SOME ALGEBRAIC STRUCTURE IN THE DUAL OF A COMPACT GROUP

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Throughout this paper, G will denote a compact (Hausdorff) topological group with identity e. When G is abelian, there is no difficulty in relating the group multiplication in G to the multiplication in the dual of G since characters are homomorphisms with respect to pointwise multiplication and pointwise multiplication of characters yields another character. However, in the non-abelian case, there are two multiplications associated with the dual of G: (1) representations are homomorphisms with respect to composition multiplication, and (2) the tensor product of representations yields another representation. This investigation has its beginnings in an attempt to relate the group multiplication in G to the tensor multiplication in the dual of G. After some basic relations are obtained, we show that several known results for locally compact abelian groups which relate algebraic properties of the dual to topological-algebraic properties of the group have analogues that hold for compact groups.

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1. Preliminaries. Our aim in this section is to introduce a dual object $G^\wedge$ for G and to endow $G^\wedge$ with an algebraic structure. The ideas and results presented in this section are known; a more complete description of $G^\wedge$ and proofs of the results can be found in (**7** and **8**, or **15**).

By a *representation* U of G we mean a strongly continuous homomorphism of G into the unitary operators on some Hilbert space; write $U(x)$ for the value of U at $x \in G$. Let $\mathcal{UF}(G)$ and $\mathcal{UI}(G)$ denote the finite-dimensional and the irreducible representations of G, respectively. Recall that $\mathcal{UI}(G) \subset \mathcal{UF}(G)$. Given $U, V \in \mathcal{UF}(G)$, then new representations belonging to $\mathcal{UF}(G)$ can be formed: (1) the direct sum $U \oplus V$, (2) the tensor product $U \otimes V$, and (3) a conjugate U' which is defined in terms of U and an adjoint operation on the representation space of U.

For $U, V \in \mathcal{UF}(G)$ we write $U \sim V$ if U and V are unitarily equivalent. Let $G^\wedge$ denote $\mathcal{UI}(G)$ modulo the equivalence relation $\sim$. If $\gamma \in G^\wedge$, write

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U^γ or V^γ for a representative of γ, $\mathscr{H}_\gamma$ for the representation space of U^γ (assume that there is only one Hilbert space of each finite dimension), and I_γ for the identity operator on $\mathscr{H}_\gamma$. Let the trivial representation and its equivalence class be denoted by 1.

We now give $G^\wedge$ an algebraic structure which makes $G^\wedge$ a hypergroup (**1**); see Helgason (**5**, § 2). If $\gamma \in G^\wedge$, set $\gamma' = \{V \in \mathscr{U}\mathscr{I}(G)\colon V \sim (U^\gamma)'\}$; then $\gamma' \in G^\wedge$ and γ' is called the *conjugate* of γ. If $\gamma, \delta \in G^\wedge$, then $U^\gamma \otimes U^\delta \in \mathscr{U}\mathscr{F}(G)$ and, hence, there is a unique (modulo $\sim$) decomposition, $U^\gamma \otimes U^\delta = m_1 V^{\alpha_1} \oplus m_2 V^{\alpha_2} \oplus \ldots \oplus m_p V^{\alpha_p}$, where each m_i is a positive integer and each $\alpha_i \in G^\wedge$. Define the $\times$-*multiplication* of γ and δ by $\gamma \times \delta = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$. For $\gamma \in G^\wedge$ and Γ, Δ subsets of $G^\wedge$, we define Γ', $\gamma \times \Gamma$ and $\Gamma \times \Delta$ in the obvious manner. Then $\times$-multiplication is commutative and associative, $1 \times \gamma = \gamma$, and $1 \in \gamma \times \gamma'$ for all $\gamma \in G^\wedge$. In the abelian case, $\times$-multiplication and conjugation reduce to pointwise multiplication and inversion.

2. Algebraic structure of $G^\wedge$. In this section we prove some elementary results, which will be used later, on the algebraic structure of $G^\wedge$. We begin by introducing some notation.

Discussion 2.1. If $\Gamma \subset G^\wedge$ and Γ is closed under conjugation and $\times$-multiplication, we write $\Gamma \leqq G^\wedge$; such a set is called a normal subhypergroup by Helgason (**5**, § 2). For $\Gamma \subset G^\wedge$ let $\langle \Gamma \rangle$ denote the smallest subset of $G^\wedge$ containing Γ that is closed under conjugation and $\times$-multiplication. Similarly, for $B \subset G$ let $\langle B \rangle$ denote the smallest closed normal subgroup of G containing B. Note that if $\Gamma \leqq G^\wedge$, then $\Gamma = \Gamma'$ and $\Gamma \times \Gamma = \Gamma$.

PROPOSITION 2.2. *Let* $\Gamma_1 \leqq G^\wedge$ *and* $\Gamma_2 \leqq G^\wedge$; *then* $\langle \Gamma_1 \cup \Gamma_2 \rangle = \Gamma_1 \times \Gamma_2$.

Proof. Since $1 \in \Gamma_1 \cap \Gamma_2$, we have that $\Gamma_1 \cup \Gamma_2 \subset \Gamma_1 \times \Gamma_2$. Hence $\langle \Gamma_1 \cup \Gamma_2 \rangle \subset \langle \Gamma_1 \times \Gamma_2 \rangle$. Clearly, $\Gamma_1 \times \Gamma_2 \subset \langle \Gamma_1 \cup \Gamma_2 \rangle$, so that $\langle \Gamma_1 \times \Gamma_2 \rangle = \langle \Gamma_1 \cup \Gamma_2 \rangle$. Hence, it suffices to show that $\langle \Gamma_1 \times \Gamma_2 \rangle = \Gamma_1 \times \Gamma_2$.

