, we may assume that:

- $(\tilde{w}_n, \tilde{z}_n) \rightharpoonup (\tilde{w}_0, \tilde{z}_0)$ weakly in E_o;

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- $\tilde{w}_n \to \tilde{w}_0$ strongly in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$, for all $p \le r < p^*$;
- $\tilde{z}_n \to \tilde{z}_0$ strongly in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, for all $q \le s < q^*$.

Then we have

$$\lim_{n\to+\infty}\int_{B_R(0)}(|\tilde{w}_n|^p + |\tilde{z}_n|^q)\,dx = \lim_{n\to+\infty}\int_{B_R(y_n)}(|w_n|^p + |z_n|^q)\,dx \ge \frac{\eta}{2} > 0,$$

which implies that $(\tilde{w}_0, \tilde{z}_0) \neq (0,0)$. Arguing as in Case 1, we get a contradiction.

Since (5.2) holds, it follows from Lemma 1.5 that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|w_n|^r\,dx = 0 \quad \text{and} \quad \lim_{n\to+\infty}\int_{\mathbb{R}^N}|z_n|^s\,dx = 0. \tag{5.3}$$

By using (2.2) and (5.3), we can conclude that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}F(\xi^{1/p}w_n)\,dx = \lim_{n\to+\infty}\int_{\mathbb{R}^N}G(\xi^{1/q}z_n)\,dx = 0 \quad \text{for all } \xi > 0. \tag{5.4}$$

Since $(u_n, v_n)_n \subset \mathcal{N}_o$, it follows from Lemma 4.2 that

$$I_o(u_n, v_n) \ge I_o(t^{1/p}u_n, t^{1/q}v_n) \quad \text{for all } t \ge 0. \tag{5.5}$$

Taking $t = \xi/K_n$ and combining (5.4) and (5.5), we deduce that

$$c_{\mathcal{N}_o} + o_n(1) = I_o(u_n, v_n) \ge I_o(\xi^{1/p}w_n, \xi^{1/q}z_n) \ge \varrho\xi + o_n(1),$$

which is a contradiction for $\xi > 0$ sufficiently large. Therefore, $(u_n, v_n)_n$ is bounded in E_o. □

REMARK 5.2. One can use the same ideas discussed in the proof of Proposition 5.1 in order to conclude that the energy functional I_o is coercive over the Nehari manifold $\mathcal{N}_o$.

In view of Proposition 5.1 we may assume, up to a subsequence, that:

- $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E_o;
- $u_n \to u_0$ strongly in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$, for all $p \le r < p^*$;
- $v_n \to v_0$ strongly in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, for all $q \le s < q^*$;
- $u_n(x) \to u_0(x)$ and $v_n(x) \to v_0(x)$, almost everywhere in $\mathbb{R}^N$.

By adapting some arguments discussed in [43], we have

$$\nabla u_n(x) \to \nabla u_0(x) \quad \text{and} \quad \nabla v_n(x) \to \nabla v_0(x) \quad \text{a.e. } x \in \mathbb{R}^N,$$

$$|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \rightharpoonup |\nabla u_0|^{p-2}\frac{\partial u_0}{\partial x_i}, \quad \text{weakly in } (L^p(\mathbb{R}^N))^*,$$

$$|\nabla v_n|^{q-2}\frac{\partial v_n}{\partial x_i} \rightharpoonup |\nabla v_0|^{q-2}\frac{\partial v_0}{\partial x_i}, \quad \text{weakly in } (L^q(\mathbb{R}^N))^*,$$

for all $1 \le i \le N$. Since $C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ is dense in the space E_o, it follows by standard arguments that $I_o'(u_0, v_0) = 0$, that is, (u_0, v_0) is a solution of system (S_o).

Thanks to the next result, we obtain a nontrivial solution for system (S_o).

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Proposition 5.3. *Suppose* (V_1)–(V_3) *and* (F_1)–(F_3). *Let* $(u_n, v_n)_n \subset \mathcal{N}_o$ *be the minimizing sequence satisfying* (5.1). *Then there exist a sequence* $(y_n)_n \subset \mathbb{R}^N$ *and constants* $R, \eta > 0$ *such that* $|y_n| \to \infty$ *as* $n \to \infty$, *and*

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} (|u_n|^p + |v_n|^q)\, dx \geq \eta > 0. \tag{5.6}$$

Proof. Arguing by contradiction, we suppose that (5.6) does not hold. Then

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^p\, dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^q\, dx = 0,$$

for any $R > 0$. Hence, applying Lemma 1.5, we conclude that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^r\, dx = 0 \quad \text{and} \quad \lim_{n\to+\infty} \int_{\mathbb{R}^N} |v_n|^s\, dx = 0. \tag{5.7}$$

