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Density-dependent incompressible viscous fluids in critical spaces

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We study the unique solvability of density-dependent incompressible Navier–Stokes equations in the whole space $\mathbb{R}^N$ ($N \geqslant 2$). The celebrated results by Fujita and Kato devoted to the constant density case are generalized to the case when the initial density is close to a constant: we find local well posedness for large initial velocity, and global well posedness for initial velocity small with respect to the viscosity. Our functional setting is very close to the one used by Fujita and Kato.

1. Introduction

In 1934, Leray stated in [18] the existence of global weak solutions with finite energy

$$E(t) \stackrel{\text{def}}{=} \|u(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \leqslant E(0) \tag{1.1}$$

for incompressible Navier–Stokes equations with constant density,

$$\left.\begin{aligned} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \Pi = f, \\ \operatorname{div} v = 0, \\ v_{|t=0} = v_0. \end{aligned}\right\} \tag{1.2}$$

Since then, despite the large amount of literature devoted to (1.2), the question of uniqueness for finite-energy solutions when $N \geqslant 3$ has remained an (outstanding) open problem. On the other hand, uniqueness may be shown in smaller classes of functions in which global existence has not been proved. Our starting point is the following *classical result*.

Classical result. *Let $E \subset \mathcal{S}'(\mathbb{R}^N)$ and $F \subset C(\mathbb{R}^+; E)$ be two functional spaces whose norm is invariant for all $\ell > 0$ by the transformation*

$$v_0(x) \mapsto \ell v_0(\ell x), \qquad v(t, x) \mapsto \ell v(\ell^2 t, \ell x). \tag{1.3}$$

For $T > 0$, let F_T denote the local version of F pertaining to functions defined on $[0, T]$. Under appropriate compatibility conditions on E and F, the following result holds true. For any data $v_0 \in E$, there exists $T > 0$ such that (1.2) has a unique local solution $v \in F_T$. If, in addition, $\|v_0\|_E \ll \mu$, then that solution is global.

Remark that the scaling condition (1.3) is exactly the one which leaves (1.2) invariant. In dimension $N = 2$, the energy defined in (1.1) is invariant by (1.3). As a consequence, the weak Leray solutions are actually unique in this particular case.

In dimension $N = 3$, the first example of spaces (E, F) for which the 'classical result' holds has been given by Fujita and Kato in [16]. In their paper, E is the homogeneous Sobolev space $\dot{H}^{1/2}$ and

$$F = \{u \in C(\mathbb{R}^+; \dot{H}^{1/2}) \mid t^{1/4}\nabla u \in C(\mathbb{R}^+; L^2) \text{ and } t^{1/4}\nabla u \to_{t\to 0} 0\}.$$

The reader is referred to [2] or [20] for more examples of appropriate spaces (E, F).

Though exciting from a mathematical viewpoint, studying (1.2) is somewhat disconnected to applications in fluid mechanics. Indeed, a 'real fluid' is hardly homogeneous or incompressible. We here aim at investigating the robustness of the 'classical result' for incompressible fluids with *variable* density. A similar concern is also relevant for compressible fluids (see [6–8, 10] for more details).

The equations we are interested in read

$$\left.\begin{aligned} \partial_t\rho + \operatorname{div}\rho u &= 0,\\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu\Delta u + \nabla\Pi &= \rho f,\\ \operatorname{div} u &= 0,\\ (\rho, u)_{|t=0} &= (\rho_0, u_0), \end{aligned}\right\} \tag{1.4}$$

where $\rho = \rho(t, x) \in \mathbb{R}^+$ stands for the density and $u = u(t, x) \in \mathbb{R}^N$ for the velocity field. The term $\nabla\Pi$ (namely the gradient of the pressure) may be seen as the Lagrange multiplier associated to the constraint $\operatorname{div} u = 0$. The initial conditions (ρ_0, u_0) and the external force f are prescribed. For the sake of simplicity, we shall assume throughout that x belongs to the whole space $\mathbb{R}^N$. Slight changes in the proofs would give similar results for x belonging to the torus $\mathbb{T}^N$.

It turns out that Leray's approach is still relevant for (1.4): assuming that $\rho_0 \in L^\infty$ is non-negative and that $u_0 \in L^2$, one can prove the existence of global weak solutions (ρ, u) with finite energy (for the sake of simplicity, we take $f = 0$),

$$\|\rho^{1/2}u(t)\|_{L^2}^2 + 2\mu\int_0^t \|\nabla u(\tau)\|_{L^2}^2\,\mathrm{d}\tau \leqslant \|\rho_0^{1/2}u_0\|_{L^2}^2.$$

In dimension $N = 2$, one can further get a pseudo-conservation law involving the norm of u in $L^\infty(0, T; H^1)$ and of ∇u in $L^2(0, T; H^1)$. This provides smoother weak solutions. Even in this latter case, however, the problem of uniqueness has not been solved. We refer to [1] and [19] for an overview of results on weak solutions. Some recent improvements have been obtained by Desjardins in [13–15].

On the other hand, the unique solvability of (1.4) in a bounded domain Ω with Dirichlet boundary conditions and smooth data has been known for a long time. The most complete results seem to have been obtained by Ladyzhenskaya and Solonnikov in [17]. There, it is assumed that the initial velocity u_0 belongs to the Besov space $B^{2-2/q}_{q,q}$ $(q > N,\ N = 2, 3)$ and that ρ_0 belongs to $C^1(\bar{\Omega})$ and is positive. As far as we know, the regularity requirements have not been improved since.

We aim at finding a framework as general as possible for which unique solvability of (1.4) may be stated. We would also like this framework to be compatible with the

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'classical result' in the case where ρ is a positive constant. Scaling considerations should help us to determine the relevant functional setting.

