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ENERGY INTENSITY OF INERTIAL WAVES IN A SPHERE

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Abstract

The decay at large wavenumbers of the energy density in an inertial wave generated in a sphere by an arbitrary initial disturbance is determined as a first step to a comparison with the general theory of Phillips [17] for a statistically steady field of random inertial waves in an arbitrary cavity.

1. Introduction

Just over a century ago, Lord Kelvin [11] established that enclosed rotating fluids with uniform density and angular velocity (Ω) can sustain oscillations that arise from the restoring action of the Coriolis force.

The first studies of such oscillations were in geodynamics. The stability of the shape of the earth, and of other self-gravitating rotating bodies, was extensively studied by Poincaré [18] and Liapounoff [13] by treating the body as a uniform fluid oscillating about a uniform angular velocity. The oscillations of astronomically observed latitudes, of about 427 day period (Chandler [3]), and the slow precession of about 26,000 year period due to the sun and moon were investigated on the same basis, but with allowance for a solid outer shell (Hough [9], Poincaré [19]). A synopsis of the early work, in particular on the stability of the Jacobi ellipsoids (and of the conjecture that the pear-shaped modes might precede ultimate fission into two separate bodies, and thence explain binary stars), is given by Lyttleton [14].

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Subsequently, the possibility of inertial oscillations in the atmosphere was explored by Bjerknes and Solberg [1]. A spate of studies of the inertial effects pertaining to the atmosphere and oceans followed that is largely summarized in Greenspan's book [6] on rotating fluids. In this context, the radial thinness generally leads, in the first approximation, to somewhat specialized effectively two-dimensional modes. Other applications are the dynamics of fluid-filled rotating projectiles (reviewed by Rumyantsev [20]) and Malkus's proposal that the westward drift of the geomagnetic field stems from Alfvén waves in the liquid core that are governed by equations equivalent to those for inertial waves (Malkus [15], Hide [8], Wood [26]). The geodynamic context is pertinent also in that a satisfactory marriage remains to be made (Smith [21]) of elastic oscillations of the mantle (which are amenable to spherical harmonic analysis) and the inertial oscillations of the liquid core (which are not).

Unfortunately, the analysis of inertial oscillations in an enclosure has awkward complexities occasioned by the equation for the spatial variation of modes (varying say as $e^{i\omega t}$) being hyperbolic when $|\omega| < 2\Omega$. For example, the eigenfrequencies for a sphere or for a cylindrical can rotating about its central axis form a dense set in the range $|\omega| < 2\Omega$ and periodic forcing at any non-resonant frequency in this range leads to eigenexpansions whose convergence involves small divisors. Furthermore, forced oscillations in the cylindrical can that vary as $e^{ik\phi}$ with the azimuthal angle ϕ can have discontinuities on the characteristics through the corners. The characteristics are circular cones symmetric about the can's axis with a semi-angle $\alpha = \text{arsin}(\omega/2\Omega)$. When the forcing frequency is non-resonant and is such that $(\tan\alpha) \times$ (the can's height/width ratio) is a rational number, the characteristics cones from the corners close after repeated reflexions at the boundaries of the can. The relevant eigenexpansions then converge, but they generally yield discontinuities in the velocity or velocity gradient (Wood [25], McEwan [16]). The patterns of these discontinuity surfaces vary erratically as the forcing frequency changes. Allied discontinuities, this time in the free modes, are believed to arise also when the effectively two-dimensional modes of Laplace's tidal equations for a spherical annulus (Haurwitz [7]) are perturbed to allow for the annular thickness (Stewartson and Rickard [23]). A velocity discontinuity appears to develop at the two (critical) circles where the characteristic cones touch the inner spherical boundary and this discontinuity then propagates around the annulus by repeated reflexion of these characteristics at the spherical boundaries. Ray methods have indicated other discontinuous modes (coupled with continuous eigenfrequency bands) in a spherical annulus (Bretherton [2], Israeli [10], Stewartson [22]) and also in non-spherical axisymmetric containers (Wood [28]).

Real inertial waves are of course variously modified by stratification, non-linearity and viscosity. But the inherent fine structure presumably persists whenever these effects are all small, and it is of some interest therefore to explore a statistical approach. One such approach is to expand the flow variables in orthogonal series appropriate to the fluid boundaries and to treat the coefficients of the (truncated) series as the coordinates of a point in a dynamical system (Kraichnan [12], Frederiksen and Sawford [5]). Probability distributions for the coefficients can then be adduced from statistical mechanics and can be used, for example, to evaluate the expectation of integrated flow quantities that depend only on these coefficients. This approach requires the governing (fluid dynamic) equations to be non-linear otherwise the motion of the effective phase points with time is insufficiently ergodic. A second approach which applies within the linear approximation but requires the motion to have a length scale small relative to that of the enclosure is to treat the motion as an assembly of locally plane waves (Phillips [17]). At the boundaries, the waves reflect about the angular velocity (rather than the normal to the boundary) and change wavenumber. After many reflexions in an enclosure whose boundaries are not everywhere either perpendicular or normal to Ω, a particular wave acquires a randomness of location and wavenumber. Phillips envisaged a statistically steady field of random inertial waves energized at low wavenumbers in a viscous fluid with a view to deriving a decay law for the energy spectrum. To achieve this he proposed that the net energy transferred from the scalar wavenumbers $K < K_0$ to the higher scalar wavenumbers $K > K_0$ was equal, for any large enough K_0, to the energy dissipated throughout the enclosure at the higher wavenumbers $K > K_0$. This energy transfer is effected (linearly) by reflexion, rather than by non-linear interaction. Otherwise, the balance proposed is like that advanced to derive Kolmogorov's decay law for turbulence.

