Probability in the Engineering and Informational Sciences, **29**, 2015, 385–397.
doi:10.1017/S0269964815000054

EXACT DISTRIBUTION OF INTERMITTENTLY CHANGING POSITIVE AND NEGATIVE COMPOUND POISSON PROCESS DRIVEN BY AN ALTERNATING RENEWAL PROCESS AND RELATED FUNCTIONS

YIFAN XU
Department of Epidemiology and Biostatistics,
Case Western Reserve University,
Cleveland, OH 44106,
USA
E-mail: yifan.xu@case.edu

SHYAMAL K. DE
School of Mathematical Sciences,
National Institute of Science Education and Research,
Bhubaneswar 751005, Odisha,
India
E-mail: sde@niser.ac.in

SHELEMYAHU ZACKS
Department of Mathematical Sciences, Binghamton University
Binghamton, NY 13902,
USA
E-mail: shelly@math.binghamton.edu

Alternating renewal processes have been widely used to model social and scientific phenomenal where independent "on" and "off" states alternate. In this paper, we study a model where the value of a process cumulates and declines according to two modes of compound Poisson processes with respect to an underlying alternating renewal process. The model discussed in the present paper can be used as a revenue management model applied to inventory or to finance. The exact distribution of the process is derived as well as the double Laplace transform with respect to the level and time of the process.

1. INTRODUCTION

Brownian motions whose trends follow a generalized telegrapher process were studied by Di Crescenzo and Zacks [5] and Di Crescenzo, Martinucci, and Zacks [4]. Applications of such processes were mentioned in finance, physics, and other areas. In the present paper, we study up and down compound Poisson processes (CPPs), which are driven by generalized

 0269-9648/15 $25.00

telegrapher processes. The reader is referred to Zacks [6] for the properties of a generalized telegrapher process.

To strengthen the motivation, let us introduce our model as a revenue management model applied to inventory. That is, there are two periods: stocking period and selling period that alternate according to two modes of CPPs. During the stocking period the purchase price is low so that one only buys and during the selling period the price is high so that one only sells. Another example of our model would be a storage model where goods arrive or leave a warehouse according to two alternating phases. In cases where arrival and departure processes consist of sequences of demands of independent identically distributed amounts where each demand occurs on Poisson clock, it is appropriate to model the arriving and departure processes by CPPs. The underlying alternating phases can be thought as an alternating renewal process. Motivated by this model we study a general "up and down" stochastic process $\{Y(t), t \geq 0\}$ which, for a random amount of time U, follows a CPP $\{Y_1(t), t \geq 0\}$ and then for a random amount of time V, follows a negative CPP $\{-Y_2(t), t \geq 0\}$, intermittently. Formally, $\{Y_i(t), t \geq 0\}, i = 1, 2$ are defined as

$$Y_i(t) = \sum_{n=0}^{N_i(t)} X_n^{(i)}, \quad i = 1, 2, \tag{1.1}$$

where $\{N_i(t), t \geq 0\}$ is a Poisson counting process with intensity $\lambda_i > 0$, and $\{X_n^{(i)}, n \geq 1\}$ are i.i.d. positive random variables having an absolutely continuous distribution F_i, with density f_i. We also set $X_0^{(i)} \equiv 0$. $\{X_n^{(i)}, n \geq 1\}$ and $\{N_i(t), t \geq 0\}$ are mutually independent. In addition, let $\{U_1, V_1, U_2, V_2, \ldots\}$ be an alternating renewal process. U_i and V_i $(i \geq 1)$ are absolutely continuous positive random variables, having distribution functions F_U and F_V, with density functions f_U and f_V, respectively. Moreover, let $\tau_0 = 0$, and

$$\tau_n = \sum_{i=1}^{n} (U_i + V_i), \quad n \geq 1, \tag{1.2}$$

be renewal epochs after the nth renewal cycle. Each renewal cycle consists of one "up" period U_i followed by one "down" period V_i.

To put notations into perspective, goods (or data) arrive at a central storage (or process) unit during U_i and leave during V_i. The arrival and departure epochs follow two independent Poisson counting processes $N_1(t)$ and $N_2(t)$, and the amount of item (data) arriving or leaving at each epoch follows positive random variables $X_n^{(1)}$ and $X_n^{(2)}$ respectively (see Figure 1).

Let us denote $\mathbf{1}_{\{\}}$ as the indicator function. The process $Y(t)$ is defined formally as

$$\begin{aligned} Y(t) = \sum_{n=1}^{\infty} \Big[& \mathbf{1}_{\{\tau_{n-1} \leq t < \tau_{n-1} + U_n\}} \big(Y(\tau_{n-1}-) + Y_1(t - \tau_{n-1}) \big) \\ & + \mathbf{1}_{\{\tau_{n-1} + U_n \leq t < \tau_n\}} \big(Y((\tau_{n-1} + U_n)-) - Y_2(t - \tau_{n-1} - U_n) \big) \Big], \end{aligned} \tag{1.3}$$

for $t > 0$ and $Y(0) = 0$. Notice that $Y(0) = 0$ is not an essential assumption. Figure 1 illustrates a typical sample path of $Y(t)$.

