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Homogenization of two-phase flows in porous media with hysteresis in the capillary relation

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The homogenization problem is considered for the equations of two-phase flow in porous media with a periodic or random small-scale structure of inhomogeneities. The capillary relation between saturation and the drop in pressures at microscales accounts for hysteresis and dynamic memory effects. Homogenized equations are derived, and convergence of solutions to the solution of the homogenized problem is proved. Properties of averaged capillary relation are described in the particular case of a two-component porous medium.

1 Introduction

Two-phase flows in porous media can be modelled by the equations (see Collins [11]):

$$m\frac{\partial s}{\partial t} = \frac{\partial}{\partial x}K^W(s)\left(\frac{\partial p^W}{\partial x} - f^W\right), \tag{1.1 a}$$

$$-m\frac{\partial s}{\partial t} = \frac{\partial}{\partial x}K^N(s)\left(\frac{\partial p^N}{\partial x} - f^N\right), \tag{1.1 b}$$

where the indices W and N relate to wetting and nonwetting fluids, $p^W = p^W(x,t)$ and $p^N = p^N(x,t)$ are the corresponding pressures, $s = s(x,t)$ stands for the saturation of the porous medium with respect to the wetting liquid, m is the porosity, and $K^W(s)$ and $K^N(s)$ denote the permeabilities of the phases. The functions f^W and f^N are given external forces.

In current models of unsaturated flow (see Coillins [11]), equations (1.1 *a*) and (1.1 *b*) are usually supplemented by a capillary relation that takes the difference in pressures, $p := p^N - p^W$, to be a single-valued function of saturation at every point of the medium. In reality, the observed relation between p and s is much more complicated. It demonstrates memory effects and, in particular, hysteresis. To account for these properties, the following model of capillary relation has been developed by Beliaev & Hassanizadeh [5]:

$$\frac{\partial}{\partial t}s = \Psi\left(s, p^N - p^W\right) \tag{1.2}$$

with a function $\Psi(s,p)$ on $[s_-, s_+] \times \mathbf{R}$ which monotonically decreases with respect to

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$p \in (-\infty, +\infty)$ and equals zero in the zone $P^c_{im}(s) \leqslant p \leqslant P^c_{dr}(s)$, where $P^c_{dr}(\cdot)$ and $P^c_{im}(\cdot)$ are two different functions of saturation, which are called the capillary pressures for the processes of drainage and imbibition, respectively (see Fig. 1).

Equation (1.2) implies that all equilibrium states in the s–p plane occupy the zone between the curves $p = P^c_{im}(s)$ and $p = P^c_{dr}(s)$. Processes with saturation increasing in time (imbibition) can be represented in this plane by various curves below the graph of $P^c_{im}(s)$, and processes with $\dot{s} < 0$ (drainage) are all above the curve $p = P^c_{dr}(s)$ (see Fig. 2). Passages from drainage to imbibition (and *vice versa*) are described by so-called 'scanning curves' in the same plane. In the model (1.2) of the capillary relation, all these intermediate processes occur with constant saturation and can be represented by vertical straight lines. This kind of relation between s and p is known as 'play-type' hysteresis (see Visintin [17]). More precisely, if $s = S^{(L)}_t(s_0, \{p(\cdot)\})$ stands for the solution of the equation

$$L\dot{s} = \Psi\,(s,\, p(t))$$

with initial saturation s_0 and given $p = p(t)$, then the limit of $S^{(L)}_t$ as $L \to 0+$ is a memory-dependent operator $(s_0,\, p(t)) \mapsto s(t)$ which is called play-type hysteresis. In this limit, all the continuous curves $(s(t), p(t))$ in the s–p plane occupy the zone $\{(s, p) : \Psi(s, p) = 0\}$, and trajectories with $\dot{s} > 0$ (resp., $\dot{s} < 0$) coincide with the curve $p = P^c_{im}(s)$ (resp., $p = P^c_{dr}(s)$). The operator $S^{(L)}_t$ with $L > 0$ corresponds to play-type hysteresis coupled with dynamic memory effects. The porous medium equation with non-hysteretic but memory-dependent capillary relation has been considered recently by Cuesta *et al.* [4] in view of properties of particular solutions. Dynamic effects have been described there by equation (1.2) with a linear function of p in place of Ψ.

Since the limit $L \to 0+$ is not considered in this paper, we have posed $L = 1$ in (1.2) without loss of generality. It should be noted that the value of L is usually negligible in applications, but passing to the limit $L \to 0$ with rigorous justification is not a trivial task. Moreover, the limit problem is much more difficult from the mathematical point of view than the 'regularized' one. On the other hand, as shown by Beliaev & Schotting [6], a small non-zero value of L is desirable for numerical studies of the equations; hence it is reasonable to keep this damping coefficient in the capillary relation.

If the porous medium is inhomogeneous, then the porosity, the permeabilities and the capillary function Ψ depend explicitly upon spatial variable x. Homogenization theory deals with the case when the spatial scale of inhomogeneities in the medium, ε, is much smaller than the size of the porous domain. The small-scale structure is usually represented by periodic or stochastic dependence of the coefficients in equations on the 'fast' variable $\xi = x/\varepsilon$. Then the solutions of equations (1.1 *a*), (1.1 *b*) and (1.2) with appropriate initial data and boundary conditions depend on the scaling parameter ε. The aim of homogenization is to find the leading term of the asymptotics for the solutions as $\varepsilon \to 0$.

The main result of the paper is a rigorous justification of the following homogenized equations for the leading terms of the pressures, p^N_* and p^W_*, and saturation s^*:

$$m^* \frac{\partial}{\partial t} s^* = \frac{\partial}{\partial x} K^W_* \left(\frac{\partial}{\partial x} p^W_* - f^W \right), \tag{1.3 a}$$

$$-m^* \frac{\partial}{\partial t} s^* = \frac{\partial}{\partial x} K^N_* \left(\frac{\partial}{\partial x} p^N_* - f^N \right), \tag{1.3 b}$$

where the homogenized porosity m^* is a constant or, in macroscopically inhomogeneous media, a fixed function of the spatial variables, but s^*, K_*^N and K_*^W are memory-dependent on $p_* = p_*^N - p_*^W$.

The homogenization problem for two-phase flows in porous media with traditional capillary relation had been considered, from the mathematical point of view, by Mikelić [15] in periodic case and by Bourgeat *et al.* [7] for random structures. The resulting homogenized problem turns out to be of the same kind as the original one. It includes (1.3 *a*) and (1.3 *b*) with some effective single-valued capillary relation. In the case of highly contrasting local properties of the porous medium, different homogenized models are available. In particular, a memory-dependent capillary relation in homogenized system has been obtained as a result of homogenization for a specially chosen range of porous medium parameters and ε (see, for instance, papers by Bourgeat *et al.* [8] or by Bourgeat & Panfilov [10]).

The homogenization problem for two-phase porous flows, either with capillary hysteresis or without it, is complicated by the fact that the range of saturation is bounded from above and below, and the coefficients of the equations become degenerate as the saturation value approaches each of its bounds. This results in difficulties with well-posedness of the problem for fixed ε; hence the taking of the scaling parameter to zero is a formal procedure and does not, in general, have a rigorous justification. The existence of weak solutions for two-phase flows with single-valued capillary relation has been established (see, for instance, Antontsev *et al.* [3], Kröner & Luckhaus [13] and Alt & Di Benedetto [2]), but uniqueness is not proved yet, except in some particular cases. If hysteresis is included, the situation with well-posedness is no better.

To avoid those troubles with the well-posedness of the problem under consideration, we impose a restriction on the initial saturation and assume that it lies in some interval $[s_-, s_+]$ which is uniformly separated from the unphysical values. In this interval, it may be assumed that the coefficients are not degenerate. This provides a possibility of proving local-in-time existence and uniqueness of the solution until the instance when saturation attains a bounding value somewhere in the porous domain. As a result, the homogenization procedure is justified for some interval of time that does not depend upon ε but may depend upon the initial data and parameters of the medium.

The structure of the paper is the following. First, we prove well-posedness of the problem under consideration for a fixed value of ε (§ 2). Then, in § 3, we prove a homogenization theorem for periodic inhomogeneous media. This is done by means of an approach based on the notion of two-scale convergence which has been developed by Nguetseng [16] and Allaire [1] for media with a periodic structure. For the sake of simplicity we deal with one-dimensional case, and only briefly discuss generalization of the results to multi-dimensional problems. After this, in § 4, the same homogenized equations are introduced in a particular case of two-component porous medium, to obtain simple explicit expressions and to get a feel for the properties of the model. Finally, in § 5, the homogenization is done for a randomly inhomogeneous porous medium. In contrast with the periodic case, where space dimension is not too important, the generalization of the 'stochastic' proof for multi-dimensional spaces is not straightforward, and we present a proof of homogenization theorem in one dimension only.

2 Results on the solvability of the problem

For the sake of definiteness, we consider (1.1 *a*), (1.1 *b*) and (1.2) on the interval $]-l, +l[\subset \mathbf{R}$ and impose Dirichlet boundary conditions at $x = \pm l$ for both pressures:

$$p^N\big|_{x=\pm l} = q^N_\pm(t), \qquad p^W\big|_{x=\pm l} = q^W_\pm(t). \tag{2.1}$$

We also assume that the saturation is given at initial instant:

$$s(x, 0) = s_0(x). \tag{2.2}$$

Before presenting any results on the asymptotic behaviour of solutions, we should be convinced that the solution of this problem exists for any fixed value of the scaling parameter ε. Below we introduce assumptions on the functions and parameters from this problem which are sufficient for its solvability. The assumptions read

(i) $m(\cdot)$ is measurable on $[-l, +l]$, and $0 \leqslant m(\cdot) \leqslant \delta$ for some $\delta > 0$ everywhere on $[-l, +l]$;

(ii) the functions $K^N(\cdot, s)$ and $K^W(\cdot, s)$ are measurable on $[-l, +l]$ for any s, Lipschitz continuous with respect to s on some fixed interval $s_- \leqslant s \leqslant s_+$ for any x with uniformly bounded Lipschitz constant and, for some $\delta > 0$, $1/\delta \geqslant K^{N(W)} \geqslant \delta$ on $[-l, +l] \times [s_-, s_+]$;

(iii) the functions $\Psi(\cdot, s, p)$ are measurable on $[-l, +l]$ for any s and p, bounded and Lipschitz continuous with respect to s and p on $[s_-, s_+] \times [-M, +M]$ for any $M > 0$ uniformly over range of x, and monotonically decreasing with respect to p for all s and x;

(iv) $s_0(\cdot)$ is measurable, and $s_0(x) \in [s_- + \delta, s_+ - \delta]$ for some $\delta > 0$ almost everywhere on $[-l, +l]$;

(v) the external forces f^N and f^W are continuous functions of $t \geqslant 0$ with values in $L_2([-l, +l])$; boundary data $q^N_\pm$ and $q^W_\pm$ are also continuous with respect to $t \geqslant 0$.