Obviously, system (1.4) is invariant for all $\ell > 0$ under the change

$$\left.\begin{aligned} (\rho_0(x), u_0(x)) &\mapsto (\rho_0(\ell x), \ell u_0(\ell x)), \\ (\rho(t,x), u(t,x), \Pi(t,x)) &\mapsto (\rho(\ell^2 t, \ell x), \ell u(\ell^2 t, \ell x), \ell^2 \Pi(\ell^2 t, \ell x)). \end{aligned}\right\} \tag{1.5}$$

If we use the Sobolev spaces setting, we are induced to choose initial data (ρ_0, u_0) such that $\nabla \rho_0$ and u_0 belong to $\dot{H}^{N/2-1}$. As system (1.4) degenerates if ρ vanishes or becomes unbounded, it seems reasonable to assume, in addition, that $\rho_0^{\pm} \in L^\infty$. For technical reasons (and possibly more serious ones), we shall assume that u_0 belongs to the Besov space $\dot{B}^{N/2-1}_{2,1}$ rather than to $\dot{H}^{N/2-1}$. On the other hand, in dimension $N \geqslant 3$, the regularity assumption $\nabla\rho_0 \in \dot{H}^{N/2-1}$ may be weakened to $\nabla\rho_0 \in \dot{B}^{N/2-1}_{2,\infty}$. In dimension $N = 2$, one has to assume that $\nabla\rho_0 \in \dot{B}^0_{2,1}$.

We here recall that homogeneous Besov spaces $\dot{B}^s_{2,r}$ ($r \in [1, +\infty]$, $|s| < \frac{1}{2}N$ if $r > 1$, $|s| \leqslant \frac{1}{2}N$ otherwise) may be defined as the completion of C_0^∞ for the norm

$$\|u\|_{\dot{B}^s_{2,r}} \stackrel{\text{def}}{=} \left(\sum_{q\in\mathbb{Z}} \left(\int_{2^{q-1} < |\xi| \leqslant 2^q} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, \mathrm{d}\xi \right)^{r/2} \right)^{1/r}$$

(with the usual change if $r = +\infty$), so that $\dot{H}^s \equiv \dot{B}^s_{2,2}$. In § 2 we shall give an equivalent definition of Besov spaces through the use of Littlewood–Paley decomposition.

Let us emphasize that, by taking $\rho_0 \in L^\infty$, $\nabla\rho_0 \in \dot{B}^{N/2-1}_{2,r}$ and $u_0 \in \dot{B}^{N/2-1}_{2,1}$ is coherent with (1.5). One further assumes that the forcing term f belongs to $L^1(\mathbb{R}^+; \dot{B}^{N/2-1}_{2,1})$.

In the present paper, we restrict ourselves to small perturbations of an initial constant density state: ρ_0 is close to a constant (say 1). The study of more drastic perturbations of homogeneous Navier–Stokes equations will be the object of a forthcoming paper.

Denote

$$a \stackrel{\text{def}}{=} 1/\rho - 1.$$

For fluids with positive density, system (1.4) can be rewritten as

$$\left.\begin{aligned} \partial_t a + u \cdot \nabla a &= 0, \\ \partial_t u + u \cdot \nabla u - \mu(1+a)\Delta u + (1+a)\nabla \Pi &= f, \\ \operatorname{div} u &= 0, \\ (a, u)_{|t=0} &= (a_0, u_0). \end{aligned}\right\} \tag{1.6}$$

Our main result follows.

THEOREM 1.1. *Let r be in $[1, +\infty]$ if $N \geqslant 3$ and $r = 1$ if $N = 2$. There exists a constant c, depending only on N, and such that, for any $u_0 \in \dot{B}^{N/2-1}_{2,1}$ with* $\operatorname{div} u_0 = 0$, $f \in L^1(\mathbb{R}^+; \dot{B}^{N/2-1}_{2,1})$ *with* $\operatorname{div} f = 0$, *and* $a_0 \in \dot{B}^{N/2}_{2,r} \cap L^\infty$ *with*

$$\|a_0\|_{\dot{B}^{N/2}_{2,\infty} \cap L^\infty} \leqslant c \qquad (\|a_0\|_{\dot{B}^1_{2,1}} \leqslant c \quad \text{if } N = 2),$$

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there is a $T \in (0, \infty]$ *such that system (1.6) has a unique solution* $(a, u, \nabla\Pi)$ *with*

$$a \in C_b([0,T); \dot{B}^{N/2}_{2,r}) \cap L^\infty(0,T; L^\infty),$$
$$u \in C_b([0,T); \dot{B}^{N/2-1}_{2,1}) \cap L^1(0,T; \dot{B}^{N/2+1}_{2,1})$$

and

$$\nabla\Pi \in L^1(0,T; \dot{B}^{N/2-1}_{2,1}).$$

If, in addition, we have

$$\|u_0\|_{\dot{B}^{N/2-1}_{2,1}} + \|f\|_{L^1(\dot{B}^{N/2-1}_{2,1})} \leqslant c'\mu$$

for some constant c' *depending only on* N*, one can take* $T = +\infty$.

Remark 1.2. The assumption $\operatorname{div} f = 0$ may be relaxed (see theorem 5.1 below).

Remark 1.3. One can further state a result of continuity with respect to the data. It means that (1.4) is well posed in the sense of Hadamard for data with critical regularity.

Remark 1.4. In dimension $N = 2$, the above result means that system (1.4) is not far from being well posed for $u_0 \in L^2$ and ρ_0 close to a constant in $H^1 \cap L^\infty$.

Remark 1.5. Similar results may be proved for fluids with variable positive viscosity $\mu = \mu(\rho)$ provided that the function μ is conveniently smooth.

Remark 1.6. In the present paper, we restrict ourselves to Besov spaces built on L^2 that are closely related to Sobolev spaces, and thus to the energy. Our approach therefore gives a natural extension of Fujita and Kato's results to density-dependent fluids. On the other hand, analogous results are very likely to hold true in spaces built on L^p. We expect a statement similar to theorem 1.1 to be true for $u_0 \in \dot{B}^{N/p-1}_{p,1}$, $a_0 \in \dot{B}^{N/p}_{p,\infty} \cap L^\infty$ and $f \in L^1(\mathbb{R}^+; \dot{B}^{N/p-1}_{p,1})$ (see § 2 below for the definition of those spaces). Owing to the nonlinear terms, however, one probably has to assume that $p < 2N$ to get existence, and $p \leqslant N$ to get uniqueness. A similar restriction appeared in [7] for compressible fluids.

The paper is structured as follows. In the first section, we recall a few results on Besov spaces. In § 3, we give estimates for the linearized equations. Section 4 is devoted to the proof of uniqueness. In § 5, we concentrate on the existence part of theorem 1.1. A technical commutation lemma is postponed in the Appendix.