The linearity allows this latter theory to be compared with particular solutions. In this note, the evolution of an arbitrary initial disturbance in a sphere is considered for an inviscid fluid as a preliminary step to such a comparison. The energy transfer hypothesis holds in this case (in the degenerate form that the energy transfer from scalar wavenumbers $K < K_0$ is zero for large K_0) but it does not determine the energy distribution for large K. Phillips' derivation appears to imply that the energy density I_k defined in Section 3 is constant for large scalar wavenumbers K. In fact, this energy density decays as K^{-11} for large K and azimuthal wavenumbers $k \geqslant 3$ of $O(1)$ and varies as K^{-4} for large K and $k = O(K)$. A further feature is that the (local) energy density I_k exhibits a degree of spatial non-uniformity that is likely to survive small viscous forces and is at variance with the spatial homogeneity assumed by Phillips. In particular, the energy density is singular at the critical circles on the sphere and when $k = O(K)$ the energy density is exponentially smaller within a radius $O(k/K)$ from the axis than it is elsewhere.

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2. Formulation

We consider small disturbances to an inviscid fluid with angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ generated in a sphere $r^2 + z^2 = 1$ by an arbitrary initial velocity $\mathbf{u}_0$ with a length scale $O(1)$. Relative to a frame with angular velocity $\boldsymbol{\Omega}$, the disturbance velocity $\mathbf{u}$ and a suitably scaled associated pressure Q satisfy the equations

$$\partial\mathbf{u}/\partial t + 2\Omega \times \mathbf{u} = -\nabla Q, \tag{2.1}$$

$$\operatorname{div}\mathbf{u} = 0. \tag{2.2}$$

For general axisymmetric containers, there is evidence that the eigenfrequencies may include continuous frequency bands (Stewartson [22], Israeli [10], Wood [28]), but for the sphere, the eigenfrequencies ω_N are discrete and the disturbance velocity can be represented by

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}_0(\mathbf{x}) + \sum_{\substack{N=-\infty \\ N\neq 0}}^{\infty} c_N \mathbf{U}_N(\mathbf{x}) e^{i\omega_N t} \tag{2.3}$$

(Greenspan [6]), where the c_N are constants. The specification of the steady axisymmetric velocity $\mathbf{U}_0$ is not needed here as our prime concern is with the high order modes. The amplitude c_N of the other modes, for which $\omega_N \neq 0$, is given by

$$c_N = \int_S \overline{\mathbf{U}}_N \cdot \mathbf{u}_0 \, dv \Big/ \int_S \overline{\mathbf{U}}_N \cdot \mathbf{U}_N \, dv. \tag{2.4}$$

The relevant modes have been described by Greenspan [6]. We recall here that the modal velocity $\mathbf{U}_N$ is related to the concomitant pressure Q_N by

$$\mathbf{U}_N = (-i\omega_N \mathbf{C}_N + 2\Omega \times \mathbf{C}_N)/\left(4\Omega^2 - \omega_N^2\right), \tag{2.5}$$

where

$$\mathbf{C}_N = \nabla Q_N - 4\Omega^2\omega_N^{-2} Q_{N_z}\hat{\mathbf{z}}, \tag{2.6}$$

whilst Q_N satisfies the differential equation

$$Q_{N_{rr}} + r^{-1}Q_{N_r} - k^2 r^{-2} Q_N = \left(4\Omega^2\omega_N^{-2} - 1\right)Q_{N_{zz}} \tag{2.7}$$

and is defined explicitly for $N \neq 0$ by

$$Q_N = P_n^{|k|}(\cos\Theta) P_n^{|k|}(\cos\Phi) e^{ik\phi}, \qquad 0 \leqslant |k| < n, \tag{2.8}$$

where

$$r\cos\theta = \sin\Theta\sin\Phi, \qquad z\sin\theta = \cos\Theta\cos\Phi,$$
$$0 \leqslant \Theta \leqslant \tfrac{1}{2}\pi - \theta \leqslant \Phi \leqslant \tfrac{1}{2}\pi + \theta, \tag{2.9}$$

$$\sin\theta = \omega_N/2\Omega, \qquad |\theta| < \pi/2. \tag{2.10}$$

The k and n here are integers and ϕ is the azimuthal angle. The eigenfrequencies ω_N ($= \omega_{kmn}$ say) are given by the roots (for $m = 1, 2, \ldots, n - |k|$) of

$$kP_n^k(\sin\theta) = \cos\theta\, dP_n^k(\sin\theta)/d\theta. \tag{2.11}$$

Modes with $\omega_N > 0$ will be assigned index numbers $N > 0$ and conjugate modes will be assigned index numbers N and $-N$, so that

$$\omega_{-N}(= \omega_{-kmn}) = -\omega_N, \qquad \mathbf{U}_{-N} = \bar{\mathbf{U}}_N, \qquad c_{-N} = \bar{c}_N. \tag{2.12}$$

We shall be concerned with the higher order modes with $n \gg 1$. When $k^{-1} = O(n^{-1})$, the behaviour of these modes follows from the asymptotic approximation

$$P_n^{|k|}(\cos\psi) \sim n_0^{-1/3}\Delta^{1/2}\big(4\zeta/(\lambda^2 - \sin^2\psi)\big)^{1/4}\operatorname{Ai}\big(n_0^{2/3}\zeta\big), \qquad 0 \leqslant \psi \leqslant \pi/2, \tag{2.13}$$

where $n_0 = n + \frac{1}{2}$, $\lambda = |k|/n_0$,

$$\Delta = \Gamma(n + |k| + 1)/\Gamma(n - |k| + 1), \tag{2.14}$$

$$2/3\zeta^{3/2} = \lambda \operatorname{arcosh}\Big[\lambda(1 - \lambda^2)^{-1/2}\cot\psi\Big] - \operatorname{arcosh}\Big[(1 - \lambda^2)^{-1/2}\cos\psi\Big] \tag{2.15}$$

