Mathematical Idealization

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Mathematical idealizations are scientific representations that result from assumptions that are believed to be false, and where mathematics plays a crucial role. I propose a two stage account of how to rank mathematical idealizations that is largely inspired by the semantic view of scientific theories. The paper concludes by considering how this approach to idealization allows for a limited form of scientific realism.

1. Introduction. For the purposes of this paper, I will say that a representation results from *idealization* when the steps leading up to the representation involve deliberate falsification, that is, assumptions are invoked that the agents constructing the representation believe to be false (cf. Jones 2005). And an idealization will be *mathematical* just in case these assumptions, or the resulting representation, involve mathematics in some crucial way. Here we find a technique for arriving at representations that is used across the sciences, but seems largely absent in nonscientific contexts.

My definition of mathematical idealization leads to many further questions, among them which scientific representations actually are mathematical idealizations in this sense. But to sidestep concerns about the vagueness of this definition, I will draw on a case of a simple mathematical idealization with which many of us will be familiar, but which is still complicated enough to raise the salient issues. My example involves replacing a difference equation, that is, an equation put in terms of discrete differences, with a differential equation. What is sometimes called Newton's law of cooling states that the amount of heat per unit of time that passes from a warmer plate 2 to a cooler plate 1 is

$$\Delta Q/\Delta t = (\kappa A|T_2 - T_1|)/d,$$

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where T_2 and T_1 are the respective temperatures of the plates, A is their area, d their distance from each other, and κ is the thermal conductivity of the material. That is, the change in heat per unit of time decreases with the distance between two plates, and increases with the temperature difference between the two plates, their area, and the properties of the materials captured by their thermal conductivity. Now I do not want to claim that no amount of mathematical idealization went into producing this representation, but will emphasize only that it is formulated in terms of finite differences of heat over finite periods of time. For this reason, it stands much closer to experimental practices than my second equation, the one-dimensional heat equation:

$$\alpha^2 u_{xx} = u_t,$$

where $\alpha^2 = \kappa/\rho s$, κ is again the thermal conductivity of the material, ρ its density, and s the specific heat of the material. Here $u(x, t)$ tracks the temperature of point x at time t, and the subscripts indicate partial differentiation with respect to that variable. That is, $u_t = \partial/\partial t\, u(x, t)$. So we have left behind finite differences between magnitudes across finite times, and arrived at specific assignments of magnitudes to points at each time.

It should be clear in what respects the heat equation results from mathematical idealization. First, it is clearly mathematical, as it is hard to see how the same representation could result without invoking mathematics. To be sure, some will dispute this claim, and to fully resolve the issue more must be said about how representations are individuated. Second, the representation results from idealization, that is, deliberate falsification. While I do not have space to review the details, the derivation of the one-dimensional heat equation from Newton's law of cooling requires the assumption that various limits are well defined. This, in turn, seems to require the assumption that the material being investigated is continuous. But we have good evidence that none of the materials that we use this equation to study are in fact continuous. Iron bars, for example, are made up of iron atoms, and presumably some impurities. The microstructure of an iron bar is incredibly complicated, and varies from bar to bar in unpredictable ways. We have every reason to believe, then, that the bar is not continuous. Despite this, we make an assumption to the contrary in the course of producing the heat equation.

The problem with mathematical idealization should now be clear. What guarantee is there that the results of employing these false assumptions will be representations? Or, more precisely, as representations may be ranked in terms of their accuracy and adequacy, why should making false assumptions contribute to the production of *good* representations?[1] The mystery is especially urgent in these sorts of cases as it looks like the only motivation for making false assumptions is so that we get, in the end, a mathematical equation that we can more easily work with. Morrison (2000) emphasizes, for example, how scientific theories can be unified mathematically. In the case of the heat equation, the result is ideally placed to be solved using the techniques available for such partial differential equations. But why should this sort of mathematical tractability have any sort of contact with the way the world is? A deep philosophical mystery lurks on the horizon.