(a) Let $\gamma \in \Gamma_1 \times \Gamma_2$; then there are $\sigma_i \in \Gamma_i$ $(i = 1, 2)$ such that $U^{\sigma_1} \otimes U^{\sigma_2} = U^\gamma \oplus V$, where $V \in \mathscr{U}\mathscr{F}(G)$. It can be verified that $U^{\sigma_1'} \otimes U^{\sigma_2'} \sim U^{\gamma'} \oplus V'$. Hence, $\gamma' \in \sigma_1' \times \sigma_2' \subset \Gamma_1 \times \Gamma_2$. Therefore, $\Gamma_1 \times \Gamma_2$ is closed under conjugation.

(b) Let $\gamma, \delta \in \Gamma_1 \times \Gamma_2$; then there are $\sigma_i, \tau_i \in \Gamma_i$ $(i = 1, 2)$ such that $U^{\sigma_1} \otimes U^{\sigma_2} = U \oplus V$ and $U^{\tau_1} \otimes U^{\tau_2} = U^\delta \oplus W$, where $V, W \in \mathscr{U}\mathscr{F}(G)$. It can be verified that $(U^{\sigma_1} \otimes U^{\tau_1}) \otimes (U^{\sigma_2} \otimes U^{\tau_2}) \sim (U^\gamma \otimes U^\delta) \oplus Y$, where $Y \in \mathscr{U}\mathscr{F}(G)$. Hence $\gamma \times \delta \subset (\sigma_1 \times \tau_1) \times (\sigma_2 \times \tau_2) \subset \Gamma_1 \times \Gamma_2$. Therefore, $\Gamma_1 \times \Gamma_2$ is closed under $\times$-multiplication.

It follows that $\langle \{\gamma_1, \gamma_2\} \rangle = \langle \gamma_1 \rangle \times \langle \gamma_2 \rangle$ for $\gamma_1, \gamma_2 \in G^\wedge$.

COROLLARY 2.3. *Let* Γ_1 *and* Γ_2 *be finite sets with* $\Gamma_1 \leqq G^\wedge$ *and* $\Gamma_2 \leqq G^\wedge$; *then* $\langle \Gamma_1 \cup \Gamma_2 \rangle$ *is finite.*

Proof. Since Γ_1 and Γ_2 are finite and $\times$-multiplication is finite-valued, $\Gamma_1 \times \Gamma_2$ is finite.

A member b of a locally compact topological group B is called compact if the smallest closed subgroup of B containing b is compact. When B is abelian, the set of compact elements of B is a closed subgroup; see (**7**, §§ (9.9), (9.10), and (9.26)). Motivated by the fact that $\times$-multiplication is commutative, we are led to the following definition.

Definition 2.4. Let $\gamma \in G^\wedge$; then γ is called *compact* or *torsion* if there is a finite set Γ such that $\gamma \in \Gamma \leqq G^\wedge$. Let $\mathrm{Cm}(G^\wedge)$ denote the set of compact elements of $G^\wedge$.

PROPOSITION 2.5. *The compact elements of $G^\wedge$ are closed under conjugation and $\times$-multiplication; that is,* $\mathrm{Cm}(G^\wedge) \leqq G^\wedge$.

Proof. Let $\gamma \in \mathrm{Cm}(G^\wedge)$; then there is a finite set Γ such that $\gamma \in \Gamma \leqq G^\wedge$. Thus, $\gamma' \in \Gamma' = \Gamma$; therefore, $\gamma' \in \mathrm{Cm}(G^\wedge)$. Let $\gamma_i \in \mathrm{Cm}(G^\wedge)$ $(i = 1, 2)$; then there are finite sets Γ_i such that $\gamma_i \in \Gamma_i \leqq G^\wedge$ $(i = 1, 2)$. Thus, $\gamma_1 \times \gamma_2 \subset \langle \Gamma_1 \cup \Gamma_2 \rangle$ which is finite by Corollary 2.3; therefore, $\gamma_1 \times \gamma_2 \subset \mathrm{Cm}(G^\wedge)$.

For locally compact abelian groups, the concept of annihilator as introduced by Pontrjagin is a useful tool in relating structure on the dual group to structure on the group. Helgason (**5**, § 2) has given an analogue of annihilator for compact groups and has obtained some basic properties which we include below. Other concepts of a duality between closed (normal) subgroups and certain representation-associated sets have been studied; see, for example, van Kampen (**11**), Nakayama (**16**), and Hochschild and Mostow (**9**). Propositions 2.7, 2.8, and 2.10 are closely related to van Kampen's results.

Definition 2.6. Let $B \subset G$ and $\Gamma \subset G^\wedge$. Set

$$\mathbf{A}(G, \Gamma) = \{x \in G\colon U^\gamma(x) = I_\gamma \text{ for all } \gamma \in \Gamma\}$$

and

$$\mathbf{A}(G^\wedge, B) = \{\gamma \in G^\wedge\colon U^\gamma(x) = I_\gamma \text{ for all } x \in B\}.$$

We call $\mathbf{A}(G, \Gamma)$ the *annihilator of* Γ *in* G and call $\mathbf{A}(G^\wedge, B)$ the *annihilator of* B *in* $G^\wedge$.