1.1. Notation

Throughout the paper, C stands for a 'harmless constant', the precise meaning of which will be clear from the context. We shall sometimes alternatively use the notation $A \lesssim B$ instead of $A \leqslant CB$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

Let X be a Banach space. For $p \in [1, +\infty]$, the notation $L^p(0,T;X)$ stands for the set of measurable functions on $(0,T)$ with values in X, such that $t \mapsto \|f(t)\|_X$ belongs to $L^p(0,T)$. We denote by $C([0,T);X)$ the set of continuous functions on $[0,T)$ with values in X, and set $C_b([0,T);X) \stackrel{\text{def}}{=} C([0,T);X) \cap L^\infty(0,T;X)$.

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2. Basic results on Besov spaces

Homogeneous Littlewood–Paley decomposition relies upon a dyadic partition of unity: let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be supported in, say, $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^N,\ \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ and such that

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \mathrm{i}\xi \neq 0.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks as follows:

$$\Delta_q u \stackrel{\text{def}}{=} \varphi(2^{-q}D)u = 2^{qN} \int_{\mathbb{R}^N} h(2^q y)u(x-y)\,\mathrm{d}y.$$

We shall also use the following low-frequency cut-off:

$$S_q u \stackrel{\text{def}}{=} \sum_{k\leqslant q-1} \Delta_k u = \chi(2^{-q}D)u, \quad \text{with } \chi \stackrel{\text{def}}{=} 1 - \sum_{q\geqslant 0} \varphi(2^{-q}\cdot).$$

The formal decomposition

$$u = \sum_{q\in\mathbb{Z}} \Delta_q u \tag{2.1}$$

holds true modulo polynomials: if $u \in \mathcal{S}'(\mathbb{R}^N)$, then $\sum_{q\in\mathbb{Z}} \Delta_q u$ converges modulo $\mathcal{P}[\mathbb{R}^N]$ and (2.1) holds in $\mathcal{S}'(\mathbb{R}^N)/\mathcal{P}[\mathbb{R}^N]$ (see [21]). Furthermore, the above dyadic decomposition has nice properties of quasi-orthogonality,

$$\Delta_k\Delta_q u \equiv 0 \quad \text{if } |k-q| \geqslant 2 \quad \text{and} \quad \Delta_k(S_{q-1}u\Delta_q u) \equiv 0 \quad \text{if } |k-q| \geqslant 5. \tag{2.2}$$

Homogeneous Besov spaces may be defined through the Littlewood–Paley decomposition.

Definition 2.1. For $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ and $u \in \mathcal{S}'(\mathbb{R}^N)$, we set

$$\|u\|_{\dot B^s_{p,r}} \stackrel{\text{def}}{=} \biggl(\sum_{q\in\mathbb{Z}} 2^{rsq}\|\Delta_q u\|^r_{L^p}\biggr)^{1/r},$$

with the usual change if $r = +\infty$, and we denote

$$\dot{\mathcal{B}}^s_{p,r} = \{u \in \mathcal{S} \mid \|u\|_{\dot B^s_{p,r}} < +\infty\}.$$

For $s < N/p$ ($s \leqslant N/p$ if $r = 1$), we then define $\dot B^s_{p,r}$ as the completion of $\dot{\mathcal{B}}^s_{p,r}$ for $\|\cdot\|_{\dot B^s_{p,r}}$. If $m \in \mathbb{N}$ and $N/p + m \leqslant s < N/p + m + 1$ ($N/p + m < s \leqslant N/p + m + 1$ if $r = 1$), then $\dot B^s_{p,r}$ is defined as the subset of distributions $u \in \mathcal{S}'$ such that $\partial^\alpha u \in \dot B^{s-m}_{p,r}$ whenever $|\alpha| = m$.

Of course, the topology of $\dot B^s_{p,r}$ does not depend on the choice of the Littlewood–Paley decomposition. When $p = 2$, it coincides with the one given in the introduction.

Remark 2.2. The space $\dot B^s_{p,\infty}$ defined above is slightly smaller than the one defined in [21] (which is not the completion of $\mathcal{S}$ for $\|\cdot\|_{\dot B^s_{p,\infty}}$). Our choice is motivated by getting a more concise statement in theorem 1.1: with the definition of [21], we would not have obtained that a is *continuous on* $[0,T]$ with values in $\dot B^{N/2}_{2,\infty}$.

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Let us now state some classical properties for those Besov spaces.

Proposition 2.3. *The following properties hold.*

(i) *Derivatives: there exists a universal constant C such that*

$$C^{-1}\|u\|_{\dot B^s_{p,r}} \leqslant \|\nabla u\|_{\dot B^{s-1}_{p,r}} \leqslant C\|u\|_{\dot B^s_{p,r}}.$$

(ii) *Sobolev embeddings: if $p_1 < p_2$ and $r_1 \leqslant r_2$, then*

$$\dot B^s_{p_1,r_1} \hookrightarrow \dot B^{s-N(1/p_1-1/p_2)}_{p_2,r_2}.$$

(iii) *Algebraic properties: for $s>0$, $\dot B^s_{p,r}\cap L^\infty$ is an algebra. Moreover, for any $p\in[1,+\infty]$, then*

$$\dot B^{N/p}_{p,1} \hookrightarrow \dot B^{N/p}_{p,\infty}\cap L^\infty,$$

and $\dot B^{N/p}_{p,1}$ is an algebra if p is finite.

(iv) *Real interpolation: $(\dot B^{s_1}_{p,r},\dot B^{s_2}_{p,r})_{\theta,r'} = \dot B^{\theta s_1+(1-\theta)s_2}_{p,r'}$.*

Remark 2.4. When manipulating homogeneous spaces, one has to be careful that, owing to the low frequencies, the inclusion $\dot B^{s+\epsilon}_{p,r}\hookrightarrow \dot B^s_{p,r}$ ($\epsilon>0$) is false!

The usual product is continuous in many Besov spaces. The following proposition (the proof of which may be found in [22, § 4.4] (see, in particular, inequality (28) on p. 174)) will be very useful.