We can see the problem as arising from two commitments: (1) Good scientific representations are related to the truth about the physical world and (2) mathematical tractability is unrelated to the way the physical world happens to be. Cartwright (1983, 1989, 1999), among others, has presented an account of scientific representation that calls (1) into question. Steiner (2005), on the other hand, is willing to reject (2). I do not want to reject either of them. In what follows I sketch an account of scientific representation that retains a link between good scientific representations and truth, and that also respects the gap between facts about mathematical tractability and the nature of the physical world.

2. Representation. The first step is to get clearer on exactly what a scientific representation is. I largely follow the semantic view of scientific representation, according to which a representation is a mathematical model, or a set of mathematical models (van Fraassen 1991, Chapter 1; da Costa and French 2003). In aligning my proposal with the semantic view, however, I am not taking on all the commitments that are associated with it. In particular, I see no hope of providing what is sometimes called a naturalistic account of scientific representation that dispenses with the beliefs and intentions of the scientists doing the representing (Suárez 2003). Instead, I will fix the content of a scientific representation using both the features of the mathematical models and the contents of the beliefs and intentions of the agents doing the representing. This might seem to lead to a regress. For if I appeal to the beliefs and intentions, that is, the representations, of the scientists in explaining scientific representation, then what is responsible for the content of these beliefs and intentions? This regress results only if we assume that all representations are scientific representations. Against this assumption, I posit a sort of nonscientific representation that does not involve mathematical models in the way that

1. Here I am assuming that there is an informative account of what makes a representation good. Contessa (2007) argues for this claim in apparent opposition to Suárez (2004).

scientific representation does, and that can be used to help clarify how scientific representation works.

A second point of difference with many advocates of the semantic view is that I explicitly identify my models with wholly mathematical models. The models involved in scientific representation, according to all advocates of the semantic view, are the models encountered in model theory. That is, they are *n*-tuples, where in the first position is a set of entities known as the domain, and the remaining positions are subsets of the domain, or subsets of the Cartesian product of the domain, and so on.[2] In saying that my models are wholly mathematical I mean to indicate that the entities in the domain are mathematical entities like real numbers and pure sets. Some defenders of the semantic view seem to hesitate to take this step, perhaps because they do not believe in mathematical entities, or because they worry that a wholly mathematical model will have a hard time representing a physical situation.

This is not the place to engage with anti-platonists in the philosophy of mathematics, but the second worry about how a wholly mathematical model can represent a physical situation is an important one to address. My proposal is that a wholly mathematical model represents a physical situation in virtue of a structure-preserving mapping like an isomorphism or an homomorphism between the physical situation and the mathematical model. Crucially, though, this mapping is picked out using physical magnitudes like temperature and position. It is here that the beliefs of agents are essential, for it is these mental states that fix which physical magnitudes are associated with which parts of the mathematical model. A scientific representation of a particular physical situation, then, requires not only a mathematical model, but also the belief that there is a homomorphism (or isomorphism) that takes physical magnitudes P_1, P_2, etc. to positions M_1, M_2, etc. in the mathematical model. Such a representation will be true just in case there is in fact such a mapping, and false in all other cases. The appeal to particular physical magnitudes is essential here, for otherwise there will always be a mapping with the appropriate structural features once certain minimal cardinality conditions are met (cf. Demopoulos 2003).

To see what account of representation I am sketching here, let us return to the heat equation, but now thought of as a fully realistic representation. In such a case, we think of the equation as cutting down the complete class of models reflecting all logically possible combinations of position, time, and temperature to those that the equation will permit. Each such model will have as its domain all the triples of real numbers, and its

2. Thomson-Jones (2006) provides a useful overview of the various purposes that philosophers have had when invoking these models.

second position will have an admissible trajectory that selects a series of triples of position, time, and temperature that are consistent with the heat equation. Note that only a small fraction of these will fit with initial and boundary conditions that we can actually work with. In the class of models, we will find not only these accessible trajectories, as we might call them, but also all the trajectories that fit with the equation under any logically possible initial and boundary conditions. For this class of models to become a representation of some iron bar, a scientist must believe that there is an isomorphism between the temperature states of the iron bar over time that preserves the position, time and temperature magnitudes instantiated in the iron bar. If there is such an isomorphism, then the representation is true. If not, then it is false.