Notice that the processes $Y_1(t)$ and $Y_2(t)$ might have no increase over some time period. Thus,

$$P\{Y_1(U_1) = 0\} = E\{e^{-\lambda_1 U_1}\} = M_U(-\lambda_1) \tag{1.4}$$

where M_U is the moment generating function of U. Similarly, $P\{Y_2(V_1) = 0\} = M_V(-\lambda_2)$.

https://doi.org/10.1017/S0269964815000054 Published online by Cambridge University Press

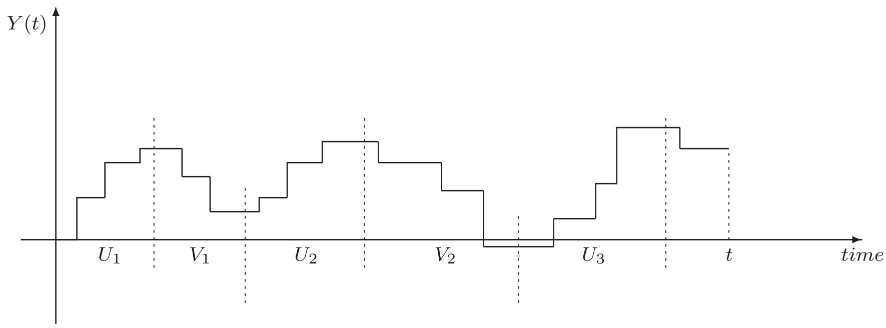

FIGURE 1. One possible sample path of $Y(t)$ up to time t. $(U_1, V_1, U_2, V_2, \ldots)$ is an alternating renewal process. Each increasing piece is a stochastic copy of the CPP $Y_1(t)$ with positive jumps, while decreasing pieces are stochastic copies of $-Y_2(t)$, a CPP with negative jumps.

The rest of the paper is structured as follows. In Section 2, we derive the cumulative distribution function (c.d.f.) of $Y(t)$ in details in terms of the distributions of $Y_i(t), i = 1, 2$. In Section 3, we find the double Laplace transform of $Y(t)$. Finally, in Section 4, we present numerical results for the special case where $\{X_n^{(i)}, n \geq 1\}$ and $\{U_i\}$, $\{V_i\}$ are exponentially distributed.

2. DISTRIBUTION OF $Y(T)$

In this section, we derive the distribution function and the moments of $Y(t)$. Recall from Eq. (1.2) that τ_n denotes the time epoch at the end of the nth renewal interval. Define the following random intervals:

$$I_n^+ = [\tau_{n-1}, \tau_{n-1} + U_n), \tag{2.1}$$

and

$$I_n^- = [\tau_{n-1} + U_n, \tau_n), \tag{2.2}$$

On I_n^+, $Y(t)$ is always non-decreasing and develops like a stochastic copy of $Y_1(t)$, while on I_n^- it is always non-increasing and develops like a stochastic copy of $-Y_2(t)$. Intervals I_n^+ will be referred to as "up" intervals, and I_n^- "as down" intervals. Let us define $W(t)$ to be the total "on" time before a fixed time t. Formally,

$$W(t) = \int_0^t \mathbf{1}_{\{s \in \bigcup_{n=1}^{\infty} I_n^+\}} ds. \tag{2.3}$$

Since CPP's $Y_1(t)$ and $Y_2(t)$ are Lévy processes, that is, processes with stationary independent increments, we have

$$Y(t) \sim Y_1(W(t)) - Y_2(t - W(t)). \tag{2.4}$$

Thus, the c.d.f. of $Y(t)$ at t can be written as

$$H_Y(y;t) := P\{Y(t) \leq y\} = P\{Y_1(W(t)) \leq y + Y_2(t - W(t))\}. \tag{2.5}$$

Zacks [7] derived the distribution of $W(t)$, and it was applied in many papers such as Di Crescenzo et al. [3] and Boxma et al. [1]. To derive the c.d.f. of $Y(t)$, we now outline the derivation of the exact distribution of total "on" time $W(t)$ before t in the following subsection.

https://doi.org/10.1017/S0269964815000054 Published online by Cambridge University Press

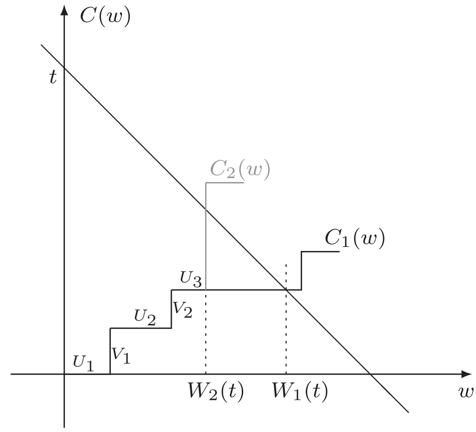

Figure 2. Two possible cases of $C(w)$, namely $C_1(w)$ and $C_2(w)$ crossing the linear boundary $\beta_t(w) = t - w$. Case 1: $C_1(W_1(t)) = t - W_1(t)$. Case 2: $C_2(W_2(t)) > t - W_2(t)$. Notice that in Case 1, $W_1(t) \in I^+$, and in Case 2, $W_2(t) \in I^-$.

