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COMPARISONS ON LARGEST ORDER STATISTICS FROM HETEROGENEOUS GAMMA SAMPLES

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This paper deals with stochastic comparisons of the largest order statistics arising from two sets of independent and heterogeneous gamma samples. It is shown that the weak supermajorization order between the vectors of scale parameters together with the weak submajorization order between the vectors of shape parameters imply the reversed hazard rate ordering between the corresponding maximum order statistics. We also establish sufficient conditions of the usual stochastic ordering in terms of the p-larger order between the vectors of scale parameters and the weak submajorization order between the vectors of shape parameters. Numerical examples and applications in auction theory and reliability engineering are provided to illustrate these results.

Keywords: gamma distribution, largest order statistics, majorization, p-larger order, reversed hazard rate order, usual stochastic order

1. INTRODUCTION

Order statistics play a critical role in many research areas such as statistical inference, operations research, reliability theory and applied probability. For instance, the kth order statistic $X_{k:n}$ from random sample $X_1, \ldots, X_n$ corresponds to the lifetime of an $(n-k+1)$-out-of-n system in the area of reliability engineering, which is a rather renowned structure of system in fault-tolerant systems that have been studied extensively. In particular, $X_{n:n}$ and $X_{1:n}$ represent the lifetimes of parallel and series systems, respectively. There have been a great many papers appearing on various aspects of order statistics when the observations are independent and identically distributed (i.i.d.); however, for the case of independent but not identically distributed (i.n.i.d.) observations, not too much work is available in the literature due to the complexity of the distribution theory; see, for example, [1,10] for comprehensive discussions on this topic.

Pledger and Proschan [23] might be the first to investigate stochastic comparisons of order statistics arising from i.n.i.d. exponential random variables. Along this line, many researchers have focused their attention on the topic of stochastic properties of order statistics stemming from exponential samples (see [7,11,13,17,22,30,31]). However, there is not much work dealing with the gamma distribution, which is one of the most commonly used distributions in many areas including actuarial science and reliability engineering. A random variable X is said to have a gamma distribution with the shape parameter $r>0$ and the scale parameter $\lambda > 0$ if its probability density function is given as

$$f(x;r,\lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad x > 0. \tag{1}$$

It is an extremely flexible distribution with decreasing, constant and increasing failure rates when $0<r<1$, $r=1$ and $r>1$, respectively. Obviously, it contains exponential distribution as a special case when $r=1$. Gamma distribution is also widely used to describe the lifetime of components in shock model or undergoing minimal repairs (see [18,25,29]).

Let $X_1, X_2, \ldots, X_n$ $[Y_1, Y_2, \ldots, Y_n]$ be a batch of independent gamma random variables with the common shape parameter r and different scale parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ $[\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*]$, respectively. Let $Z_1, Z_2, \ldots, Z_n$ be another set of independent gamma random variables with the common shape parameter r and the common scale parameter λ. For the case of two heterogeneous gamma samples, Sun and Zhang [27] proved that

$$\begin{aligned}
&r > 1 \quad \text{and} \quad (\lambda_1, \lambda_2, \ldots, \lambda_n) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \Longrightarrow X_{1:n} \leq_{\text{st}} Y_{1:n}, \\
&r \leq 1 \quad \text{and} \quad (\lambda_1, \lambda_2, \ldots, \lambda_n) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \Longrightarrow (X_{1:n}, \ldots, X_{n:n}) \geq_{\mathbf{st}} (Y_{1:n}, \ldots, Y_{n:n}), \\
&\forall\, r > 0, \quad (\lambda_1, \lambda_2, \ldots, \lambda_n) \overset{\text{m}}{\succeq} (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \Longrightarrow X_{n:n} \geq_{\text{st}} Y_{n:n},
\end{aligned} \tag{2}$$

where "$\overset{\text{m}}{\succeq}$" denotes the majorization order, "$\geq_{\text{st}}$" denotes the usual stochastic ordering and "$\geq_{\mathbf{st}}$" denotes the multivariate usual stochastic ordering. For the sake of briefness, explicit definitions of related orders used here as well as in the following text will be given in Section 2. Khaledi et al. [14] relaxed the condition of (2) as

$$\forall\, r > 0, \quad (\lambda_1, \lambda_2, \ldots, \lambda_n) \overset{\text{p}}{\succeq} (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \Longrightarrow X_{n:n} \geq_{\text{st}} Y_{n:n}. \tag{3}$$

Misra and Misra [20] further strengthened (2) by showing that

$$\forall\, r > 0, \quad (\lambda_1, \lambda_2, \ldots, \lambda_n) \overset{\text{w}}{\succeq} (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \Longrightarrow X_{n:n} \geq_{\text{rh}} Y_{n:n}. \tag{4}$$

For the case of two independent heterogeneous and homogenous gamma samples, Balakrishnan and Zhao [4] and Zhao and Balakrishnan [32] investigated the hazard rate ordering and the likelihood ratio ordering for the maximum order statistics under the condition that $0 < r \leq 1$, respectively.

However, all of the results mentioned above rely on the assumption that all of the shape parameters are same. To the best of the authors' knowledge, little work has been done for stochastic comparisons on the largest order statistics from independent and heterogeneous gamma samples when the shape parameters are different from each other except Zhao and Zhang [33] and Zhang and Zhao [28]. In these two papers, the authors dealt with the ordering properties of the maxima of two independent gamma random variables with both different shape and scale parameters by means of the likelihood ratio ordering and the hazard rate ordering. In this paper, we shall investigate ordering properties of the maximum order statistics arising from heterogeneous gamma random variables with different shape and scale parameters when the sample size is such that $n \geq 3$. Sufficient conditions will be established in terms of the majorization-type orders for the reversed hazard rate order and the usual stochastic order. Our results will generalize the aforementioned results in (2)–(4) to some extent.

The rest of the paper is organized as follows. Section 2 introduces some pertinent definitions of stochastic orderings and majorization-type orders. In Section 3, sufficient conditions on vectors of the shape and scale parameters are given to stochastically compare the largest order statistic between two sets of independent and heterogeneous gamma samples in terms of the reversed hazard rate ordering. Stochastic comparisons based on the usual stochastic ordering are carried out in Section 4. Section 5 presents some practical applications where our results can be applied. Concluding remarks can be found in Section 6.

2. PRELIMINARIES

Throughout this paper, the term increasing is used for monotone nondecreasing and decreasing is used for monotone nonincreasing. We use "$\overset{\text{sgn}}{=}$" to denote that both sides of the equality have the same sign. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_{++} = (0, +\infty)$. Define two $2 \times n$ matrix spaces as

$$\mathcal{S}_n = \left\{ (\boldsymbol{a}, \boldsymbol{b}) = \begin{pmatrix} a_1, \dots, a_n \\ b_1, \dots, b_n \end{pmatrix} : a_i, b_j > 0, (a_i - a_j)(b_i - b_j) \geq 0, i, j = 1, 2, \dots, n \right\}$$

and

$$\mathcal{U}_n = \left\{ (\boldsymbol{a}, \boldsymbol{b}) = \begin{pmatrix} a_1, \dots, a_n \\ b_1, \dots, b_n \end{pmatrix} : a_i, b_j > 0, (a_i - a_j)(b_i - b_j) \leq 0, i, j = 1, 2, \dots, n \right\}.$$

We first recall the definitions of some useful stochastic orderings used in the sequel.

Definition 2.1: *For two random variables X and Y with density functions f_X and f_Y, and distribution functions F_X and F_Y, respectively, let $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$ be their corresponding survival functions. Then, X is said to be smaller than Y in the*

i. *likelihood ratio order (denoted by $X \leq_{\text{lr}} Y$) if $f_Y(x)/f_X(x)$ is increasing in $x \in \mathbb{R}$;*
ii. *hazard rate order (denoted by $X \leq_{\text{hr}} Y$) if $\overline{F}_Y(x)/\overline{F}_X(x)$ is increasing in $x \in \mathbb{R}$;*
iii. *reversed hazard rate order (denoted by $X \leq_{\text{rh}} Y$) if $F_Y(x)/F_X(x)$ is increasing in $x \in \mathbb{R}$; and*
iv. *usual stochastic order (denoted by $X \leq_{\text{st}} Y$) if $\overline{F}_X(x) \leq \overline{F}_Y(x)$ for all $x \in \mathbb{R}$.*

It is well known that the likelihood ratio order implies both the reversed hazard rate order and the hazard rate order, which further imply the usual stochastic order. However, the reversed statements do not hold in general. For comprehensive discussions and applications on these stochastic orders, one may refer to the excellent monographs by Shaked and Shanthikumar [26] and Müller and Stoyan [21].

Majorization is quite helpful in deriving inequalities arising from the areas of applied probability and reliability theory. Let $x_{1:n} \leq \cdots \leq x_{n:n}$ be the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, \dots, x_n)$.