PROPOSITION 2.7. (i) *Let $\Gamma \subset G^\wedge$; then $\mathbf{A}(G, \Gamma)$ is a closed normal subgroup of G.*

(ii) *Let $B \subset G$; then $\mathbf{A}(G^\wedge, B) \leqq G^\wedge$.*

(iii) *Let $\Gamma_1 \subset \Gamma_2 \subset G^\wedge$; then*

$$\{e\} = \mathbf{A}(G, G^\wedge) \subset \mathbf{A}(G, \Gamma_2) \subset \mathbf{A}(G, \Gamma_1) \subset \mathbf{A}(G, \{1\}) = G.$$

(iv) *Let $B_1 \subset B_2 \subset G$; then*

$$\{1\} = \mathbf{A}(G^\wedge, G) \subset \mathbf{A}(G^\wedge, B_2) \subset \mathbf{A}(G^\wedge, B_1) \subset \mathbf{A}(G^\wedge, \{e\}) = G^\wedge.$$

(v) *Let $\Gamma \subset G^\wedge$; then $\mathbf{A}(G, \Gamma) = \mathbf{A}(G, \langle \Gamma \rangle)$.*

(vi) *Let $B \subset G$; then $\mathbf{A}(G^\wedge, B) = \mathbf{A}(G^\wedge, \langle B \rangle)$.*

(vii) *Let $\Gamma \leqq G^\wedge$; then $\Gamma = \mathbf{A}(G^\wedge, \mathbf{A}(G, \Gamma))$.*

(viii) *Let B be a closed normal subgroup of G; then $B = \mathbf{A}(G, \mathbf{A}(G^\wedge, B))$.*

Proof. These statements are proved in (**5**, § 2; **8**); only statements (vii) and (viii) require more than a routine verification.

As in the abelian case, $(G/H)^\wedge$ can be identified with $\mathbf{A}(G^\wedge, H)$, where H is a closed normal subgroup of G; see (**7**, § (23.25)).

PROPOSITION 2.8. *Let H be a closed normal subgroup of G; then there is a one-to-one mapping π of $\mathbf{A}(G^\wedge, H)$ onto $(G/H)^\wedge$ which preserves conjugation and $\times$-multiplication. Moreover, for each $\gamma \in \mathbf{A}(G^\wedge, H)$ there are representatives U^γ and $W^{\pi(\gamma)}$ such that $U^\gamma(x) = W^{\pi(\gamma)}(xH)$ for all $x \in G$.*

Proof. For each $U \in \mathscr{UF}(G)$ such that U is constant on the cosets of H, let $\pi_0(U)$ be the representation of G/H given by $\pi_0(U)(xH) = U(x)$ for all $x \in G$. If $\gamma \in \mathbf{A}(G^\wedge, H)$, then U^γ is constant on the cosets of H; thus, let $\pi(\gamma)$ be the equivalence class in $(G/H)^\wedge$ that contains $\pi_0(U^\gamma)$. It is routine to verify that $\pi_0(U^\gamma) \in \mathscr{UI}(G/H)$ (so that π is well-defined), that π is a one-to-one mapping of $\mathbf{A}(G^\wedge, H)$ onto $(G/H)^\wedge$, and that π preserves the algebraic operations.

The next two propositions are analogues of known results for locally compact abelian groups; see (**7**, § (23.29)).

PROPOSITION 2.9. *Let H be a closed normal subgroup of G; then the following statements are equivalent:*
(i) *$\mathbf{A}(G^\wedge, H)$ is finite;*
(ii) *H is open;*
(iii) *$\lambda(H) > 0$, where λ is a Haar measure on G.*

Proof. (i) $\Leftrightarrow$ (ii). By (**7**, § (5.21)), H is open if and only if G/H is discrete. Since G is compact, G/H is compact; so that G/H is discrete if and only if G/H is finite. We also know that G/H is finite if and only if $(G/H)^\wedge$ is finite. By Proposition 2.8, there is one-to-one correspondence between $(G/H)^\wedge$ and $\mathbf{A}(G^\wedge, H)$; thus, $(G/H)^\wedge$ is finite if and only if $\mathbf{A}(G^\wedge, H)$ is finite.

(ii) $\Leftrightarrow$ (iii). This equivalence is well known; see (**7**, §§ (20.17), (20.2), and (5.5)).

PROPOSITION 2.10. (i) *Let $\Gamma_1 \leqq G^\wedge$ and $\Gamma_2 \leqq G^\wedge$; then $\mathbf{A}(G, \Gamma_1) \cap \mathbf{A}(G, \Gamma_2) = \mathbf{A}(G, \Gamma_1 \times \Gamma_2)$.*

(ii) *Let H_1 and H_2 be closed normal subgroups of G; then $H_1 \cap H_2 = \{e\}$ if and only if $\mathbf{A}(G^\wedge, H_1) \times \mathbf{A}(G^\wedge, H_2) = G^\wedge$.*

Proof. (i) Let $x \in \mathbf{A}(G, \Gamma_1) \cap \mathbf{A}(G, \Gamma_2)$ and $\gamma \in \Gamma_1 \times \Gamma_2$; then $U^\gamma(x)$ is a direct summand of an identity operator and, thus is itself an identity operator. Therefore, $x \in \mathbf{A}(G, \Gamma_1 \times \Gamma_2)$ which shows that $\mathbf{A}(G, \Gamma_1) \cap \mathbf{A}(G, \Gamma_2) \subset \mathbf{A}(G, \Gamma_1 \times \Gamma_2)$. Since $1 \in \Gamma_i$, we have that $\Gamma_i \subset \Gamma_1 \times \Gamma_2$ $(i = 1, 2)$; thus, $\mathbf{A}(G, \Gamma_1 \times \Gamma_2) \subset \mathbf{A}(G, \Gamma_1) \cap \mathbf{A}(G, \Gamma_2)$.

(ii) By part (i) and Proposition 2.7 we have that

$$\begin{aligned}\mathbf{A}(G, \mathbf{A}(G^\wedge, H_1) \times \mathbf{A}(G^\wedge, H_2)) &= \mathbf{A}(G, \mathbf{A}(G^\wedge, H_1)) \cap \mathbf{A}(G, \mathbf{A}(G^\wedge, H_2)) \\ &= H_1 \cap H_2.\end{aligned}$$

Proposition 2.2 shows that $\mathbf{A}(G^\wedge, H_1) \times \mathbf{A}(G^\wedge, H_2) \leqq G^\wedge$; hence, $H_1 \cap H_2 = \{e\}$ if and only if $\mathbf{A}(G^\wedge, H_1) \times \mathbf{A}(G^\wedge, H_2) = G^\wedge$ by Proposition 2.7.