This example should help to make clear what sort of beliefs on the part of scientists I must assume in order to get this account of scientific representation off the ground. They are beliefs about structure-preserving mappings between physical situations and mathematical models, where these mappings preserve the structure of the instantiations of various physical magnitudes. In recent work, Wilson (2006) has challenged some aspects of this conception of representation, but his objections demand a detailed response that is not possible here.

3. Idealization. In clarifying my account of nonidealized scientific representation in the last section I stipulated a case where the heat equation was thought to be a fully realistic representation of the temperature dynamics of an actual iron bar. But what happens when we reintroduce the fact that the representation is an idealized one, that is, one that involves deliberate falsification? If the scientist is aware that the bar is discrete, she would have to believe in the existence of a mapping that she has every reason to believe does not exist. An isomorphism that preserves temperature magnitudes will not exist between a continuous mathematical model and a discrete iron bar. So, on the simple story told in the last section, idealized scientific representation seems to be impossible. And even if we somehow make an amendment to allow these sorts of conflicting beliefs, it will turn out that all idealized scientific representations are false. An account that says only this is unacceptable because scientists continually rank idealized scientific representations, and so our account of such representations cannot assign them an equal rank.

Several routes out of this impasse suggest themselves, but I will articulate and defend only one. This is to invoke a second wholly mathematical model which will allow the scientist to clarify her beliefs in a way that can make them consistent, and that will also provide the ingredients sufficient to rank idealized representations in various respects. Let us call the class of models picked out by the heat equation the *equation model* and the second model that I am now proposing to introduce the *matching model*. As its name would suggest, the role of the matching model is to parallel perfectly all the physical features of the physical situation. In the case of an iron bar, this will involve not only the positions of the iron molecules over time, but also, say, the color of the bar, and other physical magnitudes which are not dealt with by the heat equation. Trivially, then, there will be an isomorphism between the physical system and the matching model.

What is left to fill out the outline of my two stage account of idealized representation is the connection between the equation model and the matching model. Here I want to allow for a wide variety of mathematical transformations that goes far beyond the structure preserving mappings discussed in the last section. In specifying these transformations, the intentions of the scientists are now essential. For the case of heat equation, what we need to capture is the thought that at least one of the admissible trajectories in the equation model results from smoothing out the relevant trajectory in the matching model. But smoothing out how? If we place no restrictions on this relationship, then we have no account of what makes an idealized scientific representation good. And if we place rigid restrictions, it seems that we risk labeling as bad some representations that scientists clearly think of as adequate.

My proposal is to go contextual. We bring in the goals that the scientists have in mind for their representation. In the heat equation case, the goal is most likely to be to represent the medium scale temperature dynamics of the iron bar for a short period of time. This provides for a certain threshold of error. So, in such a case, if there is a mathematical transformation from the equation model to the matching model that falls within this threshold, then we have a good or adequate idealized representation. If, despite the beliefs and intentions of the scientists, there is no such mathematical transformation, then the idealized representation is bad or inadequate.

The upshot of this proposal is that we must look at the goals that the scientists have for the representation if we are to evaluate its goodness. What is an adequate idealized representation for some purposes may be inadequate for other purposes. Obviously, the heat equation is not going to be adequate to represent the color of the iron bar as the associated equation model contains nothing relevant to color. But even though it does have features tied to temperatures, there is also not going to be an acceptable mathematical transformation that gets the temperature dynamics right on the microscale.

To summarize the two stage model proposal, then, we will say that an equation model represents a physical situation when the scientists believe both that (i) there is an isomorphism between the physical situation and a matching model and (ii) there is an acceptable mathematical transformation between the equation model and the matching model. The matching model contains wholly mathematical analogues of all the physical magnitudes in the physical situation. A mathematical transformation will be acceptable when it is consistent with the goals of the scientists in terms of scale and accuracy. Finally, we will say that such a representation is a good one when both of the beliefs involved in conditions (i) and (ii) are correct and it is a bad one to the extent that these beliefs are incorrect.