2.1. Distribution of Total "on" Time

Consider the same alternating renewal process $(U_1, V_1, U_2, V_2, \ldots)$ defined in Section 1. First, we construct a compound renewal process $C(t)$ from the above alternating renewal process. Then, $W(t)$ can be related to the first exit time of $C(t)$ hitting a certain boundary, whose distribution is easier to derive. Define $C(t)$ as follows (see Figure 2):

$$C(t) = \sum_{m=0}^{M(t)} V_m, \tag{2.6}$$

where $M(t)$ is a counting process which gives the total number of complete "up" intervals that their sum is less than or equal to t. More precisely,

$$M(t) = \max\{m \geq 0 : \sum_{j=0}^{m} U_j \leq t\}. \tag{2.7}$$

Since $U_0 \equiv 0$, $U_1 > t$ implies that $M(t) = 0$. For $m = 0, 1, \ldots$, the probability that there are exactly m complete "up" intervals is

$$P\{M(t) = m\} = P\left\{\sum_{j=0}^{m} U_j \leq t < \sum_{j=0}^{m+1} U_j\right\} = F_U^{(m)}(t) - F_U^{(m+1)}(t), \tag{2.8}$$

where $F_U^{(m)}(t)$ is the m-fold convolution of $F_U(t)$. Here, $F_U^{(0)}(t) = 1$ for $0 \leq t < \infty$. For $m \geq 1$,

$$F_U^{(m)}(t) = \int_0^t f_U(s) F_U^{(m-1)}(t-s) ds. \tag{2.9}$$

Conditioning on $M(t)$, we obtain the c.d.f. of $C(t)$ as

$$K_C(x;t) := P\{C(t) \leq x\} = \sum_{m=0}^{\infty} \left(F_U^{(m)}(t) - F_U^{(m+1)}(t)\right) F_V^{(m)}(x). \tag{2.10}$$

Note that $C(t)$ has an atom at 0, that is,

$$P\{C(t)=0\} = 1 - F_U(t). \tag{2.11}$$

Moreover, we note that $\{C(w), w > 0\}$ is a non-decreasing process, and

$$\{W(t) > w\} = \{C(w) < t - w\}$$

as it is illustrated in Figure 2. Therefore, for $0 \leq w < t$,

$$P\{W(t) > w\} = K_C(t - w; w). \tag{2.12}$$

Next, we derive the distribution function of $W(t)$ in the following proposition.

Proposition 2.1: *The distribution function of $W(t)$ is*

$$F_W(w;t) = F_U(w) - \sum_{m=1}^{\infty} \left(F_U^{(m)}(w) - F_U^{(m+1)}(w) \right) F_V^{(m)}(t - w), \tag{2.13}$$

for $0 \leq w < t$, and

$$F_W(w;t) = 1, \quad \textit{for } w \geq t. \tag{2.14}$$

Proof: The proof of Proposition 2.1 is immediately seen by applying the Eqs (2.10)–(2.12). ■

In order to develop the distribution of $Y(t)$, we also need the density function of $W(t)$ whenever it exists. Notice from the above proposition that the distribution function of $W(t)$ has a jump at $w = t$ and the amount of jump is $P(W(t) = t) = 1 - F_U(t)$. Differentiating $F_W(w;t)$, we obtain the defective density $f_W(w;t)$ of $W(t)$. For $w \in (0,t)$,

$$\begin{aligned} f_W(w;t) &= f_U(w) - \frac{d}{dw} \sum_{m=1}^{\infty} \left(F_U^{(m)}(w) - F_U^{(m+1)}(w) \right) F_V^{(m)}(t - w) \\ &= f_U(w) + \sum_{m=1}^{\infty} \left(f_U^{(m+1)}(w) - f_U^{(m)}(w) \right) F_V^{(m)}(t - w) \\ &\quad + \sum_{m=1}^{\infty} \left(F_U^{(m)}(w) - F_U^{(m+1)}(w) \right) f_V^{(m)}(t - w). \end{aligned} \tag{2.15}$$

One can see that the defective density $f_W(w,t)$ is always positive by observing the following equation:

$$\begin{aligned} & f_U(w) + \sum_{m=1}^{\infty} \left(f_U^{(m+1)}(w) - f_U^{(m)}(w) \right) F_V^{(m)}(t - w) \\ &= \sum_{m=0}^{\infty} \left(F_V^{(m)}(t - w) - F_V^{(m+1)}(t - w) \right) f_U^{(m+1)}(w). \end{aligned} \tag{2.16}$$

Since the distribution of $W(t)$ is derived now, we can easily compute all the moments of $W(t)$ as we state in the following Proposition 2.2:

Proposition 2.2: *The kth moment of $W(t)$ is*

$$E\{(W(t))^k\} = k\int_0^t u^{k-1}K_C(t-u;u)\,du. \tag{2.17}$$

Proof:

$$E\{(W(t))^k\} = t^k(1-F_U(t)) + \int_0^t w^k f_W(w;t)\,dw.$$

Expressing $w^k = k\int_0^w u^{k-1}du$, and changing the order of integrals we get

$$\begin{aligned} E\{(W(t))^k\} &= t^k(1-F_U(t)) + k\int_0^t u^{k-1}\int_u^t f_W(w;t)\,dw\,du \\ &= t^k(1-F_U(t)) + k\int_0^t u^{k-1}(P(W(t)>u) - P(W(t)\geq t))\,du \\ &= t^k(1-F_U(t)) - t^k(1-F_U(t)) + k\int_0^t u^{k-1}K_C(t-u;u)\,du. \end{aligned} \tag{2.18}$$

■

Now, we are in a position to derive the exact distribution of $Y(t)$ and its moments in the following subsection.