Definition 2.2: *A vector $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ is said to*

i. *majorize another vector $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ (written as $\boldsymbol{x} \overset{\text{m}}{\succeq} \boldsymbol{y}$) if $\sum_{i=1}^{j} x_{i:n} \leq \sum_{i=1}^{j} y_{i:n}$ for $j = 1, \dots, n-1$ and $\sum_{i=1}^{n} x_{i:n} = \sum_{i=1}^{n} y_{i:n}$;*

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ii. *weakly supermajorize another vector* $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ *(written as* $\boldsymbol{x} \overset{\mathrm{w}}{\succeq} \boldsymbol{y}$*) if* $\sum_{i=1}^{j} x_{i:n} \le \sum_{i=1}^{j} y_{i:n}$ *for* $j = 1, \ldots, n$*;*

iii. *weakly submajorize another vector* $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ *(written as* $\boldsymbol{x} \succeq_{\mathrm{w}} \boldsymbol{y}$*) if* $\sum_{i=j}^{n} x_{i:n} \ge \sum_{i=j}^{n} y_{i:n}$ *for* $j = 1, \ldots, n$*; and*

iv. *be p-larger than another vector* $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ *(written as* $\boldsymbol{x} \overset{\mathrm{p}}{\succeq} \boldsymbol{y}$*) if* $\prod_{i=1}^{j} x_{i:n} \le \prod_{i=1}^{j} y_{i:n}$ *for* $j = 1, \ldots, n$*, and the elements of both* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are nonnegative.*

For two nonnegative vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, it is evident that $\boldsymbol{x} \overset{\mathrm{m}}{\succeq} \boldsymbol{y}$ implies $\boldsymbol{x} \overset{\mathrm{w}}{\succeq} \boldsymbol{y}$, and $\boldsymbol{x} \overset{\mathrm{p}}{\succeq} \boldsymbol{y}$ is equivalent to $\log(\boldsymbol{x}) \overset{\mathrm{w}}{\succeq} \log(\boldsymbol{y})$, where $\log(\boldsymbol{x}) = (\log x_1, \ldots, \log x_n)$. It is well known that

$$\boldsymbol{x} \overset{\mathrm{m}}{\succeq} \boldsymbol{y} \Longrightarrow \boldsymbol{x} \overset{\mathrm{w}}{\succeq} \boldsymbol{y} \Longrightarrow \boldsymbol{x} \overset{\mathrm{p}}{\succeq} \boldsymbol{y}.$$

For more details on majorization orders and their applications, one may refer to Bon and Păltănea [6] and Marshall et al. [19].

The following lemma presents sufficient and necessary conditions for the preservation of a multivariate function on the supermajorization and submajorization orders.

LEMMA 2.3 [19]: *Let* ϕ *be a continuous real-valued function defined on* $\mathcal{D} = \{\boldsymbol{x} : x_1 \ge x_2 \ge \cdots \ge x_n\}$ *and differentiable on the interior of* $\mathcal{D}$*. Denote the partial derivative of* ϕ *with respect to its kth argument by* $\phi_{(k)}(\boldsymbol{z}) = \partial\phi(\boldsymbol{z})/\partial z_k$*, for* $k = 1, \ldots, n$*. Then,* $\phi(\boldsymbol{x}) \le \phi(\boldsymbol{y})$ *whenever* $\boldsymbol{x} \preceq_{\mathrm{w}} \boldsymbol{y}$ *on* $\mathcal{D}$ *if and only if* $\phi_{(1)}(\boldsymbol{z}) \ge \phi_{(2)}(\boldsymbol{z}) \ge \cdots \ge \phi_{(n)}(\boldsymbol{z}) \ge 0$*, that is, the gradient* $\nabla\phi(\boldsymbol{z}) \in \mathcal{D}_+ = \{\boldsymbol{x} : x_1 \ge x_2 \ge \cdots \ge x_n \ge 0\}$*, for all* $\boldsymbol{z}$ *in the interior of* $\mathcal{D}$*. Similarly,* $\phi(\boldsymbol{x}) \le \phi(\boldsymbol{y})$ *whenever* $\boldsymbol{x} \overset{\mathrm{w}}{\preceq} \boldsymbol{y}$ *on* $\mathcal{D}$ *if and only if* $0 \ge \phi_{(1)}(\boldsymbol{z}) \ge \phi_{(2)}(\boldsymbol{z}) \ge \cdots \ge \phi_{(n)}(\boldsymbol{z})$*, that is, the gradient* $\nabla\phi(\boldsymbol{z}) \in \mathcal{D}_- = \{\boldsymbol{x} : 0 \ge x_1 \ge x_2 \ge \cdots \ge x_n\}$*, for all* $\boldsymbol{z}$ *in the interior of* $\mathcal{D}$*.*

3. THE REVERSED HAZARD RATE ORDERING

In this section, sufficient conditions are presented with regard to the reversed hazard rate ordering between the maximum order statistics arising from two sets of independent and heterogeneous gamma samples. The following lemmas play a key role in proving the main results.

LEMMA 3.1 [5]: *Let* X *be a nonnegative random variable with distribution function* F *and let* $\mu_r = \int_0^\infty x^r \mathrm{d}F(x)$*, r = 1, 2. If* X *has the increasing hazard rate in average (IHRA), then* $\mu_2 \le 2\mu_1^2$*.*

LEMMA 3.2 [20]: *Let* W *be a random variable having the probability density function*

$$f_W(w; \alpha, y) = \begin{cases} \dfrac{(1-w)^{\alpha-1} e^{yw}}{\int_0^1 (1-t)^{\alpha-1} e^{yt} \mathrm{d}t}, & \text{if } 0 < w < 1, \\ 0, & \text{otherwise}, \end{cases}$$

where α *and* y *are positive constants. Then,* W *has the increasing hazard rate (IHR).*

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Lemma 3.3: *For $r \in \mathbb{R}_{++}$ and $x \in \mathbb{R}_{+}$, the bivariate function*

$$\Upsilon(r,x) = \frac{\int_0^1 u^{r-1}(1-u)e^{-x(1+u)}\mathrm{d}u}{\left(\int_0^1 u^{r-1}e^{-xu}\mathrm{d}u\right)^2}$$

is decreasing in $x \in \mathbb{R}_{+}$ and increasing in $r \in \mathbb{R}_{++}$.

Proof: First, we prove the decreasing property of $\Upsilon(r,x)$ in $x \in \mathbb{R}_{+}$. Note that

$$\Upsilon(r,x) = \frac{\int_0^1 u^{r-1}(1-u)e^{-x(1+u)}\mathrm{d}u}{\left(\int_0^1 u^{r-1}e^{-xu}\mathrm{d}u\right)^2} = \frac{\int_0^1 u(1-u)^{r-1}e^{xu}\mathrm{d}u}{\left(\int_0^1 (1-u)^{r-1}e^{xu}\mathrm{d}u\right)^2}.$$

By taking the derivative of $\Upsilon(r,x)$ with respect to x, we have

$$\begin{aligned}\frac{\partial\Upsilon(r,x)}{\partial x} &\overset{\text{sgn}}{=} \int_0^1 u^2(1-u)^{r-1}e^{xu}\,\mathrm{d}u\left(\int_0^1 (1-u)^{r-1}e^{xu}\,\mathrm{d}u\right)^2 \\ &\quad - 2\int_0^1 (1-u)^{r-1}e^{xu}\,\mathrm{d}u\left(\int_0^1 u(1-u)^{r-1}e^{xu}\,\mathrm{d}u\right)^2 \\ &\overset{\text{sgn}}{=} \frac{\int_0^1 u^2(1-u)^{r-1}e^{xu}\,\mathrm{d}u}{\int_0^1 (1-u)^{r-1}e^{xu}\,\mathrm{d}u} - 2\left(\frac{\int_0^1 u(1-u)^{r-1}e^{xu}\,\mathrm{d}u}{\int_0^1 (1-u)^{r-1}e^{xu}\,\mathrm{d}u}\right)^2 \le 0,\end{aligned}$$

where the last inequality holds by using (2.2) in the proof of Theorem 2.1 of Misra and Misra [20].

Now, our attention turns to the proof of the increasing property of $\Upsilon(r,x)$ with respect to $r \in \mathbb{R}_{++}$. Upon taking the derivative of $\Upsilon(r,x)$ with respect to r, we have

$$\begin{aligned}\frac{\partial\Upsilon(r,x)}{\partial r} &= \frac{\int_0^1 u^{r-1}e^{-x(1+u)}(1-u)\ln u\,\mathrm{d}u\left(\int_0^1 u^{r-1}e^{-xu}\,\mathrm{d}u\right)^2}{\left(\int_0^1 u^{r-1}e^{-xu}\,\mathrm{d}u\right)^4} \\ &\quad - \frac{2\int_0^1 u^{r-1}(1-u)e^{-x(1+u)}\,\mathrm{d}u\int_0^1 u^{r-1}e^{-xu}\,\mathrm{d}u\int_0^1 u^{r-1}e^{-xu}\ln u\,\mathrm{d}u}{\left(\int_0^1 u^{r-1}e^{-xu}\,\mathrm{d}u\right)^4} \\ &\overset{\text{sgn}}{=} \frac{\int_0^1 u^{r-1}e^{-xu}(1-u)\ln u\,\mathrm{d}u}{\int_0^1 u^{r-1}e^{-xu}\,\mathrm{d}u} - \frac{2\int_0^1 u^{r-1}e^{-xu}(1-u)\,\mathrm{d}u\int_0^1 u^{r-1}e^{-xu}\ln u\,\mathrm{d}u}{\left(\int_0^1 u^{r-1}e^{-xu}\,\mathrm{d}u\right)^2} \\ &= \mathbb{E}((1-U)\ln U) - 2\mathbb{E}(1-U)\mathbb{E}\ln U,\end{aligned}$$

where the random variable U has the density function

$$f_U(u|r,x) = \frac{u^{r-1}e^{-xu}}{\int_0^1 t^{r-1}e^{-xt}\,\mathrm{d}t}, \quad u \in (0,1).$$