4. Evaluating Representations. The main strength of the two stage proposal is that it can accommodate several different epistemic situations with regard to the mathematical transformation between the equation model and the matching model. The typical case in science is that we have a representation that is idealized, but we do not know if it is a good representation because we do not know if there is an acceptable mathematical transformation. In what is for me the best case scenario, we can actually show that such a transformation exists by showing how the equation model and the matching model are mathematically related. This is the sort of case that Batterman (2002a, 2002b, 2007) has discussed extensively under the heading of asymptotics, although he does not necessarily endorse my interpretation of these cases. In such cases, I claim that we start with an equation that we have good reason to think accurately describes some features of the matching model. Based on mathematical manipulations, for example, setting a given quantity to 0, we can extract a new equation that then can be used to pick out what I have been calling the equation model. The ultimate check on these manipulations is that we are able to prove that the new equation and the original equation will agree on certain magnitudes within certain constraints.

Such insight into the mathematical transformation between the equation model and the matching model is rarely available. A kind of case that is one step down in terms of epistemic security, but still quite optimal, is the focus of work by McMullin (1985) and Laymon (1995). Both sketch out a process where a scientist starts from an idealized equation and proceeds to add in increasingly realistic correction terms. This rarely achieves the sort of explicit mathematical derivation that we get from asymptotic explanations, but can still give the scientists reason to believe that an acceptable mathematical transformation does in fact exist for their equation model. For example, an equation for an ideal pendulum can be corrected with terms reflecting the resistance of the medium. As these corrections are made, the gap between the equation models and the matching model decreases, at least in the sense that the relevant mathematical transformations between the two become less intricate, and the accuracy of the predictions increases.

The least secure case, and by far the most common, is where the only evidence we have that an acceptable mathematical transformation exists is that our predictions based on the equation model turn out to be correct for the purposes we have in mind. In these sorts of situation, the characteristics of the matching model may be obscure, and the extent to which our equation is getting things right therefore risks being completely mysterious. But a series of successful predictions can convince us that the idealized scientific representation is a good one, despite our ignorance about exactly how the mathematical transformation between the equation model and the matching model should be specified. In such cases, further investigation will hopefully make the connection manifest, but will often in the end convince us that there is no acceptable mathematical transformation, because the matching model will turn out to look quite different than we expected.

5. Limited Realism. Although I have distinguished three sorts of epistemic situation with respect to the existence of an appropriate mathematical transformation between the equation model and the matching model, the antirealist or nonrealist about science is likely to be unsatisfied. For the discussion so far has been put in terms of a single scientific representation at a particular time for one group of scientists. Such a synchronic definition of a good representation at a time, though, is not enough to vindicate any kind of scientific realism. In addition, what is needed is an account of why scientific representations tend to get better over time. In this final section, I sketch such a diachronic account of how mathematical idealization can help us aim at truth. This sketch is not meant either to defend scientific realism or to convince antirealists to become realists. In fact, it is consistent with a realistic attitude only in special, limited situations. It is intended, then, to mark a shift away from global realist or antirealist arguments, and towards the local circumstances that warrant this or that interpretation in particular cases (Fine 1984).[3]

Suppose a community of scientists has arrived at an equation model using our heat equation and they hope that there is a mathematical transformation appropriate for temperature over medium time and distance scales for a certain range of iron bars. The matching models for each of these bars may vary considerably, and be almost completely unknown in their details to the scientists. Still, imagine that the equation model leads to impressive success in predictions, and that all of these successes fall within the envisioned time and distance scales. Our question is: Under what conditions are the scientists warranted in extending this success to

3. For a more general discussion concerning versions of the semantic view and scientific realism see Chakravartty 2001.

other time and distance scales, or to a wider range of physical systems, or even to other physical magnitudes?