We outline the formal way of solving this problem. Equations (1.1 *a*)–(1.1 *b*) can be re-written as follows:

$$m\Psi = \frac{\partial}{\partial x} K^W \left(\frac{\partial p^W}{\partial x} - f^W \right), \tag{2.3 a}$$

$$-m\Psi = \frac{\partial}{\partial x} K^N \left(\frac{\partial p^N}{\partial x} - f^N \right), \tag{2.3 b}$$

where the time derivatives of the saturation have been replaced by the capillary function Ψ due to equality (1.2). For any given field of saturation $s(\cdot, t)$, these two equations with Dirichlet boundary condition (2.1) provide an elliptic problem for the two unknown pressures.

The solution of this elliptic problem, $p^{N(W)} = P^{N(W)}(x, \{s\})$, depends non-locally on the saturation profile. Then the pressures are eliminated from Equation (1.2) which can be written as an ordinary differential equation with respect to the field of saturation in some Banach space. It reads

$$\frac{\partial s}{\partial t} = A(\{s\}), \tag{2.4}$$

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where the spatially non-local operator $s \mapsto A(\{s\})$ is formally defined by the expression

$$A(\{s\})(x) := \Psi\left(s(x),\, P^N(x, \{s\}) - P^W(x, \{s\})\right). \tag{2.5}$$

Finally, (2.4) with the initial condition (2.2) determines the evolution of the saturation in time.

Assumptions (i)–(iii) provide well-posedness of the elliptic problem (2.3 *a*)–(2.3 *b*) with boundary conditions (2.1) for any given measurable function $s(\cdot)$ with values in the interval $[s_-, s_+]$. The weak formulation of the problem reads: find $p^{N,W} \in W^{1,2}([-l, +l])$ obeying (2.1) and satisfying the following equality for any two test functions φ^N, $\varphi^W \in C_0^\infty([-l, +l])$:

$$\int_{-l}^{+l} \left\{ m(\cdot)\Psi(\cdot,\cdot, p^N - p^W)\left(\varphi^W - \varphi^N\right) + K^W(\cdot,\cdot)\left(\frac{\partial p^W}{\partial x} - f^W\right)\frac{\partial \varphi^W}{\partial x} \right.$$
$$\left. + K^N(\cdot,\cdot)\left(\frac{\partial p^N}{\partial x} - f^N\right)\frac{\partial \varphi^N}{\partial x} \right\} dx = 0. \tag{2.6}$$

Note that $K^{N(W)}(x, s)$ and $\Psi(x, s, p)$ are Carathéodory functions and, consequently, $K^{N(W)}(x, s(x))$ and $\Psi(x, s(x), p(x))$ are measurable in x. This makes all terms of the identity (2.6) correctly defined.

Properties of solution of this problem are outlined in the following.

Proposition 2.1 (A priori estimates). *Let conditions* (i)–(iii) *be satisfied, and the saturation* $s = s(x)$ *in the elliptic problem* (2.3 *a*)–(2.3 *b*) *with boundary conditions* (2.1) *be an arbitrary measurable function, such that* $s_- \leqslant s(\cdot) \leqslant s_+$ *a.e. Then solution* $p^{N(W)} = P^{N(W)}(x, \{s\})$ *of this problem satisfies the estimate*

$$\left\|\frac{\partial P^N(x,\{s\})}{\partial x}\right\| + \left\|\frac{\partial P^W(x,\{s\})}{\partial x}\right\| + \max_x |P^N(x,\{s\})|$$
$$+ \max_x |P^W(x,\{s\})| \leqslant M, \tag{2.7}$$

where $\|\cdot\|$ *stands for the norm in* $L_2([-l, +l])$, *and the positive constant* M *does not depend upon* s. *It is also independent of the boundary data and external forces within the set* $\|f^N\| + \|f^W\| + |q_-^N| + |q_-^W| + |q_+^N| + |q_+^W| \leqslant M_1$ *for a fixed* $M_1 > 0$.

Furthermore, if $p_1^{N,W} = P^{N,W}(\cdot, \{s_1\})$ *and* $p_2^{N,W} = P^{N,W}(\cdot, \{s_2\})$ *are solutions of the elliptic problem under consideration with input data* $(s_1, f_1^N, f_1^W, q_{1\pm}^N, q_{1\pm}^W)$ *and* $(s_2, f_2^N, f_2^W, q_{2\pm}^N, q_{2\pm}^W)$, *respectively, then*

$$\left\|\frac{\partial(p_1^N - p_2^N)}{\partial x}\right\| + \left\|\frac{\partial(p_1^W - p_2^W)}{\partial x}\right\| + \max_x |p_1^N - p_2^N| + \max_x |p_1^W - p_2^W|$$
$$\leqslant C\Big(|q_{1+}^N - q_{2+}^N| + |q_{1-}^N - q_{2-}^N| + |q_{1+}^W - q_{2+}^W| + |q_{1-}^W - q_{2-}^W|$$
$$+ \|f_1^N - f_2^N\| + \|f_1^W - f_2^W\| + \operatorname*{ess\,sup}_x |s_1 - s_2|\Big), \tag{2.8}$$

where the constant C *is also independent of* s, *the boundary data and the external forces within the above set.*

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The proof is given in the Appendix. As a consequence of this proposition, we get

Theorem 2.2 *Under assumptions* (i)–(v)*, there exists a unique local-in-time solution* $\left(p^N, p^W, s\right)$ *satisfying* (2.1)–(2.4) *on some interval of time* $[0, T]$*, and this solution possesses the following properties:*

(1) $s \in C^1(0, T; L_\infty([-l, +l]))$;

(2) $p^N, p^W \in C(0, T; W^{1,2}([-l, +l])) \subset C([-l, +l] \times [0, T])$.

Proof of Theorem 2.2 The operator $s \mapsto A(\{s\})$ defined by (2.5) in $L_\infty([-l, +l])$ is Lipschitz continuous with respect to s and continuous with respect to time t which enters in A via boundary data and external forces. The continuity conditions (v) provide the estimate $\|f^N\| + \|f^W\| + |q_+^N| + |q_+^W| + |q_-^N| + |q_-^W| \leqslant M_1$ for a fixed M_1 and for all t in some interval $[0, T_1]$. Due to the first part of Proposition 2.1, the drop in pressures, $P(\cdot, \{s\}) = P^N(\cdot, \{s\}) - P^W(\cdot, \{s\})$, has to be also bounded by some M; then properties (iii) of the capillary function Ψ provide that the Lipschitz constant of $A(\{s\})$ can be chosen independently of $s(\cdot)$ and $t \in [0, T_1]$. Then, starting with initial saturation $s_0(x)$ that satisfies condition (iv) and looking for a solution of the ordinary differential equation (2.4) in $L_\infty([-l, +l])$, we refer to standard results on existence and uniqueness of local-in-time solution, which exists at least until $s(x, t) \in [s_-, s_+]$ for all x. The existence time T can be estimated from below by $\min\{T_1, 1/C_1, \delta/C_2\}$ where C_1 is an upper bound for the Lipschitz constant of A with respect to s, δ is the minimal distance from s_0 to the boundaries of the interval $[s_-, s_+]$ and C_2 is the upper bound of $|\dot{s}|$ which is not greater than upper bound of $|\Psi(\cdot, s, p)|$ over all $s \in [s_-, s_+]$ and $p \in [-M, +M]$. Obviously, $s(\cdot, \cdot) \in C^1(0, T; L_\infty([-l, +l]))$ and $p^{N(W)} \in C(0, T; W^{1,2}([-l, +l]))$. □

Remark 2.3 If a family of functions $K^{N(M)}$, Ψ and s_0 is considered, and assumptions (i)–(iv) are satisfied for representatives of this family uniformly, then the constants M and C from Proposition 2.1 and time T from Theorem 2.2 can be chosen independently of the representative. As a result, the norms of solutions s, p^N and p^W in corresponding spaces are uniformly bounded.

Remark 2.4 Theorem 2.2 and Proposition 2.1 hold if the problem (2.1)–(2.4) and conditions (i)–(iv) are generalized so that the function $s(x, t)$ takes its values in a bounded subset of Banach space instead of interval $[s_-, s_+]$. The only problem is measurability of $K^{N(W)}(x, s(x))$ and $\Psi(x, s(x), p(x))$ in x that is needed for the correct weak formulation of the elliptic problem with respect to the pressures. If, for instance, the above Banach space is separable or the function $x \mapsto s(x)$ is continuous, then measurability holds.

Remark 2.5 If Neumann boundary conditions are employed instead of (2.1), then the pressure fields p^N and p^W do not possess the Lipschitz property with respect to the profile of saturation and, moreover, they may be non-unique because of the nonstrict monotonicity of the capillary function. Nevertheless, the operator $s \mapsto A(\{s\})$ is Lipschitz continuous, although the proof of this fact is not straightforward. Thus, Theorem 2.2

can be generalized to the case of a Neumann boundary value problem. For a particular example of the capillary function Ψ, this has been done by Beliaev & Schotting [6].

Remark 2.6 The multi-dimensional generalization of Proposition 2.1 and Theorem 2.2 is trivial. The only trouble is the proof of continuity of the pressures in the spatial variable x because $W^{1,2}$ is not embedded in the space of continuous functions C if the dimension is greater than 1. Then, in order to prove continuity of the pressures, one may use regularity results for elliptic partial differential operators (see the book by Ladyzhenskaya & Ural'tseva [14, Theorem 13.1, p. 199]).

3 Homogenization of periodic structures

Small-scale periodicity of a porous medium implies that its constitutive functions m, K^N, K^W and Ψ depend periodically on the variable $\xi = x/\varepsilon$. Let $\Pi = [0, 1]$ be the periodicity cell with respect to ξ. Of course, these functions could have different periods, but this quasi-periodic situation is, in some sense, included in more general theory of stochastic homogenization which will be considered later. Concerning the initial saturation, we assume that it depends upon both variables, x and ξ, and the dependence on ξ is Π-periodic. In the case of macroscopically inhomogeneous media, an explicit dependence of m, K^N, K^W and Ψ on x, in addition to ξ, would also be possible, but we ignore it. Generalization of all further results for this case is straightforward.