Proposition 2.5. *Let $1\leqslant r,p,p_1,p_2\leqslant+\infty$. The following inequalities hold true:*

$$\|uv\|_{\dot B^s_{p,r}} \lesssim \|u\|_{L^\infty}\|v\|_{\dot B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{\dot B^s_{p,r}} \quad \text{if } s>0, \tag{2.3}$$

$$\|uv\|_{\dot B^{s_1+s_2-N/p}_{p,r}} \lesssim \|u\|_{\dot B^{s_1}_{p,r}}\|v\|_{\dot B^{s_2}_{p,\infty}} \quad \text{if } s_1,s_2<\frac{N}{p} \quad\text{and}\quad s_1+s_2+N\min\Bigl(0,1-\frac{2}{p}\Bigr)>0, \tag{2.4}$$

$$\|uv\|_{\dot B^s_{p,r}} \lesssim \|u\|_{\dot B^s_{p,r}}\|v\|_{\dot B^{N/p}_{p,\infty}\cap L^\infty} \quad \text{if } |s|<N/p. \tag{2.5}$$

The limit case $s_1+s_2=0$ in (2.4) is of interest. When $p\geqslant 2$, we have

$$\|uv\|_{\dot B^{-N/p}_{p,\infty}} \lesssim \|u\|_{\dot B^s_{p,1}}\|v\|_{\dot B^{-s}_{p,\infty}} \tag{2.6}$$

whenever s is in the range $(-N/p,N/p]$ (see, for example, [22]).

The study of non-stationary partial differential equations requires spaces of type $L^\rho_T(X)\stackrel{\text{def}}{=}L^\rho(0,T;X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equations through the Littlewood–Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain bounds in spaces that are not of type $L^\rho(0,T;\dot B^s_{p,r})$ (except if $r=\rho$). That remark naturally leads to the following definition (introduced in [5]).

DEFINITION 2.6. Let $(r,\rho,p)\in[1,+\infty]^3$, $T\in\,]0,+\infty]$ and $s\in\mathbb{R}$. We set

$$\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})}\stackrel{\text{def}}{=}\biggl(\sum_{q\in\mathbb{Z}}2^{qrs}\biggl(\int_0^T\|\Delta_q u(t)\|^\rho_{L^p}\,\mathrm{d}t\biggr)^{r/\rho}\biggr)^{1/r},$$

with the usual change if $r=+\infty$.

Let us remark that, by virtue of Minkowski's inequality, we have

$$\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})}\leqslant\|u\|_{L^\rho_T(\dot{B}^s_{p,r})}\quad\text{if }\rho\leqslant r,$$
$$\|u\|_{L^\rho_T(\dot{B}^s_{p,r})}\leqslant\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})}\quad\text{if }\rho\geqslant r.$$

Let $\theta\in[0,1]$, The following interpolation inequality holds,

$$\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})}\leqslant\|u\|^\theta_{\tilde{L}^{\rho_1}_T(\dot{B}^{s_1}_{p,r})}\|u\|^{1-\theta}_{\tilde{L}^{\rho_2}_T(\dot{B}^{s_2}_{p,r})},\tag{2.7}$$

whenever $1/\rho=\theta/\rho_1+(1-\theta)/\rho_2$ and $s=\theta s_1+(1-\theta)s_2$.

Let us state some estimates for the product in those spaces, the proof of which is a straightforward adaptation of the one for usual Besov spaces (see [5, 22]).

PROPOSITION 2.7. *If $s>0$, $r\in[1,+\infty]$ and $1/\rho_2+1/\rho_3=1/\rho_1+1/\rho_4=1/\rho\leqslant 1$, then*

$$\|uv\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})}\lesssim\|u\|_{L^{\rho_1}_T(L^\infty)}\|v\|_{\tilde{L}^{\rho_4}_T(\dot{B}^s_{p,r})}+\|v\|_{L^{\rho_2}_T(L^\infty)}\|u\|_{\tilde{L}^{\rho_3}_T(\dot{B}^s_{p,r})}.$$

If $s_1,s_2<N/p$ ($s_1,s_2\leqslant N/p$ if $r=1$), $s_1+s_2+N\min(0,1-2/p)>0$ and $1/\rho_1+1/\rho_2=1/\rho\leqslant 1$, then

$$\|uv\|_{\tilde{L}^\rho_T(\dot{B}^{s_1+s_2-N/p}_{p,r})}\lesssim\|u\|_{\tilde{L}^{\rho_1}_T(\dot{B}^{s_1}_{p,r})}\|v\|_{\tilde{L}^{\rho_2}_T(\dot{B}^{s_2}_{p,r})}.$$

The analogous result of the endpoint estimate (2.6) reads (for $p\geqslant 2$)

$$\|uv\|_{\tilde{L}^\rho_T(\dot{B}^{-N/p}_{p,\infty})}\lesssim\|u\|_{\tilde{L}^{\rho_1}_T(\dot{B}^s_{p,1})}\|v\|_{\tilde{L}^{\rho_2}_T(\dot{B}^{-s}_{p,\infty})}\tag{2.8}$$

whenever s is in the range $(-N/p,N/p]$ and $1/\rho_1+1/\rho_2=1/\rho\leqslant 1$.

Proof. For the sake of completeness, let us prove (2.8). First remark that, for any $\sigma<0$ and $\rho\in[1,+\infty]$,

$$\sup_q 2^{q\sigma}\|S_q v\|_{L^\rho_T(L^\infty)}\lesssim\|v\|_{\tilde{L}^\rho_T(\dot{B}^\sigma_{\infty,\infty})}.\tag{2.9}$$

Indeed, since $\sigma<0$, we can write

$$\|S_q v\|_{L^\rho_T(L^\infty)}\leqslant\sum_{q'\leqslant q-1}2^{-q'\sigma}(2^{q'\sigma}\|\Delta_{q'}v\|_{L^\rho_T(L^\infty)})\lesssim 2^{-q\sigma}\sup_{q'}2^{q'\sigma}\|\Delta_{q'}v\|_{L^\rho_T(L^\infty)}.$$

Now, the proof of (2.8) lies on elementary paradifferential calculus. Recall that paradifferential calculus was introduced by Bony in [3]. The paraproduct between u and v is defined by

$$T_u v\stackrel{\text{def}}{=}\sum_{q\in\mathbb{Z}}S_{q-1}u\Delta_q v.$$

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We have the following so-called Bony's decomposition (modulo a polynomial):

$$uv = T_u v + T_v u + R(u,v), \quad \text{with } R(u,v) \stackrel{\text{def}}{=} \sum_{q\in\mathbb{Z}} \Delta_q u(\Delta_{q-1}+\Delta_q+\Delta_{q+1})v.$$

Therefore, we only have to prove that the two paraproducts, $T_u v$ and $T_v u$, and the remainder $R(u,v)$ satisfy (2.8).