Using (2.1) and Lemma 3.1, we can deduce that

$$\begin{aligned} 0 &= \left\langle I_o'(u_n, v_n), \left(\frac{1}{p}u_n, \frac{1}{q}v_n\right)\right\rangle \\ &\geq (\varrho - \varepsilon)(\|u_n\|^p_{a_o,p} + \|v_n\|^q_{b_o,q}) - C_\varepsilon(\|u_n\|^r_r + \|v_n\|^s_s). \end{aligned} \tag{5.8}$$

Taking $\varepsilon > 0$ sufficiently small such that $\varrho - \varepsilon > 0$, it follows from (5.7) and (5.8) that

$$0 \geq (\varrho - \varepsilon)(\|u_n\|^p_{a_o,p} + \|v_n\|^q_{b_o,q}) + o_n(1),$$

which implies that $\|(u_n, v_n)\|_o \to 0$ as $n \to +\infty$. However, since $I_o(u_n, v_n) \to c_{\mathcal{N}_o} > 0$ and I_o is continuous, the minimizing sequence $(u_n, v_n)_n$ cannot converge to zero strongly in E_o. This is a contradiction, proving that (5.6) holds. This proves the desired result. □

Proposition 5.4. *Suppose* (V_1)–(V_3) *and* (F_1)–(F_3). *Then there exists a ground state solution for system* (S_o).

Proof. Let (u_0, v_0) be the critical point of the energy functional I. We split the proof into two cases.

Case 1. $(u_0, v_0) \neq (0, 0)$.

If $(u_0, v_0) \neq (0, 0)$, then we have a nontrivial solution for system (S_o). It remains to prove that (u_0, v_0) is in fact a ground state. Notice that $(u_0, v_0) \in \mathcal{N}_o$. Thus, $c_{\mathcal{N}_o} \leq I_o(u_0, v_0)$. On the other hand, using (3.2), (5.1) and Fatou's lemma, we can deduce that

$$\begin{aligned} c_{\mathcal{N}_o} + o_n(1) &= I_o(u_n, v_n) - \left\langle I_o'(u_n, v_n), \left(\frac{1}{p}u_n, \frac{1}{q}v_n\right)\right\rangle \\ &= \int_{\mathbb{R}^N} \left[\frac{1}{p}(f(u_n)u_n - pF(u_n)) + \frac{1}{q}(g(v_n)v_n - qG(v_n))\right] dx \\ &\geq \int_{\mathbb{R}^N} \left[\frac{1}{p}(f(u_0)u_0 - pF(u_0)) + \frac{1}{q}(g(v_0)v_0 - qG(v_0))\right] dx + o_n(1) \\ &= I_o(u_0, v_0) - \left\langle I_o'(u_0, v_0), \left(\frac{1}{p}u_0, \frac{1}{q}v_0\right)\right\rangle + o_n(1) \\ &= I_o(u_n, v_n) + o_n(1), \end{aligned}$$

which implies that $c_{\mathcal{N}_o} \geq I_o(u_0, v_0)$. Therefore, $I_o(u_0, v_0) = c_{\mathcal{N}_o}$, that is, (u_0, v_0) is a ground state solution for system (S_o).

Case 2. $(u_0, v_0) = (0, 0)$.

In light of Proposition 5.3, there exist a sequence $(y_n)_n \subset \mathbb{R}^N$ and constants $R, \eta > 0$ such that

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} (|u_n|^p + |v_n|^q)\, dx \geq \eta > 0. \tag{5.9}$$

Without loss of generality we assume that $(y_n)_n \subset \mathbb{Z}^N$. Let us define the shift sequence $(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x + y_n), v_n(x + y_n))$. Since $a_o(\cdot)$, $b_o(\cdot)$ and $\lambda_o(\cdot)$ are periodic, we can use the invariance of the energy functional I_o, to deduce that

$$\|(u_n, v_n)\|_o = \|(\tilde{u}_n, \tilde{v}_n)\|_o \quad \text{and} \quad I_o(u_n, v_n) = I_o(\tilde{u}_n, \tilde{v}_n) \to c_{\mathcal{N}_o}.$$

Moreover, arguing as before, we can conclude that $(\tilde{u}_n, \tilde{v}_n)_n$ is a bounded sequence in E_o. Thus, up to a subsequence, we may assume that:

- $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}_0, \tilde{v}_0)$ weakly in E_o;
- $\tilde{u}_n \to \tilde{u}_0$ strongly in $L^r_{\mathrm{loc}}(\mathbb{R}^N)$, for all $p \leq r < p^*$;
- $\tilde{v}_n \to \tilde{v}_0$ strongly in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, for all $q \leq s < q^*$.