An interesting result, which is a generalization of a theorem of Burnside (**2**, p. 299) follows from the previous proposition.

PROPOSITION 2.11. *Let $V \in \mathscr{U}\mathscr{F}(G)$ and set $\Gamma = \{\gamma \in G^\wedge\colon U^\gamma$ is a direct summand of $V\}$; then V is one-to-one if and only if $\langle\Gamma\rangle = G^\wedge$.*

Proof. Since $V \in \mathscr{U}\mathscr{F}(G)$, we have that

$$V = m_1 U^{\gamma_1} \oplus m_2 U^{\gamma_2} \oplus \ldots \oplus m_p U^{\gamma_p},$$

where the m_i are positive integers and the γ_i are distinct members of $G^\wedge$; then $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_p\}$. Note that $\mathrm{Ker}(V) = \bigcap\{\mathrm{Ker}(U^{\gamma_i})\colon 1 \leqq i \leqq p\}$ and $\mathrm{Ker}\ (U^\gamma) = \mathbf{A}(G, \{\gamma\})$ for $\gamma \in G^\wedge$. Hence

$$\mathrm{Ker}(V) = \bigcap\{\mathbf{A}(G, \{\gamma_i\})\colon 1 \leqq i \leqq p\} = \mathbf{A}(G, \langle\gamma_1\rangle \times \langle\gamma_2\rangle \times \ldots \times \langle\gamma_p\rangle)$$

by Proposition 2.10. Thus, Proposition 2.2 shows that $\mathrm{Ker}(V) = \mathbf{A}(G, \langle\Gamma\rangle)$. Therefore, V is one-to-one if and only if $\langle\Gamma\rangle = G^\wedge$ by Proposition 2.7.

The previous result shows that V is faithful if and only if the tensor powers of V and V' exhaust $G^\wedge$.

One might ask, hoping that complete analogues of all locally compact abelian results for annihilators were true, if $H^\wedge$ can be identified with $G^\wedge/\mathbf{A}(G^\wedge, H)$ for H a closed subgroup of G; see (**7**, § (24.11)). However, it is not clear how to define $G^\wedge/\mathbf{A}(G^\wedge, H)$ since there is no "nice" concept of coset in $G^\wedge$ as shown by the following proposition. A partial result in this direction, Proposition 2.13, suffices for this paper.

PROPOSITION 2.12. *Let $\gamma, \delta \in G^\wedge$ and $\Gamma \leqq G^\wedge$; then the following two statements are equivalent:*

(i) $\gamma \times \Gamma = \delta \times \Gamma$;

(ii) $\gamma \in \delta \times \Gamma$ *and* $\delta \in \gamma \times \Gamma$.

The following statement implies the preceding two statements:

(iii) $\gamma \times \delta' \subset \Gamma$.

Statement (i) *does not imply statement* (iii).

Proof. (i) $\Rightarrow$ (ii) Since $\Gamma \leqq G^\wedge$, we have $1 \in \Gamma$; thus, (ii) clearly follows from (i).

(ii) $\Rightarrow$ (i) Since $\gamma \in \delta \times \Gamma$ and $\Gamma \leqq G^\wedge$, we have that $\gamma \times \Gamma \subset \delta \times \Gamma \times \Gamma = \delta \times \Gamma$. Similarly, $\delta \in \gamma \times \Gamma$ implies that $\delta \times \Gamma \subset \gamma \times \Gamma$. Therefore, $\gamma \times \Gamma = \delta \times \Gamma$.

(iii) $\Rightarrow$ (ii) Since $\gamma \times \delta' \subset \Gamma$, we have that $\gamma \times \delta' \times \delta \subset \Gamma \times \delta = \delta \times \Gamma$. Hence, $\gamma \in \delta \times \Gamma$ since $1 \in \delta' \times \delta$. Since $\gamma \times \delta' \subset \Gamma$ and $\Gamma \leqq G^\wedge$, we have

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that $\gamma' \times \gamma \times \delta' \subset \gamma' \times \Gamma = (\gamma \times \Gamma)'$. Hence $\delta' \in (\gamma \times \Gamma)'$ since $1 \in \gamma' \times \gamma$; therefore $\delta \in \gamma \times \Gamma$.

(i) $\not\Rightarrow$ (iii) Let G be the symmetric group on three letters; then $G^\wedge$ consists of two 1-dimensional classes, 1 and β, and one 2-dimensional class, γ; see (**4**, p. 225). Let $\Gamma = \{1, \beta\}$, then $\Gamma \leqq G^\wedge$; clearly, $\gamma \times \Gamma = \gamma \times \Gamma$. Using characters, it can be shown that $\gamma \in \gamma \times \gamma'$. However, $\gamma \notin \Gamma$; therefore $\gamma \times \gamma' \not\subset \Gamma$.

PROPOSITION 2.13. *Let H be a closed normal subgroup of G. Define a function ϕ from $G^\wedge$ to the family of finite subsets of $H^\wedge$ by $\phi(\gamma) = \{\rho \in H^\wedge$: U^ρ is a direct summand of $U^\gamma|H\}$ for all $\gamma \in G^\wedge$.*

(i) *If Γ is a finite subset of $G^\wedge$, then $\phi(\Gamma)$ is a finite subset of $H^\wedge$.*

(ii) *If $\Gamma \leqq G^\wedge$, then $\phi(\Gamma) \leqq H^\wedge$.*

(iii) *Let $\gamma \in G^\wedge$; then $\phi(\gamma) = \{1\}$ if and only if $\gamma \in \mathrm{A}(G^\wedge, H)$.*

(iv) *If $\gamma \in \mathrm{Cm}(G^\wedge)$, then $\phi(\gamma) \subset \mathrm{Cm}(H^\wedge)$.*

3. Relations between the algebraic structure of $G^\wedge$ and the topological algebraic structure of G. This section contains the principal results of the paper.