For $T_u v$, we write, using (2.2) and Hölder's inequality,

$$\begin{aligned}
\|\Delta_q T_u v\|_{L^\rho_T(L^p)} &\leqslant \sum_{|q'-q|\leqslant 4} \|\Delta_q(S_{q'-1}u\Delta_{q'}v)\|_{L^\rho_T(L^p)}\\
&\lesssim \sum_{|q'-q|\leqslant 4} \|S_{q'-1}u\|_{L^{\rho_1}_T(L^\infty)}\|\Delta_{q'}v\|_{L^{\rho_2}_T(L^p)}\\
&\lesssim 2^{qN/p}\Big(\sup_{q'} 2^{q'(s-N/p)}\|S_{q'-1}u\|_{L^{\rho_1}_T(L^\infty)}\Big)\Big(\sup_{q'} 2^{-q's}\|\Delta_{q'}v\|_{L^{\rho_2}_T(L^p)}\Big),
\end{aligned}$$

which, according to (2.9), yields

$$\|\Delta_q T_u v\|_{L^\rho_T(L^\infty)} \lesssim \|u\|_{\tilde{L}^{\rho_1}_T(\dot{B}^{s-N/p}_{\infty,\infty})}\|u\|_{\tilde{L}^{\rho_2}_T(\dot{B}^{-s}_{p,\infty})}.$$

Since, obviously,

$$\|u\|_{\tilde{L}^{\rho_1}_T(\dot{B}^{s-N/p}_{\infty,\infty})} \leqslant \|u\|_{\tilde{L}^{\rho_1}_T(\dot{B}^{s}_{p,1})},$$

we obtain the desired estimate. Note that, in the limit case $s=N/p$, the arguments below still apply, provided that $\|u\|_{\tilde{L}^{\rho_1}_T(\dot{B}^{s-N/p}_{\infty,\infty})}$ has been replaced by $\|u\|_{L^{\rho_1}_T(L^\infty)}$.

The symmetric term $T_v u$ may be treated similarly.

Let us study the term $R(u,v)$. As $p\geqslant 2$, by virtue of Bernstein's inequality, we have

$$\|\Delta_q R(u,v)\|_{L^p} \lesssim 2^{qN/p}\|R(u,v)\|_{L^{p/2}}.$$

Therefore, thanks to the Hölder inequality,

$$\begin{aligned}
2^{-qN/p}\|\Delta_q R(u,v)\|_{L^\rho_T(L^p)} &\lesssim \sum_{i=1}^{3}\sum_q [(2^{qs}\|\Delta_q u\|_{L^{\rho_1}_T(L^p)})(2^{-(q+i)s}\|\Delta_q v\|_{L^{\rho_2}_T(L^p)})]\\
&\lesssim \Big(\sum_q 2^{qs}\|\Delta_q u\|_{L^{\rho_1}_T(L^p)}\Big)\Big(\sup_q 2^{-qs}\|\Delta_q v\|_{L^{\rho_2}_T(L^p)}\Big),
\end{aligned}$$

which completes the proof of (2.8). □

In proposition 2.3 (iv) and in (2.7), we saw how to interpolate between spaces having different regularity indices. In § 4.2, the question of how far from $\tilde{L}^1_T(\dot{B}^s_{p,\infty})$ the space $L^1_T(\dot{B}^s_{p,1})$ is will arise. The answer is given by the following proposition.

PROPOSITION 2.8. *For any* $(p,\rho)\in[1,+\infty]^2$, $s\in\mathbb{R}$ *and* $\epsilon\in(0,1]$, *we have*

$$\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,1})} \lesssim \frac{\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,\infty})}}{\epsilon}\log\bigg(\mathrm{e}+\frac{\|u\|_{\tilde{L}^\rho_T(\dot{B}^{s-\epsilon}_{p,\infty})}+\|u\|_{\tilde{L}^\rho_T(\dot{B}^{s+\epsilon}_{p,\infty})}}{\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,\infty})}}\bigg).$$

Proof. It merely stems from a judicious splitting into low, medium and high frequencies. Indeed, for any $m \in \mathbb{N}^\star$, we have

$$\begin{aligned}
&\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,1})}\\
&\quad = \sum_{q=-\infty}^{-m} (2^{q(s-\epsilon)}\|\Delta_q u\|_{L^\rho_T(L^p)})2^{q\epsilon}\\
&\qquad + \sum_{q=1-m}^{m-1} 2^{qs}\|\Delta_q u\|_{L^\rho_T(L^p)} + \sum_{q=m}^{\infty} (2^{q(s+\epsilon)}\|\Delta_q u\|_{L^\rho_T(L^p)})2^{-q\epsilon}\\
&\quad \leqslant \left(\sum_{q=-\infty}^{-m} 2^{q\epsilon}\right) \sup_q 2^{q(s-\epsilon)}\|\Delta_q u\|_{L^\rho_T(L^p)}\\
&\qquad + (2m-1)\sup_q 2^{qs}\|\Delta_q u\|_{L^\rho_T(L^p)} + \left(\sum_{q=m}^{\infty} 2^{-q\epsilon}\right)\sup_q 2^{q(s+\epsilon)}\|\Delta_q u\|_{L^\rho_T(L^p)}\\
&\quad \leqslant (2m-1)\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,\infty})} + \frac{2^{1-m\epsilon}}{1-2^{-\epsilon}}(\|u\|_{\tilde{L}^\rho_T(\dot{B}^{s-\epsilon}_{p,\infty})} + \|u\|_{\tilde{L}^\rho_T(\dot{B}^{s+\epsilon}_{p,\infty})}).
\end{aligned}$$

Choosing m as the integer part of

$$\frac{1}{\epsilon}\log_2\left(\frac{\|u\|_{\tilde{L}^\rho_T(\dot{B}^{s-\epsilon}_{p,\infty})} + \|u\|_{\tilde{L}^\rho_T(\dot{B}^{s+\epsilon}_{p,\infty})}}{\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,\infty})}}\right)$$

yields the desired inequality. □

3. The linearized equations

3.1. The transport equation

Here, we recall some estimates for the following linear transport equation:

$$\left.\begin{aligned} \partial_t f + \operatorname{div}(vf) = F,\\ f_{|t=0} = f_0.\end{aligned}\right\} \qquad (\mathcal{T})$$

The following result suffices for our purposes.