Let $W = 1 - U$. The density function of W can be written as follows:

$$f_W(w|r,x) = \frac{(1-w)^{r-1}e^{-x(1-w)}}{\int_0^1 t^{r-1}e^{-xt}\,\mathrm{d}t} = \frac{(1-w)^{r-1}e^{-x}e^{xw}}{\int_0^1 (1-u)^{r-1}e^{-x}e^{xu}\,\mathrm{d}u}.$$

According to Lemma 3.2, we know that W is IHR. Let $Y = -\ln U$. Then, the density function of Y can be written as follows:

$$f_Y(y|r,x) = \frac{e^{-ry-xe^{-y}}}{\int_0^1 t^{r-1}e^{-xt}\,\mathrm{d}t}.$$

By taking twice derivative of $\ln f_Y(y|r,x)$ with respect to y, we get

$$\frac{d^2 \ln f_Y(y|r,x)}{dy^2} = -xe^{-y} \le 0,$$

which means the density function of random variable Y is logconcave, and hence, Y has IHR. Therefore, both of $1-U$ and $-\ln U$ have IHR. Based on Lemma 3.1, it then follows that

$$\mathbb{E}(1-U)^2 \le 2(\mathbb{E}(1-U))^2 \quad \text{and} \quad \mathbb{E}(\ln U)^2 \le 2(\mathbb{E}\ln U)^2. \tag{5}$$

Upon applying Cauchy–Schwarz inequality to (5), we have

$$(\mathbb{E}(1-U)\ln U)^2 \le \mathbb{E}(1-U)^2\mathbb{E}(\ln U)^2,$$

which leads to

$$(\mathbb{E}(1-U)\ln U)^2 \le \mathbb{E}(1-U)^2\mathbb{E}(\ln U)^2 \le 4(\mathbb{E}(1-U))^2(\mathbb{E}\ln U)^2,$$

that is,

$$2\mathbb{E}(1-U)\mathbb{E}\ln U \le \mathbb{E}(1-U)\ln U. \tag{6}$$

In accordance with (6), we know that $\partial\Upsilon(r,x)/\partial r$ is nonnegative. Thus, one can see that the function $\Upsilon(r,x)$ is increasing $r \in \mathbb{R}_{++}$. To sum up, the proof is completed. ■

Next, the reversed hazard rate ordering is established in the following theorem under the weak supermajorization order between the vectors of scale parameters.

Theorem 3.4: *Let $X_1, X_2, \ldots, X_n$ be independent gamma random variables with the shape parameter vector $\boldsymbol{r}$ and the scale parameter vector $\boldsymbol{\lambda}$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent gamma random variables with the vector of shape parameter $\boldsymbol{r}$ and the vector of scale parameter $\boldsymbol{\lambda}^*$. Suppose that $(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*) \in \mathcal{S}_n$ and $(\boldsymbol{r}, \boldsymbol{\lambda}) \in \mathcal{U}_n$. Then, it holds that*

$$\boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow X_{n:n} \ge_{\mathrm{rh}} Y_{n:n}.$$

Proof: Without loss of generality, we assume that $r_1 \ge r_2 \ge \cdots \ge r_n > 0$, $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and $\lambda_1^* \le \lambda_2^* \le \cdots \le \lambda_n^*$. Denote by $\tilde{r}_{X_{n:n}}(t)$ and $\tilde{r}_{Y_{n:n}}(t)$, the reversed hazard rate functions of $X_{n:n}$ and $Y_{n:n}$, respectively. Then, we have

$$\tilde{r}_{X_{n:n}}(t) = \frac{1}{t}\sum_{i=1}^{n} \frac{e^{-\lambda_i t}}{\int_0^1 u^{r_i-1} e^{-\lambda_i t u}\,\mathrm{d}u} \quad \text{and} \quad \tilde{r}_{Y_{n:n}}(t) = \frac{1}{t}\sum_{i=1}^{n} \frac{e^{-\lambda_i^* t}}{\int_0^1 u^{r_i-1} e^{-\lambda_i^* t u}\,\mathrm{d}u}.$$

Let $x_i = \lambda_i t$ and $x_i^* = \lambda_i^* t$, $i = 1, 2, \ldots, n$. Then, we know that $x_1 \le x_2 \le \cdots \le x_n$, $x_1^* \le x_2^* \le \cdots \le x_n^*$and $(x_1, x_2, \ldots, x_n) \overset{\mathrm{w}}{\succeq} (x_1^*, x_2^*, \ldots, x_n^*)$. It suffices to prove that

$$\sum_{i=1}^{n} \frac{e^{-x_i}}{\int_0^1 u^{r_i-1} e^{-x_i u}\,\mathrm{d}u} \ge \sum_{i=1}^{n} \frac{e^{-x_i^*}}{\int_0^1 u^{r_i-1} e^{-x_i^* u}\,\mathrm{d}u}.$$

In light of Lemma 2.3, we need to show that the derivative functions

$$\frac{\partial \Psi}{\partial x_k}(x_n, x_{n-1}, \ldots, x_1), \quad k = 1, 2, \ldots, n$$

of differentiable function $\Psi : \mathcal{D}_{\overleftarrow{\boldsymbol{x}}} = \{(x_n, x_{n-1}, \ldots, x_1) : x_n \ge x_{n-1} \ge \cdots \ge x_1\} \to \mathbb{R}_{++}$ given by

$$\Psi(x_n, x_{n-1}, \ldots, x_1) = \sum_{i=1}^{n} \frac{e^{-x_i}}{\int_0^1 u^{r_i-1} e^{-x_i u}\,\mathrm{d}u}$$

satisfy that

$$0 \ge \frac{\partial \Psi}{\partial x_j}(x_n, x_{n-1}, \ldots, x_1) \ge \frac{\partial \Psi}{\partial x_i}(x_n, x_{n-1}, \ldots, x_1) \quad \text{for all } n \ge j \ge i \ge 1.$$

Taking the derivative of $\Psi(x_n, x_{n-1}, \ldots, x_1)$ with respect to x_i gives rise to

$$\begin{aligned} \frac{\partial \Psi}{\partial x_i}(x_n, x_{n-1}, \ldots, x_1) &= \frac{-e^{-x_i}\int_0^1 u^{r_i-1} e^{-x_i u}\,\mathrm{d}u + e^{-x_i}\int_0^1 u^{r_i} e^{-x_i u}\,\mathrm{d}u}{\left(\int_0^1 u^{r_i-1} e^{-x_i u}\,\mathrm{d}u\right)^2} \\ &= -\frac{\int_0^1 u^{r_i-1}(1-u) e^{-x_i(1+u)}\,\mathrm{d}u}{\left(\int_0^1 u^{r_i-1} e^{-x_i u}\,\mathrm{d}u\right)^2} \le 0. \end{aligned}$$

Similarly,

$$\frac{\partial \Psi}{\partial x_j}(x_n, x_{n-1}, \ldots, x_1) = -\frac{\int_0^1 u^{r_j-1}(1-u) e^{-x_j(1+u)}\,\mathrm{d}u}{\left(\int_0^1 u^{r_j-1} e^{-x_j u}\,\mathrm{d}u\right)^2}.$$

Note that

$$\begin{aligned}
&\frac{\partial \Psi}{\partial x_i}(x_n, x_{n-1}, \ldots, x_1) - \frac{\partial \Psi}{\partial x_j}(x_n, x_{n-1}, \ldots, x_1) \\
&= \frac{\int_0^1 u^{r_j-1}(1-u)e^{-x_j(1+u)}\,\mathrm{d}u}{\left(\int_0^1 u^{r_j-1}e^{-x_j u}\,\mathrm{d}u\right)^2} - \frac{\int_0^1 u^{r_i-1}(1-u)e^{-x_i(1+u)}\,\mathrm{d}u}{\left(\int_0^1 u^{r_i-1}e^{-x_i u}\,\mathrm{d}u\right)^2} \\
&= \frac{\int_0^1 u^{r_j-1}(1-u)e^{-x_j(1+u)}\,\mathrm{d}u}{\left(\int_0^1 u^{r_j-1}e^{-x_j u}\,\mathrm{d}u\right)^2} - \frac{\int_0^1 u^{r_i-1}(1-u)e^{-x_j(1+u)}\,\mathrm{d}u}{\left(\int_0^1 u^{r_i-1}e^{-x_j u}\,\mathrm{d}u\right)^2} \\
&\quad + \left[\frac{\int_0^1 u^{r_i-1}(1-u)e^{-x_j(1+u)}\,\mathrm{d}u}{\left(\int_0^1 u^{r_i-1}e^{-x_j u}\,\mathrm{d}u\right)^2} - \frac{\int_0^1 u^{r_i-1}(1-u)e^{-x_i(1+u)}\,\mathrm{d}u}{\left(\int_0^1 u^{r_i-1}e^{-x_i u}\,\mathrm{d}u\right)^2}\right] \\
&\leq 0,
\end{aligned}$$

where the inequality is based on Lemma 3.3 for $0 < r_j \leq r_i$ and $x_i \leq x_j$, for all $1 \leq i \leq j \leq n$. Hence, the theorem is proved. ■

Remark 3.5: It is worth noting that Theorem 3.4 partially generalizes (4) [20]

In the next, we implement stochastic comparisons under different shape parameters by means of the reversed hazard rate ordering.