2.2. The Distribution Function and Moments of $Y(t)$

In order to find the exact distribution of $Y(t)$, let us first state the distribution of $Y_i(t)$, $i = 1, 2$. Since $N_i(t)$ and $X_n^{(i)}$ are independent for $i = 1, 2$, and $n = 1, 2, \ldots$, conditioning on $N_i(t)$ yields the distribution function of $Y_i(t)$ as

$$H_i(y;t) = e^{-\lambda_i t} + \sum_{m=1}^{\infty} p(m;\lambda_i t)F_i^{(m)}(y), \tag{2.19}$$

for $y \geq 0$, where

$$p(m;\lambda_i t) = e^{-\lambda_i t}\frac{(\lambda_i t)^m}{m!}$$

is the probability mass function of Poisson($\lambda_i t$). The defective density function of $Y_i(t)$ is

$$h_i(y;t) = \sum_{m=1}^{\infty} p(m;\lambda_i t)f_i^{(m)}(y), \quad \text{for } y > 0. \tag{2.20}$$

$H_i(y;t)$ has a discontinuity at $y = 0$. $H_i(0-;t) = 0$ and $H_i(0;t) = e^{-\lambda_i t}$.

Since $\{U_n\}$, $\{V_n\}$ are independent of $Y_i(t), i = 1, 2$, $W(t)$ is independent of $Y_1(t)$ and $Y_2(t)$. Therefore, from Eq. (2.5), we have

$$P\{Y(t) \leq y | W(t) = w\} = P\{Y_1(w) \leq y + Y_2(t-w)\}. \tag{2.21}$$

Now, conditioning on $Y_2(t)$ and then using the independence of $Y_1(t)$ and $Y_2(t)$, we obtain

$$P\{Y_1(w) \leq y + Y_2(t-w)\} = e^{-\lambda_2(t-w)}H_1(y;w) + \int_0^{\infty} h_2(u;t-w)H_1(y+u;w)\,du. \tag{2.22}$$

The exponential term on the right-hand side of the above equation appears since $Y_2(t)$ has an atom at 0. Finally, from Eqs (2.21) and (2.22), the distribution function of $Y(t)$ is derived

https://doi.org/10.1017/S0269964815000054 Published online by Cambridge University Press

in the following proposition by taking expectation of $P\{Y(t) \leq y | W(t) = w\}$ with respect to the distribution of $W(t)$ and using the fact that $W(t)$ has an atom at t by Proposition 2.1.

PROPOSITION 2.3: *The distribution function of $Y(t)$ is given by*

$$H_Y(y;t) = (1 - F_U(t))H_1(y;t) + \int_0^t f_W(w;t)\left[e^{-\lambda_2(t-w)}H_1(y;w) + \int_0^\infty h_2(u;t-w)H_1(y+u;w)\,du\right]dw \tag{2.23}$$

An interesting observation from the above proposition is that the distribution of $Y(t)$ has an atom at 0. Below, we show that the distribution function $H_Y(y;t)$ of $Y(t)$ has a discontinuity (atom) at $y = 0$. Moreover, we find the defective density function of $Y(t)$. Since $H_i(y;t) = 0$ for all $y < 0$, $i = 1, 2$, we can write $H_Y(y;t)$ more explicitly as

$$\begin{aligned} H_Y(y;t) = \mathbf{1}_{\{y<0\}}&\left(\int_0^t f_W(w;t)\int_{-y}^\infty h_2(u;t-w)H_1(y+u;w)\,dudw\right) \\ &+ \mathbf{1}_{\{y\geq 0\}}\left((1-F_U(t))H_1(y;t) + \int_0^t f_W(w;t)\left[e^{-\lambda_2(t-w)}H_1(y;w).\right.\right. \\ &\left.\left. + \int_0^\infty h_2(u;t-w)H_1(y+u;w)\,du\right]dw\right). \end{aligned} \tag{2.24}$$

From Eq. (2.24), we obtain

$$H_Y(0-;t) = \int_0^t f_W(w;t)\int_0^\infty h_2(u;t-w)H_1(u;w)\,dw. \tag{2.25}$$

On the other hand, using the Eqs (2.24) and (2.25) and the fact that $H_1(y;t)$ has an atom at 0 of amount $e^{-\lambda_1 t}$, we write

$$H_Y(0;t) = (1-F_U(t))\,e^{-\lambda_1 t} + \int_0^t f_W(w;t)\,e^{-\lambda_2(t-w)-\lambda_1 w}dw + H_Y(0-;t). \tag{2.26}$$

Thus, $H_Y(y;t)$ has a discontinuity at $y = 0$ of size

$$(1-F_U(t))\,e^{-\lambda_1 t} + \int_0^t f_W(w;t)\,e^{-\lambda_2(t-w)-\lambda_1 w}dw. \tag{2.27}$$

Differentiating $H_Y(y;t)$ and using the fact that $H_i(y;t)$, $i = 1, 2$, has an atom at 0, we obtain the defective density of $Y(t)$ as

$$\begin{aligned} h_Y(y;t) = \mathbf{1}_{\{y<0\}}&\left(\int_0^t f_W(w;t)\left[e^{-\lambda_1 w}h_2(-y;t-w) + \int_{-y}^\infty h_2(u;t-w)h_1(y+u;w)\,du\right]dw\right) \\ &+ \mathbf{1}_{\{y\geq 0\}}\left((1-F_U(t))h_1(y;t) + \int_0^t f_W(w;t)\left[e^{-\lambda_2(t-w)}h_1(y;w)\right.\right. \\ &\left.\left. + \int_0^\infty h_2(u;t-w)h_1(y+u;w)\,du\right]dw\right). \end{aligned} \tag{2.28}$$

Next, we derive the moments of $Y(t)$ in the following proposition.

https://doi.org/10.1017/S0269964815000054 Published online by Cambridge University Press