Thus, the problem under consideration consists of (1.1 *a*), (1.1 *b*), (1.2), boundary conditions (2.1) and initial condition (2.2), where $m = m(x/\varepsilon)$, $K^{N(W)} = K^{N(W)}(x/\varepsilon, s)$, $\Psi = \Psi(x/\varepsilon, s, p^N - p^W)$ and $s_0 = s_0(x, x/\varepsilon)$ are Π-periodic with respect to x/ε. We assume that representatives of this family satisfy conditions (i)–(iv) for any $\varepsilon > 0$. Then Theorem 2.2 provides existence and uniqueness of solution p^N_ε, p^W_ε, s^ε of the problem on some interval of time independent of ε.

In addition to (i)–(v), we require that the initial saturation $s_0(\cdot,\cdot) \in C([-l,+l]; L_\infty(\Pi))$. Our reason for this restriction is the need for weak convergence of functions $x \mapsto \phi(x, x/\varepsilon)$ to the mean value of $\phi(x,\cdot)$ over the periodicity cell Π. As shown by Allaire [1] (Lemmas 5.5 and 5.6, p.1514), functions $\phi = \phi(x,\xi)$ from the Banach space $C([-l,+l]; L_\infty(\Pi))$ satisfy this property.

As soon as the solution is determined for all $\varepsilon > 0$, we are able to investigate its asymptotic behaviour for $\varepsilon \to 0$. In this respect, the following results hold:

Theorem 3.1 *Under the given set of assumptions, there exist functions* p^N_*, p^W_* : $[-l,+l] \times [0,T] \to \mathbf{R}$ *and* σ: $[-l,+l] \times [0,T] \times \Pi \to \mathbf{R}$ *such that*

(1) $s^\varepsilon(x,t) - \sigma(x,t,x/\varepsilon)$ *converges to* 0 *in* $C^1(0, T; L_\infty([-l, +l]))$ *strongly;*

(2) p^N_ε *and* p^W_ε *converge to* p^N_* *and* p^W_*, *respectively, in* $C([-l, +l] \times [0, T])$ *strongly;*

(3) σ *satisfies the equation* (3.1) *below with the initial condition* $\sigma(x, 0, \xi) = s_0(x, \xi)$*;*

(4) p^N_* *and* p^W_* *satisfy the homogenized equations* (3.2 *a*)–(3.2 *b*) *with the same Dirichlet boundary conditions* (2.1),

where the system of homogenized equations includes an ordinary differential equation for the local saturation σ

$$\frac{\partial}{\partial t}\sigma(x,t,\cdot) = \Psi\left(\cdot,\, \sigma,\, p_*(x,t)\right), \qquad p_* := p_*^N - p_*^W, \tag{3.1}$$

and two elliptic equations for the limiting pressures coupled by lower-order terms:

$$m^*\Psi^*\left(\{\sigma(x,t,\cdot)\}, p_*(x,t)\right) = \frac{\partial}{\partial x}K_*^W\left(\{\sigma(x,t,\cdot)\}\right)\left(\frac{\partial p_*^W}{\partial x} - f^W\right), \tag{3.2 a}$$

$$-m^*\Psi^*\left(\{\sigma(x,t,\cdot)\}, p_*(x,t)\right) = \frac{\partial}{\partial x}K_*^N\left(\{\sigma(x,t,\cdot)\}\right)\left(\frac{\partial p_*^N}{\partial x} - f^N\right), \tag{3.2 b}$$

where

$$m^* = \langle m(\cdot)\rangle := \int_{\Pi} m(\xi)d\xi, \tag{3.3}$$

$$\Psi^*\left(\{\sigma(x,t,\cdot)\}, p_*\right) = \langle m(\cdot)\rangle^{-1}\left\langle m(\cdot)\Psi\left(\cdot, \sigma(x,t,\cdot), p_*\right)\right\rangle, \tag{3.4}$$

and explicit formulas for K_*^N and K_*^W available in one dimension are

$$K_*^{N(W)}\left(\{\sigma(x,t,\cdot)\}\right) = \left\langle \left(K^{N(W)}(\cdot,\, \sigma(x,\, t,\, \cdot))\right)^{-1}\right\rangle^{-1}. \tag{3.5}$$

Equations (3.2 *a*), (3.2 *b*) and (3.1) are of the same kind as (2.3 *a*), (2.3 *b*) and (2.4), apart the fact that the local saturation $\sigma(x,t,\cdot)$ is a continuous function with values in the Banach space $L_\infty(\Pi)$, whereas the saturation $s(x,t)$ takes values in $\mathbf{R}$. Accounting for Remark 2.4, one may justify conditions (i)–(v) for the homogenized problem, and conclude that it is also solvable and has a unique solution on the same interval of time $[0,\, T]$.

One may introduce a function

$$s^*(\{\sigma(x,t,\cdot)\}) = \langle m(\cdot)\rangle^{-1}\langle m(\cdot)\sigma(x,\, t,\, \cdot)\rangle \tag{3.6}$$

for the averaged saturation and transform equations (3.2 *a*)–(3.2 *b*) to the traditional form (1.3 *a*)–(1.3 *b*). Since the solution σ of (3.1) depends nonlocally upon $p_* := p_*^N - p_*^W$, then formulae (3.6) and (3.5) determine $K_*^{N(W)}$ and s^* as memory-dependent functions of p_*.

Proof of Theorem 3.1 Since the saturation s^ε is a differentiable function of t with values in $L_\infty([-l,+l])$, and its derivative is bounded uniformly with respect to ε, then it represents a sequence of equipotentially continuous functions on $[0,\, T]$. Consequently, using Proposition 2.1, we conclude that $p_\varepsilon^N(x,\, t)$ and $p_\varepsilon^W(x,\, t)$ are also uniformly bounded and equipotentially continuous functions of t with values in the Sobolev space $W^{1,2}([-l,+l])$. They are also equipotentially continuous in x because the embedding $C([-l,+l]) \subset W^{1,2}([-l,+l])$ is compact. By the Arzela theorem, any uniformly bounded and equicontinuous sequence is compact, and we are able to extract a subsequence of p_ε^N, p_ε^W which converges to some p_*^N, $p_*^W \in C([-l,+l] \times [0,\, T])$. These two limit functions obviously satisfy the same boundary conditions (2.1) as the pressures for $\varepsilon > 0$. For any t, the subsequences p_*^N and p_*^W are also weakly compact in $W^{1,2}([-l,\, +l])$. Therefore, p_*^N, $p_*^W \in W^{1,2}([-l,\, +l])$ for any $t \in [0,\, T]$. Later on, we shall prove that they are independent of the subsequence, but now this is not assumed.

Fixing the above subsequence, we define a function $\sigma(x,t,\xi)$ as the solution of the

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ordinary differential equation (3.1) with the initial condition $\sigma(x,0,\xi) = s_0(x,\xi)$. The function $s^\varepsilon(x,t)$ satisfies almost the same equation as (3.1) with x/ε in place of $(\cdot)$ and $p_\varepsilon^N(x,t) - p_\varepsilon^W(x,t)$ substituted instead of $p_*(x,t)$. Accounting for the Lipschitz properties of $\Psi(\xi,\cdot,\cdot)$, the proof of item (1) of the theorem is straightforward. It is also easy to prove that, if $s_0 \in C([-l,+l]; L_\infty(\Pi))$, then so is $\sigma(\cdot,t,\cdot)$ for any $t \geqslant 0$.

We have to prove that the limit pressures satisfy equations (3.2 *a*) and (3.2 *b*), and this will complete the proof because the solution of the homogenized problem is unique and, therefore, the family of pressures $p_\varepsilon^{N(W)}$ has a unique limit point as $\varepsilon \to 0$. In doing so, we take test functions for the integral identity (2.6) in the form $\varphi_\varepsilon^{N(W)}(x) = \varphi^{N(W)}(x) + \varepsilon\phi^{N(W)}(x,x/\varepsilon)$ where $\varphi^{N(W)}(x)$ and $\phi^{N(W)}(x,\xi)$ are smooth functions vanishing at $x = \pm l$, and $\phi^{N(W)}(x,\xi)$ are Π-periodic in ξ. Then we take the limit $\varepsilon \to 0$.

For the first term, we obtain

$$\lim_{\varepsilon\to 0}\int_{-l}^{+l} m\left(\frac{x}{\varepsilon}\right)\Psi\left(\frac{x}{\varepsilon}, s^\varepsilon(x,t), p_\varepsilon^N(x,t) - p_\varepsilon^W(x,t)\right)\left(\varphi_\varepsilon^W(x) - \varphi_\varepsilon^N(x)\right)dx$$
$$= \lim_{\varepsilon\to 0}\int_{-l}^{+l} m\left(\frac{x}{\varepsilon}\right)\Psi\left(\frac{x}{\varepsilon}, \sigma\left(x,t,\frac{x}{\varepsilon}\right), p_*(x,t)\right)\left(\varphi^W(x) - \varphi^N(x)\right)dx$$
$$= \int_{-l}^{+l} m^*\Psi^*\left(\{\sigma(x,t,\cdot)\}, p_*(x,t)\right)\left(\varphi^W(x) - \varphi^N(x)\right)dx,$$

where the first equality holds due to strong convergence of $\varphi_\varepsilon^{N(W)}$, $p_\varepsilon^{N(W)}$ and $s^\varepsilon - \sigma$ to $\varphi^{N(W)}$, $p_*^{N(W)}$ and zero, respectively, and the last equality is valid due to above remark on weak convergence for oscillating functions from $C([-l,+l]; L_\infty(\Pi))$.