THEOREM 3.6: *Let $X_1, X_2, \ldots, X_n$ be independent gamma random variables with the shape parameter vector $\boldsymbol{r} = (r_1, \ldots, r_n)$ and the scale parameter vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent gamma random variables with the shape parameter vector $\boldsymbol{r^*} = (r_1^*, \ldots, r_n^*)$ and the scale parameter vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Suppose that $(\boldsymbol{r}, \boldsymbol{r^*}) \in \mathcal{S}_n$ and $(\boldsymbol{r}, \boldsymbol{\lambda}) \in \mathcal{U}_n$. Then,*

$$\boldsymbol{r} \succeq_{\mathrm{w}} \boldsymbol{r^*} \Longrightarrow X_{n:n} \geq_{\mathrm{rh}} Y_{n:n}.$$

PROOF: Without loss of generality, it is supposed that $r_1 \geq r_2 \geq \cdots \geq r_n$, $r_1^* \geq r_2^* \geq \cdots \geq r_n^*$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Let $\tilde{r}_{X_{n:n}}(t)[\tilde{r}_{Y_{n:n}}(t)]$ be the reversed hazard rate function of $X_{n:n}[Y_{n:n}]$. Then,

$$\tilde{r}_{X_{n:n}}(t) = \sum_{i=1}^{n} \frac{e^{-\lambda_i t}}{\int_0^1 t u^{r_i-1} e^{-\lambda_i t u}\,\mathrm{d}u} \quad \text{and} \quad \tilde{r}_{Y_{n:n}}(t) = \sum_{i=1}^{n} \frac{e^{-\lambda_i t}}{\int_0^1 t u^{r_i^*-1} e^{-\lambda_i t u}\,\mathrm{d}u}.$$

To get $\tilde{r}_{X_{n:n}}(t) \geq \tilde{r}_{Y_{n:n}}(t)$, we need to prove

$$K(r_1, r_2, \ldots, r_n) = \sum_{i=1}^{n} \frac{1}{\int_0^1 u^{r_i-1} e^{y_i(1-u)}\,\mathrm{d}u} \geq \sum_{i=1}^{n} \frac{1}{\int_0^1 u^{r_i^*-1} e^{y_i(1-u)}\,\mathrm{d}u} = K(r_1^*, r_2^*, \ldots, r_n^*),$$

where $y_i = \lambda_i t$ and satisfy the restriction $y_1 \leq y_2 \leq \cdots \leq y_n$. According to Lemma 2.3, we need to prove that the derivative function

$$\frac{\partial K}{\partial r_i}(r_1, r_2, \ldots, r_n), \quad i = 1, 2, \ldots, n$$

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of differentiable function $K : \mathcal{D}_{\boldsymbol{r}} = \{(r_1, r_2, \ldots, r_n) : r_1 \geq r_2 \geq \cdots \geq r_n\} \to \mathbb{R}_{++}$ is decreasing in $i = 1, 2, \ldots, n$ and positive, that is,

$$\frac{\partial K}{\partial r_i}(r_1, r_2, \ldots, r_n) \geq \frac{\partial K}{\partial r_j}(r_1, r_2, \ldots, r_n) \geq 0 \quad \text{for all } n \geq j \geq i \geq 1.$$

To reach the above result, we just need to show that

$$k(r_i, y_i) = \frac{\int_0^1 u^{r_i-1} e^{y_i(1-u)} \ln u \, \mathrm{d}u}{\left(\int_0^1 u^{r_i-1} e^{y_i(1-u)} \, \mathrm{d}u\right)^2} \leq \frac{\int_0^1 u^{r_j-1} e^{y_j(1-u)} \ln u \, \mathrm{d}u}{\left(\int_0^1 u^{r_j-1} e^{y_j(1-u)} \, \mathrm{d}u\right)^2} = k(r_j, y_j),$$

for all pairs $n \geq j \geq i \geq 1$. Note that

$$k(r_i, y_i) - k(r_j, y_j) = k(r_i, y_i) - k(r_j, y_i) + k(r_j, y_i) - k(r_j, y_j). \tag{7}$$

Therefore, it is sufficient for us to prove that $k(r, y)$ is decreasing in $r \in \mathbb{R}_{++}$ and increasing in $y \in \mathbb{R}_{++}$, respectively, to make sure that equation (7) is nonpositive. Firstly, the decreasing property of $k(r, y)$ with respect to r is proved by Theorem 3.2 of Zhang and Zhao [29]. Next, we will show $k(r, y)$ is increasing in $y \in \mathbb{R}_{++}$. Observe that

$$k(r, y) = \frac{\int_0^1 u^{r-1} e^{y(1-u)} \ln u \, \mathrm{d}u}{\left(\int_0^1 u^{r-1} e^{y(1-u)} \, \mathrm{d}u\right)^2} = \frac{\int_0^1 (1-u)^{r-1} e^{yu} \ln(1-u) \, \mathrm{d}u}{\left(\int_0^1 (1-u)^{r-1} e^{yu} \, \mathrm{d}u\right)^2}$$

and

$$\ln[-k(r, y)] = \ln\left[-\int_0^1 (1-u)^{r-1} e^{yu} \ln(1-u) \, \mathrm{d}u\right] - 2 \ln\left[\int_0^1 (1-u)^{r-1} e^{yu} \, \mathrm{d}u\right].$$

By taking the derivative of $\ln[-k(r, y)]$ with respect to y, one can see

$$\begin{aligned}
\frac{\partial \ln[-k(r, y)]}{\partial y} &= \frac{\int_0^1 u(1-u)^{r-1} e^{yu} \ln(1-u) \, \mathrm{d}u}{\int_0^1 (1-u)^{r-1} e^{yu} \ln(1-u) \, \mathrm{d}u} - \frac{2\int_0^1 u(1-u)^{r-1} e^{yu} \, \mathrm{d}u}{\int_0^1 (1-u)^{r-1} e^{yu} \, \mathrm{d}u} \\
&= \Bigg[\frac{\int_0^1 u(1-u)^{r-1} e^{yu} \ln(1-u) \, \mathrm{d}u}{\int_0^1 (1-u)^{r-1} e^{yu} \, \mathrm{d}u} \\
&\quad - \frac{2\int_0^1 (1-u)^{r-1} e^{yu} \ln(1-u) \mathrm{d}u \int_0^1 u(1-u)^{r-1} e^{yu} \mathrm{d}u}{\left(\int_0^1 (1-u)^{r-1} e^{yu} \mathrm{d}u\right)^2}\Bigg] \\
&\quad \times \frac{\int_0^1 (1-u)^{r-1} e^{yu} \mathrm{d}u}{\int_0^1 (1-u)^{r-1} e^{yu} \ln(1-u) \mathrm{d}u} \\
&= \frac{\int_0^1 (1-u)^{r-1} e^{yu} \mathrm{d}u}{\int_0^1 (1-u)^{r-1} e^{yu} \ln(1-u) \mathrm{d}u} \times \left[\mathbb{E}(U \ln(1-U)) - 2\mathbb{E}U \mathbb{E} \ln(1-U)\right],
\end{aligned} \tag{8}$$

where the random variable U has the density function defined as follows:

$$f_U(u|r, y) = \frac{(1-u)^{r-1} e^{yu}}{\int_0^1 (1-u)^{r-1} e^{yu} \, \mathrm{d}u}, \quad u \in (0, 1).$$

According to Lemma 3.2, U has IHR, and hence, the random variable $-\ln(1-U)$ also has IHR. Then, it follows that

$$\mathbb{E}U^2 \leq 2(\mathbb{E}U)^2 \quad \text{and} \quad \mathbb{E}(\ln(1-U))^2 \leq 2(\mathbb{E}\ln(1-U))^2. \tag{9}$$

Based on Cauchy–Schwarz inequality, it holds

$$(\mathbb{E}U\ln(1-U))^2 \leq \mathbb{E}U^2\mathbb{E}(\ln(1-U))^2. \tag{10}$$

Combining (9) with (10), we have

$$(\mathbb{E}U\ln(1-U))^2 \leq \mathbb{E}U^2\mathbb{E}(\ln(1-U))^2 \leq 4(\mathbb{E}U)^2(\mathbb{E}\ln(1-U))^2,$$

which is

$$2\mathbb{E}U\mathbb{E}\ln(1-U) \leq \mathbb{E}U\ln(1-U)$$

and this implies that the right hand of (8) is nonnegative. Therefore, $k(r,y)$ is increasing in $y \in \mathbb{R}_{++}$. To sum up, the proof is completed. ■

Next, we present the main result of this section in terms of the reversed hazard rate ordering under some mild and sufficient conditions.

Theorem 3.7: *Let $X_1, X_2, \ldots, X_n$ be independent gamma random variables with the shape parameter vector $\boldsymbol{r} = (r_1, \ldots, r_n)$ and the scale parameter vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent gamma random variables with the shape parameter vector $\boldsymbol{r^*} = (r_1^*, \ldots, r_n^*)$ and the scale parameter vector $\boldsymbol{\lambda^*} = (\lambda_1^*, \ldots, \lambda_n^*)$. Suppose that $(\boldsymbol{\lambda}, \boldsymbol{\lambda^*}) \in \mathcal{S}_n$, $(\boldsymbol{r}, \boldsymbol{r^*}) \in \mathcal{S}_n$ and $(\boldsymbol{r}, \boldsymbol{\lambda}) \in \mathcal{U}_n$. Then,*

$$\boldsymbol{\lambda} \overset{\text{w}}{\succeq} \boldsymbol{\lambda^*}, \quad \boldsymbol{r} \succeq_{\text{w}} \boldsymbol{r^*} \Longrightarrow X_{n:n} \geq_{\text{rh}} Y_{n:n}.$$

Proof: Without loss of generality, assume that $r_1 \geq r_2 \geq \cdots \geq r_n > 0$, $r_1^* \geq r_2^* \geq \cdots \geq r_n^* > 0$, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*$. Let $Z_1, Z_2, \ldots, Z_n$ be a set of independent gamma random variables with Z_i having the shape and scale parameters r_i and λ_i^*, respectively. In accordance with Theorem 3.4, it can be obtained that $X_{n:n} \geq_{\text{rh}} Z_{n:n}$. On the other hand, we can see that $Z_{n:n} \geq_{\text{rh}} Y_{n:n}$ by using Theorem 3.6. Hence, it holds that $X_{n:n} \geq_{\text{rh}} Y_{n:n}$. ■

Now, we will give a numerical example to illustrate the results of Theorems 3.4, 3.6 and 3.7.