(Thorne [24]). Hence a transition region of width $O(n^{-2/3})$ is located near the spheroid

$$r^2\lambda^{-2}\cos^2\theta + z^2(1 - \lambda^2)^{-1}\sin^2\theta = 1 \tag{2.16}$$

defined by $\sin\Theta = \lambda$. Between this transition region and the sphere, the modes comprise modulated waves with velocity $O(\Delta)$ represented by

$$Q_N \sim A_N(\cos n_0\nu_1 + \sin n_0\nu_2)e^{ik\phi}, \tag{2.17}$$

where

$$\nu_1 = \chi(\cos\Theta) - \chi(\cos\Phi), \qquad \nu_2 = \chi(\cos\Theta) + \chi(\cos\Phi), \tag{2.18}$$

$$A_N = \Delta(\pi n)^{-1}(\sin^2\Theta - \lambda^2)^{-1/4}(\sin^2\Phi - \lambda^2)^{-1/4}, \tag{2.19}$$

$$\chi(\cos\psi) = \arccos\Big[(1 - \lambda^2)^{-1/2}\cos\psi\Big] - \lambda\arccos\Big[\lambda(1 - \lambda^2)^{-1/2}\cot\psi\Big]. \tag{2.20}$$

Between the transition region and the axis, the velocity is $O(\Delta e^{-|O(n)|})$ and the fluid is relatively still. The velocity in the transition zone itself is $O(\Delta n^{1/6})$. The eigenfrequencies ω_N (> 0) are determined by

$$\chi(\omega_N/2\Omega) = n_0^{-1}\Big[(m + 1/4)\pi \pm \operatorname{arcot}\big\{\lambda^{-2}\big(1 - 1/4\omega_N^2\Omega^{-2}\big) - 1\big\}^{1/2}\Big] + O(n^{-2}), \qquad k \geqslant 0, \quad |\operatorname{arcot}| < \pi/2, \tag{2.21}$$

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provided that $\omega_N \neq 2\Omega\sqrt{1-\lambda^2}$, where m is any integer such that $1 \ll m \leqslant \frac{1}{2}[n - |k|] + \frac{1}{4}[1-(-1)^{n-k}]$. The relatively few remaining frequencies ω_N, for which $\omega_N \simeq 2\Omega\sqrt{1-\lambda^2}$, are given to leading order by

$$\omega_N/2\Omega = (1-\lambda^2)^{1/2} - n_0^{2/3}\left[4(1-\lambda^2)\right]^{1/6}\lambda^{-4/3}\gamma_i, \qquad (2.22)$$

where the γ_i are the zeros $\gamma = \gamma_i$ of $\mathrm{Ai}(\gamma)$. Hence the positive eigenfrequencies for a given k and n are distributed with a spacing at most $O(n^{-2/3})$ over the range $(0, 2\Omega\sqrt{1-\lambda^2})$. When $\omega_N \simeq 0$, the spheroid (2.16) that locates the transition zone is close to the cylinder $r = \lambda$. The spheroids for given k and n and various m all intersect the sphere where $r = \lambda$, but, as $\omega_N(m)$ increases, they bulge progressively until, for $\omega_N \simeq 2\Omega\sqrt{1-\lambda^2}$, they lie close to the sphere. So the modulated waves for a given k and n occupy the region $n^{2/3}r \gg n^{2/3}\lambda$ outside the cylinder $r = \lambda$ when $\omega_N \simeq 0$ and recede progressively to the sphere as ω_N increases. When $\omega_N/2\Omega = \sqrt{1-\lambda^2} + O(n^{-2/3})$, the modulated waves are eliminated, and the erstwhile transition zone becomes a thin layer over the whole sphere $r^2 + z^2 = 1$, outside which the velocity is small of $O(\Delta e^{-|O(n)|})$. Any particular point r, ϕ, z is in a modulated wave for those modes for which

$$|k|/n < r - |O(n^{-2/3})|,$$
$$\frac{1}{2}(1-\lambda) + \frac{1}{4n}\left[1-(-1)^{n-k}\right] \leqslant m/n < \pi^{-1}\chi(R) + |O(n^{-2/3})|, \qquad (2.23)$$

where

$$R = (r^2-\lambda^2)^{1/2}\left[r^2 - \lambda^2(1-\lambda^2)^{-1}z^2\right]^{-1/2} \qquad (2.24)$$

and the point is in a transition zone if

$$|k|n^{-1} - r = O(n^{-2/3}), \qquad mn^{-1} - \pi^{-1}\chi(R) = O(n^{-2/3}). \qquad (2.25)$$

When $|k| \ll n$, the behaviour of the high order modes follows from the approximation

$$P_n^{|k|}(\cos\psi) \sim \Delta^{1/2}(-1)^{|k|}(\psi/\sin\psi)^{1/2}J_{|k|}\left((n+\tfrac{1}{2})\psi\right), \qquad 0 < \psi \leqslant \pi/2 \qquad (2.26)$$

(Erdélyi [4]). Where $r^{-1} = O(1)$, the modes again comprise modulated waves with velocity $O(\Delta)$ represented by (2.17) but with the phases and amplitudes now given by

$$\nu_1 = \Theta - \Phi, \qquad \nu_2 = \Theta + \Phi + \lambda\pi, \qquad (2.27)$$

$$A_N = \Delta/\pi n(r\cos\theta)^{1/2}. \qquad (2.28)$$

Near the z-axis, where $r = O(n^{-1})$, the velocities are $O(\Delta n^{1/2})$. The eigenfrequencies $\omega_N > 0$ for $k \ll n$ are given by

$$\omega_N = 2\Omega \sin\left(\tfrac{1}{2}\pi - \gamma n_0^{-1}\right), \tag{2.29}$$

where the γ are given by

$$\gamma = \left(m - \tfrac{3}{4} + \tfrac{1}{2}k\right)\pi, \qquad 1 \ll m \leqslant \tfrac{1}{2}(n - |k|) + \tfrac{1}{4}\left(1 - (-1)^{n-k}\right), \tag{2.30}$$

for $\omega_N \not\simeq 2\Omega$ and the γ are the roots $\gamma = \gamma_t$ of

$$\gamma J'_{|k|}(\gamma) = -kJ_{|k|}(\gamma) \tag{2.31}$$

for $\omega_N \simeq 2\Omega$.