PROPOSITION 3.1. *Let $r \in [1,+\infty]$ and s be such that $|s| < 1 + \frac{1}{2}N$. Let v be a solenoidal vector field such that ∇v belongs to $L^1(0,T;\dot{B}^{N/2}_{2,r} \cap L^\infty)$. Suppose that $f_0 \in \dot{B}^s_{2,r}$, $F \in L^1(0,T;\dot{B}^s_{2,r})$ and that $f \in L^\infty(0,T;\dot{B}^s_{2,r}) \cap C([0,T];\mathcal{S}')$ solves $(\mathcal{T})$.*

There exists a constant C, depending only on s and N, and such that the following inequality holds true,

$$\|f\|_{\tilde{L}^\infty_t(\dot{B}^s_{2,r})} \leqslant \mathrm{e}^{CV(t)}\left(\|f_0\|_{\dot{B}^s_{2,r}} + \int_0^t \mathrm{e}^{-CV(\tau)}\|F(\tau)\|_{\dot{B}^s_{2,r}}\,\mathrm{d}\tau\right), \qquad (3.1)$$

with

$$V(t) = \int_0^t \|\nabla v(\tau)\|_{B^{N/2}_{2,r}\cap L^\infty}\,\mathrm{d}\tau.$$

Moreover, f belongs to $C([0,T];\dot{B}^s_{2,r})$.

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Estimate (3.1), with the Sobolev space $\dot H^s$ instead of $\dot B^s_{2,r}$, is standard. Estimates for general non-homogeneous Besov norms have been stated in [9, proposition A.1]. Slight changes in the proof given there yield estimates in homogeneous norms.

3.2. The linearized momentum equation

When the density is close to a constant, we are led to study the following linearized momentum equation:

$$\left.\begin{aligned} \partial_t u + v\cdot\nabla u - \mu\Delta u + \nabla\Pi &= f,\\ \operatorname{div} u &= 0,\\ u_{|t=0} &= u_0. \end{aligned}\right\} \tag{3.2}$$

For that system, we have the following.

Proposition 3.2. *Let $s\in(-\frac12 N, 2+\frac12 N)$, $r\in[1,+\infty]$, u_0 be a divergence-free vector field with coefficients in $\dot B^{s-1}_{2,r}$ and f be a time-dependent vector field with coefficients in $\tilde L^1_T(\dot B^{s-1}_{2,r})$. Let u, v be two divergence-free time-dependent vector fields such that $\nabla v\in L^1(0,T;\dot B^{N/2}_{2,r}\cap L^\infty)$ and $u\in C([0,T];\dot B^{s-1}_{2,r})\cap\tilde L^1_T(\dot B^{s+1}_{2,r})$. In addition, assume that (3.2) is fulfilled for some distribution Π.*

There exists $C=C(s,N)$ such that the following estimate holds:

$$\begin{aligned} &\|u\|_{\tilde L^\infty_T(\dot B^{s-1}_{2,r})} + \mu\|u\|_{\tilde L^1_T(\dot B^{s+1}_{2,r})} + \|\nabla\Pi\|_{\tilde L^1_T(\dot B^{s-1}_{2,r})}\\ &\qquad\leqslant \exp(C\|\nabla v\|_{L^1_T(\dot B^{N/2}_{2,r}\cap L^\infty)})(\|u_0\|_{\dot B^{s-1}_{2,r}} + C\|f\|_{\tilde L^1_T(\dot B^{s-1}_{2,r})}). \end{aligned}$$

Proof. Applying Δ_q to system (3.2) yields

$$\partial_t\Delta_q u + v\cdot\nabla\Delta_q u - \mu\Delta\Delta_q u + \nabla\Delta_q\Pi = \Delta_q f + [v,\Delta_q]\cdot\nabla u.$$

Let $\mathcal P$ denote the L^2 projector on divergence-free vector fields. Take the L^2 scalar product of the above equality with $\Delta_q u$. After some obvious computations based on integration by parts or Bernstein's inequality, we gather that, for some universal positive constant κ,

$$\frac12\frac{\mathrm d}{\mathrm dt}\|\Delta_q u\|^2_{L^2} + \kappa\mu 2^{2q}\|\Delta_q u\|^2_{L^2} \leqslant \|\Delta_q u\|_{L^2}(\|\Delta_q\mathcal P f\|_{L^2} + \|[v,\Delta_q]\cdot\nabla u\|_{L^2}),$$

whence

$$\begin{aligned} &2^{q(s-1)}\|\Delta_q u\|_{L^\infty_T(L^2)} + \kappa\mu 2^{q(s+1)}\|\Delta_q u\|_{L^1_T(L^2)}\\ &\qquad\leqslant 2^{q(s-1)}\|\Delta_q u_0\|_{L^2} + 2^{q(s-1)}\|\Delta_q\mathcal P f\|_{L^1_T(L^2)} + 2^{q(s-1)}\|[v,\Delta_q]\cdot\nabla u\|_{L^1_T(L^2)}. \end{aligned}$$

Take the $\ell^r(\mathbb Z)$ norm of the above inequality. Making use of Minkowski's inequality, we end up with

$$\begin{aligned} &\|u\|_{\tilde L^\infty_T(\dot B^{s-1}_{2,r})} + \kappa\mu\|u\|_{\tilde L^1_T(\dot B^{s+1}_{2,r})}\\ &\qquad\leqslant \|u_0\|_{\dot B^{s-1}_{p,r}} + \|f\|_{\tilde L^1_T(\dot B^{s-1}_{2,r})} + \int_0^T\biggl(\sum_q\|[v,\Delta_q]\cdot\nabla u\|^r_{L^2}2^{rq(s-1)}\biggr)^{1/r}\mathrm dt. \end{aligned}$$