Example 3.8: *Let $\boldsymbol{X} = (X_1, X_2, X_3)$ be a vector of independent gamma random variables with the shape parameters $(r_1, r_2, r_3) = (2.1, 1.6, 0.1)$ and the scale parameters $(\lambda_1, \lambda_2, \lambda_3) = (0.8, 1.3, 2.4)$, let $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$ be another set of independent gamma random variables with the shape parameters $(r_1^*, r_2^*, r_3^*) = (1.8, 1.5, 0.3)$ and the scale parameters $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1, 1.2, 3.2)$, and let $\boldsymbol{Z} = (Z_1, Z_2, Z_3)$ be a set of independent gamma random variables with the shape parameters (r_1, r_2, r_3) and the scale parameters $(\lambda_1^*, \lambda_2^*, \lambda_3^*)$. It can be seen that $(2.1, 1.6, 0.1) \succeq_{\text{w}} (1.8, 1.5, 0.3)$ and $(0.8, 1.3, 2.4) \overset{\text{w}}{\succeq} (1, 1.2, 3.2)$, which is in accordance with the conditions of Theorem 3.7. Let $u \in (0,1)$ and $t = -\log(u)$. Figure 1 plots the difference of the reversed hazard rate functions $\tilde{r}_{X_{3:3}}(t(u)) - \tilde{r}_{Z_{3:3}}(t(u))$ and $\tilde{r}_{Z_{3:3}}(t(u)) - \tilde{r}_{Y_{3:3}}(t(u))$ for $t = -\log(u)$ and $u \in (0,1)$. We can observe that the inequality chain $\tilde{r}_{Y_{3:3}}(t(u)) \leq$*

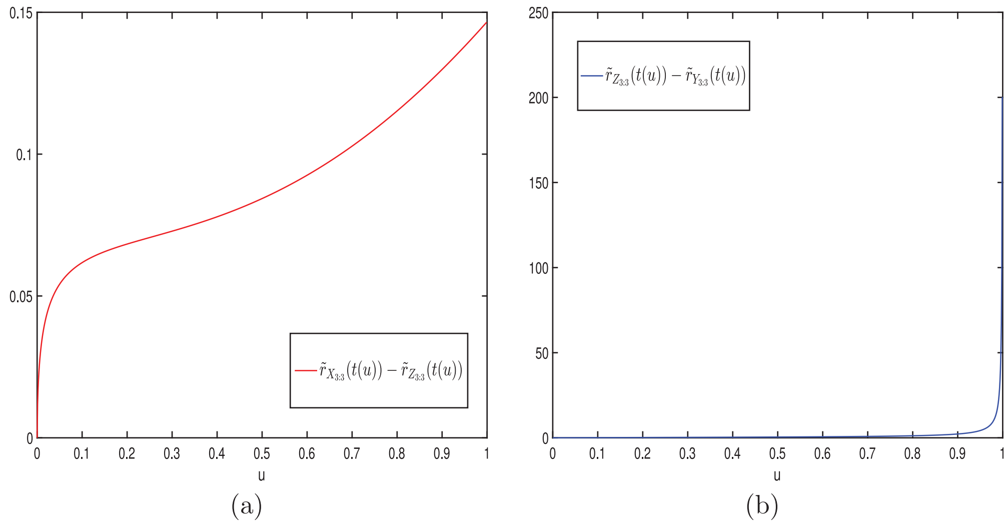

FIGURE 1. (a) Plot of the difference function $\tilde{r}_{X_{3:3}}(t(u)) - \tilde{r}_{Z_{3:3}}(t(u))$ for $u \in (0,1)$. (b) Plot of the difference function $\tilde{r}_{Z_{3:3}}(t(u)) - \tilde{r}_{Y_{3:3}}(t(u))$ for $u \in (0,1)$.

$\tilde{r}_{Z_{3:3}}(t(u)) \leq \tilde{r}_{X_{3:3}}(t(u))$ *always holds for* $u \in (0,1)$*, which confirms the statement of the theoretical results established in Theorems 3.4, 3.6 and 3.7.*

One may wonder whether the likelihood ratio ordering holds under the assumptions of Theorem 3.7. For the special case of $n = 2$, the result has been verified by Zhao and Zhang [33] when the submajorization order "$x \succeq_{\rm w}$" is replaced by "$\overset{\rm m}{\succeq}$" between the vectors of shape parameters with other conditions unchanged. However, the likelihood ratio ordering may not hold when $n \geq 3$. The following example is provided to explain this point.

EXAMPLE 3.9: *Let* $\boldsymbol{X} = (X_1, X_2, X_3)$ *be a group of independent gamma random variables with the shape parameters* $(r_1, r_2, r_3) = (0.9, 0.5, 0.1)$ *and the scale parameters* $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 1, 99)$*, and let* $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$ *be another group of independent gamma random variables with the shape parameters* $(r_1^*, r_2^*, r_3^*) = (0.5, 0.4, 0.05)$ *and the scale parameters* $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1, 10, 91)$*. We then have* $(0.9, 0.5, 0.1) \succeq_{\rm w} (0.5, 0.4, 0.05)$ *and* $(0.1, 1, 99) \overset{\rm w}{\succeq} (1, 10, 91)$*. Figure 2 plots the ratio function of the density functions* $f_{X_{3:3}}(t)$ *and* $f_{Y_{3:3}}(t)$*. It can be seen that the function* $f_{X_{3:3}}(t)/f_{Y_{3:3}}(t)$ *is not monotonic in* $t \in \mathbb{R}_+$*, which means that the likelihood ratio ordering does not hold between* $X_{3:3}$ *and* $Y_{3:3}$*.*

Note that the conditions in Theorem 3.7 require that the shape parameters are arranged in the opposite direction to the scale parameters. It is natural to ask whether these requirements are necessary. The following example tells us that these restrictions cannot be ignored.

EXAMPLE 3.10: *Let* $\boldsymbol{X} = (X_1, X_2, X_3)$ *be a vector of independent gamma random variables with the shape parameters* $(r_1, r_2, r_3) = (0.99, 0.3, 0.5)$ *and the scale parameters* $(\lambda_1, \lambda_2, \lambda_3) = (2.5, 0.8, 1.3)$*, and let* $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$ *be another vector of independent gamma random variables with the shape parameters* $(r_1^*, r_2^*, r_3^*) = (0.8, 0.6, 0.2)$ *and the scale parameters* $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1, 3.2, 1.2)$*. Then, we have* $(0.99, 0.3, 0.5) \succeq_{\rm w} (0.8, 0.6, 0.2)$

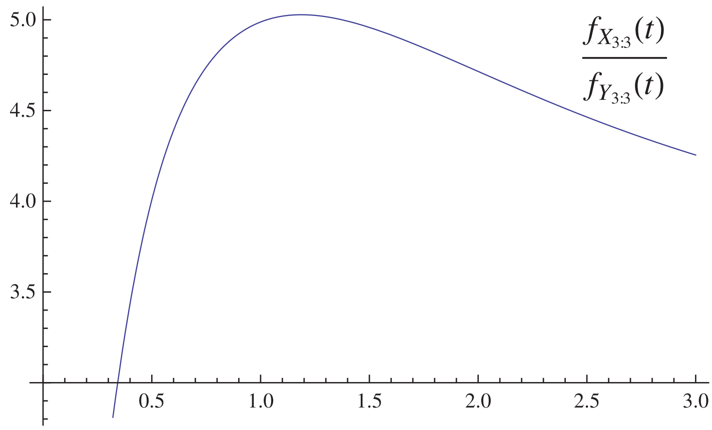

FIGURE **2.** Plot of the ratio function between $f_{X_{3:3}}(t)$ and $f_{Y_{3:3}}(t)$ when $(r_1, r_2, r_3) = (0.9, 0.5, 0.1)$, $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 1, 99)$, $(r_1^*, r_2^*, r_3^*) = (0.5, 0.4, 0.05)$ and $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1, 10, 91)$.