In either of the cases $k^{-1} = O(n^{-1})$ or $k \ll n$, each modulated wave (given by (2.17)) with $N > 0$, comprises four travelling waves with phases $\omega_N t + k\phi \pm n_0 \nu_i$, $i = 1, 2$, and wavenumbers

$$\mathbf{K}_{i\pm} = -\left(\pm n_0 \nu_{i_r}, kr^{-1}, \pm n_0 \nu_{i_z}\right). \tag{2.32}$$

These waves have the familiar dispersion relations

$$\omega_N K_{i\pm} = 2\,|\boldsymbol{\Omega}\cdot\mathbf{K}_{i\pm}| \tag{2.33}$$

and have group velocities

$$c_{gi\pm} = \omega_N^3\left(\pm\nu_{i_r}, \lambda r^{-1}, \mp\cot^2\theta\,\nu_{i_z}\right)/4n\Omega^2\nu_{i_z}^2. \tag{2.34}$$

A further common property is that the velocity of the modulated wave is amplified by a factor $O(n)$ within distances $O(n^{-2})$ in a direction normal to the boundary from the critical circles at the boundary (at which the group velocity is tangential).

3. Energy density and energy flux

The mean kinetic energy ρE and the mean energy flux $\mathbf{F}$ can be defined by

$$\rho E = \lim_{T\to\infty} T^{-1}\int_t^{t+T} \tfrac{1}{2}\rho \mathbf{u}^2\,dt, \qquad \mathbf{F} = \lim_{T\to\infty} T^{-1}\int_t^{t+T} Q\mathbf{u}\,dt, \tag{3.1}$$

where ρ is the density. Hence

$$E = \tfrac{1}{2}\mathbf{U}_0^2 + \sum_{N>0} |c_N|^2 \overline{\mathbf{U}}_N \cdot \mathbf{U}_N, \tag{3.2}$$

$$\mathbf{F} = Q_0\mathbf{U}_0 + \sum_{N>0} |c_N|^2\,\mathrm{Re}\left(\overline{Q}_N \mathbf{U}_N\right) \tag{3.3}$$

and both quantities are independent of ϕ and t.

We now consider the parts of E and $\mathbf{F}$ due to the high order modes, with $n \geqslant \mathbf{n}$ say. All these modes are quasi-harmonic functions of the Φ coordinate. So, an

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energy density and flux density can be defined for them by

$$\bar{E}(\mathbf{x}) = \sum_{\substack{N\geq 0\\ n\geqslant \mathbf{n}}} E_N, \qquad \bar{\mathbf{F}}(\mathbf{x}) = \sum_{\substack{N\geq 0\\ n\geqslant \mathbf{n}}} \mathbf{F}_N, \tag{3.4}$$

where

$$E_N = (\delta V)^{-1}\int_{\delta V} |c_N|^2 \bar{\mathbf{U}}_N \cdot \mathbf{U}_N \, dv, \qquad \mathbf{F}_N = (\delta C)^{-1}\int_{\delta C} |c_N|^2 \operatorname{Re}\left(\bar{Q}_N \mathbf{U}_N\right) dS \tag{3.5}$$

and δV and δC denote respectively a suitable small sphere and small disc with normal $\mathbf{x}$ (and their measures). From (2.5) and (2.6), we have that

$$\bar{\mathbf{U}}_N \cdot \mathbf{U}_N = \omega_N^{-4} \tan^4\theta\left\{\left(\omega_N^2 + 4\Omega^2\right)\left(Q_{N_r}^2 + k^2 r^{-2} Q_N^2\right) + 8\omega_N \Omega k r^{-1} Q_N Q_{N_r}\right\} + \omega_N^{-2} Q_{N_z}^2. \tag{3.6}$$

If the Nth mode gives rise to a modulated wave represented by (2.17) at the point $\mathbf{x}$, this expression reduces to leading order to

$$\bar{\mathbf{U}}_N \cdot \mathbf{U}_N = \omega_N^{-2} \sec^2\theta\, n^2 A_N^2\left(\nu_{1_z}^2 + \nu_{2_z}^2 + O_N\right), \tag{3.7}$$

where O_N is $O(1)$ and is linear in all of the terms $\exp in_0(\pm\nu_i \pm \nu_j)$, $i, j = 1, 2$, save the constant term. Hence, since the amplitude A_N does not vary rapidly with $\mathbf{x}$, the energy density is given approximately by

$$E_N = E_{N1} + E_{N2}, \tag{3.8}$$

where

$$E_{Ni} = \omega_N^{-2} \sec^2\theta\, n^2 |c_N|^2 A_N^2 \nu_{i_z}^2. \tag{3.9}$$

A similar calculation shows that the flux density is given by

$$\mathbf{F}_N = \tfrac{1}{2}\left\{E_{N1}\left(\mathbf{c}_{g1+} + \mathbf{c}_{g1-}\right) + \mathbf{E}_{N2}\left(\mathbf{c}_{g2+} + \mathbf{c}_{g2-}\right)\right\}, \tag{3.10}$$

as is to be expected.

In passing, it might be added that if the mean helicity is defined by

$$H = \lim_{T\to\infty} T^{-1}\int_t^{t+T} \mathbf{u}\cdot \operatorname{curl}\mathbf{u}\, dt, \tag{3.11}$$

it reduces, without approximation, to

$$H = \mathbf{U}_0 \cdot \operatorname{curl}\mathbf{U}_0, \tag{3.12}$$

and is thus unaffected by the unsteady modes.