PROPOSITION 2.4: *The kth moment of $Y(t)$ is*

$$E\{(Y(t))^k\} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \int_0^t f_W(w;t)\mu_{k-j}^{(1)}(w)\mu_j^{(2)}(t-w)\,dw. \quad \textbf{(2.29)}$$

PROOF: Since $W(t)$, $Y_1(t)$ and $Y_2(t)$ are independent, we have

$$\begin{aligned} E\{(Y(t))^k | W(t) = w\} &= E\{(Y_1(w) - Y_2(t-w))^k\} \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} E\{(Y_1(w))^{k-j}\} E\{(Y_2(t-w))^j\}. \end{aligned} \quad \textbf{(2.30)}$$

For $i = 1, 2$, denote the moment generating function (m.g.f) of $X_1^{(i)}$ and $Y_i(t)$ by $M_X^{(i)}(s)$ and $M_i(s;t)$, respectively. It is not hard to see

$$M_i(s;t) = e^{-\lambda_i t(1 - M_X^{(i)}(s))}. \quad \textbf{(2.31)}$$

Therefore, assuming $M_X^{(i)}$ is known, the moments of $Y_i(t)$, denoted by $\mu_l^{(i)}(t) = E\{(Y_i(t))^l\}, l = 0, 1, 2, \ldots$, can be found through Eq. (2.31). Finally, using the equality $E\{(Y(t))^k\} = E\{E\{(Y(t))^k | W(t)\}\}$ we complete the proof of Proposition 2.4. ■

We remark that both the distribution function and the defective density function of $Y(t)$ can be computed numerically via some numerical integration method. Moreover, all moments of $Y(t)$ can also be computed from Proposition 2.4. We illustrate this in Section 4 through an example.

3. LAPLACE TRANSFORMS

In this section, we derive the double Laplace transform of $W(t)$ and of $Y(t)$. Cohen [2] derived the Laplace Stieltjes transform of $W(t)I_{\{t \in I^+\}}$. Here, we extend the result to the entire time line.

Denote the Laplace transforms of U_1, V_1, $X_1^{(1)}$ and $X_1^{(2)}$ by

$$\begin{aligned} \phi_U(s) &:= \int_0^\infty e^{-st} f_U(t)\,dt, \qquad \phi_V(s) := \int_0^\infty e^{-st} f_V(t)\,dt, \\ \phi_1(s) &:= \int_0^\infty e^{-st} f_1(t)\,dt, \qquad \phi_2(s) := \int_0^\infty e^{-st} f_2(t)\,dt, \end{aligned} \quad \textbf{(3.1)}$$

respectively. Let

$$\phi_W(s,\rho) = \int_0^\infty e^{-st} E\left\{e^{-\rho W(t)}\right\} dt \quad \textbf{(3.2)}$$

be the double Laplace transform of $W(t)$. Substituting Eqs (2.15) and (2.16) gives

$$\begin{aligned} \phi_W(s,\rho) &= \sum_{n=0}^{\infty} \int_{t=0}^{\infty} e^{-st} \int_{v=0-}^{t} e^{-\rho(t-v)} \left(F_U^{(n)}(t-v) - F_U^{(n+1)}(t-v)\right) f_V^{(n)}(v)dv\,dt \\ &\quad + \sum_{n=0}^{\infty} \int_{t=0}^{\infty} e^{-st} \int_{u=0-}^{t} e^{-\rho u} \left(F_V^{(n)}(t-u) - F_V^{(n+1)}(t-u)\right) f_U^{(n+1)}(u)\,du\,dt. \end{aligned} \quad \textbf{(3.3)}$$

https://doi.org/10.1017/S0269964815000054 Published online by Cambridge University Press

For $n = 0$, the first renewal cycle covers t. We have

$$\int_0^\infty e^{-st} E\left\{e^{-\rho W(t)} I_{\{t\in I_1^+\}}\right\} dt = \int_0^\infty e^{-st} e^{-\rho t}(1 - F_U(t))\, dt = \frac{1 - \phi_U(s+\rho)}{s+\rho} \quad \textbf{(3.4)}$$

and

$$\int_0^\infty e^{-st} E\left\{e^{-\rho W(t)} I_{\{t\in I_1^-\}}\right\} dt = \int_0^\infty e^{-st} \int_0^t e^{-\rho u}(1 - F_V(t-u))\, du\, dt$$
$$= \frac{1}{s}\big(1 - \phi_V(s)\big)\phi_U(s+\rho). \quad \textbf{(3.5)}$$

Recall that the Laplace transform of the n-fold convolution of a positive function equals the nth power of the Laplace transform of the function. Therefore, the two parts on the right-hand side of Eq. (3.3) are simplified by separating the case $n = 0$ and changing the order of integrations. Some calculation yields

$$\sum_{n=0}^\infty \int_{t=0}^\infty e^{-st} \int_{v=0-}^t e^{-\rho(t-v)} \left[F_U^{(n)}(t-v) - F_U^{(n+1)}(t-v)\right] dF_V^{(n)}(v)\, dt$$
$$= \frac{1 - \phi_U(s+\rho)}{s+\rho} + \sum_{n=1}^\infty \int_{v=0}^\infty \int_{t=v}^\infty e^{-st} e^{-\rho(t-v)} \left[F_U^{(n)}(t-v) - F_U^{(n+1)}(t-v)\right] dt\, dF_V^{(n)}(v)$$
$$= \frac{1}{s+\rho} \cdot \frac{1 - \phi_U(s+\rho)}{1 - \phi_U(s+\rho)\phi_V(s)}. \quad \textbf{(3.6)}$$

