for $n \geqslant 1$. Then

$$P_{n+1}(X) = \prod\left(X - \sqrt{a_{n+1}} - \left(\pm\sqrt{a_1} + \pm\sqrt{a_2} + \ldots + \pm\sqrt{a_n}\right)\right)$$
$$\times \prod\left(X + \sqrt{a_{n+1}} - \left(\pm\sqrt{a_1} + \pm\sqrt{a_2} + \ldots + \pm\sqrt{a_n}\right)\right)$$
$$= P_n\left(X - \sqrt{a_{n+1}}\right)P_n\left(X + \sqrt{a_{n+1}}\right).$$

Let $u = X - \sqrt{a_{n+1}}$ and $v = X + \sqrt{a_{n+1}}$. Note that $P_n(u)P_n(v)$ is a symmetric polynomial in u, v with rational coefficients. Hence $P_n(u)P_n(v)$ is also a polynomial in uv, $u + v$ with rational coefficients. Since $uv = X^2 - a_{n+1}$ and $u + v = 2X$, $P_n(u)P_n(v)$ is a polynomial in X with rational coefficients. So $P_{n+1}(X) \in \mathbb{Q}[x]$. Hence $(*)$ is proved.

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NGUYEN XUAN THO
Hanoi University of Science and Technology, Hanoi, Vietnam
e-mail: *tho.nguyenxuan1@hust.edu.vn*

106.06 An optimisation problem involving right circular cones

The reader may be familiar with the following optimisation problem:

> Given a circular metal sheet of radius R, a sector with central angle θ is cut out of it and rolled into a right circular cone of negligible thickness. For what value of θ is the volume of the cone the greatest?

If r and h are the radius of the base and height, respectively, of the right circular cone formed then

$$2\pi r = R\theta$$

and

$$\pi r\sqrt{r^2 + h^2} = \frac{R^2\theta}{2},$$

whence

$$h = R\sqrt{1 - \left(\frac{r}{R}\right)^2} = R\sqrt{1 - \left(\frac{\theta}{2\pi}\right)^2}$$

and the volume V of the cone is given by

$$V = \frac{\pi r^2 h}{3} = \left(\frac{\pi R^3}{3}\right) \cdot \left(\frac{\theta}{2\pi}\right)^2 \cdot \sqrt{1 - \left(\frac{\theta}{2\pi}\right)^2},$$

which is maximised when $\left(\frac{\theta}{2\pi}\right)^2 \cdot \sqrt{1 - \left(\frac{\theta}{2\pi}\right)^2}$ is maximised. Therefore, set $x = \left(\frac{\theta}{2\pi}\right)^2$, observe that $0 < x < 1$ and we can derive

$$x\sqrt{1 - x} = 2\sqrt{\frac{x}{2} \cdot \frac{x}{2} \cdot (1 - x)} \leqslant 2\sqrt{\left(\frac{\frac{1}{2}x + \frac{1}{2}x + 1 - x}{3}\right)^3} = \frac{2}{3\sqrt{3}},$$

where we have applied the well-known arithmetic mean – geometric mean (A.M – G.M) inequality to the product under the radical sign. Thus V attains a maximum if, and only if, there is equality in the A.M – G.M inequality, which happens when $x = 2/3$, and this readily shows

$$\theta = 2\pi\sqrt{\frac{2}{3}} \approx 294^\circ.$$

Now we look at a variant of this problem. Given a circular metal sheet of radius R, a sector with central angle θ is cut out of it. If we form two right circular cones from this sector and the complementary sector with central angle $2\pi - \theta$, what should be the value of θ so that the sum of the volumes of the two cones is a relative extremum?

In this case the total volume V is given by

$$V = \frac{\pi R^3}{3}\left[t^2\sqrt{1 - t^2} + (1 - t)^2\sqrt{1 - (1 - t)^2}\right],$$

where $t = \theta/2\pi$. (We have assumed that the thickness of the cones is negligible.) To determine the maximum or minimum value of V we need to ascertain the maximum or minimum value of

$$f(t) = t^2\sqrt{1 - t^2} + (1 - t)^2\sqrt{1 - (1 - t)^2}$$

when $0 < t < 1$. Observe that $f(t) = f(1 - t)$ and f is differentiable on $(0, 1)$ with

$$f'(t) = \frac{t(2 - 3t^2)}{\sqrt{1 - t^2}} - \frac{(1 - t)(2 - 3(1 - t)^2)}{\sqrt{1 - (1 - t)^2}}.$$

Evidently, $f'\left(\frac{1}{2}\right) = 0$ and to find the other zeros we simplify the expression to obtain that $f'(t) = 0$ implies

$$g(t) = (2t - 1)\left(18t^6 - 54t^5 + 24t^4 + 42t^3 - 42t^2 + 12t - 1\right) = 0.$$

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But it should be noted that a root α of g is not a stationary point of f unless

$$\left(2 - 3\alpha^2\right)\left(2 - 3(1 - \alpha)^2\right) > 0.$$

In view of this, we are only interested in real roots α of g which satisfy

$$1 - \sqrt{\frac{2}{3}} < \alpha < \sqrt{\frac{2}{3}}.$$

Let $g_1(t) = 18t^6 - 54t^5 + 24t^4 + 42t^3 - 42t^2 + 12t - 1$. We will show that the roots of $g_1(t)$ in $(0, 1)$ are irrational. To see this consider the polynomial

$$h(u) = u^6 g_1\left(\frac{1}{u}\right) = -u^6 + 12u^5 - 42u^4 + 42u^3 + 24u^2 - 54u + 18.$$