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The commutator may be bounded according to lemma A.1 in the Appendix, provided that $|s-1| < \frac{1}{2}N + 1$. We eventually find that

$$\|u\|_{\tilde{L}^\infty_T(\dot{B}^{s-1}_{2,r})} + \kappa\mu\|u\|_{\tilde{L}^1_T(\dot{B}^{s+1}_{2,r})} \leqslant \|u_0\|_{\dot{B}^{s-1}_{2,r}} + \|\mathcal{P}f\|_{\tilde{L}^1_T(\dot{B}^{s-1}_{2,r})} + C\int_0^T \|\nabla v\|_{\dot{B}^{N/2}_{2,r}\cap L^\infty}\|u\|_{\dot{B}^{s-1}_{2,r}}\,\mathrm{d}t. \tag{3.3}$$

There remains to estimate the pressure term. We first apply div to system (3.2) and get $\Delta\Pi = \operatorname{div} f - \operatorname{div}(v\cdot\nabla u)$. The usual product laws in Besov spaces do not allow us to get estimates for s describing the whole range $(-\frac{1}{2}N, 2+\frac{1}{2}N)$. We shall make use of Bony's decomposition. Remembering that $\operatorname{div} u = \operatorname{div} v = 0$, we get

$$\Delta\pi = \operatorname{div} f - \sum_{1\leqslant i,\, j\leqslant N}(T_{\partial_i v^j}\partial_j u^i - T_{\partial_j u^i}\partial_i v^j - \partial_i\partial_j R(u^i, v^j)).$$

Then basic continuity results for the paraproduct (see, for example, [22, § 4.4]) yield

$$\|\Delta\pi\|_{\tilde{L}^1_T(\dot{B}^{s-2}_{2,r})} \leqslant \|\mathcal{Q}f\|_{\tilde{L}^1_T(\dot{B}^{s-1}_{2,r})} + C\int_0^T \|u\|_{\dot{B}^{s-1}_{2,r}}\|\nabla v\|_{\dot{B}^{N/2}_{2,r}\cap L^\infty}\,\mathrm{d}t,$$

where def $= QI - \mathcal{P}$.

Adding the latter inequality to (3.3) and applying Gronwall's lemma completes the proof. □

4. Uniqueness results

In this section, we study the problem of uniqueness for a solution with critical regularity. For some reason, which will be explained in § 4.2, the statement (and the proof) of the uniqueness result is different depending on $N \geqslant 3$ or $N = 2$, the latter case being far more technical.

4.1. Case where $N \geqslant 3$

We shall prove the following proposition.

PROPOSITION 4.1. *Let $(a^1, u^1, \nabla\Pi^1)$ and $(a^2, u^2, \nabla\Pi^2)$ solve (1.6) with the same data $a_0 \in \dot{B}^{N/2}_{2,\infty}\cap L^\infty$, $u_0 \in \dot{B}^{N/2-1}_{2,1}$ with $\operatorname{div} u_0 = 0$, and $f \in L^1_{\mathrm{loc}}([0,T^\star); \dot{B}^{N/2-1}_{2,1})$ such that $\mathcal{Q}f \in L^1_{\mathrm{loc}}([0,T^\star); \dot{B}^{N/2-2}_{2,1})$. Suppose that, for $i = 1, 2$,*

$$\begin{aligned} a^i &\in C([0,T^\star); \dot{B}^{N/2}_{2,\infty}) \cap L^\infty(0,T^\star; L^\infty), \\ u^i &\in C([0,T^\star); \dot{B}^{N/2-1}_{2,1}) \cap L^1_{\mathrm{loc}}([0,T^\star); \dot{B}^{N/2+1}_{2,1}), \\ \nabla\Pi^i &\in L^1_{\mathrm{loc}}([0,T^\star); \dot{B}^{N/2-1}_{2,1}). \end{aligned}$$

Then there exists a constant $c > 0$, depending only on N and such that if

$$\|a^1\|_{L^\infty_{T^\star}(\dot{B}^{N/2}_{2,\infty}\cap L^\infty)} \leqslant c, \tag{4.1}$$

then $(a^2, u^2, \nabla\Pi^2) \equiv (a^1, u^1, \nabla\Pi^1)$.

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Proof. We remark that the L^∞ norm of a^1 and a^2 is conserved. Moreover, as

$$\|a_0\|_{\dot B^{N/2}_{2,\infty}} \leqslant c,$$

inequality (3.1) insures that the smallness condition (4.1) is also fulfilled by a^2 (with $2c$ instead of c) on some non-trivial finite time-interval $[0,T]$.

The equations for

$$(\delta a, \delta u, \nabla\delta\Pi) \stackrel{\text{def}}{=} (a^2-a^1, u^2-u^1, \nabla\Pi^2-\nabla\Pi^1)$$

read

$$\left.\begin{aligned} \partial_t\delta a + u^2\cdot\nabla\delta a &= -\delta u\cdot\nabla a^1,\\ \partial_t\delta u + u^2\cdot\nabla\delta u - \mu\Delta\delta u + \nabla\delta\Pi &= -\delta u\cdot\nabla u^1 + a^1(\mu\Delta\delta u - \nabla\delta\Pi)\\ &\qquad + \delta a(\mu\Delta u^2 - \nabla\Pi^2). \end{aligned}\right\} \tag{4.2}$$

As usual when proving uniqueness, the presence of a transport equation is responsible for the loss of one derivative in the estimates involving δa. This induces us to bound $(\delta a, \delta u, \nabla\delta\Pi)$ in the space $F_T^{N/2-1}$, where

$$F_T^s \stackrel{\text{def}}{=} C([0,T];\dot B^s_{2,\infty}) \times (L^1_T(\dot B^{s+1}_{2,1}) \cap C([0,T];\dot B^{s-1}_{2,1}))^N \times (L^1_T(\dot B^{s-1}_{2,1}))^N,$$

endowed with the norm

$$\|(a,u,\nabla\Pi)\|_{F^s_T} \stackrel{\text{def}}{=} \mu\|a\|_{L^\infty_T(\dot B^s_{2,\infty})} + \mu\|u\|_{L^1_T(\dot B^{s+1}_{2,1})} + \|u\|_{L^\infty_T(\dot B^{s-1}_{2,1})} + \|\nabla\Pi\|_{L^1_T(\dot B^{s-1}_{2,1})}.$$