Moreover, $(\tilde{u}, \tilde{v})$ is a critical point of I_o. By using (5.9) one sees that

$$\liminf_{n\to\infty} \int_{B_R(0)} (|\tilde{u}_n|^p + |\tilde{v}_n|^q)\, dx = \liminf_{n\to\infty} \int_{B_R(y_n)} (|u_n|^p + |v_n|^q)\, dx \geq \eta > 0.$$

Therefore, $(\tilde{u}, \tilde{v}) \neq (0, 0)$ is a solution for system (S_o). The conclusion follows by arguing as in the proof of Case 1. This concludes the proof. □

PROPOSITION 5.5. *Suppose (V_1)–(V_3) and (F_1)–(F_3). Assume also that (F_4) holds and $\lambda_o(x) \geq 0$ for all $x \in \mathbb{R}^N$. Then there exists a nonnegative ground state for system (S_o).*

PROOF. Let (u_0, v_0) be the ground state solution obtained in Proposition 5.4. Then from Lemma 4.2 there exists a unique $t_0 > 0$ such that $(t_0^{1/p}|u_0|, t_0^{1/q}|v_0|) \in \mathcal{N}_o$. Since $\lambda_o(x) \geq 0$, it follows from (F_4) that $I_o(t_0^{1/p}|u_0|, t_0^{1/q}|v_0|) \leq I_o(t_0^{1/p}u_0, t_0^{1/q}v_0)$. Thus, since $(u_0, v_0) \in \mathcal{N}_o$ we have

$$I_o(t_0^{1/p}|u_0|, t_0^{1/q}|v_0|) \leq \max_{t\geq 0} I_o(t^{1/p}u_0, t^{1/q}v_0) = I_o(u_0, v_0) = c_{\mathcal{N}_o}.$$

Therefore, $(t_0^{1/p}|u_0|, t_0^{1/q}|v_0|) \in \mathcal{N}_o$ is a ground state solution for system (S_o). □

At this point, we have obtained a nonnegative ground state solution $(u, v) \in E_o$ for system (S_o). However, this solution could be semitrivial, that is, of type $(u, 0)$ or $(0, v)$. The next step is to prove that if (V_3') holds, then for some $\eta_0 > 0$ the ground state cannot be semitrivial.

PROPOSITION 5.6. *Suppose (V_1), (V_2) and (F_1)–(F_4). Assume also that (V_3') holds and $\lambda_o(x) \geq 0$ for all $x \in \mathbb{R}^N$. Then any ground state solution $(u, v) \in E_o$ for system (S_o) satisfies $u \neq 0$ and $v \neq 0$ for all $\eta_0 > 0$ large enough.*

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Proof. If we consider $\lambda_o(x) = 0$, for all $x \in \mathbb{R}^N$, then we have the uncoupled equation

$$-\Delta_p u + a_o(x)|u|^{p-2}u = f(u), \quad x \in \mathbb{R}^N. \tag{S_{a_o}}$$

Let $I_{a_o} : E_{a_o,p} \to \mathbb{R}$ be the energy functional associated to (S_{a_o}), defined by

$$I_{a_o}(u) = \frac{1}{p}\|u\|^p_{a_o,p} - \int_{\mathbb{R}^N} F(u)\,dx.$$

The Nehari manifold associated to (S_{a_o}) is given by

$$\mathcal{N}_{a_o} = \{u \in E_{a_o,p}\backslash\{0\} : \langle I'_{a_o}(u), u\rangle = 0\}.$$

The minimal energy level for Problem (S_{a_o}) is denoted by $c_{\mathcal{N}_{a_o}}$. Note that the same arguments used in this work holds true for (S_{a_o}). Thus, let $u_0 \in \mathcal{N}_{a_o}$ be a positive ground state solution for (S_{a_o}). By similar arguments used in the proof of Lemma 4.2 we can deduce that:

- $I_{a_o}(tu_0)$ is increasing for $0 < t < 1$;
- $I_{a_o}(tu_0)$ is decreasing for $t > 1$;
- $I_{a_o}(tu_0) \to -\infty$, as $t \to +\infty$.

Therefore, $\max_{t\geq 0} I_{a_o}(tu_0) = I_{a_o}(u_0)$. Analogously, we can introduce I_{b_o} and $\mathcal{N}_{b_o}$ and conclude that there exists a positive ground state solution $v_0 \in \mathcal{N}_{b_o}$ for the uncoupled equation

$$-\Delta_q u + b_o(x)|v|^{q-2}v = g(v), \quad x \in \mathbb{R}^N. \tag{S_{b_o}}$$