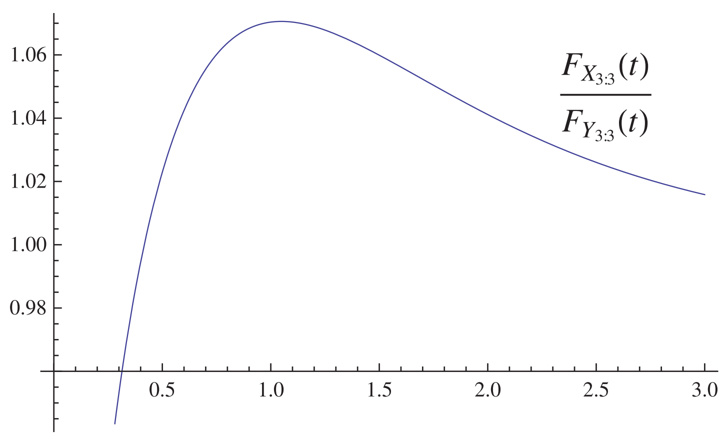

FIGURE **3.** Plot of the ratio $F_{X_{3:3}}(t)$ and $F_{Y_{3:3}}(t)$ when $(r_1, r_2, r_3) = (0.99, 0.3, 0.5)$, $(\lambda_1, \lambda_2, \lambda_3) = (2.5, 0.8, 1.3)$, $(r_1^*, r_2^*, r_3^*) = (0.8, 0.6, 0.2)$ and $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1, 3.2, 1.2)$.

and $(2.5, 0.8, 1.3) \overset{\text{w}}{\succeq} (1, 3.2, 1.2)$. *However,* $(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*) \notin \mathcal{S}_n$ *and* $(\boldsymbol{r}, \boldsymbol{r}^*) \notin \mathcal{S}_n$. *Figure 3 plots the ratio function of distribution functions* $F_{X_{3:3}}(t)$ *and* $F_{Y_{3:3}}(t)$. *It can be seen that the function* $F_{X_{3:3}}(t)/F_{Y_{3:3}}(t)$ *is not monotonic in* $t \in \mathbb{R}_+$, *which means the reversed hazard rate ordering does not hold between* $X_{3:3}$ *and* $Y_{3:3}$.

4. THE USUAL STOCHASTIC ORDERING

In this section, we carry out stochastic comparisons on the largest order statistics arising from two sets of independent and heterogeneous gamma random variables in terms of the usual stochastic ordering. Firstly, two useful lemmas are given as follows, which are very helpful to the proofs of the main results.

LEMMA 4.1: *For any* $\lambda \in \mathbb{R}_{++}$*, the function*

$$\psi(r) = r - \frac{\int_0^1 \lambda u^r e^{-\lambda u}\,\mathrm{d}u}{\int_0^1 u^{r-1} e^{-\lambda u}\,\mathrm{d}u}$$

is increasing in $r \in \mathbb{R}_{++}$.

PROOF: Note that

$$\psi(r) = r - \frac{\int_0^1 \lambda u^r e^{-\lambda u}\,\mathrm{d}u}{\int_0^1 u^{r-1} e^{-\lambda u}\,\mathrm{d}u} = r - \frac{r\int_0^1 u^{r-1} e^{-\lambda u}\,\mathrm{d}u - e^{-\lambda}}{\int_0^1 u^{r-1} e^{-\lambda u}\,\mathrm{d}u} =: \frac{1}{\eta(r)},$$

where

$$\eta(r) = \int_0^1 u^{r-1} e^{\lambda(1-u)}\,\mathrm{d}u.$$

Since $\eta'(r) \le 0$, we know that $\eta(r)$ is decreasing in $r \in \mathbb{R}_{++}$. Hence, it can been seen that $\psi(r)$ is increasing in $r \in \mathbb{R}_{++}$. ■

LEMMA 4.2: *For any* $r \in \mathbb{R}_{++}$*, the function*

$$\phi(\lambda) = \frac{\int_0^1 \lambda u^r e^{-\lambda u}\,\mathrm{d}u}{\int_0^1 u^{r-1} e^{-\lambda u}\,\mathrm{d}u}$$

is increasing in $\lambda \in \mathbb{R}_{++}$.

PROOF: Note that

$$\phi(\lambda) = \frac{\int_0^1 \lambda u^r e^{-\lambda u}\,\mathrm{d}u}{\int_0^1 u^{r-1} e^{-\lambda u}\,\mathrm{d}u} = \frac{r\int_0^1 u^{r-1} e^{-\lambda u}\,\mathrm{d}u - e^{-\lambda}}{\int_0^1 u^{r-1} e^{-\lambda u}\,\mathrm{d}u} =: r - \frac{1}{\varphi(\lambda)},$$

where

$$\varphi(\lambda) = \int_0^1 u^{r-1} e^{\lambda(1-u)}\,\mathrm{d}u.$$

Then, the increasing property of $\phi(\lambda)$ can be acquired due to the fact that $\varphi'(\lambda) \ge 0$. ■

Now, we show that the p-larger order between the vectors of scale parameters implies the usual stochastic ordering of the maximum order statistics from gamma samples.

THEOREM 4.3: *Let* $X_1, X_2, \ldots, X_n$ *be independent gamma random variables with the shape parameter* $\boldsymbol{r} = (r_1, \ldots, r_n)$ *and the scale parameter* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$*. Let* $Y_1, Y_2, \ldots, Y_n$ *be another set of independent gamma random variables with the shape parameter* $\boldsymbol{r} = (r_1, \ldots, r_n)$ *and the scale parameter* $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$*. Suppose that* $(\boldsymbol{\lambda}, \boldsymbol{\lambda}^*) \in \mathcal{S}_n$ *and* $(\boldsymbol{r}, \boldsymbol{\lambda}) \in \mathcal{U}_n$*. Then, we have*

$$\boldsymbol{\lambda} \overset{\mathrm{p}}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow X_{n:n} \ge_{\mathrm{st}} Y_{n:n}.$$

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Proof: Without loss of generality, we assume that $r_1 \geq r_2 \geq \cdots \geq r_n > 0$, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*$. The distribution function of $X_{n:n}$ can be written as follows:

$$F_{X_{n:n}}(t) = \prod_{i=1}^{n} \int_0^t \frac{\lambda_i^{r_i}}{\Gamma(r_i)} u^{r_i-1} e^{-\lambda_i u}\,\mathrm{d}u = \prod_{i=1}^{n} \int_0^1 \frac{(\lambda_i t)^{r_i}}{\Gamma(r_i)} u^{r_i-1} e^{-\lambda_i t u}\,\mathrm{d}u.$$

Similarly,

$$F_{Y_{n:n}}(t) = \prod_{i=1}^{n} \int_0^1 \frac{(\lambda_i^* t)^{r_i}}{\Gamma(r_i)} u^{r_i-1} e^{-\lambda_i^* t u}\,\mathrm{d}u.$$

To reach the desired result, we need to prove that $F_{X_{n:n}}(t) \leq F_{Y_{n:n}}(t)$, which is equivalent to

$$\prod_{i=1}^{n} \int_0^1 \frac{x_i^{r_i}}{\Gamma(r_i)} u^{r_i-1} e^{-x_i u}\,\mathrm{d}u \leq \prod_{i=1}^{n} \int_0^1 \frac{(x_i^*)^{r_i}}{\Gamma(r_i)} u^{r_i-1} e^{-x_i^* u}\,\mathrm{d}u \tag{11}$$

under the transformations $x_i = \lambda_i t$ and $x_i^* = \lambda_i^* t$, and the conditions $x_1 \leq x_2 \leq \cdots \leq x_n$, $x_1^* \leq x_2^* \leq \cdots \leq x_n^*$ and $(x_1, x_2, \ldots, x_n) \overset{\mathrm{p}}{\succeq} (x_1^*, x_2^*, \ldots, x_n^*)$. Let $y_i = \log x_i$ and $y_i^* = \log x_i^*$. Then, (11) is equivalent to proving

$$-\prod_{i=1}^{n} \int_0^1 \frac{e^{r_i y_i}}{\Gamma(r_i)} u^{r_i-1} e^{-e^{y_i} u}\,\mathrm{d}u \geq -\prod_{i=1}^{n} \int_0^1 \frac{e^{r_i y_i^*}}{\Gamma(r_i)} u^{r_i-1} e^{-e^{y_i^*} u}\,\mathrm{d}u \tag{12}$$

under the conditions $y_1 \leq y_2 \leq \cdots \leq y_n$, $y_1^* \leq y_2^* \leq \cdots \leq y_n^*$ and $(y_1, y_2, \ldots, y_n) \overset{\mathrm{w}}{\succeq} (y_1^*, y_2^*, \ldots, y_n^*)$. Upon using Lemma 2.3, we need to prove that the derivative functions

$$\frac{\partial \Phi}{\partial y_k}(y_n, y_{n-1}, \ldots, y_1), \quad k = 1, 2, \ldots, n$$

of differentiable function $\Phi : \mathcal{D}_{\overleftarrow{\boldsymbol{y}}} = \{(y_n, y_{n-1}, \ldots, y_1) : y_n \geq y_{n-1} \geq \cdots \geq y_1\} \to (-\infty, 0)$ given by

$$\Phi(y_n, y_{n-1}, \ldots, y_1) = -\prod_{i=1}^{n} \int_0^1 \frac{e^{r_i y_i}}{\Gamma(r_i)} u^{r_i-1} e^{-e^{y_i} u}\,\mathrm{d}u$$

satisfy that

$$0 \geq \frac{\partial \Phi}{\partial y_j}(y_n, y_{n-1}, \ldots, y_1) \geq \frac{\partial \Phi}{\partial y_i}(y_n, y_{n-1}, \ldots, y_1) \quad \text{for all } n \geq j \geq i \geq 1.$$

Observe that, for $1 \leq i \leq n$,

$$\begin{aligned}\frac{\partial \Phi}{\partial y_i}(y_n, y_{n-1}, \ldots, y_1) &= \Phi(y_n, y_{n-1}, \ldots, y_1) \frac{\int_0^1 \frac{u^{r_i-1}}{\Gamma(r_i)} \left(r_i e^{r_i y_i} e^{-e^{y_i} u} - e^{r_i y_i} e^{-e^{y_i} u} e^{y_i} u \right) \mathrm{d}u}{\int_0^1 \frac{e^{r_i y_i}}{\Gamma(r_i)} u^{r_i-1} e^{-e^{y_i} u}\,\mathrm{d}u} \\ &= \Phi(y_n, y_{n-1}, \ldots, y_1) \left(r_i - \frac{\int_0^1 e^{y_i} u^{r_i} e^{-e^{y_i} u}\,\mathrm{d}u}{\int_0^1 u^{r_i-1} e^{-e^{y_i} u}\,\mathrm{d}u} \right) \\ &= \frac{\Phi(y_n, y_{n-1}, \ldots, y_1)}{\int_0^1 u^{r_i-1} e^{e^{y_i}(1-u)}\,\mathrm{d}u} \leq 0.\end{aligned}$$