The spatial variation of the energy density E_N depends merely on the velocity amplitude of the Nth mode which has been outlined above (for $n \gg 1$). The salient relative magnitudes are summarized in Table 1. Further details are given in previous papers (Wood [27], [28]).

TABLE 1. Relative magnitudes of E_N

	Still zone	Transition zone	Modulated wave	Critical circle
$k^{-1} = O(n^{-1})$	$e^{-\|O(n)\|}$	$O(n^{1/3})$	$O(1)$	$O(n^2)$
$k \ll n$		Axial zone ($r = O(n^{-1})$) $O(n)$	$O(1)$	$O(n^2)$

The distribution of $\bar{E}(\mathbf{x})$ among the various modes at a given point depends on c_N, which proves to have the form

$$|c_N|^2 = \Delta^{-2}C\{G_1 + (-1)^{n+k}G_2\}^2, \tag{3.13}$$

where

$$C = \begin{cases} 2\Omega^2\pi k^{-2}n^{-3}\cot^4\theta\cos^2\theta(1-\lambda^2)^{-1/2}(\cos^2\theta - \lambda^2)^{-1/2}, \\ k^{-1} = O(n^{-1}), \\ \frac{1}{8}\Omega^{-6}\pi^3 n^{-2k-5}\cot^4\theta\cos\theta, \qquad k = 1, 2, \\ \frac{1}{4}\Omega^{-6}\pi^2 n^{-10}\cot^4\theta, \qquad k = 0, 3 \leqslant k \ll n, \end{cases} \tag{3.14}$$

and the G_i are each of the form $A(\theta)k^2 + B(\theta)$. The $A(\theta)$ and $B(\theta)$ here are linear combinations of various Fourier coefficients with respect to ϕ of the initial velocity and its gradient. These formulae are derived, and the various G_1 and G_2 defined, in the appendix. For $m, n \gg 1$ and $k^{-1} = O(n^{-1})$, the terms of E_{Ni} vary with N to leading order as follows:

$$\theta = \theta(\lambda, \mu), \qquad \omega_N = \omega_N(\theta), \qquad \nu_i = \nu_i(\lambda, \theta, r, z), \tag{3.15}$$

$$|c_N|^2 A_N^2 = n^{-5} f(\lambda, \theta, r, z)\{G_1(k,\theta) + (-1)^{n+k}G_2(k,\theta)\}^2, \tag{3.16}$$

where $\mu = m/n_0$ (and $\lambda = |k|/n_0$). Since these terms vary relatively slowly with m and n, the corresponding contribution ($\bar{E}_M$ say) to the mean energy density $\bar{E}(\mathbf{x})$ can be approximated to leading order in the form

$$\bar{E}_M = \sum_k \iint (E_{N1}^* + E_{N2}^*)n\,dn\,d\mu, \tag{3.17}$$

where the superposition incorporates the appropriate modes and the E_{Ni}^* are equal to the previous E_{Ni} with the $|c_N|^2$ now replaced by

$$c_N^{*2} = \Delta^{-2}C\{G_1^2 + G_2^2\}. \tag{3.18}$$

In terms of local wavenumbers this becomes

$$\bar{E}_M = \iint I(K, \theta')K^2 \sin\theta'\,dK\,d\theta', \tag{3.19}$$

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where

$$I = \sum_k \sum_{i=1,2} \frac{nE^*_{Ni}}{K^2 \sin\theta'} \left| \frac{\partial(n,\mu)}{\partial(K,\theta')} \right| \tag{3.20}$$

and K, θ' are spherical coordinates in wavenumber space given by

$$K = |\mathbf{K}_i|, \qquad \theta' = \pi/2 - \theta, \tag{3.21}$$

the third angular coordinate being $\phi' = \operatorname{artan}(\lambda/r\nu_{i_r})$. To leading order,

$$\frac{\partial(n,\mu)}{\partial(K,\theta')} = \cos\theta' \Big/ \frac{\partial(n_0 \nu_{i_z})}{\partial n_0} \frac{\partial\theta'}{\partial\mu} \tag{3.22}$$

$$= \pi^{-1} \tan\theta(\cos^2\theta - \lambda^2)^{1/2} / \left(\nu_{i_z} - \lambda \partial\nu_{i_z}/\partial\lambda\right). \tag{3.23}$$

Hence we arrive at an energy density in wavenumber space of

$$I(K,\theta') = \sum_k I_k, \tag{3.24}$$

where

$$I_k = \sum_{i=1,2} \frac{n \tan\theta E^*_{Ni}(\cos^2\theta - \lambda^2)^{1/2}}{\pi K^2 \cos\theta\left(\nu_{i_z} - \lambda\partial\nu_{i_z}/\partial\lambda\right)}. \tag{3.25}$$

Substitution from (3.9), (3.13), (3.14) and (2.18) then yields

$$I_k = \frac{\cos^2\theta\left[G_1^2 + G_2^2\right]\{A(\Theta,\Phi) + A(\Phi,\Theta)\}}{2\pi^2 k^2 K^4 (1-\lambda^2)^{1/2} \sin^4\theta(\cos^2\Theta - \cos^2\Phi)^3}, \qquad \text{for } k^{-1} = O(n^{-1}), \tag{3.26}$$

where

$$A(\Theta,\Phi) = \frac{\cos\Phi\sqrt{\sin^2\Theta - \lambda^2} + \cos\Theta\sqrt{\sin^2\Phi - \lambda^2}}{\sin^2\Theta\cos\Phi\sqrt{\sin^2\Phi - \lambda^2} + \sin^2\Phi\cos\Theta\sqrt{\sin^2\Theta - \lambda^2}}. \tag{3.27}$$

For $k = O(1)$, the energy density I_k can be defined in the same way, and we find that

$$I_k = \frac{\beta \sin^{2\sigma+1}\theta\left[G_1^2 + G_2^2\right]\left[\sin^{\sigma+1}(\Theta+\Phi) + \sin^{\sigma+1}(\Theta-\Phi)\right]}{16\pi r \Omega^8 K^{\sigma+2} \cos^{\sigma+4}\theta(\cos^2\Theta - \cos^2\Phi)^{\sigma+1}}, \tag{3.28}$$

where $\beta = 1$, $\sigma = 9$ for $k = 0$ and $k \geqslant 3$ and $\beta = \frac{1}{2}\pi\cos\theta$, $\sigma = 2k + 4$ for $k = 1$, 2.