Similarly,

$$\sum_{n=0}^\infty \int_{t=0}^\infty e^{-st} \int_{u=0-}^t e^{-\rho u} \left[F_V^{(n)}(t-u) - F_V^{(n+1)}(t-u)\right] dF_U^{(n+1)}(u)\, dt$$
$$= \frac{1}{s}\big(1 - \phi_V(s)\big)\phi_U(s+\rho) + \sum_{n=1}^\infty \int_{u=0}^\infty e^{-(s+\rho)u} \int_{t=0}^\infty e^{-st} \left[F_V^{(n)}(t) - F_V^{(n+1)}(t)\right] dt\, dF_U^{(n+1)}(u)$$
$$= \frac{\phi_U(s+\rho)}{s} \cdot \frac{1 - \phi_V(s)}{1 - \phi_V(s)\phi_U(s+\rho)}. \quad \textbf{(3.7)}$$

Thus, $\phi_W(s,\rho)$ is the sum of (3.6) and (3.7) as given in Proposition 3.1 below. Now, let us find the double Laplace transform of $Y(t)$ defined as

$$\phi_Y(s,\rho) := \int_0^\infty e^{-st} E\left\{e^{-\rho Y(t)}\right\} dt.$$

The Laplace transform of $Y(t)$ is

$$E\left\{e^{-\rho Y(t)}\right\} = E\left\{e^{-\rho\left(Y_1(W(t)) - Y_2(t-W(t))\right)}\right\}.$$

The Laplace transform of $Y_i(t)$ is

$$E\{e^{-\rho Y_i(t)}\} = e^{-\lambda_i t(1-\phi_i(\rho))}, \quad i = 1, 2. \quad \textbf{(3.8)}$$

https://doi.org/10.1017/S0269964815000054 Published online by Cambridge University Press

Therefore, for $0 < w \le t$,

$$E\left\{e^{-\rho Y(t)}|W(t)=w\right\} = E\{e^{-\rho Y_1(w)}\}E\{e^{\rho Y_2(t-w)}\}$$
$$= \exp\left\{-\lambda_1 w(1-\phi_1(\rho)) - \lambda_2(t-w)(1-\phi_2(-\rho))\right\}$$
$$= \exp\left\{-w\Big(\lambda_1(1-\phi_1(\rho)) - \lambda_2(1-\phi_2(-\rho))\Big) - \lambda_2 t\Big(1-\phi_2(-\rho)\Big)\right\}. \quad \textbf{(3.9)}$$

Let $a := \lambda_1(1-\phi_1(\rho)) - \lambda_2(1-\phi_2(-\rho))$ and $b := \lambda_2(1-\phi_2(-\rho))$. Taking expectation of $E\left\{e^{-\rho Y(t)}|W(t)=w\right\}$ with respect to the distribution of $W(t)$, we obtain the Laplace transform of $Y(t)$ as $e^{-bt}E\{-aW(t)\}$. Therefore, the double Laplace transform of $Y(t)$ is

$$\phi_Y(s,\rho) = \int_0^\infty e^{-st}E\left\{e^{-\rho Y(t)}\right\}dt = \int_0^\infty e^{-(s+b)t}E\left\{e^{-aW(t)}\right\}dt.$$

From the definition (3.2), it follows that

$$\phi_Y(s,\rho) = \phi_W\Big(s+\lambda_2(1-\phi_2(-\rho)), \lambda_1(1-\phi_1(\rho)) - \lambda_2(1-\phi_2(-\rho))\Big). \quad \textbf{(3.10)}$$

We summarize these results in the following proposition.

Proposition 3.1: *The double Laplace transforms of $W(t)$ and $Y(t)$ are*

$$\phi_W(s,\rho) = \frac{s+\phi_U(s+\rho)\left(\rho-(s+\rho)\phi_V(s)\right)}{s(s+\rho)\left(1-\phi_U(s+\rho)\phi_V(s)\right)} \quad \textbf{(3.11)}$$

and

$$\phi_Y(s,\rho) = \phi_W\Big(s+\zeta_2(-\rho), \zeta_1(\rho)-\zeta_2(-\rho)\Big), \quad \textbf{(3.12)}$$

where

$$\zeta_i(u) = \lambda_i(1-\phi_i(u)), i = 1,2.$$

4. NUMERICAL RESULTS

In this section, we present the formula for distribution function $H_Y(y;t)$ of $Y(t)$ when length of the "up" periods $\{U_n\}$ and the "down" periods $\{V_n\}$ are exponentially distributed with rate parameters θ_u and θ_v, respectively, and $\{X_n^{(1)}\}$ and $\{X_n^{(2)}\}$ follow exponential distributions with rate parameters θ_1 and θ_2, respectively. The random variables $U_n, V_n, X_n^{(1)}$, and $X_n^{(2)}$ are mutually independent for $n = 1, 2, \ldots$. Let $g(.;m,\theta)$ and $G(.;m,\theta)$ be the probability density function and the c.d.f. of a gamma random variable respectively, where m and θ represent the shape and the rate parameter of gamma random variable, respectively. Let us denote the c.d.f. of Poisson(λ) distribution by $\mathcal{P}(.;\lambda)$. Following Eq. (2.15), we obtain for $0 \le w < t$,