The next two terms are quite similar, and one of them reads

$$\lim_{\varepsilon\to 0}\int_{-l}^{+l} K^W\left(\frac{x}{\varepsilon}, s^\varepsilon(x,t)\right)\left(\frac{\partial p_\varepsilon^W(x,t)}{\partial x} - f^W\right)\frac{\partial \varphi_\varepsilon^W(x)}{\partial x}dx$$
$$= \lim_{\varepsilon\to 0}\int_{-l}^{+l} K^W\left(\frac{x}{\varepsilon}, \sigma\left(x,t,\frac{x}{\varepsilon}\right)\right)\left(\frac{\partial p_\varepsilon^W(x,t)}{\partial x} - f^W\right)\left(\frac{\partial \varphi^W(x)}{\partial x} + \frac{\partial \phi^W(x,\frac{x}{\varepsilon})}{\partial \xi}\right)dx. \quad (3.7)$$

Here we have come to the central point of the proof. To investigate the asymptotic behavior of the last integral, we are going to make use of an approach based on the notion of two-scale convergence. It was introduced by Nguetseng [16] and developed by Allaire [1] in the framework of homogenization theory of periodic structures. For the reader's convenience, we recall the following:

Definition 3.2 (Allaire [1, p. 1485]). *A bounded sequence $u_\varepsilon \in L_2([-l,+l])$ is called two-scale convergent to $u(\cdot,\cdot) \in L_2([-l,+l] \times \Pi)$ if*

$$\lim_{\varepsilon\to 0}\int_{-l}^{+l} u_\varepsilon(x)\phi(x,x/\varepsilon)dx = \int_\Pi\int_{-l}^{+l} u(x,\xi)\phi(x,\xi)dxd\xi \quad (3.8)$$

for any smooth function $\phi(x,\xi)$ which is Π-periodic in ξ.

It should be mentioned that test functions in Definition 3.2 may possess a less regularity. In particular, if equality (3.8) holds for smooth test functions, then it is also valid on $C([-l,+l];L_\infty(\Pi))$ (see Theorem 1.8 by Allaire [1, p. 1488]).

The main result of two-scale approach to homogenization is the compactness property of bounded sequences in L_2 with respect to two-scale convergence. More detailed information, related to proof of homogenization theorem 3.1, is given in the following.

Proposition 3.3 (Allaire [1, p. 1491]). *If u_ε and $\partial u_\varepsilon/\partial x$ are bounded sequences in $L_2([-l,+l])$, then there exist $u\in W^{1,2}([-l,+l])$, $v\in L_2([-l,+l];W^{1,2}_{per}(\Pi))$ and a subsequence $\varepsilon\to 0$ such that u_ε two-scale converges to u and $\partial u_\varepsilon/\partial x$ two-scale converges to $\partial u(x)/\partial x+\partial v(x,\xi)/\partial\xi$ along this subsequence.*

We apply the notion of two-scale convergence and Proposition 3.3 to determine the limit in (3.7). Since the sequence p^W_ε is bounded in $W^{1,2}([-l,+l])$ for any fixed t, then one may extract a two-scale convergent subsequence for it and its gradient. The two-scale limit of p^W_ε is, of course, p^W_*. Let $Q^W\in L_2([-l,+l];W^{1,2}_{per}(\Pi))$ be a function such that the gradients of p^W_ε two-scale converge to $\partial p^W_*/\partial x+\partial Q^W/\partial\xi$. Therefore, the limit in (3.7) equals

$$\int_{-l}^{+l}\int_{\Pi} K^W(\xi,\sigma(x,t,\xi))\left(\frac{\partial p^W_*(x,t)}{\partial x}-f^W+\frac{\partial Q^W}{\partial\xi}\right)\left(\frac{\partial\varphi^W(x)}{\partial x}+\frac{\partial\phi^W(x,\xi)}{\partial\xi}\right)dxd\xi$$

and analogous expression arises from the term for non-wetting phase.

Collecting the above preliminary calculations, we obtain the following integral identity which is satisfied with all admissible test functions $\varphi^{N(W)}(x)$ and $\phi^{N(W)}(x,\xi)$:

$$\begin{aligned}&\int_{-l}^{+l} m^*\Psi^*\left(\{\sigma(x,t,\cdot)\},p^N_*(x,t)-p^W_*(x,t)\right)\left(\varphi^W(x)-\varphi^N(x)\right)dx\\&\quad+\int_{-l}^{+l}\int_{\Pi}\left\{K^W(\xi,\sigma(x,t,\xi))\left(\frac{\partial p^W_*(x,t)}{\partial x}-f^W+\frac{\partial Q^W}{\partial\xi}\right)\left(\frac{\partial\varphi^W(x)}{\partial x}+\frac{\partial\phi^W(x,\xi)}{\partial\xi}\right)\right.\\&\quad\left.+K^N(\xi,\sigma(x,t,\xi))\left(\frac{\partial p^N_*(x,t)}{\partial x}-f^N+\frac{\partial Q^N}{\partial\xi}\right)\left(\frac{\partial\varphi^N(x)}{\partial x}+\frac{\partial\phi^N(x,\xi)}{\partial\xi}\right)\right\}dxd\xi=0.\end{aligned}\quad(3.9)$$

Setting here $\varphi^N=\varphi^W=0$ and taking into account the arbitrariness of test functions ϕ^N and ϕ^W, we obtain

$$\frac{\partial}{\partial\xi}K^{N(W)}(\xi,\sigma(x,t,\xi))\left(\frac{\partial p^{N(W)}_*(x,t)}{\partial x}-f^{N(W)}+\frac{\partial Q^{N(W)}}{\partial\xi}\right)=0.\quad(3.10)$$

For any function $\sigma(x,t,\xi)$ and for any fixed x and t, these equations provide two independent elliptic problems in $W^{1,2}_{per}(\Pi)$ for the auxiliary functions Q^N and Q^W. In one dimension, these problems can be solved explicitly, and the following formula holds:

$$\int_{\Pi} K^{N(W)}(\xi, \sigma(x,t,\xi)) \left(\frac{\partial p_*^{N(W)}(x,t)}{\partial x} - f^{N(W)} + \frac{\partial Q^{N(W)}}{\partial \xi} \right) d\xi$$
$$= K_*^{N(W)}\left(\{\sigma(x,t,\cdot)\}\right) \left(\frac{\partial p_*^{N(W)}(x,t)}{\partial x} - f^{N(W)} \right), \qquad (3.11)$$

where $K_*^{N(W)}$ is given by equality (3.5). In the multi-dimensional case, an explicit solution of the auxiliary problem (3.10) is not available in general, and an equality of the same kind as (3.11) works as a definition of effective permeability tensors.

With formula (3.11), setting $\phi^{N(W)} = 0$ in identity (3.9) and coming back to arbitrary $\varphi^{N(W)}$, we get the equality

$$\int_{-l}^{+l} m^* \Psi^* \left(\{\sigma(x,t,\cdot)\}, p_*^N(x,t) - p_*^W(x,t)\right) \left(\varphi^W(x) - \varphi^N(x)\right) dx$$
$$+ \int_{-l}^{+l} \left\{ K_*^W\left(\{\sigma(x,t,\cdot)\}\right) \left(\frac{\partial p_*^W(x,t)}{\partial x} - f^W \right) \frac{\partial \varphi^W(x)}{\partial x} \right.$$
$$\left. + K_*^N\left(\{\sigma(x,t,\cdot)\}\right) \left(\frac{\partial p_*^N(x,t)}{\partial x} - f^N \right) \frac{\partial \varphi^N(x)}{\partial x} \right\} dx = 0. \qquad (3.12)$$

This identity is the integral form of the homogenized elliptic problem (3.2 *a*)–(3.2 *b*) for the limiting pressures. □

4 The particular case of a two-component medium

In the homogenized model of flow in porous medium, the relation between saturation and drop in pressures includes hysteresis and dynamic memory. This relation is presented in the form of a memory-dependent operator $(s_0, p_*) \mapsto \sigma$ which is defined by means of the ordinary differential equation (3.1). The total saturation s^* and effective permeabilities $K_*^{N(W)}$ are functions of the local saturation field σ. Their dependence on p_* captures a partial integral information about the properties of the homogenized system, and one has to operate with σ for a full description of the system.

Graphically, the homogenized hysteresis in each point x should be represented by capillary curves in an infinite-dimensional p_*–σ 'plane' where the last variable takes values in some Banach space of periodic functions. The equilibrium states of the system occupy the zone $\{(p_*, \sigma(\cdot,\xi)) : \Psi(\xi, \sigma(\cdot,\xi), p_*) = 0 \ \forall \xi \in \Pi\}$, and capillary curves for equilibrium processes are all inside this set. External points are accessible to fast processes when the dynamic effect may not be neglected.

Hysteresis in an infinite dimensional Banach space is not a suitable model in view of applications, numerical studies and experimental validation of parameters. That is why we are going to consider a simplified model which corresponds to the case of a two-component porous medium and allows us to reduce the general homogenized capillary equation to a relation for a finite number of real-valued variables.

To introduce the two-component structure of the medium, let I and $II = \mathbf{R} \backslash I$ be

Π-periodic subsets of the space, and let us take constitutive functions $m(\xi)$, $K^{N(W)}(\xi,\cdot)$ and $\Psi(\xi,\cdot,\cdot)$ in the following form:

$$m(\xi) := m_I \mathbf{1}_I(\xi) + m_{II}\mathbf{1}_{II}(\xi),$$

$$K^{N(W)}(\xi,\, s) = K_I^{N(W)}(s)\mathbf{1}_I(\xi) + K_{II}^{N(W)}(s)\mathbf{1}_{II}(\xi),$$

$$\Psi(\xi,\, s,\, p) = \Psi_I(s,\, p)\mathbf{1}_I(\xi) + \Psi_{II}(s,\, p)\mathbf{1}_{II}(\xi),$$

where $\mathbf{1}_I$ and $\mathbf{1}_{II}$ are indicators of I and II. The domains I and II represent two homogeneous porous materials with different porosities, phase permeabilities, etc., and the porous medium is a periodic mixture of these two. One may easily generalize this simplified example to multi-component porous media with any finite number of components. Looking forward, we should also mention that periodicity of the medium is not essential, and the same simplification works in the case of random structures.

In addition, we consider the initial function $s_0(x,\,\xi)$ to be of the following particular type:

$$s_0(x,\,\xi) = s_0^I(x)\mathbf{1}_I(\xi) + s_0^{II}(x)\mathbf{1}_{II}(\xi). \tag{4.1}$$

This restriction allows us to reduce the homogenized problem to a system of equations with a finite number of dependent variables because the class of piece-wise constant in ξ functions $\sigma(x,t,\xi)$ is invariant under the transformations defined by the ordinary differential equation (3.1) if the capillary function Ψ is piece-wise constant in ξ. In other words, if the initial local saturation satisfies (4.1) then the solution of (3.1) has the form

$$\sigma(x,\,t,\,\xi) = \sigma_I(x,t)\mathbf{1}_I(\xi) + \sigma_{II}(x,t)\mathbf{1}_{II}(\xi),$$

and the functions σ_I and σ_{II} satisfy the equations

$$\begin{cases} \partial\sigma_I/\partial t &= \Psi_I\,(\sigma_I,\, p_*(x,t)) \\ \partial\sigma_{II}/\partial t &= \Psi_{II}\,(\sigma_{II},\, p_*(x,t)) \end{cases} \tag{4.2}$$

with initial data $s_0^I(x)$ and $s_0^{II}(x)$. Of course, the functions σ_I and σ_{II} are limiting individual saturations of materials I and II respectively.