The minimal energy level for problem (S_{b_o}) is denoted by $c_{\mathcal{N}_{b_o}}$. Moreover, $\max_{t\geq 0} I_{b_o}(tv_0) = I_{b_o}(v_0)$. It follows from Lemma 4.2 that there exists $t_0 > 0$ such that $(t_0^{1/p}u_0, t_0^{1/q}v_0) \in \mathcal{N}_o$. In fact, we observe that $t_0 \in (0, 1]$ and

$$t_0 \leq \tilde{t}_0 := \frac{q\delta\max(\alpha/p, \beta/q)}{1 - q\delta\max(\alpha/p, \beta/q)} \frac{\int_{\mathbb{R}^N}[f(u_0)u_0 + g(v_0)v_0]\,dx}{\int_{B_R(0)} \lambda(x)|u_0|^\alpha|v_0|^\beta\,dx}.$$

Moreover, using (V'_3), we deduce that

$$c_{\mathcal{N}_o} \leq I_o(t_0^{1/p}u_0, t_0^{1/q}v_0) \leq t_0\Big(1 - \frac{1}{\theta}\Big)\Big(\frac{1}{p}\|u_0\|^p_{a_o,p} + \frac{1}{q}\|v_0\|^q_{b_o,q}\Big)$$

holds true for some $\theta > 1$. Taking into account hypothesis (V_3) and Lemma 3.1, it follows that

$$c_{\mathcal{N}_o} \leq \frac{\tilde{t}_0}{\eta_0}\Big(1 - \frac{1}{\theta}\Big)\Big(\frac{1}{p}\|u_0\|^p_{a_o,p} + \frac{1}{q}\|v_0\|^q_{b_o,q}\Big).$$

In particular, for $\eta_0 > 0$ large enough we have $c_{\mathcal{N}_o} < \min\{c_{\mathcal{N}_{a_o}}, c_{\mathcal{N}_{b_o}}\}$. Therefore, in this case, if $I_o(u, v) = c_{\mathcal{N}_o}$, then we have $u \neq 0$ and $v \neq 0$. This concludes the proof. □

Proposition 5.7. *Suppose (V_1), (V_2) and (F_1)–(F_4). Assume also that (V'_3) holds. Then there exists a positive ground state for system (S_o), for all $\eta_0 > 0$ large enough.*

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Proof. According to Proposition 5.5, we obtain a nonnegative ground state solution (u, v) for the problem $(S_\circ)$. By using standard arguments on regularity for elliptic equations of quasilinear elliptic type we mention that the functions u, v belong to $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, that is, we know that u, v are Hölder continuous functions; see [23, 24]. The main point here is to prove that u, v belong to the $C^{1,\alpha}(B_R(0))$ class for any $R > 0$. It follows from Proposition 5.4 that (u, v) is not trivial. In view of Proposition 5.6 we mention also that the pair (u, v) is not semitrivial, that is, the sets $\{x \in \mathbb{R}^N : u(x) = 0\}$ and $\{x \in \mathbb{R}^N : v(x) = 0\}$ are different from the whole space $\mathbb{R}^N$. Thus, we see that

$$\begin{cases} -\Delta_p u + a_\circ(x) u^{p-1} \geq 0, & x \in \mathbb{R}^N, \\ u \in E_{a_\circ,p} \cap C^{1,\alpha}, \ u \neq 0, \end{cases}$$

and

$$\begin{cases} -\Delta_q v + b_\circ(x) v^{p-1} \geq 0, & x \in \mathbb{R}^N, \\ v \in E_{b_\circ,q} \cap C^{1,\alpha}, \ v \neq 0. \end{cases}$$

Here we mention that $s \to \beta_1(s) := a_\circ(x) s^{p-1}$ and $s \to \beta_2(x) := b_\circ(x) s^{q-1}$ are nondecreasing functions for each $s > 0$ and $x \in \mathbb{R}^N$. By applying the strong maximum principle [33] we infer that $u > 0$ and $v > 0$ in $\mathbb{R}^N$. This concludes the proof. □

Proof of Theorem 1.1. It follows from Propositions 5.4, 5.5, 5.6 and 5.7. □

6. Proof of Theorem 1.2

In this section we are concerned with the existence of ground states for system (S), when the potentials are asymptotically periodic. Analogously to the periodic case, we introduce the Nehari manifold associated to system (S) defined by

$$\mathcal{N} := \left\{ (u, v) \in E \backslash \{(0, 0)\} : \left\langle I'(u, v), \left(\frac{1}{p} u, \frac{1}{q} v \right) \right\rangle = 0 \right\},$$

and the ground state energy level

$$c_{\mathcal{N}} := \inf_{(u,v) \in \mathcal{N}} I(u, v).$$

We point out that all results obtained in Section 4 remains true in the asymptotically periodic case. Thus, $\mathcal{N}$ is a C^1-manifold and for any $(u, v) \in E \backslash \{(0, 0)\}$ there exists a unique $t_0 > 0$, depending only on (u, v), such that

$$(t_0^{1/p} u, t_0^{1/q} v) \in \mathcal{N} \quad \text{and} \quad I(t_0^{1/p} u, t_0^{1/q} v) = \max_{t \geq 0} I(t^{1/p} u, t^{1/q} v). \tag{6.1}$$

In order to get a ground state solution for System (S) we establish a relation between the energy levels $c_{\mathcal{N}_\circ}$ and $c_{\mathcal{N}}$.