$$
\begin{aligned}
f_W(w;t) &= g(w;1,\theta_u) + \sum_{m=1}^{\infty} \{g(w;m+1,\theta_u) - g(w;m,\theta_u)\}\, G(t-w;m,\theta_v) \\
&\quad + \sum_{m=1}^{\infty} \{G(w;m,\theta_u) - G(w;m+1,\theta_u)\}\, g(t-w;m,\theta_v) \\
&= \theta_u e^{-\theta_u w} + \theta_u \sum_{m=1}^{\infty} \{p(m;\theta_u w) - p(m-1;\theta_u w)\}(1 - \mathcal{P}(m-1;(t-w)\theta_v)) \\
&\quad + \theta_v \sum_{m=1}^{\infty} p(m;\theta_u w) p(m-1;(t-w)\theta_v) \\
&= e^{-\theta_u w - \theta_v (t-w)} + \theta_u \sum_{m=1}^{\infty} p(m;\theta_u w) p(m;\theta_v (t-w)) \\
&\quad + \theta_v \sum_{m=1}^{\infty} p(m;\theta_u w) p(m-1;(t-w)\theta_v) \\
&= e^{-\theta_u w - \theta_v (t-w)} + \sum_{m=1}^{\infty} p(m;\theta_u w) \{\theta_u p(m;\theta_v (t-w)) + \theta_v p(m-1;(t-w)\theta_v)\}
\end{aligned}
\tag{4.1}
$$

and $P\,(W(t) = t) = e^{-\theta_u t}$. The defective density function of $Y_2(t)$ is given by

$$
h_2(y;t) = \sum_{m=1}^{\infty} p(m;\lambda_2 t) g(y;m,\theta_2) = \theta_2 \sum_{m=1}^{\infty} p(m;\lambda_2 t) p(m-1;\theta_2 y). \tag{4.2}
$$

for $y > 0$, and $P(Y_2(t) = 0) = e^{-\lambda_2 t}$. The distribution function of $Y_1(t)$ is given by

$$
H_1(y;t) = e^{-\lambda_1 t} + \sum_{m=1}^{\infty} p(m;\lambda_1 t) G(y;m,\theta_1) = 1 - \sum_{m=1}^{\infty} p(m;\lambda_1 t) \mathcal{P}(m-1;\theta_1 y). \tag{4.3}
$$

Finally, the distribution function of $Y(t)$ is given by Eq. (2.24) where the jump at 0 is

$$
\Delta H_Y(0;t) = e^{-(\theta_u + \lambda_1)t} + \int_0^t f_W(w;t)\, e^{-\lambda_2 (t-w) - \lambda_1 w} dw. \tag{4.4}
$$

In Figures 3 and 4, we plot the c.d.f of $Y(t)$ for different values of t and the rate parameters. The plots clearly show the fact that the distribution functions of $Y(t)$ are discontinuous at 0.

In addition to deriving the distribution function of $Y(t)$, we also derive the moments of $Y(t)$ and then study the shape of the distribution of $Y(t)$. In Table 1, we present mean $\mu(t)$, standard deviation $\sigma(t)$, measure of skewness $\gamma_1(t)$, and kurtosis $\gamma_2(t)$ of $Y(t)$ for different rate parameters of exponential distribution. Recall that

$$
\gamma_1(t) = \frac{E(Y(t) - \mu(t))^3}{\sigma(t)^3} \quad \text{and} \quad \gamma_2(t) = \frac{E(Y(t) - \mu(t))^4}{\sigma(t)^4} - 3.
$$

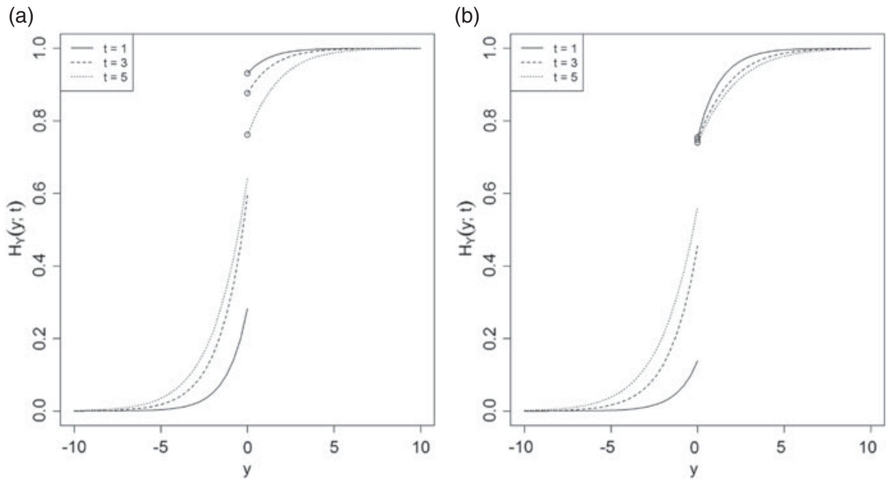

Figure 3. (a) The distribution functions of $Y(t)$ for $\lambda_1 = \lambda_2 = 0.5$, $\theta_1 = \theta_2 = 1$, and $\theta_u = \theta_v = 2$. (b) The distribution functions of $Y(t)$ for $\lambda_1 = \lambda_2 = 0.5$, $\theta_1 = \theta_2 = 1$, and $\theta_u = \theta_v = 0.5$.