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Thus, it follows that, for $n \geq j \geq i \geq 1$,

$$\begin{aligned}
&\frac{\partial \Phi}{\partial y_i}(y_n, y_{n-1}, \ldots, y_1) - \frac{\partial \Phi}{\partial y_j}(y_n, y_{n-1}, \ldots, y_1) \\
&\overset{\text{sgn}}{=} r_j - \frac{\int_0^1 e^{y_j} u^{r_j} e^{-e^{y_j} u}\,\mathrm{d}u}{\int_0^1 u^{r_j - 1} e^{-e^{y_j} u}\,\mathrm{d}u} - r_i + \frac{\int_0^1 e^{y_i} u^{r_i} e^{-e^{y_i} u}\,\mathrm{d}u}{\int_0^1 u^{r_i - 1} e^{-e^{y_i} u}\,\mathrm{d}u} \\
&= r_j - \frac{\int_0^1 e^{y_i} u^{r_j} e^{-e^{y_i} u}\,\mathrm{d}u}{\int_0^1 u^{r_j - 1} e^{-e^{y_i} u}\,\mathrm{d}u} - \left(r_i - \frac{\int_0^1 e^{y_i} u^{r_i} e^{-e^{y_i} u}\,\mathrm{d}u}{\int_0^1 u^{r_i - 1} e^{-e^{y_i} u}\,\mathrm{d}u} \right) \\
&\quad + \frac{\int_0^1 e^{y_i} u^{r_j} e^{-e^{y_i} u}\,\mathrm{d}u}{\int_0^1 u^{r_j - 1} e^{-e^{y_i} u}\,\mathrm{d}u} - \frac{\int_0^1 e^{y_j} u^{r_j} e^{-e^{y_j} u}\,\mathrm{d}u}{\int_0^1 u^{r_j - 1} e^{-e^{y_j} u}\,\mathrm{d}u} \\
&=: A + B,
\end{aligned}$$

where

$$A = r_j - \frac{\int_0^1 e^{y_i} u^{r_j} e^{-e^{y_i} u}\,\mathrm{d}u}{\int_0^1 u^{r_j - 1} e^{-e^{y_i} u}\,\mathrm{d}u} - \left(r_i - \frac{\int_0^1 e^{y_i} u^{r_i} e^{-e^{y_i} u}\,\mathrm{d}u}{\int_0^1 u^{r_i - 1} e^{-e^{y_i} u}\,\mathrm{d}u} \right)$$

and

$$B = \frac{\int_0^1 e^{y_i} u^{r_j} e^{-e^{y_i} u}\,\mathrm{d}u}{\int_0^1 u^{r_j - 1} e^{-e^{y_i} u}\,\mathrm{d}u} - \frac{\int_0^1 e^{y_j} u^{r_j} e^{-e^{y_j} u}\,\mathrm{d}u}{\int_0^1 u^{r_j - 1} e^{-e^{y_j} u}\,\mathrm{d}u}.$$

Then, it can be verified that $A \leq 0$ by applying Lemma 4.1 and $B \leq 0$ according to Lemma 4.2, respectively. Thus, the desired result can be obtained from Lemma 2.3. ■

Remark 4.4: It can be seen that Theorem 4.3 extends (3) [14]

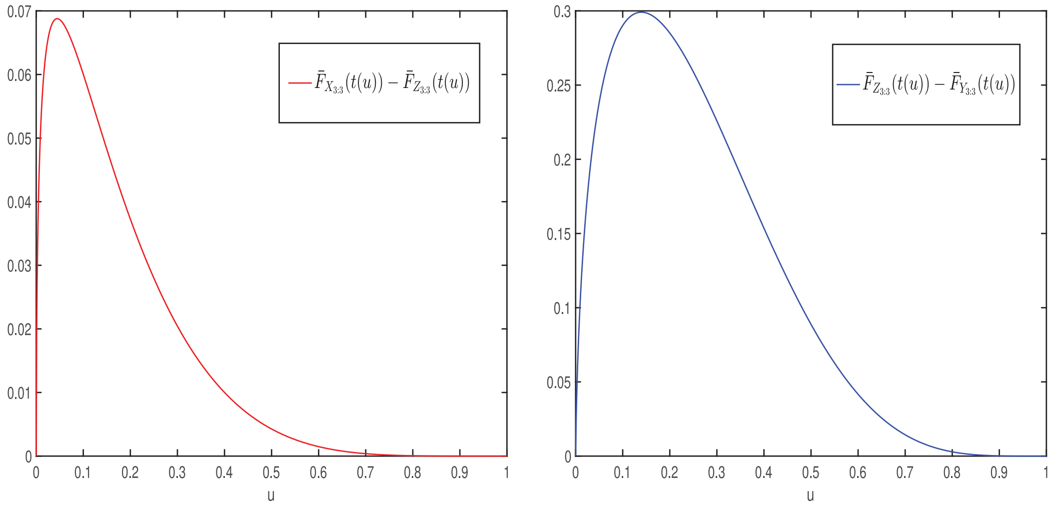
Now, we will present the main result in this section in terms of the usual stochastic ordering when both sets of the shape and scale parameters are heterogeneous.

THEOREM 4.5: *Let $X_1, X_2, \ldots, X_n$ be independent gamma random variables with the shape parameter $\boldsymbol{r} = (r_1, \ldots, r_n)$ and the scale parameter $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent gamma random variables with the shape parameter $\boldsymbol{r^*} = (r_1^*, \ldots, r_n^*)$ and the scale parameter $\boldsymbol{\lambda^*} = (\lambda_1^*, \ldots, \lambda_n^*)$. Suppose that $(\boldsymbol{\lambda}, \boldsymbol{\lambda^*}) \in \mathcal{S}_n$, $(\boldsymbol{r}, \boldsymbol{r^*}) \in \mathcal{S}_n$ and $(\boldsymbol{r}, \boldsymbol{\lambda}) \in \mathcal{U}_n$. Then,*

$$\boldsymbol{\lambda} \overset{\text{p}}{\succeq} \boldsymbol{\lambda^*}, \quad \boldsymbol{r} \succeq_{\text{w}} \boldsymbol{r^*} \Longrightarrow X_{n:n} \geq_{\text{st}} Y_{n:n}.$$

PROOF: Without loss of generality, it is assumed that $r_1 \geq r_2 \geq \cdots \geq r_n > 0$, $r_1^* \geq r_2^* \geq \cdots \geq r_n^* > 0$, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*$. Let $Z_1, Z_2, \ldots, Z_n$ be a set of independent gamma random variables with Z_i having the shape and the scale parameters r_i and λ_i^*, respectively. In accordance with Theorem 4.3, we have $X_{n:n} \geq_{\text{st}} Z_{n:n}$. Also, we can see that $Z_{n:n} \geq_{\text{rh}} Y_{n:n}$ by applying Theorem 3.6, which implies that $Z_{n:n} \geq_{\text{st}} Y_{n:n}$. Hence, it follows that $X_{n:n} \geq_{\text{st}} Y_{n:n}$. ■

To display the validity of Theorems 4.3 and 4.5, we will present a numerical example as follows.



(a) (b)

Figure 4. (a) Plot of the difference function $\overline{F}_{X_{3:3}}(t(u)) - \overline{F}_{Z_{3:3}}(t(u))$ for $u \in (0,1)$. (b) Plot of the difference function $\overline{F}_{Z_{3:3}}(t(u)) - \overline{F}_{Y_{3:3}}(t(u))$ for $u \in (0,1)$.

Example 4.6: *Let* $\boldsymbol{X} = (X_1, X_2, X_3)$ *be a collection of independent gamma random variables with the shape parameters* $(r_1, r_2, r_3) = (2.9, 1.7, 0.3)$ *and the scale parameters* $(\lambda_1, \lambda_2, \lambda_3) = (0.9, 2.7, 3)$, *and let* $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$ *be another collection of independent gamma random variables with the shape parameters* $(r_1^*, r_2^*, r_3^*) = (1.8, 1.6, 1.1)$ *and the scale parameters* $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1, 2.5, 3.2)$. *Let* $\boldsymbol{Z} = (Z_1, Z_2, Z_3)$ *be a collection of independent gamma random variables with the shape parameters* (r_1, r_2, r_3) *and the scale parameters* $(\lambda_1^*, \lambda_2^*, \lambda_3^*)$. *Note that* $(2.9, 1.7, 0.3) \succeq_{\mathrm{w}} (1.8, 1.6, 1.1)$, $(0.9, 2.7, 3) \overset{\mathrm{p}}{\succeq} (1, 2.5, 3.2)$ *and* $(0.9, 2.7, 3) \underset{\mathrm{w}}{\not\succeq} (1, 2.5, 3.2)$. *Let* $u \in (0,1)$ *and* $t = -\log(u)$. *Figure 4 plots the difference of the survival functions* $\overline{F}_{X_{3:3}}(t(u)) - \overline{F}_{Z_{3:3}}(t(u))$ *and* $\overline{F}_{Z_{3:3}}(t(u)) - \overline{F}_{Y_{3:3}}(t(u))$, *respectively. It can be observed that* $\overline{F}_{Y_{3:3}}(t(u)) \le \overline{F}_{Z_{3:3}}(t(u)) \le \overline{F}_{X_{3:3}}(t(u))$ *always holds for* $u \in (0,1)$, *which proves the effectiveness of both Theorems 4.3 and 4.5.*

One may wonder whether the hazard rate ordering holds under the assumptions of Theorem 4.5. A counterexample is presented as follows.