As a result, capillary hysteresis for any point x can be represented by curves in a three-dimensional space with coordinates p_*, σ_I and σ_{II}. If the main drainage and imbibition curves of both materials look like those in Figure 2, then the equilibrium zone in this space looks like a 'caterpillar' with its head at the point of full saturation $p_* = 0$, $\sigma_I = \sigma_{II} = 1$, and the tail stretched to the completely dry state $p_* = +\infty$, $\sigma_I = \sigma_{II} = 0$. (In fact, one should use irreducible values of saturation instead of 1 and 0, but we neglect these details for the sake of brevity).

We introduce the volume fractions of components by the formulae $c_I = \langle \mathbf{1}_I(\cdot)\rangle$ and $c_{II} = \langle \mathbf{1}_{II}(\cdot)\rangle$. Then the constitutive parameters in the homogenized equations (3.2 *a*) and (3.2 *b*) are represented by the following equalities:

$$m^* = c_I m_I + c_{II} m_{II},$$

$$K_*^{N(W)} = K_*^{N(W)}(\sigma_I,\sigma_{II}) = \frac{K_I^{N(W)}(\sigma_I)K_{II}^{N(W)}(\sigma_{II})}{c_{II}K_I^{N(W)}(\sigma_I) + c_I K_{II}^{N(W)}(\sigma_{II})}, \tag{4.3}$$

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$$\Psi^* = \Psi^*(\sigma_I, \sigma_{II}, p_*) = \frac{c_I m_I \Psi_I(\sigma_I, p_*) + c_{II} m_{II} \Psi_{II}(\sigma_{II}, p_*)}{c_I m_I + c_{II} m_{II}},$$

and the total saturation s^* is defined by the relation

$$s^* = \frac{c_I m_I \sigma_I(x,t) + c_{II} m_{II} \sigma_{II}(x,t)}{c_I m_I + c_{II} m_{II}}. \tag{4.4}$$

In applications, capillary curves are usually plotted in the plane of total saturation versus drop in pressures. In order to describe qualitative behaviour of these curves, we restrict the observation to trajectories of equilibrium processes. We also suppose that the main drainage and imbibition capillary pressures for both components of the porous medium, $P_{dr}^{I(II)}$ and $P_{im}^{I(II)}$, are monotonic functions of the individual saturations and introduce inverse functions $\sigma_{I(II)} = \Gamma_{dr}^{I(II)}(p_*)$ and $\sigma_{I(II)} = \Gamma_{im}^{I(II)}(p_*)$. In terms of these functions, the range of admissible individual saturations for a fixed p_* is the rectangle $\left[\Gamma_{im}^I(p_*), \Gamma_{dr}^I(p_*)\right] \times \left[\Gamma_{im}^{II}(p_*), \Gamma_{dr}^{II}(p_*)\right]$ in the plane (σ_I, σ_{II}). Then the set of equilibrium values of the total saturation, in accordance with formula (4.4), is determined by the inequalities

$$\frac{c_I m_I \Gamma_{im}^I(p_*) + c_{II} m_{II} \Gamma_{im}^{II}(p_*)}{c_I m_I + c_{II} m_{II}} \leqslant s^* \leqslant \frac{c_I m_I \Gamma_{dr}^I(p_*) + c_{II} m_{II} \Gamma_{dr}^{II}(p_*)}{c_I m_I + c_{II} m_{II}}.$$

This set is shown in Fig. 3. The individual capillary curves of the components are plotted on the same picture in order to make clear the sense of formula (4.4). For every internal point (p_*, s^*) of the hysteretic zone on the p_*–s^* plane, one may find different admissible pairs of individual saturations σ_I and σ_{II} providing the same value of the total saturation. The boundaries of the zone correspond to the case when the individual states of both materials, (p_*, σ_I) and (p_*, σ_{II}), are uniquely defined. If, for instance, an equilibrium drainage process occurs in both materials, then the point (p_*, s^*) on the plane p_* versus s^* moves to the left along the upper boundary of the homogenized hysteretic zone, and vice versa.

In contrast with the simplest model of hysteresis in the original problem, the scanning curves in the p_*–s^* plane are not represented by vertical straight lines, although each scanning curve has a vertical part. We illustrate this property of the homogenized model by an example shown in Fig. 3. There the total drainage is interrupted at some instance, and p_* begins to decrease in time. At the beginning, the point (p_*, s^*) leaves the total drainage curve and moves downward vertically because both individual saturations are constant until the passage from drainage to imbibition, and their arithmetic mean, s^*, is also constant. When p_* reaches the largest value of two individual capillary pressures, $P_{im}^I(\sigma_I)$ or $P_{im}^{II}(\sigma_{II})$, the intermediate process in one of the components is replaced by imbibition, and the corresponding individual saturation will begin to increase while the other is still fixed. Then the total saturation s^* increases, and the scanning curve on the p_*–s^* plane becomes inclined. As soon as p_* reaches the imbibition capillary pressure of the other material, the point (p_*, s^*) reaches the total imbibition curve and moves along it after this. Thus, the scanning curves in the homogenized model are vertical at the beginning only, and thereafter have a finite slope. Passages from total imbibition to drainage occur in a similar way; hence scanning curves in homogenized model are irreversible.

The slope and irreversibility of scanning curves make the model much more acceptable

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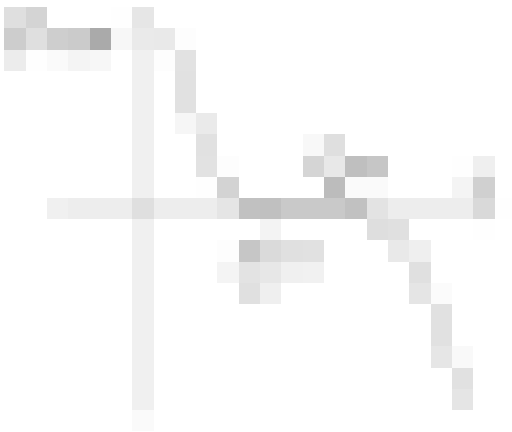

FIGURE 1. Capillary function for play-type hysteresis.

FIGURE 2. Capillary curves for the simple model of hysteresis.

in view of real capillary properties of porous media than the simple model with vertical scanning curves. The only serious complication is the need to operate with an additional dependent variable, because both individual saturations should be involved, but this is a necessary cost if we are to get a more detailed description of the physical system.

An important peculiarity of homogenized hysteresis relates to the range of admissible individual saturations. For each point x, this range is the projection of the hysteretic zone in the three-dimensional space $(p_*, \sigma_I, \sigma_{II})$ onto the plane (σ_I, σ_{II}). This projection does not cover the whole square $[0, 1] \times [0, 1]$ and does not have a rectangular shape.

FIGURE 3. Capillary hysteresis for two-component porous media.

Therefore, the individual saturations σ_I and σ_{II} are not completely independent. (This conclusion relates to the case when the dynamic effect is neglected and the system is in equilibrium at every point x. The set of non-equilibrium states is not restricted by the hysteretic zone, and, if the dynamic terms are included, all the points of the square are admissible.) The intersection of the hysteretic zone with a plane $p_* = const$ is the rectangle $\left[\Gamma^I_{im}(p_*), \Gamma^I_{dr}(p_*)\right] \times \left[\Gamma^{II}_{im}(p_*), \Gamma^{II}_{dr}(p_*)\right]$; hence the same set of admissible individual saturations may be obtained as the union of all these rectangles with p_* varying from 0 to $+\infty$. The result is shown in Fig. 4. The range of admissible saturations is the 'lens' between the curves $P^I_{dr}(\sigma_I) = P^{II}_{im}(\sigma_{II})$ and $P^I_{im}(\sigma_I) = P^{II}_{dr}(\sigma_{II})$.

For any point of this lens there exists p_* such that the point $(p_*, \sigma_I, \sigma_{II})$ belongs to the hysteretic zone. Of course, this p_* is not unique. Its range is $\max\left\{P^I_{im}(\sigma_I), P^{II}_{im}(\sigma_{II})\right\} \leqslant p_* \leqslant \min\left\{P^I_{dr}(\sigma_I), P^{II}_{dr}(\sigma_{II})\right\}$. If p_* increases (decreases) and leaves this range, then one of individual saturations or both of them begin to decrease (increase). This results in a displacement of the point (σ_I, σ_{II}) in some direction. Therefore, each point in the lens of admissible saturations may be supplemented with two arrows to indicate possible directions of the displacement for increasing and decreasing drop in pressures p_* as shown in Fig. 4.

An interesting property of the set of admissible saturations is the existence of trapping domains. The minimal of these is bounded by the curves $P^I_{dr}(\sigma_I) = P^{II}_{dr}(\sigma_{II})$ and $P^I_{im}(\sigma_I) = P^{II}_{im}(\sigma_{II})$. Once being captured there, a point will never go out.

5 Homogenization theorem for random structures

To define random statistically homogeneous fields in place of coefficients of the problem under consideration, we utilize an approach that is usual in the homogenization theory. This approach is sufficiently generic, and it provides a way to describe periodic and

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FIGURE 4. Equilibrium states for two-component porous media.

quasi-periodic structures as particular examples. In this respect, we are following the book by Jikov *et al.* [12, 10, ch. VII and VIII], where one can find proofs omitted here and informative examples.

Let $(\Omega, \mathscr{B}, \mu)$ be a probability space with σ-algebra $\mathscr{B}$ of measurable subsets and probability measure μ. Then random functions of $x \in \mathbf{R}$ are functions on $\Omega \times \mathbf{R}$. Statistical homogeneity implies that there exists a family of mappings $T_x : \Omega \mapsto \Omega$, $x \in \mathbf{R}$, conserving the measure μ on Ω and obeying the following group property: for any x', $x \in \mathbf{R}$ and $\omega \in \Omega$

$$T_{x'} \circ T_x \omega = T_{x'+x}\omega, \quad T_0\omega \equiv \omega.$$

This family is supposed to be measurable on $\Omega \times \mathbf{R}$, and the functions $\omega \mapsto T_x\omega$ are assumed measurable on Ω for any $x \in \mathbf{R}$. A measurable function q on $\Omega \times \mathbf{R}$ is called a statistically homogeneous random field if it has a form $q = Q(T_x\omega)$.