Discussion 3.1. Let $\mathrm{Cp}(G)$ denote the component of the identity in G; then $\mathrm{Cp}(G)$ is a closed normal subgroup of G(**7**, § (7.1)) and $G/\mathrm{Cp}(G)$ is totally disconnected (**7**, § (7.3)). A topological space Y is called 0-dimensional if the clopen sets form a basis for the topology of Y. In a locally compact, Hausdorff space, totally disconnected and 0-dimensional are equivalent (**7**, § (3.5)).

A useful fact concerning $G^\wedge$ in the abelian case is: if z is a complex number with $|z| = 1$ and $0 < |z - 1| < 1$, then there is a positive integer n such that $|z^n - 1| > 1$. A non-abelian analogue of this fact is given by the following technical proposition which is used to prove Theorem 3.3.

PROPOSITION 3.2. *Let $U \in \mathscr{UF}(G)$; let B be a subgroup of G and suppose that there is a $b \in B$ with $0 < ||U(b) - I|| < 1$, where I is the identity operator on the representation space $\mathscr{H}$ of U and $||\cdot||$ denotes the operator norm. Then there is a $V \in \mathscr{UF}(G)$ and a $b' \in B$ such that $V \sim U$ and $||V(b') - I|| > 1$.*

Proof. Let n be the dimension of $\mathscr{H}$ and let $\mathscr{E}$ be an orthonormal basis for $\mathscr{H}$. By matrix theory (**4**, p. 255), there is a unitary operator T on $\mathscr{H}$ such that $T^{-1}U(b)T$ is diagonal when considered as a matrix relative to $\mathscr{E}$. Set $V(x) = T^{-1}U(x)T$ for all $x \in G$, then $V \in \mathscr{UF}(G)$ and $V \sim U$. Relative to $\mathscr{E}$ we have that $V(b) = \mathrm{diag}(v_1, v_2, \ldots, v_n)$, where each $|v_i| = 1$ since $V(b)$ is unitary. For each positive integer m set $A_m = V(b^m) - I = V(b)^m - I$. Then A_m is normal and, relative to $\mathscr{E}$, we have that $A_m = \mathrm{diag}(v_1{}^m - 1, v_2{}^m - 1, \ldots, v_n{}^m - 1)$. Since the bounded linear operators on $\mathscr{H}$ form a C^*-algebra (**15**, p. 309), we have that $||A_m||$ is the spectral radius $(A_m) = \sup\{|v_i{}^m - 1|: 1 \leqq i \leqq n\}$. Note that $||A_1|| = ||V(b) - I|| = ||U(b) - I||$. Since $0 < ||U(b) - I|| < 1$, there is a $j \in \{1, 2, \ldots, n\}$ such that $0 < |v_j - 1| < 1$. This inequality and the fact that $|v_j| = 1$ imply that there is a positive integer k such that $|v_j^k - 1| > 1$. Therefore, $||A_k|| > 1$ and we let $b' = b^k$.

Theorem 3.3 and its corollary are analogues of known results for locally compact abelian groups; see (**7**, §§ (24.17) and (24.18)).

THEOREM 3.3. (i) $\mathrm{Cm}(G^\wedge) = \mathrm{A}(G^\wedge, \mathrm{Cp}(G))$.
(ii) $\mathrm{Cp}(G) = \mathrm{A}(G, \mathrm{Cm}(G^\wedge))$.

Proof. (i) (a) First, assume that G is 0-dimensional. Let $\gamma \in G^\wedge$ and set $N = \{x \in G\colon \, ||U^\gamma(x) - I_\gamma|| < 1\}$; note that N does not depend on the representative U^γ but only on γ. Since N is a neighbourhood of the identity e, there is an open normal subgroup H contained in N (**7**, § (7.7)). Suppose that there is an $h' \in H$ such that $U^\gamma(h') \neq I_\gamma$; then Proposition 3.2 implies that there is a V^γ and an $h'' \in H$ such that $||V^\gamma(h'') - I_\gamma|| > 1$. This is a contradiction to $H \subset N$; therefore, $U^\gamma(h) = I_\gamma$ for all $h \in H$. Hence, $\gamma \in \mathrm{A}(G^\wedge, H)$ which is finite since H is open (Proposition 2.9). Thus, $\mathrm{A}(G^\wedge, H) \leqq G^\wedge$ implies that $\gamma \in \mathrm{Cm}(G^\wedge)$. Therefore, $G^\wedge = \mathrm{Cm}(G^\wedge)$.

(b) Second, assume that G is connected. Suppose that there is a $\gamma \in \mathrm{Cm}(G^\wedge)$ such that $\gamma \neq 1$; then there is a finite set Γ such that $\gamma \in \Gamma \leqq G^\wedge$. Propositions 2.7 and 2.9 show that the proper subgroup $\mathrm{A}(G, \Gamma)$ is open which is a contradiction to G being connected. Therefore, $\mathrm{Cm}(G^\wedge) = \{1\}$.

(c) Now consider the general case. Since $G/\mathrm{Cp}(G)$ is a 0-dimensional compact group, part (a) shows that $[G/\mathrm{Cp}(G)]^\wedge = \mathrm{Cm}([G/\mathrm{Cp}(G)]^\wedge)$. By Proposition 2.8, $[G/\mathrm{Cp}(G)]^\wedge$ can be identified with $\mathrm{A}(G^\wedge, \mathrm{Cp}(G))$ so that $\mathrm{A}(G^\wedge, \mathrm{Cp}(G)) \subset \mathrm{Cm}(G^\wedge)$. Suppose that there is a $\gamma \in \mathrm{Cm}(G^\wedge)\backslash \mathrm{A}(G^\wedge, \mathrm{Cp}(G))$. Define a function ϕ from $G^\wedge$ into the family of finite subsets of $\mathrm{Cp}(G)^\wedge$ as in Proposition 2.13. Then $\{1\} \neq \phi(\gamma) \subset \mathrm{Cm}(\mathrm{Cp}(G)^\wedge)$ which, in view of part (b), is a contradiction to the compact group $\mathrm{Cp}(G)$ being connected. Therefore

$$\mathrm{A}(G^\wedge, \mathrm{Cp}(G)) = \mathrm{Cm}(G^\wedge).$$

(ii) Statement (ii) is obtained from statement (i) by taking annihilators in G.