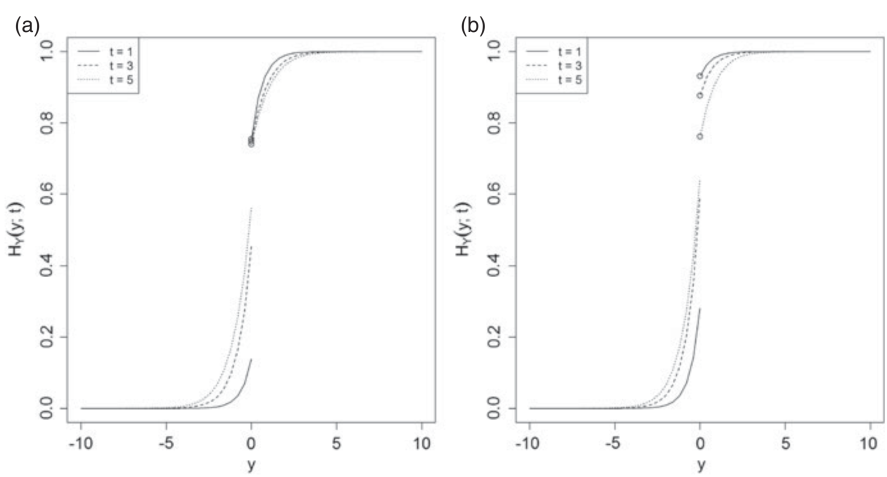

Figure 4. (a) The distribution functions of $Y(t)$ for $\lambda_1 = \lambda_2 = 0.5$, $\theta_1 = \theta_2 = 2$, and $\theta_u = \theta_v = 0.5$. (b) The distribution functions of $Y(t)$ for $\lambda_1 = \lambda_2 = 0.5$, $\theta_1 = \theta_2 = 2$, and $\theta_u = \theta_v = 2$.

Note that, for our exponential example, we can write the kth moment of $Y_i(t)$, $i = 1, 2$, as

$$\begin{aligned} \mu_k^{(i)}(t) = E\left(Y_i(t)^k\right) &= \int_0^\infty y^k h_i(y;t)\,dy \\ &= \sum_{m=1}^{\infty} p(m;\lambda_i t) \int_0^\infty y^k p(m-1;\theta_i y)\,dy \\ &= \frac{k!}{\theta_i^{k+1}} \sum_{m=1}^{\infty} \binom{k+m-1}{k} p(m;\lambda_i t), \end{aligned} \tag{4.5}$$

https://doi.org/10.1017/S0269964815000054 Published online by Cambridge University Press

Table 1. The mean, standard deviation, coefficients of skewness, and kurtosis of $Y(t)$ for different values of parameters

Parameters	Time	$\mu(t)$	$\sigma(t)$	$\gamma_1(t)$	$\gamma_2(t)$
$\lambda_1 = \lambda_2 = 0.5,\ \theta_1 = \theta_2 = 1,\ \theta_u = \theta_v = 2$	$t = 1$	0.0550	0.9603	0.3961	14.2859
	$t = 3$	0.1213	1.8191	0.1168	3.9472
	$t = 5$	0.1249	2.3552	0.0546	2.3292
$\lambda_1 = \lambda_2 = 0.5,\ \theta_1 = \theta_2 = 1,\ \theta_u = \theta_v = 0.5$	$t = 1$	0.0128	0.6524	0.3627	34.9260
	$t = 3$	0.1404	1.6803	0.2252	5.7384
	$t = 5$	0.2914	2.4416	0.1236	2.6357
$\lambda_1 = \lambda_2 = 0.5,\ \theta_1 = \theta_2 = 2,\ \theta_u = \theta_v = 0.5$	$t = 1$	0.0064	0.3262	0.3628	34.9279
	$t = 3$	0.0702	0.8417	0.2292	5.7942
	$t = 5$	0.1457	1.2238	0.1254	2.6689
$\lambda_1 = \lambda_2 = 0.5,\ \theta_1 = \theta_2 = 2,\ \theta_u = \theta_v = 2$	$t = 1$	0.0275	0.4801	0.3961	14.2866
	$t = 3$	0.0606	0.9105	0.1191	3.9741
	$t = 5$	0.0624	1.1807	0.0560	2.3677

for $k = 1, 2, \ldots$, and $\mu_0^{(i)}(t) = 1$. Finally, the moments of $Y(t)$ are computed using Eq. (2.29) in proposition 2.4.

Acknowledgments

The authors gratefully acknowledge the comments and suggestions of Professor Onno Boxma and Professor David Perry. The authors are thankful to the referee for the suggestions that helped us revise this article.

References

1. Boxma, O., Perry, D. & Zacks, S. (2014). *A fluid EOQ model of perishable items with intermittent high and low demand rates*. Mathematics of Operations Research.
2. Cohen, J.W. (1982). *The single server queue*. Amsterdam: Elsevier Science Publishers.
3. Di Crescenzo, A., Iuliano, A., Martinucci, B. & Zacks, S. (2013). *Generalized telegraph process with random jumps*. *Journal of Applied Probability* 50: 450–463.
4. Di Crescenzo, A., Martinucci, B. & Zacks, S. (2014). On the geometric Brownian motion with alternating trends. In *Mathematical and Statistical Methods for Actuarial Sciences and Finance* (A. Perma and M. Sibillo, M., Eds). New York: Springer.
5. Di Crescenzo, A. & Zacks, S. (2013). Probability law and flow function of Brownian motion driven by a generalized telegraph process. *Methodology and Computing in Applied Probability* 1–20.
6. Zacks, S. (2004). Generalized integrated telegrapher process and the distribution of related stopping times. *Journal of Applied Probability* 41: 497–507.
7. Zacks, S. (2012). Distribution of the total time in a mode of an alternating renewal process with applications. *Sequential Analysis* 31: 397–408.