Lemma 6.1. *Under the assumptions (V_1)–(V_6) and (F_1)–(F_3),*

$$c_{\mathcal{N}} < c_{\mathcal{N}_\circ}.$$

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PROOF. Let $(u, v) \in \mathcal{N}_o$ be the nonnegative ground state solution for system (S_o) obtained in the previous section. In light of assumption (V_4), we can deduce that

$$\int_{\mathbb{R}^N} [(a(x) - a_o(x))u^p + (b(x) - b_o(x))v^q + (\lambda_o(x) - \lambda(x))uv]\, dx < 0, \tag{6.2}$$

By using (6.1) we get a $t_0 > 0$ such that $(t_0^{1/p}u, t_0^{1/q}v) \in \mathcal{N}$. Thus, it follows from (6.2) that

$$I(t_0^{1/p}u, t_0^{1/q}v) - I_o(t_0^{1/p}u, t_0^{1/q}v) < 0.$$

Therefore, since $(u, v) \in \mathcal{N}_o$ we conclude that

$$c_{\mathcal{N}} \le I(t_0^{1/p}u, t_0^{1/q}v) < I_o(t_0^{1/p}u, t_0^{1/q}v) \le \max_{t\ge 0} I_o(t^{1/p}u, t^{1/q}v) = I_o(u, v) = c_{\mathcal{N}_o},$$

which concludes the proof. □

Let us consider a Palais–Smale sequence $(u_n, v_n)_n \subset \mathcal{N}$ to $c_{\mathcal{N}}$, that is,

$$I(u_n, v_n) \to c_{\mathcal{N}} \quad \text{and} \quad I'(u_n, v_n) \to 0 \quad \text{as } n \to +\infty. \tag{6.3}$$

PROPOSITION 6.2. *Suppose that assumptions (V_1)–(V_6) and (F_1)–(F_3) hold. Then any sequence $(u_n, v_n)_n$ satisfying (6.3) is bounded in E.*

PROOF. Here we just give a sketch since the proof is similar to that of Proposition 5.1. Arguing by contradiction, we suppose that $\|(u_n, v_n)\| = \|u_n\|_{a,p} + \|v_n\|_{b,q} \to +\infty$ as $n \to +\infty$. We define $w_n = u_n/K_n^{1/p}$ and $z_n = v_n/K_n^{1/q}$, where $K_n := \|u_n\|_{a,p}^p + \|v_n\|_{b,q}^q$. Thus, $(w_n, z_n)_n$ is bounded in E. We may assume up to a subsequence that $(w_n, z_n) \rightharpoonup (w_0, z_0)$ weakly in E. If $(w_0, z_0) \neq (0, 0)$, then we get a contradiction in the same way as in Case 1 of Proposition 5.1. If $(w_0, v_0) = (0, 0)$, then we claim that for any $R > 0$ we have

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} (|w_n|^p + |z_n|^q)\, dx = 0. \tag{6.4}$$

If (6.4) does not hold, then there exist a sequence $(y_n)_n \subset \mathbb{Z}^N$ and $R, \eta > 0$ such that

$$\lim_{n\to+\infty} \int_{B_R(y_n)} (|w_n|^p + |z_n|^q)\, dx \ge \eta > 0. \tag{6.5}$$

We define the shift sequence $(\tilde{w}_n(x), \tilde{z}_n(x)) = (w_n(x + y_n), z_n(x + y_n))$. Since $E_a \hookrightarrow W^{1,p}(\mathbb{R}^N)$ and $E_b \hookrightarrow W^{1,q}(\mathbb{R}^N)$, we deduce that $(\tilde{w}_n, \tilde{z}_n)_n$ is bounded in E due the fact that $(w_n, z_n)_n$ is bounded, and thus up to a subsequence that $(\tilde{w}_n, \tilde{z}_n) \rightharpoonup (\tilde{w}_0, \tilde{z}_0)$. By using (6.5) we conclude that $(\tilde{w}_0, \tilde{z}_0) \neq (0, 0)$ and we get a contradiction as in Case 1 of Proposition 5.1. Therefore, (6.4) holds and the conclusion follows as in Case 2 of Proposition 5.1. □

In view of the preceding proposition, we may assume, up to a subsequence, that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E. By a standard density argument we can conclude that (u_0, v_0) is a critical point of I. The main difficulty here is to prove that (u_0, v_0) is a nontrivial solution, since we do not have the invariance by translations of the energy functional in this case.