COROLLARY 3.4. (i) *G is connected if and only if* $\mathrm{Cm}(G^\wedge) = \{1\}$.
(ii) *G is 0-dimensional if and only if* $\mathrm{Cm}(G^\wedge) = G^\wedge$.

We now proceed to give a relationship between $\times$-multiplication in $G^\wedge$ and the group multiplication in G. It will be shown that for $\gamma \in G^\wedge$, the tensor powers of U^γ and $U^{\gamma'}$ yield no new irreducible representations after a finite power if and only if $U^\gamma(x)$ has finite order in the group of unitary operators on $\mathscr{H}_\gamma$ for each $x \in G$. Thus, in a certain sense, periodic with respect to $\times$-multiplication is equivalent to periodic with respect to composition.

Definition 3.5. Let $\gamma \in G^\wedge$; then γ is called *finite* (*infinite*) if $U^\gamma(G)$ is a finite (infinite) group. For a topological space Y, let $\omega(Y)$ denote the least cardinal number of a basis for Y. For a set S, let $\mathrm{card}(S)$ denote the cardinal number of S.

The following theorem is borrowed from (**8**); the heart of its proof is contained in (**10**).

THEOREM 3.6. *If G is infinite, then* $\operatorname{card}(G^\wedge) = \omega(G)$.

Discussion 3.7. Let $\gamma \in G^\wedge$ and set $K_\gamma = \{x \in G\colon U^\gamma(x) = I_\gamma\}$. Since $\mathscr{H}_\gamma$ is finite-dimensional and U^γ is strongly continuous, we know that U^γ is continuous. Then G being compact implies that U^γ is an open mapping and G/K_γ is homeomorphic and isomorphic to $U^\gamma(G)$ (**7**, § (5.39) (j)).

THEOREM 3.8. *Let $\gamma \in G^\wedge$.*
(i) *If γ is finite, then* $\operatorname{card}(\langle\gamma\rangle)$ *is finite.*
(ii) *If γ is infinite, then* $\operatorname{card}(\langle\gamma\rangle) = \aleph_0$.

Proof. From Discussion 3.7, we know that $\omega(G/K_\gamma) = \omega(U^\gamma(G))$ and γ is finite if and only if G/K_γ is a finite group. By Proposition 2.7, we have that $\langle\gamma\rangle = \mathbf{A}(G^\wedge, \mathbf{A}(G, \{\gamma\})) = \mathbf{A}(G^\wedge, K_\gamma)$. Thus, $\operatorname{card}(\langle\gamma\rangle) = \operatorname{card}((G/K_\gamma)^\wedge)$ since there is a one-to-one correspondence between $\mathbf{A}(G^\wedge, K_\gamma)$ and $(G/K_\gamma)^\wedge$ (Proposition 2.8). (i) Since G/K_γ is finite only if $(G/K_\gamma)^\wedge$ is finite, we have that $\operatorname{card}(\langle\gamma\rangle)$ is finite if γ is finite. (ii) If γ is infinite, then G/K_γ is an infinite group and $\omega(G/K_\gamma) = \operatorname{card}((G/K_\gamma)^\wedge)$ by Theorem 3.6. Therefore, $\operatorname{card}(\langle\gamma\rangle) = \omega(U^\gamma(G)) = \aleph_0$.

THEOREM 3.9. *Let $\gamma \in G^\wedge$; then the following statements are equivalent:*
(i) *γ is finite;*
(ii) $\gamma \in \operatorname{Cm}(G^\wedge)$;
(iii) *There is a positive integer m such that $U^\gamma(x^m) = U^\gamma(x)^m = I_\gamma$ for all $x \in G$.*

Proof. (i) is equivalent to (ii) by Theorem 3.8. (i) $\Rightarrow$ (iii) Let $m = \operatorname{card}(U^\gamma(G))$ which is finite. The order of each element in $U^\gamma(G)$ divides m, hence $U^\gamma(x)^m = I_\gamma$ for all $x \in G$.

(iii) $\Rightarrow$ (i) By hypothesis, $U^\gamma(G)$ is a periodic subgroup of the group of unitary operators on $\mathscr{H}_\gamma$ with bounded order; a theorem of Burnside (**2**, p. 493) states that $U^\gamma(G)$ is finite.

COROLLARY 3.10. *Let $\gamma \in \operatorname{Cm}(G^\wedge)$; then $\langle\gamma\rangle$ is the smallest subset of $G^\wedge$ containing γ that is closed under $\times$-multiplication.*

Proof. Since $\gamma \in \operatorname{Cm}(G^\wedge)$, there is a positive integer m such that $U^\gamma(x^m) = I_\gamma$ for all $x \in G$. Hence

$$\int_G \operatorname{trace} U^\gamma(x^m)\, d\lambda(x) = \lambda(G) \operatorname{dimension} \mathscr{H}_\gamma \neq 0,$$

where λ is a Haar measure on G. Then (**17**, proof of Lemma 8) there is a positive integer k such that $\int_G[\operatorname{trace} U^\gamma(x)]^k\, d\lambda(x) \neq 0$. Therefore (see **8**), $U^{\gamma'}$ is a direct summand of the $k-1$ tensor power of U^γ.

The following theorem is a generalization of a known result (**7**, §§ (24.25) and (24.26)) in the abelian case.