The answer that our definition from Section 4 suggests is that we look to the nature of the mathematical transformation between the equation model and the relevant matching models. In the heat equation case, we expect the extension to shorter times and to longer times to fail, although for very different reasons. Over shorter times, the particle-particle interactions that are tracked in the matching model will become more significant to the temperature dynamics. As our idealizations have erased anything corresponding to these aspects of the iron bar, and our mathematical transformation ignores them, we have good reason to think that the equation will fail over shorter times. In the long term, other issues become problematic, and these are again issues that are ignored by our process of idealization and the associated mathematical transformation. In this case, the problem is that heat loss to the environment becomes a dominating factor as the time scale is increased.

A somewhat trivial positive case of extending the range of application of such a mathematical idealization is when we keep the time and distance scales fixed, but shift from iron bars to bars made up of similar materials, for example, steel. Here, based on a prior understanding of the materials, we see that the cases are relevantly similar. And relevance is determinable in advance when we see the way the parts of one kind of bar admit of a matching model which stands in the right kind of mathematical relationship to the equation model. The same thing can happen in a more rarefied kind of case where we see that the same equation can work for different physical magnitudes. For example, the same mathematical equations that govern certain electromagnetic systems also fit certain fluid mechanics systems. By arranging one such system appropriately, we can see what will happen in the other. Again, what is needed for us to be confident that such a substitution will work is that the matching models of the systems with the two magnitudes of interest be mathematically similar in a way that connects to the needed mathematical transformation between the equation model and the matching model.

Understanding the mathematical transformation between the equation model and the matching models, then, is crucial not only in determining that a given scientific representation is a good one, but also in suggesting which new situations require new equation models, and which do not. When we have good reason to think that the same mathematical transformation will not work, a radically new kind of equation may be warranted that may not look anything like our original equation. For example, new complications are needed to understand how an iron bar will radiate heat into its environment. There is no reason to expect that an equation that could handle such radiation would mesh easily with our original heat equation. To be sure, if they are both good mathematical idealizations of the same physical system, there will be appropriate mathematical transformations from each equation model to the matching model. But this is a very weak condition. Almost any pair of equation models can be grouped together this way by finding a carefully constructed matching model.

Our account of idealization gives us the resources to respond to some recent criticisms of scientific realism based on the prevalence of mathematical idealization in scientific practice. The realist, we are told, cannot accommodate the wide range of conflicting models that are used to treat the same physical system. Traditional realism held out the hope that a single theory, thought of as a list of equations or more recently as a set of models picked out by a list of equations, could truly represent a class of physical systems. But this hope has proven illusory. For example, in response to McMullin's account of idealization, Morrison has complained that "the successful use of models does not involve refinements to a unique idealized representation of some phenomena or group of properties, but rather a proliferation of structures, each of which is used for different purposes. Indeed in many cases we do not have the requisite information to determine the degree of approximation that the model bears to the real system" (Morrison 2005, 170). If the models conflict, then of course they cannot all truly represent the physical systems. But is there any reason why a realist must restrict herself to theories in this sense? The realist need only insist that the goal of science is truth and that we have good reason to think that our representations are getting better at capturing truth over time. This fits perfectly with our scientific practice, at any given time, requiring several different, even conflicting, mathematical models that are used to provide good representations of this or that aspect of a physical system. The connections between these models may be clearly understood theoretically, to the point where we can explain why this model works for this purpose, whereas this model must be applied only for these other purposes. But even in the absence of these clarifications, we can often be confident that we have a good representation of this aspect using this model, and some other aspect using some other model. A limited realist position, then, does not try to explain away these appearances, but insists that it is consistent with adequate and improvable scientific representations.

The costs of this approach should be obvious. For unlike the traditional realist who tried to get arguments that applied to science as a whole, the limited realism I am suggesting requires a case by case consideration of exactly what evidence we have that our representations are good ones. This piecemeal approach, however, promises to fit in better with the judgments of scientists themselves, but still holds out the hope that scientists are discovering at least a part of the truth about a mind independent physical world.

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