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Proposition 6.3. *Suppose that* (V_1)–(V_6) *and* (F_1)–(F_3) *hold. Then the weak limit* (u_0, v_0) *is nontrivial.*

Proof. We suppose by contradiction that $(u_0, v_0) = (0, 0)$. Thus, we have:

- $u_n \to u_0$ strongly in $L^r_{\text{loc}}(\mathbb{R}^N)$, for all $p \le r < p^*$;
- $v_n \to v_0$ strongly in $L^s_{\text{loc}}(\mathbb{R}^N)$, for all $q \le s < q^*$;
- $u_n(x) \to u_0(x)$ and $v_n(x) \to v_0(x)$, almost everywhere in $\mathbb{R}^N$.

It follows by Assumption (V_4) that for any $\varepsilon > 0$ there exists $R > 0$ such that

$$|a_o(x) - a(x)| < \varepsilon, \quad |b_o(x) - b(x)| < \varepsilon, \quad |\lambda(x) - \lambda_o(x)| < \varepsilon, \tag{6.6}$$

for all $x \in B_R(0)^c$. Using (6.6) and the local convergence, we deduce that

$$\begin{aligned} \left|\int_{\mathbb{R}^N} (a_o(x) - a(x))|u_n|^p\,dx\right| &\le \int_{B_R(0)} |a_o(x) - a(x)||u_n|^p\,dx + C\varepsilon \int_{B_R(0)^c} |u_n|^p\,dx \\ &\le (\|a_o\|_\infty + \|a\|_\infty)\varepsilon + C\varepsilon, \end{aligned} \tag{6.7}$$

for all $n \ge n_0$. Analogously we get

$$\left|\int_{\mathbb{R}^N} (b_o(x) - b(x))|v_n|^q\,dx\right| \le (\|b_o\|_\infty + \|b\|_\infty)\varepsilon + C\varepsilon. \tag{6.8}$$

Moreover, using the Hölder inequality with $\alpha/p + \beta/q = 1$, we deduce that

$$\left|\int_{\mathbb{R}^N} (\lambda(x) - \lambda_o(x))|u_n|^\alpha |v_n|^\beta\,dx\right| \le (\|\lambda\|_\infty + \|\lambda_o\|_\infty)\varepsilon + C\varepsilon. \tag{6.9}$$

Combining (6.7), (6.8) and (6.9), we conclude that

$$I_o(u_n, v_n) - I(u_n, v_n) = o_n(1)$$

and

$$\left\langle I_o'(u_n, v_n) - I'(u_n, v_n), \left(\frac{1}{p}u_n, \frac{1}{q}v_n\right)\right\rangle = o_n(1),$$

which jointly with (6.3) imply that

$$I_o(u_n, v_n) = c_{\mathcal{N}} + o_n(1) \quad \text{and} \quad \left\langle I_o'(u_n, v_n), \left(\frac{1}{p}u_n, \frac{1}{q}v_n\right)\right\rangle = o_n(1). \tag{6.10}$$

In light of Lemma 3.1, we get a sequence $(t_n)_n \subset (0, +\infty)$ such that $(t_n^{1/p}u_n, t_n^{1/q}v_n)_n \subset \mathcal{N}_o$.

Claim 1. $\limsup_{n\to+\infty} t_n \le 1$.

We suppose by contradiction that the claim does not hold, that is, there exists $\varepsilon_0 > 0$ such that, up to a subsequence, we have $t_n \ge 1 + \varepsilon_0$, for all $n \in \mathbb{N}$. By using (6.10) and the fact that $(t_n^{1/p}u_n, t_n^{1/q}v_n)_n \subset \mathcal{N}_o$ we obtain

$$\frac{1}{p}\int_{\mathbb{R}^N}\left(\frac{f(t_n^{1/p}u_n)}{t_n^{1-1/p}}u_n - f(u_n)u_n\right)dx + \frac{1}{q}\int_{\mathbb{R}^N}\left(\frac{g(t_n^{1/q}v_n)}{t_n^{1-1/q}}v_n - g(v_n)v_n\right)dx = o_n(1).$$

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Since $t_n \geq 1 + \varepsilon_0$, it follows from (4.3) that

$$\frac{1}{p}\int_{\mathbb{R}^N}\Big(\frac{f((1+\varepsilon_0)^{1/p}u_n)}{(1+\varepsilon_0)^{1-1/p}}u_n - f(u_n)u_n\Big) + \frac{1}{q}\int_{\mathbb{R}^N}\Big(\frac{g((1+\varepsilon_0)^{1/q}v_n)}{(1+\varepsilon_0)^{1-1/q}}v_n - g(v_n)v_n\Big) \leq o_n(1).$$