Let us illustrate this property with a trivial example. To this end, consider the interval [0, 1] with Borel σ-algebra of measurable subsets and standard Lebesgue measure on it as the probability space Ω. Then define the mappings T_x by the formula $T_x\omega = x+\omega (\mathrm{mod}\, 1)$. In this case, realizations of homogeneous functions $x \mapsto Q(T_x\omega)$ have to be periodic functions of $x \in \mathbf{R}$ with period 1. The randomness of these periodic functions is that their phase shift ω is not fixed but is distributed uniformly over the periodicity cell.

The family of mappings T_x on Ω is a dynamical system with 'time variable' x. We are supposing *ergodicity* of this system in the following sense: any function $Q \in L_1(\Omega, \mu)$ and bounded domain $D \subset \mathbf{R}$ are assumed to satisfy a.s. (almost surely) the equality

$$\lim_{\varepsilon \to 0} \frac{1}{|D|} \int_D Q(T_{x/\varepsilon}\omega) dx = \int_\Omega Q(\omega)\mu(d\omega), \tag{5.1}$$

where $|D|$ is the volume of D. In the framework of the homogenization theory, the ergodicity is not a necessary condition. It is introduced here in order to simplify the formulation of our results. The general approach to homogenization of partial differential equations with non-ergodic random coefficients has been done by Bourgeat *et al.* [9].

Thus, to introduce small-scaled random structure for the porous medium, we take the family of constitutive functions in the form $m = m(T_{x/\varepsilon}\omega)$, $K^{N(M)} = K^{N(M)}(T_{x/\varepsilon}\omega, s)$ and $\Psi = \Psi(T_{x/\varepsilon}\omega, s, p)$. In order to get a well-posed problem, we suppose that properties (i)–(v) are satisfied uniformly over ranges ε and ω. A special remark should be made with respect to measurability of these functions in x. From the definition of statistically homogeneous field and Fubini's theorem, it follows that realizations $Q(T_x\omega)$ of homogeneous functions $Q \in L_p(\Omega)$, $p \geqslant 1$, are measurable in x and belong to $L_p^{loc}(\mathbf{R})$ for almost all ω. Consequently, measurability of the constitutive functions in x may be assumed for all ω except for a set of zero measure on the probabilistic space Ω. It is shown by Bourgeat *et al.* [9, proof of Proposition 3.2, p. 31]; see also the more detailed proof for the periodic case by Allaire [1, Lemma 5.6, p. 1514]) that, due to continuity of $K^{N(M)}$ and Ψ in the other variables, this set can be chosen independently of s and p. Concerning the initial saturation, we assume that $s_0 = s_0(x, T_{x/\varepsilon}\omega)$ and $s_0(\cdot,\cdot) \in C([-l,+l]; L_\infty(\Omega))$. Then the function $x \mapsto s_0(x, T_{x/\varepsilon}\omega)$ is measurable a.s. due to Proposition 3.2 by Bourgeat *et al.* [9]. Therefore, the well-posedness of the problem (2.1)–(2.4) for every $\varepsilon > 0$ and almost all $\omega \in \Omega$ is provided by Theorem 2.2. The main result on homogenization of this problem is the following.

Theorem 5.1 *If assumptions* (i)–(v) *for random coefficients, boundary data, external forces and initial conditions hold uniformly over ranges of* ε *and* ω*, then there exist functions* p_*^N, $p_*^W: [-l,+l] \times [0,T] \mapsto \mathbf{R}$ *and* $\sigma: [-l,+l] \times [0,T] \times \Omega \mapsto \mathbf{R}$ *such that, with probability 1,*

(1) $s^\varepsilon(x, t) - \sigma(x, t, T_{x/\varepsilon}\omega)$ *converges to* 0 *in* $C^1(0, T; L_\infty([-l, +l]))$ *strongly;*

(2) p_ε^N *and* p_ε^W *converge to* p_*^N *and* p_*^W*, respectively, in* $C([-l, +l] \times [0, T])$ *strongly;*

(3) σ, p_*^N *and* p_*^W *satisfy the homogenized equations* (3.1), (3.2 *a*) *and* (3.2 *b*) *with initial saturation* $\sigma(x,0,\omega) = s_0(x,\omega)$ *and boundary conditions* (2.1)*, where the homogenized constitutive functions are again defined by* (3.3)–(3.5)*, but angle brackets in these formulae stand now for the integral over* Ω *with measure* μ *instead of the mean value over the periodicity cell.*

Proof of Theorem 5.1 Making use of *a priori* estimates in the same way as for the periodic case, we conclude that the family of functions $x,t \mapsto p_\varepsilon^{N(W)}$ is uniformly bounded and equipotentially continuous over ranges of the scaling parameter ε and, up to a subset of zero measure, the probabilistic variable ω. Then we are able to extract a uniformly convergent subsequence. The limit functions, $p_{*(\omega)}^N(x,t)$ and $p_{*(\omega)}^W(x,t)$, may depend upon the probabilistic variable ω. Being a pointwise limit of measurable functions, the limit pressures are also measurable in ω. Later on, we shall prove that they are independent of ω almost everywhere on Ω, but for now this may not be assumed. This is, in fact, the main cause of the difference between periodic and random cases. Otherwise we could make use of the notion of two-scale stochastic convergence in the mean by Bourgeat *et al.*[9] and the proof would be quite similar to the periodic one.

To determine the leading limiting term of the saturation, we consider the solution of the ordinary differential equation (3.1) with the function $s_0(x,\cdot)$ as initial condition. The parameter ω is involved in this initial value problem in the variables of Ψ and s_0 explicitly and, via the limit pressures, implicitly. We are going to distinguish between these different types of dependencies. Thus, we take two different probabilistic variables, ω and $\tilde{\omega}$, and set $s_0 = s_0(x, \tilde{\omega})$, $\Psi = \Psi(\tilde{\omega}, \sigma, p_{*(\omega)}(x,t))$ with $p_{*(\omega)}(x,t) := p^N_{*(\omega)}(x,t) - p^W_{*(\omega)}(x,t)$. Then the solution of equation (3.1) depends on both ω's. We denote the solution by $\sigma_{(\omega)}(x,t,\tilde{\omega})$. A more detailed notation, which indicates that the solution depends on initial data and input function $p_{*(\omega)}$, reads:

$$\sigma_{(\omega)}(x,t,\tilde{\omega}) = S_t\left(s_0(x,\tilde{\omega}),\ \tilde{\omega},\ \{p^N_{*(\omega)}(x,\cdot) - p^W_{*(\omega)}(x,\cdot)\}\right). \tag{5.2}$$

The leading term of the saturation is $\sigma_{(\omega)}\left(x,t,T_{x/\varepsilon}\omega\right)$, and a straightforward generalization of the proof from § 3 provides the uniform over $[-l,+l] \times [0,T]$ convergence to zero of $s^\varepsilon(x,t) - \sigma_{(\omega)}\left(x,t,T_{x/\varepsilon}\omega\right)$ and its time derivative.

We need some technical results on measurability and weak convergence for functions of the form $f = f(x,\omega,T_{x/\varepsilon}\omega)$. The natural fact that in the weak limit one has to take averaged value of $f(x,\omega,\tilde{\omega})$ over the second probabilistic variable holds, but needs some assumptions with respect to f. We introduce a class of weakly convergent functions by the following.

Lemma 5.2 *Let K be a compact subset of a Banach space, and $F(\tilde{\omega},\lambda)$ be a function on $\Omega \times K$ such that $F \in C(K; L_\infty(\Omega))$. Then, for any measurable mapping $\lambda : \mathbf{R} \times \Omega \mapsto K$, the function $x \mapsto F\left(T_{x/\varepsilon}\omega,\ \lambda(x,\omega)\right)$ is measurable in x with probability 1 and *-weakly converges in $L_\infty([-l,+l])$ to the mean value over the 'fast probabilistic variable', i.e. to $\langle F(\cdot,\lambda(x,\omega))\rangle$. In particular,*

$$\lim_{\varepsilon\to 0}\int_{-l}^{l} F\left(T_{x/\varepsilon}\omega,\ \lambda(x,\omega)\right)dx = \int_\Omega\int_{-l}^{l} F\left(\tilde{\omega},\ \lambda(x,\omega)\right)\mu(d\tilde{\omega})dx \tag{5.3}$$

for almost all $\omega \in \Omega$.

The proof is given in the Appendix.

The function $\sigma_{(\omega)}\left(x,t,T_{x/\varepsilon}\omega\right)$ is of the same class as introduced by Lemma 5.2. Indeed, the family of functions $t \mapsto p_{*(\omega)}(x,t)$ is equipotentially continuous and bounded uniformly over ranges of x and ω. Thus, for any fixed t it occupies a compact subset K^t of the Banach space $C([0,t])$. An explicit description of this subset is available in terms of Lipschitz constants for the constitutive functions and the continuity parameters of the external forces and boundary data. Consider the pair x and $p_{*(\omega)}(x,\cdot)$ as a measurable function $\lambda = \lambda(x,\omega)$ with values in $K := [-l,+l] \times K^t$; then the memory-dependent operator S_t from (5.2) can be taken as the function $F(\tilde{\omega},\lambda)$. Its continuity in λ is beyond any doubt. Thus the leading term of saturation satisfies the conditions of Lemma 5.2, and weakly converges to the mean value over the 'fast probabilistic variable'.

The functions $K^{N(W)}\left(T_{x/\varepsilon}\omega, \sigma_{(\omega)}(x,t,T_{x/\varepsilon}\omega)\right)$ and $\Psi\left(T_{x/\varepsilon}\omega, \sigma_{(\omega)}(x,t,T_{x/\varepsilon}\omega), p_{*(\omega)}(x,t)\right)$ are also of the same kind, and Lemma 5.2 can be applied to them. Let us denote the corresponding $*$-weak limits by s^*, $K_*^{N(W)}$ and Ψ^*. In accordance with Lemma 5.2, the same weak limits can be determined by formulae (3.4)–(3.6) where angle brackets stand for averaging over the 'fast' probabilistic variable. The last one coincides with the normal probabilistic expectation if the functions under consideration are independent of the 'slow'

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probabilistic variable. Otherwise, the effective permeabilities $K_*^{N(W)}$, averaged saturation s^* and homogenized capillary function Ψ^* may depend, besides x and t, on the probabilistic variable ω which had arise from the limit pressures and limit saturation $\sigma_{(\omega)}$. As soon as the pressures are proved independent of ω, we are able to conclude that our temporary definition of the homogenized parameters is the same as that claimed by the theorem.