The energy density so defined thus decays as K^{-4} for $k = O(n)$ and as K^{-11} for $3 \leqslant k = O(1)$. By contrast, the energy density of Phillips theory for a viscous fluid is independent of K in the appropriate inertial subrange (Phillips [17]). Phillips bases his decay law on the hypothesis that the energy flux at the boundary from wavenumbers K less than any arbitrary $K_0 \gg 1$ equals the viscous dissipation throughout the flow at wavenumbers K greater than K_0. The corresponding hypothesis in the present context is that the net flux

$$\int_{\delta S} \{\Sigma E_{N1}(\mathbf{c}_{g1+} + \mathbf{c}_{g1-}) + \Sigma E_{N2}(\mathbf{c}_{g2+} + \mathbf{c}_{g2-})\} \cdot \hat{\boldsymbol{\nu}}\, dS = 0, \qquad (3.29)$$

where δS is the surface of the sphere and the summations are over the terms E_{Ni} for which K_i $(= n\nu_{i_z} \sec\theta)$ exceeds K_0 $(\gg 1)$. This condition follows because the modal pressure $Q_N e^{-ik\phi}$ is real so that the modulated waves representing it occur in conjugate pairs. Consequently, the group velocities c_{g_i-} and c_{g_i+} have equal and opposite components in each axial plane and $(\mathbf{c}_{gi+} + \mathbf{c}_{g_i-}) \cdot \hat{\boldsymbol{\nu}} = 0$. Thus the hypothesis holds but yields no information about the decay of the energy density with K.

Two other features deserve comment. Firstly, the energy density I_k is determined by the initial velocity near the critical circles when $k \neq 1$ or 2 and by the initial velocity near the poles when $k = 1$ or 2. (This follows from the corresponding property of the modal coefficients c_N shown in the appendix.) Secondly, I_k is infinite to a high order (as $(\Phi - \Theta)^{-10}$ for $k \geqslant 3$) at the critical circles (where $\Theta = \Phi$) on the sphere.

The singularity in I_k at the critical circles can be viewed as a consequence of the relevant ray paths (of the modulated wave components of each mode) being locally tangential to the boundary. Where a ray is reflected tangentially, the neighbouring rays are much closer after reflexion than before. The phases $n\nu_t$ of the rays are interchanged, to leading order, on reflexion, apart at most from a change of sign and a multiple of π. So the crowding of the rays causes the lateral gradient of phase on the reflected ray to be much larger than on the incident ray. It happens that the lateral gradient on the incident rays is not small. Hence the lateral (*i.e.* radial) gradient of the phase on the reflected rays is large. On the tangential ray, the radial phase gradient at the boundary is infinite. One consequence is that the mode varies locally much more rapidly with distance in the axial plane than with ϕ. Accordingly, the group velocity is locally tangential to the characteristic cones for large k, just as it is when $k = O(1)$. This in turn means that tangency of the ray paths always occurs at the critical circles (irrespective of whether k is large). More importantly, the lateral compression causes the velocity amplitude of the tangentially reflected wave to become infinite at the critical circle. Thus the energy density E_{N1} becomes infinite, because ν_{1_z} is infinite. The velocity itself is finite, a zero in phase at the critical circle serving to eliminate the singularity in the amplitude. (A fuller discussion of the singularity near the critical circles of inertial oscillations represented by modulated waves is given, for an arbitrary axisymmetric enclosure, in an earlier paper (Wood [28]).) When the energy density E_{N1} is converted to the wavenumber energy density $I_k(K, \theta')$, the

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spatial singularity is exaggerated by the replacement of n^{-1} by $\nu_{1_z}/K \sin\theta'$ in the relevant powers of n^{-1}.

Finally, it should be noted that the singularity (locally) invalidates the formula (3.9) for the energy density E_{N_i}. This is most easily seen by noting that the speed of each mode varies near the critical circle as $(\Theta - \Phi)^{-1} \sin n_0\alpha(\Theta - \Phi)$, where α $(= (1 - \lambda^2 \sec^2\theta)^{1/2})$ is spatially constant and $\Theta - \Phi$ $(= \operatorname{arcos}(r\cos\theta + z\sin\theta))$ represents locally the square root of radial distance from the boundary (Wood [28]). Thus the separate averaging of the periodic component implicit in the formula (3.9) for E_{N_i} is inappropriate near the critical circle, with the result that the integral $\int\int_S E_N r\,dr\,dz$, instead of representing the total energy of the Nth mode, is infinite.

Appendix—estimates for c_N

Asymptotic approximations for the coefficients c_N $(\equiv c_{kmn})$ of the eigenexpansion for the velocity are obtained here for $n \gg 1$ and $k = O(1)$ or $O(n)$. The initial velocity is assumed, for this purpose, to be an analytic function (of cartesian coordinates).