THEOREM 3.11. (i) γ *is infinite for all* $\gamma \in G^\wedge\backslash\{1\}$ *if and only if* G *is connected.*

(ii) γ *is finite for all* $\gamma \in G^\wedge$ *if and only if* G *is* 0-*dimensional.*

Proof. Combine Theorem 3.9 and Corollary 3.4.

The previous theorem can be proved without resorting to the concepts of compact element and annihilator, as shown below.

PROPOSITION 3.12. *The following statements are equivalent:*

(i) γ *is finite for all* $\gamma \in G^\wedge$;

(ii) G *is* 0-*dimensional.*

Proof. (i) $\Rightarrow$ (ii) This proof was suggested by Professor Kenneth Ross. Let $a \in G\backslash\{e\}$; then there is a $\gamma \in G^\wedge$ such that $U^\gamma(a) \neq I_\gamma$ (**7**, § (22.12)). Hence, there are $\xi, \eta \in \mathscr{H}_\gamma$ such that $\langle U^\gamma(a)\xi, \eta\rangle \neq \langle \xi, \eta\rangle$. Set $f(x) = \langle U^\gamma(x)\xi, \eta\rangle$ for all $x \in G$; then f is continuous, $f(a) \neq f(e)$ and by hypothesis, $f(G)$ is finite. Thus there is a clopen set containing e that does not contain a. Therefore G is totally disconnected (**13**, p. 55).

(ii) $\Rightarrow$ (i) This (more difficult to prove) half of the proposition is stated and proved in (**8**).

PROPOSITION 3.13. *The following statements are equivalent:*

(i) γ *is infinite for all* $\gamma \in G^\wedge\backslash\{1\}$;

(ii) G *is connected.*

Proof. (i) $\Rightarrow$ (ii) Suppose that G is not connected; then $G/\mathrm{Cp}(G)$ is non-trivial; thus, there is a $U \in \mathscr{U}\mathscr{I}(G/\mathrm{Cp}(G))$ such that $U \neq 1$ (**7**, § (22.12)). Proposition 3.12 implies that $U(G/\mathrm{Cp}(G))$ is finite since $G/\mathrm{Cp}(G)$ is 0-dimensional. Set $V = U \circ \phi$, where ϕ is the natural homomorphism of G onto $G/\mathrm{Cp}(G)$; then it is easily checked that $V \in \mathscr{U}\mathscr{I}(G)$. Also, we have that $V \neq 1$ and $V(G) = U(G/\mathrm{Cp}(G))$ which is a contradiction to our hypothesis. Therefore, G is connected.

(ii) $\Rightarrow$ (i) Let $\gamma \in G^\wedge\backslash\{1\}$. Since U^γ is continuous, $U^\gamma(G)$ is connected; and since $\gamma \neq 1$, $U^\gamma(G)$ is not a point. Hence, $U^\gamma(G)$ is infinite since the only connected finite sets in a Hausdorff space are points.

4. Applications.

In this section we apply our principal results to obtain some information about the structure of G. These analogues stem from known results in the abelian case; see (**7** or **18**).

Rider (**17**, p. 980) defines $\Gamma \leqq G^\wedge$ to be *ordered* if there is an $\Omega \subset \Gamma$ such that Ω is closed under $\times$-multiplication, $\Omega \cap \Omega' = \{1\}$ and $\Omega \cup \Omega' = \Gamma$; see also (**18**, § 8.1.1).

PROPOSITION 4.1. *If* $G^\wedge$ *is ordered, then* G *is connected.*

Proof. It is easily seen that $G^\wedge$ is ordered only if each $\Gamma \leqq G^\wedge$ is ordered. Corollary 3.10 shows that if $\gamma \in \mathrm{Cm}(G^\wedge)\backslash\{1\}$, then $\langle\gamma\rangle$ is not ordered. Therefore, $\mathrm{Cm}(G^\wedge) = \{1\}$ and G is connected by Corollary 3.4.

We suspect that the converse of the preceding proposition is also true as in the abelian case (**18**, § 8.1.2).

PROPOSITION 4.2. *Let* $S = \{a \in G\colon \langle a\rangle = G\}$; *then*

$$S = \bigcap\{G\backslash K_\gamma\colon \gamma \in G^\wedge\backslash\{1\}\} = \{a \in G\colon U^\gamma(a) \neq I_\gamma \text{ for all } \gamma \in G^\wedge\backslash\{1\}\}.$$

Proof. Let $a \in G$ and note that $\mathbf{A}(G^\wedge, \{a\}) = \mathbf{A}(G^\wedge, \langle a\rangle)$. Hence, $\langle a\rangle = G$ if and only if $\mathbf{A}(G^\wedge, \{a\}) = \{1\}$.

For a locally compact abelian group, S is not empty if and only if G is monothetic (**7**, § (25.11)). It would be interesting to know if the structure theorems (**7**, §§ (25.14), (25.15), (25.16), and (25.17)) for compact monothetic groups have non-abelian analogues when monothetic is defined as: S is not empty.

The following result and its proof are transcribed from the abelian case (**7**, § (25.27)(a)).

COROLLARY 4.3. *Suppose that* G *is connected and has a countable base for its topology. Let* $S = \{a \in G\colon \langle a\rangle = G\}$; *then* S *is measurable and* $\lambda(S) = 1$, *where* λ *is the normalized Haar measure on* G.

Proof. Theorem 3.6 shows that $\operatorname{card}(G^\wedge) = \aleph_0$ since G is infinite and has a countable base for its topology. Write $G^\wedge = \{1, \gamma_1, \gamma_2, \ldots\}$; then Theorem 3.11 implies that each γ_n is infinite since G is connected, and hence (Discussion 3.7) each G/K_{γ_n} is infinite. Therefore, $\lambda(K_{\gamma_n}) = 0$ for each n and, therefore $\lambda(\bigcap\{G\backslash K_{\gamma_n}\colon 1 \leqq n < \infty\}) = 1$.