The last step of proof is the derivation of equations for the limit pressures. To this end, we pass to a subsequence for ε tending to zero in the integral identity (2.6) where the test functions have to be specially constructed. At this step we are essentially making use of the fact that the space dimension is 1. For any $\varphi^{N(W)} \in C_0^\infty([-l,+l])$, let us take as test functions the solutions $\varphi_\varepsilon^{N(W)}(x)$ of the following equations:

$$\frac{\partial^2}{\partial x^2}\varphi_\varepsilon^{N(W)} = \frac{\partial}{\partial x}\left(\frac{K_*^{N(W)}\left(\{\sigma_{(\omega)}(x,t,\cdot)\}\right)}{K^{N(W)}\left(T_{x/\varepsilon}\omega, \sigma_{(\omega)}(x,t,T_{x/\varepsilon}\omega)\right)}\frac{\partial \varphi^{N(W)}}{\partial x}\right)$$

with $\varphi_\varepsilon^{N(W)}(\pm l) = 0$. These equations can be solved explicitly and, accounting for the above weak convergence results, one may easily check that the functions $x \mapsto \varphi_\varepsilon^N$ and $x \mapsto \varphi_\varepsilon^W$ are bounded in $W^{1,2}([-l,+l])$ and uniformly convergent to φ^N and φ^W. Also, the following formula holds:

$$\frac{\partial}{\partial x}\varphi_\varepsilon^{N(W)} = \frac{K_*^{N(W)}\left(\{\sigma_{(\omega)}(x,t,\cdot)\}\right)}{K^{N(W)}\left(T_{x/\varepsilon}\omega, \sigma_{(\omega)}(x,t,T_{x/\varepsilon}\omega)\right)}\frac{\partial}{\partial x}\varphi^{N(W)} + \alpha(\varepsilon,t,\omega), \tag{5.4}$$

where the last term in the right-hand side is independent of x and goes to zero as $\varepsilon \to 0$.

Passing to the limit in (2.6), we first take into account the uniform convergence of pressures and saturation. This allows us to replace s^ε and $p_\varepsilon^{N(W)}$, the arguments of $K^{N(W)}$ and Ψ, by the limit fields $\sigma_{(\omega)}$ and $p_{*(\omega)}^{N(W)}$. We obtain

$$\lim_{\varepsilon\to 0}\int_{-l}^{l}\Big\{m(\cdot)\,\Psi\left(\cdot,\sigma_{(\omega)},p_{*(\omega)}\right)\left(\varphi_\varepsilon^W - \varphi_\varepsilon^N\right) + K^W\left(\cdot,\sigma_{(\omega)}\right)\left(\frac{\partial p_\varepsilon^W}{\partial x} - f^W\right)\frac{\partial \varphi_\varepsilon^W}{\partial x} + (W \to N)\Big\}dx = 0. \tag{5.5}$$

Then formula (5.4) provides the way to conclude that the terms with pressure gradients in (5.5) can be represented as products of weakly and strongly convergent factors. Therefore, passing to the limit yields the integral identity (3.12) which is equivalent to the homogenized equations (3.2 *a*) and (3.2 *b*). Since the homogenized problem has a unique solution and the probabilistic variable is not involved in it explicitly, then the limit pressures are independent of the subsequence and ω. □

6 Concluding remarks

In this paper we have presented a study of a model for two-phase flow in porous media which includes (1.2) for the capillary relation between the saturation and the drop in phase pressures. This equation takes into account a simple sort of hysteresis coupled with dynamic memory effects. The porous medium is endowed with a heterogeneous microstructure. This is described by periodic or stochastic dependence of the model coefficients on x/ε where ε is a small parameter.

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The result of homogenization is the system of equations (1.3 *a*) and (1.3 *b*) coupled with a homogenized capillary relation. The latter is expressed in terms of auxiliary variable σ which is called the local saturation. It is a function of x and t with values in some Banach space that depends on the type of microstructure. In the homogenized system (1.3 *a*)–(1.3 *b*), the total saturation s^* is a function of σ defined by formula (3.6), and the phase permeabilities, $K_*^N(\{\sigma\})$ and $K_*^W(\{\sigma\})$, are given by formula (3.5) for the one-dimensional case. The local saturation satisfies (3.1), so this ordinary differential equation provides a memory-dependent relation between σ and the drop in limiting pressures $p_*^N - p_*^W$.

Convergence of saturation and phase pressures to the solution of the homogenized problem is proved on the basis of the two-scale convergence approach for periodic porous media (Theorem 3.1). In the case of porous media with random microstructure, results on convergence are presented in Theorem 5.1. Convergence with probability 1 is established for one-dimensional flows. Properties of the homogenized model are described in the particular case of a two-component porous medium which allows us to reduce the range of local saturation σ to a two-dimensional subset of Banach space. Then the dynamic capillary relation (3.1) can be replaced by the two ordinary differential equations (4.2) for two real-valued functions. This relation demonstrates more or less realistic behaviour of the capillary curves. In particular, it accounts for the slope and irreversibility of scanning curves within the hysteretic zone in the plane of total saturation versus the drop in pressures.

Appendix

Proof of Proposition 2.1 With boundary conditions (2.1), equations (2.3 *a*) and (2.3 *b*) present an elliptic problem for both pressures. The time variable t is involved in this problem via the saturation $s(x, t)$, external forces $f^{N,W}$, and boundary data. Set $q^{N(W)}(x) := q_-^{N(W)} + (x+l)(2l)^{-1}(q_+^{N(W)} - q_-^{N(W)})$. Then take test functions $\varphi^{N(W)}$ for the integral identity (2.6) in the form $\varphi^N := p^N - q^N$ and $\varphi^W := p^W - q^W$. After some trivial transformations, this results in

$$\int_{-l}^{+l} \Big\{ m(\cdot)\left(\Psi(\cdot,\cdot,p^N - p^W) - \Psi(\cdot,\cdot,q^N - q^W)\right)\left((q^N - q^W) - (p^N - p^W)\right)$$
$$+K^W(\cdot,\cdot)\left(\frac{\partial p^W}{\partial x}\right)^2 + K^N(\cdot,\cdot)\left(\frac{\partial p^N}{\partial x}\right)^2 \Big\}\, dx$$
$$= \int_{-l}^{+l} \Big\{ m(\cdot)\Psi(\cdot,\cdot,q^N - q^W)\left((p^N - p^W) - (q^N - q^W)\right) - K^W(\cdot,\cdot) f^W \frac{\partial q^W}{\partial x}$$
$$- K^N(\cdot,\cdot) f^N \frac{\partial q^N}{\partial x} + K^W(\cdot,\cdot)\left(f^W + \frac{\partial q^W}{\partial x}\right)\frac{\partial p^W}{\partial x} + K^N(\cdot,\cdot)\left(f^N + \frac{\partial q^N}{\partial x}\right)\frac{\partial p^N}{\partial x} \Big\}\, dx.$$

The term with Ψ on the left-hand side of this equality is non-negative due to the monotonicity of the capillary function. The value of $\Psi(\cdot,\cdot,q^N - q^W)$ on the right is estimated by a constant which is independent of x and s due to condition *(iii)*. It

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depends, however, upon $|q_\pm^N - q_\pm^W|$. Therefore, we get the inequality

$$\left\|\frac{\partial p^W}{\partial x}\right\|^2 + \left\|\frac{\partial p^N}{\partial x}\right\|^2 \leqslant C_1 \max|p^N - q^N - p^W + q^W| + C_2\left(\left\|\frac{\partial p^N}{\partial x}\right\| + \left\|\frac{\partial p^W}{\partial x}\right\|\right) + C_3,$$

where C_1, C_2 and C_3 depend on $q_\pm^N$, $q_\pm^W$, $\|f^N\|$ and $\|f^W\|$. The values of the functions $p^N - q^N$ and $p^W - q^W$ on the right may be estimated by the L_2-norm of their derivatives due to the following version of Poincaré inequality:

$$\max|\phi(x)| \leqslant \min|\phi(x)| + \sqrt{2l}\left\|\frac{\partial \phi}{\partial x}\right\|,$$

where $\phi := p^N - q^N$ or $p^W - q^W$, and $\min|p^N - q^N| = \min|p^W - q^W| = 0$ due to the boundary conditions at $x = \pm l$. This is followed by estimates (2.7).

Now let (p_1^N, p_1^W) and (p_2^N, p_2^W) satisfy the integral identity (2.6) for settings $(s_1, f_1^N, f_1^W, q_{1\pm}^N, q_{1\pm}^W)$ and $(s_2, f_2^N, f_2^W, q_{2\pm}^N, q_{2\pm}^W)$ respectively. Subtracting the identities from each other and posing $\varphi^W := p_2^W - p_1^W - q_2^W + q_1^W$, $\varphi^N := p_2^N - p_1^N - q_2^N + q_1^N$, we get the equality