Since 2 divides every coefficient of h except the coefficient of u^6 and 2^2 does not divide the constant term 18, by Eisenstein's criterion it follows that h is irreducible over the rational numbers. Then since $g_1(0) \neq 0$ and h is irreducible over the rationals, so is g_1. Using a computer package which employs the Newton-Raphson Method we obtain the irrational roots of g_1 in $(0, 1)$ as

$$0.14797, \qquad 0.32401, \qquad 0.67598, \qquad 0.85202$$

up to 5 decimal places. But $0.14797 < 0.148 < 1 - \sqrt{\frac{2}{3}}$ and $\sqrt{\frac{2}{3}} < 0.817 < 0.85202$ shows that these are not zeros of f'. On the other hand,

$$1 - \sqrt{\tfrac{2}{3}} < 0.315 < 0.32401 < 0.325 < 0.675 < 0.67598 < 0.68 < \sqrt{\tfrac{2}{3}}$$

shows that

$$\left\{t \in (0, 1) \mid f'(t) = 0\right\} = \left\{0.32401,\ 0.5,\ 0.67598\right\}$$

where the roots are considered up to 5 decimal places.

Now we ascertain the nature of the stationary points of f. Let us denote the three roots of $f'(t)$ when $t \in (0, 1)$ as α_1, α_2 and α_3, where $\alpha_1 < \alpha_2 < \alpha_3$. We know that $\alpha_1 = 0.32$ (up to 2 decimal places), $\alpha_2 = 0.5$ and $\alpha_3 = 1 - \alpha_1$. Since we already know that the only roots of $f'(t)$ in the interval $t \in (0, 1)$ are α_1, α_2, α_3 and $f'(t)$ is smooth when $t \in (0, 1)$, we can (for example) check the sign of $f'(t)$ at the points $t = 0.6$ and $t = 0.8$. This gives us the signs of $f'(0.2)$, $f'(0.4)$, $f'(0.6)$ and $f'(0.8)$ as $f'(t) = -f'(1 - t)$. We obtain $f'(0.2) > 0$, $f'(0.4) < 0$, $f'(0.6) > 0$ and $f'(0.8) < 0$. We also know that the only points at which $f'(t)$ possibly changes its sign when $t \in (0, 1)$ are α_1, α_2, α_3. Therefore $f(t)$ increases when $t \in (0, \alpha_1)$, decreases when $t \in (\alpha_1, \alpha_2)$, increases when $t \in (\alpha_2, \alpha_3)$ and finally decreases when $t \in (\alpha_3, 1)$, meaning that f has local maxima at α_1 and α_3, and a local minimum at α_2.

The local maxima are also the absolute maxima but the absolute minimum does not exist as the value of $f(t)$ can be made as small as possible by making t arbitrarily close to 0 or 1. In terms of θ, the local maximum occurs at $\theta \approx 115.2°$ and the local minimum at $\theta = 180°$. The total volume at these respective values of θ are $(0.44)\dfrac{\pi R^3}{3}$ and $(0.43)\dfrac{\pi R^3}{3}$,

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approximately. When $\theta \approx 294°$, it is approximately $(0.41)\dfrac{\pi R^3}{3}$.

In conclusion, if we want to maximise the total volume of the two cones then we cut out a sector with central angle approximately 115.2° from the given metal sheet and roll the two sectors into right circular cones.

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PRITHWIJIT DE
Homi Bhabha Centre for and Science Education,
Tata Institute of Fundamental Research, Mumbai - 400088, INDIA
e-mails: *de.prithwijit@gmail.com* / *prithwijit@hbcse.tifr.res.in*

106.07 A function-based proof of the harmonic mean – geometric mean – arithmetic mean inequalities

For $a, b \in \mathbb{R}$, with $0 < b \leqslant a$, the harmonic, geometric and arithmetic means of a and b are respectively defined by

$$H(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a + b},\ G(a, b) = \sqrt{ab} \text{ and } A(a, b) = \frac{a + b}{2}.$$

Theorem: For $0 < b \leqslant a, H(a, b) \leqslant G(a, b) \leqslant A(a, b)$, that is

$$\frac{2ab}{a + b} \leqslant \sqrt{ab} \leqslant \frac{a + b}{2}.$$

Proof: If $x = \dfrac{b}{a}$, then $x \in (0, 1]$ and the inequalities to prove are

$$\frac{2x}{1 + x} \leqslant \sqrt{x} \leqslant \frac{1 + x}{2}.$$

There are easy purely algebraic proofs for these inequalities [1]. Here, instead, we propose an elementary approach based on the graph of some functions to prove them.

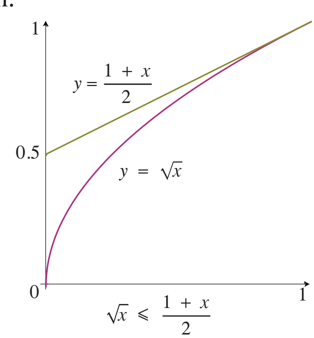

FIGURE 1: $G \leqslant A$