Arguing as in Proposition 6.2, we introduce the sequence $(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x + y_n), v_n(x + y_n))$, which is bounded in E and, up to a subsequence, $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}_0, \tilde{v}_0)$ weakly in E. Moreover, $(\tilde{u}_0, \tilde{v}_0) \neq (0, 0)$. Thus, using (4.3) and Fatou's lemma, we get

$$\begin{aligned}0 &< \frac{1}{p}\int_{\mathbb{R}^N}\Big(\frac{f((1+\varepsilon_0)^{1/p}u_0)}{(1+\varepsilon_0)^{1-1/p}}u_0 - f(u_0)u_0\Big)\\ &\quad + \frac{1}{q}\int_{\mathbb{R}^N}\Big(\frac{g((1+\varepsilon_0)^{1/q}v_0)}{(1+\varepsilon_0)^{1-1/q}}v_0 - g(v_0)v_0\Big) \leq o_n(1),\end{aligned}$$

which is not possible and concludes the proof of Claim 1.

Claim 2. There exists $n_0 \in \mathbb{N}$ such that $t_n \geq 1$, for all $n \geq n_0$.

We suppose by contradiction that $t_n < 1$ for all $n \in \mathbb{N}$. Thus, $t_n^{1/p} \leq t_n^{1/q} < 1$. Hence, using Lemma 3.3 and the fact that $(t_n^{1/p}u_n, t_n^{1/q}v_n)_n \subset \mathcal{N}_o$, we obtain

$$\begin{aligned}c_{\mathcal{N}_o} &\leq \frac{1}{p}\int_{\mathbb{R}^N}(f(t_n^{1/p}u_n)t_n^{1/p}u_n - pF(t_n^{1/p}u_n))\,dx\\ &\quad + \frac{1}{q}\int_{\mathbb{R}^N}(g(t_n^{1/q}v_n)t_n^{1/q}v_n - qG(t_n^{1/q}v_n))\,dx\\ &\leq \frac{1}{p}\int_{\mathbb{R}^N}(f(u_n)u_n - pF(u_n))\,dx + \frac{1}{q}\int_{\mathbb{R}^N}(g(v_n)v_n - qG(v_n))\,dx\\ &= c_{\mathcal{N}} + o_n(1),\end{aligned}$$

which implies that $c_{\mathcal{N}_o} \leq c_{\mathcal{N}}$. This is absurd due the fact that $c_{\mathcal{N}} < c_{\mathcal{N}_o}$; see Lemma 6.1. As a consequence, Claims 1 and 2 hold.

By using Claims 1 and 2 we can deduce that

$$\int_{\mathbb{R}^N}(F(t_n^{1/p}u_n) - F(u_n))\,dx = \int_1^{t_n^{1/p}}\int_{\mathbb{R}^N} f(\tau u_n)u_n\,dx = o_n(1), \tag{6.11}$$

$$\int_{\mathbb{R}^N}(G(t_n^{1/q}v_n) - G(v_n))\,dx = \int_1^{t_n^{1/q}}\int_{\mathbb{R}^N} g(\tau v_n)v_n\,dx = o_n(1). \tag{6.12}$$

Moreover, since $a_o, b_o \in L^\infty(\mathbb{R}^N)$ and $(u_n, v_n)_n$ is bounded in E_o we also have

$$(t_n - 1)\Big(\frac{1}{p}\|u_n\|_{a_o,p}^p + \frac{1}{q}\|v_n\|_{b_o,q}^q - \int_{\mathbb{R}^N}\lambda_o(x)|u_n|^\alpha|v_n|^\beta\,dx\Big) = o_n(1). \tag{6.13}$$

Combining (6.11), (6.12) and (6.13), we conclude that

$$I_o(t_n^{1/p}u_n, t_n^{1/q}v_n) - I_o(u_n, v_n) = o_n(1).$$

In view of (6.10), we mention also that

$$c_{\mathcal{N}_o} \leq I_o(t_n^{1/p}u_n, t_n^{1/q}v_n) = I_o(u_n, v_n) + o_n(1) = c_{\mathcal{N}} + o_n(1),$$

which contradicts Lemma 6.1. Therefore, $(u_0, v_0) \neq (0, 0)$. This concludes the proof. □