$$\begin{aligned}
&\int_{-l}^{+l}\Bigg\{ m(\cdot)\left(\Psi(\cdot, s_1, p_2^N - p_2^W) - \Psi(\cdot, s_1, p_1^N - p_1^W)\right)\left(p_1^N - p_1^W - p_2^N + p_2^W\right) \\
&\quad + K^W(\cdot, s_1)\left(\frac{\partial(p_2^W - p_1^W)}{\partial x}\right)^2 + K^N(\cdot, s_1)\left(\frac{\partial(p_2^N - p_1^N)}{\partial x}\right)^2\Bigg\}\, dx \\
&\quad = \int_{-l}^{+l}\Bigg\{ m(\cdot)\left(\Psi(\cdot, s_2, p_2^N - p_2^W) - \Psi(\cdot, s_1, p_1^N - p_1^W)\right)\left(q_2^W - q_1^W - q_2^N + q_1^N\right) \\
&\quad + m(\cdot)\left(\Psi(\cdot, s_1, p_2^N - p_2^W) - \Psi(\cdot, s_2, p_2^N - p_2^W)\right)\left(p_2^W - p_1^W - p_2^N + p_1^N\right) \\
&\quad + \left(K^W(\cdot, s_1) - K^W(\cdot, s_2)\right)\left(\frac{\partial p_2^W}{\partial x} - f_2^W\right)\left(\frac{\partial(p_2^W - p_1^W)}{\partial x} - \frac{\partial(q_2^W - q_1^W)}{\partial x}\right) \\
&\quad + \left(K^N(\cdot, s_1) - K^N(\cdot, s_2)\right)\left(\frac{\partial p_2^N}{\partial x} - f_2^N\right)\left(\frac{\partial(p_2^N - p_1^N)}{\partial x} - \frac{\partial(q_2^N - q_1^N)}{\partial x}\right) \\
&\quad + K^W(\cdot, s_1)\left(f_2^W - f_1^W + \frac{\partial(q_2^W - q_1^W)}{\partial x}\right)\frac{\partial(p_2^W - p_1^W)}{\partial x} \\
&\quad + K^N(\cdot, s_1)\left(f_2^N - f_1^N + \frac{\partial(q_2^N - q_1^N)}{\partial x}\right)\frac{\partial(p_2^N - p_1^N)}{\partial x} \\
&\quad - K^W(\cdot, s_1)\left(f_2^W - f_1^W\right)\frac{\partial(q_2^W - q_1^W)}{\partial x} - K^N(\cdot, s_1)\left(f_2^N - f_1^N\right)\frac{\partial(q_2^N - q_1^N)}{\partial x}\Bigg\}\, dx.
\end{aligned}$$

Here again the term with Ψ on the left-hand side is non-negative due to the monotonicity of Ψ. To estimate the terms on the right, we make use of the Lipschitz property of K^N,

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K^W and Ψ. Supposing that the *a priori* estimate (2.7) holds for both solutions, we obtain

$$\left\| \frac{\partial(p_2^W - p_1^W)}{\partial x} \right\|^2 + \left\| \frac{\partial(p_2^N - p_1^N)}{\partial x} \right\|^2$$
$$\leqslant C_\Psi \Big(\text{ess sup}|s_2 - s_1| \cdot |q_2^N - q_1^N - q_2^W + q_1^W|$$
$$+ \max |p_2^N - p_1^N - p_2^W + p_1^W| \cdot |q_2^N - q_1^N - q_2^W + q_1^W|$$
$$+ \text{ess sup}|s_2 - s_1| \cdot \max |p_2^N - p_1^N - p_2^W + p_1^W| \Big)$$
$$+ C_K \,\text{ess sup}|s_2 - s_1| \cdot \left(\left\| \frac{\partial(p_2^W - p_1^W)}{\partial x} \right\| + \left\| \frac{\partial(p_2^N - p_1^N)}{\partial x} \right\| \right.$$
$$\left. + \left\| \frac{\partial(q_2^W - q_1^W)}{\partial x} \right\| + \left\| \frac{\partial(q_2^N - q_1^N)}{\partial x} \right\| \right) + C \left(\|f_2^W - f_1^W\| \cdot \left\| \frac{\partial(p_2^W - p_1^W)}{\partial x} \right\| \right.$$
$$+ \left\| \frac{\partial(q_2^W - q_1^W)}{\partial x} \right\| \cdot \left\| \frac{\partial(p_2^W - p_1^W)}{\partial x} \right\| + \|f_2^W - f_1^W\| \cdot \left\| \frac{\partial(q_2^W - q_1^W)}{\partial x} \right\|$$
$$+ \|f_2^N - f_1^N\| \cdot \left\| \frac{\partial(p_2^N - p_1^N)}{\partial x} + \right\| + \left\| \frac{\partial(q_2^N - q_1^N)}{\partial x} \right\| \cdot \left\| \frac{\partial(p_2^N - p_1^N)}{\partial x} \right\|$$
$$\left. + \|f_2^N - f_1^N\| \cdot \left\| \frac{\partial(q_2^N - q_1^N)}{\partial x} \right\| \right),$$

where the numbers C_Ψ and C_K relate to the Lipschitz constants of Ψ and $K^{N,W}$, respectively. Then, making use of the Poincaré inequality, we obtain relation (2.8). Thus the proof of Proposition 2.1 is completed. □

Proof of Lemma 5.2 Assumption (5.1) implies that, for any $f \in L_p(\Omega)$, the functions $x \mapsto f(T_{x/\varepsilon}\omega)$ converge with probability 1 weakly in $L_p^{loc}(\mathbf{R})$, as $\varepsilon \to 0$, to the expectation of f (see Jikov *et al.* [12]). Some wider classes, in relation to functions $f = f(x, T_{x/\varepsilon}\omega)$, have been considered by Bourgeat *et al.* [9] and, in the framework of periodic homogenization, by Allaire [1]. Lemma 5.2 is a natural generalization of their results.

We choose a 'representative' of F such that, for any ω from a set of full measure $\Omega' \subset \Omega$, the function $\lambda \to F(\omega, \lambda)$ is continuous on K. The existence of this representative is proved by Allaire [1, Lemma 5.6, p. 1514] for subsets of Euclidean spaces in place of Ω and K, and one needs nothing but change in notations for the generalization. Then, for any natural n, we consider a partition of K into a finite number of measurable sets $\Delta_i \subset K$ of maximal diameter n^{-1}. Let us pose

$$A_i = \{(x, \omega) \in [-l, +l] \times \Omega : \quad \lambda(x, \omega) \in \Delta_i\}$$

and choose a point $(x_i, \omega_i) \in A_i$ for each non-empty A_i. Then we introduce a function f^n : $[-l, +l] \times \Omega \times \Omega \to \mathbf{R}$ by formula

$$f^n(x, \omega, \tilde{\omega}) = \sum_i F(\tilde{\omega}, \lambda(x_i, \omega_i)) \mathbf{1}_{A_i}(x, \omega)$$

where $\mathbf{1}_A$ is the indicator of the set A. Due to continuity $F(\cdot, \lambda)$ with respect to λ, the sequence f^n converges to $F(\tilde{\omega}, \lambda(x, \omega))$ as $n \to \infty$ for all $x \in [-l, +l]$, $\omega \in \Omega$ and $\tilde{\omega} \in \Omega'$, and this convergence is uniform over ranges of x and ω.

The sets A_i are measurable; hence the indicators of these sets are measurable functions on $[-l, +l] \times \Omega$. Fubini's theorem results in their measurability in x almost everywhere

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on Ω. Therefore, there exists a subset $\Omega'' \subset \Omega$ of full measure such that the functions $f^n(x, \omega, \tilde{\omega})$ are measurable in x for any $\omega \in \Omega''$ (and any $\tilde{\omega} \in \Omega'$). Being a pointwise limit of measurable functions, $F(\tilde{\omega}, \lambda(x, \omega))$ is also measurable in x everywhere on $\Omega'' \times \Omega'$. Furthermore, the statistically homogeneous functions $x \mapsto F(T_{x/\varepsilon}\tilde{\omega}, \lambda(x_i, \omega_i))$ are measurable and weakly converge to the expectation over the range of $\tilde{\omega}$ a.s., and we are able to extract a subset $\Omega''' \subset \Omega$, independent of n and i, of full measure such that the measurability and convergence hold for any $\tilde{\omega} \in \Omega'''$.

To prove (5.3), let us take an arbitrary $\omega \in \Omega' \cap \Omega'' \cap \Omega'''$ and consider the integral

$$\begin{aligned} J_\varepsilon &:= \int_{-l}^{l} F\left(T_{x/\varepsilon}\omega, \lambda(x,\omega)\right) dx - \int_{\Omega}\int_{-l}^{l} F\left(\tilde{\omega}, \lambda(x,\omega)\right) \mu(d\tilde{\omega})dx \\ &= \int_{-l}^{l} \left(F\left(T_{x/\varepsilon}\omega, \lambda(x,\omega)\right) - f^n\left(x, \omega, T_{x/\varepsilon}\omega\right)\right) dx \quad (:= J_\varepsilon^1) \\ &\quad + \int_{-l}^{l} f^n\left(x, \omega, T_{x/\varepsilon}\omega\right) dx - \int_{\Omega}\int_{-l}^{l} f^n\left(x, \omega, \tilde{\omega}\right) \mu(d\tilde{\omega})dx \quad (:= J_\varepsilon^2) \\ &\quad + \int_{\Omega}\int_{-l}^{l} \left(f^n\left(x, \omega, \tilde{\omega}\right) - F\left(\tilde{\omega}, \lambda(x,\omega)\right)\right) \mu(d\tilde{\omega})dx \quad (:= J_\varepsilon^3). \end{aligned}$$

Here we take ε to zero first, and take the limit as $n \to \infty$ after this. The term J_ε^2 goes to zero for any fixed n due to weak convergence property for statistically homogeneous functions because each term of the function $f^n\left(x, \omega, T_{x/\varepsilon}\omega\right)$ is a product of measurable in x indicator $\mathbf{1}_{A_i}(x, \omega)$ and weakly convergent factor $F(T_{x/\varepsilon}\omega, \lambda(x_i, \omega_i))$. The term J_ε^3 is independent of ε and tends to zero as $n \to \infty$ because the expression under the integral is bounded and converges to zero uniformly with respect x and ω for almost every $\tilde{\omega}$. For the term J_ε^1, we obtain

$$\left|J_\varepsilon^1\right| \leqslant \int_{-l}^{l} \sup_{x',\omega'} \left|F\left(T_{x/\varepsilon}\omega, \lambda(x',\omega')\right) - f^n\left(x', \omega', T_{x/\varepsilon}\omega\right)\right| dx.$$

Here the expression under the integral is a homogeneous function of $T_{x/\varepsilon}\omega$, and by the ergodic property (5.1), we get

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\left|J_\varepsilon^1\right| \leqslant \lim_{n\to\infty}\int_{-l}^{l}\int_{\Omega} \sup_{x',\omega'} \left|F\left(\omega, \lambda(x',\omega')\right) - f^n\left(x', \omega', \omega\right)\right| \mu(d\omega)dx = 0.$$

Therefore, equality (5.3) is established. To prove *-weak convergence of $F(T_{x/\varepsilon}\omega, \lambda(x,\omega))$ in $L_\infty([-l, +l])$, it is sufficient to note that (5.3) holds for any $[x_1, x_2] \subset [-l, +l]$ in place of $[-l, +l]$. □

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