Example 4.7: *Let* $\boldsymbol{X} = (X_1, X_2, X_3)$ *be a set of independent gamma random variables with the shape parameters* $(r_1, r_2, r_3) = (0.9, 0.5, 0.1)$ *and the scale parameters* $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 15, 100)$, *and let* $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$ *be another set of independent gamma random variables with the shape parameters* $(r_1^*, r_2^*, r_3^*) = (0.5, 0.4, 0.05)$ *and the scale parameters* $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1, 10, 91)$. *We then have* $(0.9, 0.5, 0.1) \succeq_{\mathrm{w}} (0.5, 0.4, 0.05)$ *and* $(0.1, 15, 100) \overset{\mathrm{p}}{\succeq} (1, 10, 91)$. *Figure 5 plots the ratio function of the survival functions* $\overline{F}_{X_{3:3}}(t)$ *and* $\overline{F}_{Y_{3:3}}(t)$. *Note that* $\overline{F}_{X_{3:3}}(t)/\overline{F}_{Y_{3:3}}(t)$ *is not monotonic in* $t \in \mathbb{R}_+$, *which means the hazard rate ordering does not hold between* $X_{3:3}$ *and* $Y_{3:3}$.

5. APPLICATIONS

In this section, we present two explicit scenarios where our main results in both Sections 3 and 4 can be applied to analyze effects of the heterogeneity among the shape and scale

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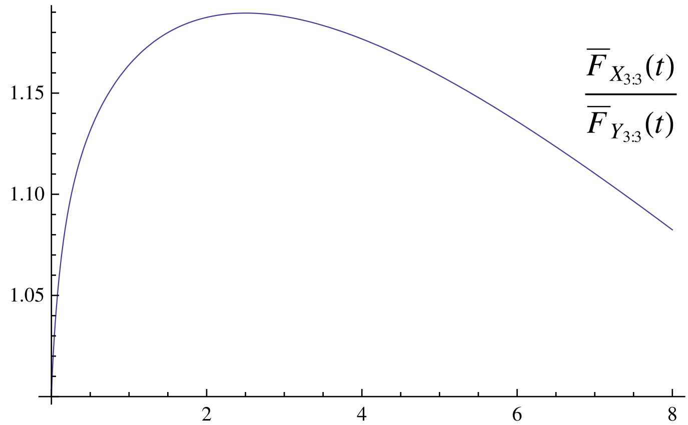

FIGURE 5. Plot of the ratio function between $\overline{F}_{X_{3:3}}(t)$ and $\overline{F}_{Y_{3:3}}(t)$ when $(r_1, r_2, r_3) = (0.9, 0.5, 0.1)$, $(\lambda_1, \lambda_2, \lambda_3) = (0.1, 15, 100)$, $(r_1^*, r_2^*, r_3^*) = (0.5, 0.4, 0.05)$ and $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (1, 10, 91)$.

parameters of gamma random variables on the stochastic properties of the maximum order statistics.

5.1. Auction Theory

Auctions have a long history and can be recorded as early as 500 BC. Nowadays, auctions play a vital role in economy and can be adopted to sell goods or services by offering them up for bid, taking bids, and then selling the item to the highest bidder. In general, auction can be classified into *open ascending price auction* (English auction), *open descending price auction* (Dutch auction), *sealed first-price auction* (FPA), *sealed second-price auction* (SPA) and so on (see the monograph Krishna [16]). Recently, there have been some recent works concentrated on the effect of bidders' asymmetries on revenue in FPA and SPA (see [8,9,12]). For example, Chen and Xu studied the effect of bidder asymmetry on the revenue in SPA with the help of the useful tool of majorization.

FPA is commonly used in practical scenarios for the sale of an item such as a piece of precious painting. For this kind of auction, all bidders submit sealed bids simultaneously so that no bidder knows the bidding price of any other participants. The highest bidder wins the item and pays the price that they submitted to the auctioneer.

Suppose that an auctioneer is ready to auction an antique by employing FPA. There are n people who do not know each other coming to bid for the item. It is assumed that the bid price for each person is a gamma random variable with different shape and scale parameters. The results established in Theorems 3.7 and 4.5 state that the more heterogeneity between the prices of bidders will lead to a higher final price (denoted by $X_{n:n}$) in the FPA, which will be beneficial for the auctioneer.

5.2. Minimal Repairs

Consider a parallel system comprised of n exponential components in a factory, and these n components have hazard rates $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. In order to improve the reliability of the system, an economical way is to conduct minimal repairs (see [25]) immediately when the original components fail. For such kind of parallel system, each of the components $X_1, \ldots, X_n$ is assumed to be allocated k_i minimal repairs in advance, for $i = 1, 2, \ldots, n$,

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where $\sum_{i=1}^{n} k_i = K$ and K is the total number of available minimal repairs. Now, an important question for system engineers is to determine how to allocate these minimal repairs among the components.

Intuitively, all minimal repairs should be put in the node with the component having the smallest hazard rate parameter (c.f. [25]). Denote by $X_i(k_i)$, the lifetime of component X_i assembled with k_i minimal repairs. It is known from the Gamma–Poisson relationship that

$$F_{X_i(k_i)}(t) = \mathbb{P}(X_i(k_i) \le t) = \sum_{j=k_i+1}^{\infty} \frac{e^{-\lambda_i t}(\lambda_i t)^j}{j!} = \int_0^t \frac{\lambda_i^{k_i+1}}{\Gamma(k_i+1)} x^{k_i} e^{-\lambda_i x}\, \mathrm{d}t.$$

Thus, X_i with k_i minimal repairs has a gamma distribution with the scale parameter λ_i and the shape parameter $k_i + 1$, that is, $\Gamma(k_i + 1, \lambda_i)$ for $i = 1, 2, \ldots, n$. Then, the lifetime of the resulting system under policy $\boldsymbol{k}$ can be expressed as the maximum of n gamma random variables with the shape and scale parameters $(k_1 + 1, k_2 + 1, \ldots, k_n + 1)$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Recently, by employing the Gamma–Poisson relationship, Zhang and Zhao [29] studied optimal allocation strategies of minimal repairs for parallel and series systems with i.i.d. components in the sense of the hazard rate, the reversed hazard rate and the likelihood ratio orderings.

Now, we assume that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. For two allocation policies $\boldsymbol{k}$ and $\boldsymbol{k}^*$ such that $k_1 \ge k_2 \ge \cdots \ge k_n$, $k_1^* \ge k_2^* \ge \cdots \ge k_n^*$ and $\boldsymbol{k} \overset{\mathrm{m}}{\succeq} \boldsymbol{k}^*$, we can conclude that the allocation policy $\boldsymbol{k}$ is better than $\boldsymbol{k}^*$ in accordance with Theorem 3.6. Therefore, it can be claimed that the optimal allocation policy for a parallel system with exponential components having hazard rates $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ must be $(K, 0, \ldots, 0)$.

6. CONCLUDING REMARKS

In this paper, we have investigated the ordering properties of the largest order statistics arising from independent and heterogeneous gamma samples. Let $X_1, X_2, \ldots, X_n$ be a set of independent gamma random variables with the vector of shape parameter $\boldsymbol{r} = (r_1, \ldots, r_n)$ and the vector of scale parameter $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Let $Y_1, Y_2, \ldots, Y_n$ be another set of independent gamma random variables with the vector of shape parameter $\boldsymbol{r^*} = (r_1^*, \ldots, r_n^*)$ and the vector of scale parameter $\boldsymbol{\lambda^*} = (\lambda_1^*, \ldots, \lambda_n^*)$. Suppose that $(\boldsymbol{\lambda}, \boldsymbol{\lambda^*}) \in \mathcal{S}_n$, $(\boldsymbol{r}, \boldsymbol{r^*}) \in \mathcal{S}_n$ and $(\boldsymbol{r}, \boldsymbol{\lambda}) \in \mathcal{U}_n$. It has been shown that, for all $1 \le i \le n$,

$$\boldsymbol{\lambda} \overset{\mathrm{w}}{\succeq} \boldsymbol{\lambda^*}, \quad \boldsymbol{r} \succeq_{\mathrm{w}} \boldsymbol{r^*} \Longrightarrow X_{n:n} \ge_{\mathrm{rh}} Y_{n:n}. \tag{13}$$

Besides, we also prove that

$$\boldsymbol{\lambda} \overset{\mathrm{p}}{\succeq} \boldsymbol{\lambda^*}, \quad \boldsymbol{r} \succeq_{\mathrm{w}} \boldsymbol{r^*} \Longrightarrow X_{n:n} \ge_{\mathrm{st}} Y_{n:n}. \tag{14}$$

We also present some real applications in auction theory, reliability system and minimal repairs to address the importance of our main results.

It is of interest to check whether similar ordering results can be obtained for the smallest order statistics under some sufficient conditions as stated in (13) and (14). On the other hand, Zhao and Zhang [33] established the likelihood ratio ordering under the assumption of (13) for $n = 2$. It is natural to examine whether their results could be extended to the case of multiple-outlier gamma models. We are currently working on these problems and hope to report some interesting findings in a future paper.

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