THEOREM **4**.4. *Suppose that there is a positive integer* M *such that* n_γ, *the dimension of* $\mathscr{H}_\gamma$, *is* $\leqq M$ *for all* $\gamma \in G^\wedge$; *then the following statements are equivalent*:

(i) G *is of bounded order*;

(ii) $G^\wedge$ *is of bounded order*; *that is, there is a positive integer* N *such that* $\operatorname{card}(\langle\gamma\rangle) \leqq N$ *for all* $\gamma \in G^\wedge$.

Proof. (i) $\Rightarrow$ (ii) Let m be the least common multiple of the orders of elements in G. Let $\gamma \in G^\wedge$; then $U^\gamma(x)^m = U^\gamma(x^m) = U^\gamma(e) = I_\gamma$ for all $x \in G$. Hence, $U^\gamma(G)$ is a periodic group with bounded order m; furthermore, $U^\gamma(G)$ is an irreducible group of linear operators on $\mathscr{H}_\gamma$. By the proof of a theorem of Burnside in the book by Curtis and Reiner (**4**, p. 251) we have that $\operatorname{card}(U^\gamma(G)) \leqq m^{n_\gamma} \leqq m^M$. Since $U^\gamma(G)$ is finite, $\operatorname{card}(U^\gamma(G)) = \operatorname{card}(G/K_\gamma) \geqq \operatorname{card}((G/K_\gamma)^\wedge) = \operatorname{card}(\mathbf{A}(G^\wedge, K_\gamma)) = \operatorname{card}(\langle\gamma\rangle)$; see the proof of Theorem 3.8. Therefore, $\operatorname{card}(\langle\gamma\rangle) \leqq m^M$.

(ii) $\Rightarrow$ (i) Let $\gamma \in G^\wedge$; then $U^\gamma(G)$ is a finite group by Theorem 3.8. Hence, we have that

$$\operatorname{card}(U^\gamma(G)) = \operatorname{card}(G/K_\gamma) = \sum\{n\rho^2\colon \rho \in (G/K_\gamma)^\wedge\} =$$
$$\sum\{n\rho^2\colon \rho \in \mathbf{A}(G^\wedge, K_\gamma)\} = \sum\{n\rho^2\colon \rho \in \langle\gamma\rangle\} \leqq M^2 \operatorname{card}(\langle\gamma\rangle) \leqq M^2 N.$$

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Thus, the set of integers $\{\operatorname{card}(U^\gamma(G))\colon \gamma \in G^\wedge\}$ is bounded by M^2N; let m be the least common multiple of these integers. Then $U^\gamma(x^m) = U^\gamma(x)^m = I_\gamma$ for all $\gamma \in G^\wedge$ and all $x \in G$. Therefore, $x^m = e$ for all $x \in G$ since $G^\wedge$ separates the points of G (**7**, § (22.12)).

Groups satisfying the hypothesis of the previous theorem are called groups with representations of bounded degree. Kaplansky (**12**) has given necessary and sufficient conditions in terms of G or the group algebra of G that G be a group with representations of bounded degree. He has also shown that if G is a finite group extension of an abelian group, then G is a group with representations of bounded degree.

The following theorem is true for locally compact abelian groups (**7**, § (24.21)). For a compact abelian group B there is a stronger theorem (**7**, § (25.9)) which shows that B is of bounded order if B is periodic. It would be interesting to know if this result is true for an arbitrary compact group.

THEOREM 4.5. *If G is periodic, then G is 0-dimensional.*

Proof. By Theorem 3.11, it suffices to show that γ is finite for each $\gamma \in G^\wedge$. Thus, let $\gamma \in G^\wedge$; then $U^\gamma(G)$ is a compact (and hence closed) periodic subgroup of the group of unitary operators on $\mathscr{H}_\gamma$ since U^γ is a continuous homomorphism. Hence, $U^\gamma(G)$ is a Lie group since it is a closed subgroup of the Lie group of unitary operators on $\mathscr{H}_\gamma$ (**3**, p. 135). Suppose that the dimension of $U^\gamma(G)$ is not zero, then there is a non-trivial homomorphism θ of the additive group of real numbers into G (**6**, p. 93) and a neighbourhood N of 0 such that θ is one-to-one on N (**6**, p. 32). This is a contradiction to $U^\gamma(G)$ being periodic; hence, $U^\gamma(G)$ must be 0-dimensional. But 0-dimensional manifolds are discrete; therefore, $U^\gamma(G)$ is finite since it is also compact.

The above argument shows that a periodic Lie group is discrete; the proof of this fact was suggested by colleagues at the University of Toronto.

Discussion 4.6. For each positive integer k set $G^{(k)} = \{x^k\colon x \in G\}$. Two extensions of the notion of divisible abelian group are plausible: (1) G is divisible if $G^{(k)} = G$ for all positive integers k, and (2) G is divisible if $\langle G^{(k)}\rangle = G$ for all positive integers k. With the aid of a theorem of Mycielski, we shall show in the following theorem that these two concepts of divisibility coincide for compact groups.

THEOREM 4.7. *The following statements are equivalent:*

(i) *G is connected;*

(ii) *$G^{(k)} = G$ for each positive integer k;*

(iii) *$\langle G^{(k)}\rangle = G$ for each positive integer k.*

Proof. (i) $\Leftrightarrow$ (ii) This equivalence has been proved by Mycielski (**14**).

(i) $\Leftrightarrow$ (iii) Theorem 3.9 shows that $\operatorname{Cm}(G^\wedge) = \bigcup\{\mathbf{A}(G^\wedge, G^{(k)})\colon k \text{ is a positive integer}\}$. Hence, G is connected if and only if $\bigcup\{\mathbf{A}(G^\wedge, G^{(k)})\colon k \text{ is a positive integer}\} = \{1\}$ by Corollary 3.4. Therefore, Proposition 2.7 implies that G is connected if and only if $\langle G^{(k)}\rangle = G$ for each positive integer k.

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