For $n \gg 1$, the inner product

$$\int_S \overline{\mathbf{U}}_N \cdot \mathbf{U}_N\,dv \sim \int_O n^2\omega_N^{-2} \sec^2\theta A_N^2\left(\nu_{1_z}^2 + \nu_{2_z}^2\right) dv, \tag{A1}$$

where O is the zone in which the modulated waves occur (and the expression (3.7) for $\overline{\mathbf{U}}_N \cdot \mathbf{U}_N$ has been used). When $k = O(1)$, the contribution to the inner product from the axial zone (where the modulated wave approximation fails) is small relative to the contribution from the modulated wave zone O. Similarly, when $k^{-1} = O(n^{-1})$ the contributions from both the inner zone (where $\mathbf{U}_N$ is exponentially small) and the transition zone are relatively small. For $k^{-1} = O(n^{-1})$, substitution from (2.9), (2.18) and (2.19) yields

$$\begin{aligned} 2\pi^2\Omega^2 \sin\theta\cos^4\theta \int\int_S \overline{\mathbf{U}}_N \cdot \mathbf{U}_N r\,dr\,dz \\ \sim \Delta^2 \int_{\pi/2-\theta}^{\pi/2+\theta} \int_{\theta_c}^{\pi/2-\theta} \sin\Theta \sin\Phi \\ \times \frac{\left[(1-\lambda^2)(\cos^2\Theta + \cos^2\Phi) - 2\cos^2\Theta\cos^2\Phi\right] d\Theta\,d\Phi}{(\cos^2\Theta - \cos^2\Phi)(\sin^2\Theta - \lambda^2)^{1/2}(\sin^2\Phi - \lambda^2)^{1/2}}, \end{aligned} \tag{A2}$$

where $\theta_c = \arcsin\lambda$ $(0 < \theta_c \leqslant \pi/2)$, and after a routine integration we find that

$$\int\int_S \overline{\mathbf{U}}_N \cdot \mathbf{U}_N r\,dr\,dz \sim (2\pi)^{-1}\Omega^{-2}\Delta^2 \sec^4\theta(\cos^2\theta - \lambda^2)^{1/2}. \tag{A3}$$

Similarly, for $k = O(1)$, we find with the aid of (2.27) and (2.28) that

$$\iint_S \overline{\mathbf{U}}_N \cdot \mathbf{U}_N r\, dr\, dz$$

$$\sim \frac{\Delta^2}{2\pi^2\Omega^2 \sin\theta \cos^4\theta} \int_{\pi/2-\theta}^{\pi/2+\theta} \int_0^{\pi/2-\theta} \frac{2 - \sec^2\Theta - \sec^2\Phi}{\sec^2\Phi - \sec^2\Theta} d\Theta\, d\Phi$$

$$\sim (2\pi)^{-1}\Omega^{-2}\Delta^2 \sec^3\theta, \tag{A4}$$

which is consistent with (A3).

To estimate the inner product $\int_S \overline{\mathbf{U}}_N \cdot \mathbf{u}_0\, dv$, we first rewrite it as

$$2\pi\omega_N^{-1}\left\{\int_0^{\pi} L(\sin\psi, \cos\psi) Q_N(\sin\psi, \cos\psi)\, d\psi \right.$$
$$\left. + \tan^2\theta \operatorname{cosec}\theta \iint_S M Q_N\, dr\, dz\right\}, \tag{A5}$$

where

$$L(r, z) = r\left[r \tan^2\theta (i u_{0k} + v_{0k} \operatorname{cosec}\theta) - i z w_{0k}\right], \tag{A6}$$

and

$$M(r, z) = -\frac{\partial}{\partial r}\left[r(i u_{0k} \operatorname{cosec}\theta + v_{0k})\right] + (i u_{0k} + v_{0k} \operatorname{cosec}\theta) k. \tag{A7}$$

The u_{0k}, v_{0k}, w_{0k} here are the azimuthal Fourier coefficients of the initial velocity $\mathbf{u}_0$ such that $\mathbf{u}_0 = \sum_{k=-\infty}^{\infty} (u_{0k}, v_{0k}, w_{0k}) e^{-ik\phi}$.

For $k = O(1)$, use of (2.8) and repeated integration by parts shows that

$$\int_0^{\pi} L(\sin\psi, \cos\psi) Q_N(\sin\psi, \cos\psi)\, d\psi = \frac{2P_n^{|k|}(\sin\theta)}{n_2 n_4 \cdots n_{2p}} \int_0^{\pi/2} L_p P_n^{|k|+p}(\cos\psi)\, d\psi, \tag{A8}$$

where $n_{2p} = (n + |k| + p)(n - |k| - p + 1)$ and

$$L_p = \sin^{|k|+p}\psi \frac{d}{d\psi}\left[L_{p-1} \sin^{-|k|-p}\psi\right], \qquad p \geqslant 1, \tag{A9}$$

$$L_0 = L(\sin\psi, \cos\psi) + (-1)^{n+k} L(\sin\psi, -\cos\psi). \tag{A10}$$

The dominant contribution to the last integral arises from the neighbourhood $\psi = O(n^{-1})$ of the axis. Assuming that $\mathbf{u}_0$ is an analytic function of position implies that, for $\psi \ll 1$, L_0 has the form

$$L_0 = \begin{cases} d \sin^{|k|+1}\psi + d_1 \sin^{|k|+2}\psi + O(\psi^{|k|+3}), & k \geqslant 1, \\ d^* \sin^3\psi + d_1^* \sin^4\psi + O(\psi^5), & k = 0. \end{cases} \tag{A11}$$

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Thence, after use of (2.26) we find from (A8) that

$$\int_0^{\pi} L(\sin\psi, \cos\psi) Q_N(\sin\psi, \cos\psi)\, d\psi = \begin{cases} O(n^{k-7/2}), & k \geqslant 1, \\ O(n^{-11/2}), & k = 0. \end{cases} \tag{A12}$$