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PROOF OF THEOREM 1.2 COMPLETED. Since (u_0, v_0) is a nontrivial critical point of I, we have that $(u_0, v_0) \in \mathcal{N}$. Hence, $c_{\mathcal{N}} \leq I(u_0, v_0)$. On the other hand, it follows from (6.3) and Fatou's lemma that

$$\begin{aligned} c_{\mathcal{N}} + o_n(1) &= \frac{1}{p}\int_{\mathbb{R}^N}(f(u_n)u_n - pF(u_n))\,dx + \frac{1}{q}\int_{\mathbb{R}^N}(g(v_n)v_n - qG(v_n))\,dx \\ &\geq \frac{1}{p}\int_{\mathbb{R}^N}(f(u_0)u_0 - pF(u_0))\,dx + \frac{1}{q}\int_{\mathbb{R}^N}(g(v_0)v_0 - qG(v_0))\,dx + o_n(1) \\ &= I(u_0, v_0) + o_n(1), \end{aligned}$$

which implies that $c_{\mathcal{N}} \geq I(u_0, v_0)$. Therefore, (u_0, v_0) is a ground state for system (S). By a similar argument used in the proof of Propositions 5.5, 5.6 and 5.7, we obtain $t_0 > 0$ such that $(t_0^{1/p}|u_0|, t_0^{1/q}|v_0|) \in \mathcal{N}$ is a positive ground state solution for system (S) for all $\eta > 0$ large enough. □

7. Final conclusions

Here we indicate some interesting questions related to this class of quasilinear Schrödinger systems.

QUESTION 1. In the present paper we make use of the differential structure of the Nehari manifold $\mathcal{N}$, which allows us to look for a minimizer of the constrained functional $I|_{\mathcal{N}}$. This was possible since we are considering differentiable nonlinearities. It should be an interesting question to consider system (1.1) when the nonlinearities f and g are just continuous functions. In this case, the Nehari manifold associated with the problem may not be smooth. In [36, 37], Szulkin and Weth introduced a method to overcome this difficulty, by proving that the Nehari manifold $\mathcal{N}$ and the unit sphere S^1 are homeomorphic. Thus, one can try to adapt the ideas introduced in [36, 37] in order to use the differential structure of the unit sphere to look for ground state solutions.

QUESTION 2. We understand that one can try to use the same approach as in this work to prove the existence of ground states for the following more general class of quasilinear elliptic systems:

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = f(u) + c(x)H_u(u, v), & x \in \mathbb{R}^N, \\ -\Delta_q v + b(x)|v|^{q-2}v = g(v) + c(x)H_v(u, v), & x \in \mathbb{R}^N, \end{cases}$$

where $a(x), b(x)$ are periodic or asymptotically periodic and f, g are continuous functions. For instance, we could assume $c \in L^\infty(\mathbb{R}^N)$ and $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the following assumptions.

(i) The function H is in C^1 class which satisfies a subcritical growth in the sense that

$$|H_u(u, v)| \leq c_1(1 + |u|^{r_1 - 1} + |v|^{r_2 - 1}) \quad \text{for all } (u, v) \in \mathbb{R} \times \mathbb{R},$$

$$|H_v(u, v)| \leq c_2(1 + |u|^{r_1 - 1} + |v|^{r_2 - 1}) \quad \text{for all } (u, v) \in \mathbb{R} \times \mathbb{R},$$

for some constants $c_1, c_2 > 0$ and $r_1 \in (p, p^*)$, $r_2 \in (q, q^*)$.

(ii) $H(t^{1/p}u, t^{1/q}v) = tH(u, v)$, for any $t \geq 0$ and for all $(u, v) \in \mathbb{R} \times \mathbb{R}$.
(iii) $|H(u, v)| \leq k(|u|^p + |v|^q)$, for all $(u, v) \in \mathbb{R} \times \mathbb{R}$, where $k > 0$ is small enough.

Typical examples for H are $H(u, v) = |u|^\alpha |v|^\beta$ for $(u, v) \in \mathbb{R} \times \mathbb{R}$, where $1 \leq \alpha < p$ and $1 \leq \beta < q$. We mention that one can try to consider more general assumptions over the coupling function than used in this paper.

Question 3. By making some modifications to the arguments employed in the proof of our main results one can also consider the following more general class of elliptic systems:

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = R_u(u, v) + c(x)H_u(u, v), & x \in \mathbb{R}^N, \\ -\Delta_q v + b(x)|v|^{q-2}v = R_v(u, v) + c(x)H_v(u, v), & x \in \mathbb{R}^N, \end{cases}$$

where $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is subcritical and belongs to the C^1 class. The coupling term H is also of the C^1 class which satisfies the assumptions mentioned above.

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J. C. DE ALBUQUERQUE, Department of Mathematics,
Federal University of Pernambuco, 50670-901 Recife – PE, Brazil
e-mail: joserre@gmail.com, jc@dmat.ufpe.br

JOÃO MARCOS DO Ó, Department of Mathematics,
University of Brasília, 7, 0910-900 Brasília – DF, Brazil
e-mail: jmbo@pq.cnpq.br

EDCARLOS D. SILVA, Institute of Mathematics and Statistics,
Federal University of Goiás, 74001-970, Goiás – GO, Brazil
e-mail: eddomingos@hotmail.com

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