To deal with the second integral in (A5), we transform to Θ, Φ coordinates and integrate by parts. Whence

$$\iint_S MQ_N\, dr\, dz \sim \Big[n_1^{-1} P_n^{|k|+1}(\cos\Phi) J_0^* + (n_1 n_2)^{-1} P_n^{|k|+2}(\cos\Phi) J_1^* + \cdots + (n_1 n_2 \cdots n_p)^{-1} P_n^{|k|+p}(\cos\Phi) J_p^* + \cdots \Big]_{\Phi=\pi/2-\theta}^{\Phi=\pi/2+\theta}, \tag{A13}$$

where

$$J_r^* = \int_0^{\pi/2-\theta} P_n^{|k|}(\cos\Theta) m_r\, d\Theta, \tag{A14}$$

and

$$m_r = -\sin^{|k|+1}\Phi \frac{\partial}{\partial\Phi}\left(\sin^{-|k|-1}\Phi\, m_{r-1}\right), \qquad r \geqslant 1, \tag{A15}$$

$$m_0 = M(\cos^2\Phi - \cos^2\Theta)/\sin\theta\cos\theta. \tag{A16}$$

Thence, after use again of (2.26), a lengthy but straightforward calculation, based on integration by parts of the integrals J_r^*, shows that

$$\iint_S MQ_N\, dr\, dz \sim \begin{cases} 2n^{-4}\left[P_n^{|k|}(\sin\theta)\right]^2 \left[G_1 + (-1)^{n+k} G_2\right], & k = 0, \geqslant 3, \\ |k| n^{-2} p_n^{|k|}(\sin\theta)\left[G_1 + (-1)^{n+k} G_2\right], & k = 1, 2, \end{cases} \tag{A17}$$

where

$$G_1 = M_{N_\Theta} - M_{N_\Phi} - (k^2 - \tfrac{1}{4}) M_N, \qquad G_2 = M_{S_\Theta} + M_{S_\Phi} - (k^2 - \tfrac{1}{4}) M_S, \qquad k = 0, \geqslant 3, \tag{A18}$$

$$G_1 = \cot\theta\left[M_{N_\Phi} - \tfrac{3}{8} M_N\right], \qquad G_2 = -\cot\theta\left[M_{S_\Phi} + \tfrac{3}{8} M_S\right], \qquad k = 1, \tag{A19}$$

$$G_1 = -(1 + \tfrac{15}{32}\cot\theta) M_{N_\Theta} + \cot\theta M N_{\Theta\Phi},$$
$$G_2 = -(1 + \tfrac{15}{32}\cot\theta) M_{S_\Theta} - \cot\Theta M_{S_{\Theta\Phi}}, \qquad k = 2, \tag{A20}$$

and

$$M_S = M(0, -1), \qquad M_N = M(0, 1), \qquad k = 1, 2,$$
$$M_S = M(\cos\theta, -\sin\theta), \qquad M_N = M(\cos\theta, \sin\theta), \qquad k = 0, \geqslant 3. \tag{A21}$$

The integrals obtained after successive integration by parts eventually become dominated by the values of their integrands near $\Theta = 0$, and they can then be readily approximated. The formulae of (A17) are different for $k = 1$ and 2, because the residual integrals contribute the leading approximations in these cases. The integral (A17) dominates the first integral of equation (A5) for all $k = O(1)$ and hence determines to leading order the inner product $\int_S \bar{\mathbf{U}}_N \cdot \mathbf{u}_0 \, dv$ (apart from the factor in (A5)).

When $k = O(n)$, it is helpful to use the pressure equation to re-express the inner product in the form

$$\iint_S \bar{\mathbf{U}}_N \cdot \mathbf{u}_0 r \, dr \, dz = ik^{-1}\omega_N^{-1} \operatorname{cosec}\theta \times \int_0^{\pi/2} L_0^*(\sin\psi, \cos\psi) Q_n(\sin\psi, \cos\psi) \, d\psi + O(k^{-2}), \tag{A22}$$

where

$$L^*(r, z) = r^2(zu_{0k_z} - rw_{0k_z}), \tag{A23}$$

$$L_0^*(r, z) = L^*(r, z) + (-1)^{k+n} L^*(r, -z). \tag{A24}$$

The dominant contribution to the integral now emanates from the transition zone on the sphere; and after a short calculation (which involves (2.26) and the fact that $\int_{-\infty}^{\infty} \mathrm{Ai}(x)\, dx = 1$) we find that (for $k^{-1} = O(n^{-1})$)

$$\iint_S \bar{\mathbf{U}}_N \cdot \mathbf{u}_0 r \, dr \, dz \sim \frac{(-1)^{m+1} i\Delta\left(G_1 + (-1)^{k+n} G_2\right)(\cos^2\theta - \lambda^2)^{1/4}}{\sqrt{2\pi}\,\Omega \sin^2\theta \cos\theta \,|k|\, n^{3/2}(1-\lambda^2)^{1/4}}, \tag{A25}$$

where (for $k^{-1} = O(n^{-1})$)

$$G_1 = L^*\left(\lambda, \sqrt{1-\lambda^2}\right), \qquad G_2 = L^*\left(\lambda, -\sqrt{1-\lambda^2}\right). \tag{A26}$$

The leading approximations to the modal coefficients c_N for $n \gg 1$ now follow from combining (A4) with (A17) and (A3) with (A25). In passing, it might be noted that, for $k = O(1)$, the c_N are determined to leading order by the initial velocity near the poles ($r = 0$, $z = \pm 1$) when $k = 1, 2$ and by the initial velocity near the critical circles ($r = \cos\theta$, $z = \pm\sin\theta$) when $k \neq 1$ or 2. On the other hand for $k^{-1} = O(n^{-1})$, the coefficients are determined to leading order by the initial velocity near the intersection ($r = \lambda$, $z = \pm\sqrt{1-\lambda^2}$) with the sphere of the caustic of the oscillatory zone.

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