We claim that $(\delta a, \delta u, \nabla\delta\Pi)$ belongs to $F_T^{N/2-1}$ (a fact that is not straightforward, since we work in homogeneous spaces). Indeed, decompose $(u^i, \nabla\Pi^i)$ into $(\bar u^i + u_L, \nabla\bar\Pi^i + \nabla\Pi_L)$, where $(u_L, \nabla\Pi_L)$ is the solution to the following Stokes equation:

$$\left.\begin{aligned} \partial_t u_L - \mu\Delta u_L + \nabla\Pi_L &= f,\\ \operatorname{div} u_L &= 0,\\ u_{|t=0} &= u_0. \end{aligned}\right\} \tag{4.3}$$

Thanks to proposition 3.2 above and proposition 2.1 in [5], we have

$$u_L \in C([0,T];\dot B^{N/2-1}_{2,1}) \cap L^1(0,T;\dot B^{N/2+1}_{2,1}) \quad\text{and}\quad \nabla\Pi_L \in L^1(0,T;\dot B^{N/2-1}_{2,1}).$$

Moreover, $(\bar u^i, \nabla\Pi^i)$ solves

$$\partial_t\bar u^i - \mu\Delta\bar u^i = \mathcal{P}(-u^i\cdot\nabla u^i + a^i(\mu\Delta u^i - \nabla\Pi^i)), \tag{4.4}$$

$$\operatorname{div}((1+a^i)\nabla\Pi^i) - \operatorname{div}\mathcal{Q}f = -\operatorname{div}(u^i\cdot\nabla u^i) - \mu\operatorname{div}(a^i\Delta u^i). \tag{4.5}$$

Proposition 2.5 ensures that the right-hand side of (4.4) belongs to $L^2(0,T;\dot B^{N/2-3}_{2,1})$, and thus to $L^1(0,T;\dot B^{N/2-3}_{2,1})$ for finite T. Therefore, if we make the hypothesis

$$\mathcal{Q}f \in L^1(0,T;\dot B^{N/2-2}_{2,1})$$

and assume, in addition, that $\|a^i\|_{L^\infty_T(\dot B^{N/2}_{2,\infty}\cap L^\infty)}$ is small, we find that

$$\nabla\Pi^i \in L^1(0,T;\dot B^{N/2-2}_{2,1}).$$

Hence the right-hand side of (4.4) belongs to $L^1(0,T;\dot B^{N/2-2}_{2,1})$.

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Therefore,

$$\bar{u}^i \in L^1(0,T;\dot{B}^{N/2}_{2,1}) \cap C([0,T];\dot{B}^{N/2-2}_{2,1}) \quad \text{and} \quad \nabla\bar{\Pi}^i \in L^1(0,T;\dot{B}^{N/2-2}_{2,1}).$$

On the other hand, $\partial_t a^i \in L^2_T(\dot{B}^{N/2-1}_{2,\infty})$, so that $(a^i - a_0) \in C([0,T];\dot{B}^{N/2-1}_{2,\infty})$. Since

$$\delta a = (a^2 - a_0) - (a^1 - a_0), \quad \delta u = \bar{u}^2 - \bar{u}^1 \quad \text{and} \quad \nabla\delta\Pi = \nabla\bar{\Pi}^2 - \nabla\bar{\Pi}^1,$$

we conclude that $(\delta a, \delta u, \nabla\delta\Pi) \in F^{N/2-1}_T$.

To get an estimate for $\|(\delta a, \delta u, \nabla\delta\Pi)\|_{F^{N/2-1}_t}$, we apply propositions 3.1 and 3.2 to system (4.2). In view of (2.4) and (2.5), we find, for $t \leqslant T$,

$$\begin{aligned}
&\|\delta\|_{L^\infty_t(\dot{B}^{N/2-2}_{2,1})}u + \mu\|\delta u\|_{L^1_t(\dot{B}^{N/2}_{2,1})} + \|\nabla\delta\Pi\|_{L^1_t(\dot{B}^{N/2-2}_{2,1})} \\
&\qquad \leqslant C\mathrm{e}^{C\|\nabla u^2\|_{L^1_t(\dot{B}^{N/2}_{2,1})}} \\
&\qquad\qquad \times [\|a^1\|_{L^\infty_t(\dot{B}^{N/2}_{2,\infty}\cap L^\infty)}(\mu\|\Delta\delta u\|_{L^1_t(\dot{B}^{N/2-2}_{2,1})} + \|\nabla\delta\Pi\|_{L^1_t(\dot{B}^{N/2-2}_{2,1})}) \\
&\qquad\qquad\quad + \|\delta\|_{L^\infty_t(\dot{B}^{N/2-1}_{2,\infty})}a(\mu\|\Delta u^2\|_{L^1_t(\dot{B}^{N/2-1}_{2,1})} + \|\nabla\Pi^2\|_{L^1_t(\dot{B}^{N/2-1}_{2,1})}) \\
&\qquad\qquad\qquad + \|\delta u\|_{L^\infty_t(\dot{B}^{N/2-2}_{2,1})}\|\nabla u^1\|_{L^1_t(\dot{B}^{N/2}_{2,1})}]
\end{aligned}$$

and

$$\|\delta a\|_{L^\infty_t(\dot{B}^{N/2-1}_{2,\infty})} \leqslant C\exp(C\|\nabla u^2\|_{L^1_t(\dot{B}^{N/2}_{2,\infty}\cap L^\infty)})\|\delta u\|_{L^1_t(\dot{B}^{N/2}_{2,1})}\|\nabla a^1\|_{L^\infty_t(\dot{B}^{N/2-1}_{2,\infty})}.$$

From the above inequalities and the smallness assumption (4.1), we get

$$\begin{aligned}
&\|(\delta a, \delta u, \nabla\delta\Pi)\|_{F^{N/2-1}_t} \\
&\quad \leqslant C\mathrm{e}^{C\|\nabla u^2\|_{L^1_t(\dot{B}^{N/2}_{2,1})}}(c + \mu\|u^2\|_{L^1_t(\dot{B}^{N/2+1}_{2,1})} + \|\nabla\Pi^2\|_{L^1_t(\dot{B}^{N/2-1}_{2,1})} \\
&\qquad\qquad + \|\nabla u^1\|_{L^1_t(\dot{B}^{N/2}_{2,1})})\|(\delta a, \delta u, \nabla\delta\Pi)\|_{F^{N/2-1}_t}.
\end{aligned}$$

We remark that $t \mapsto \|(\delta a, \delta u, \nabla\delta\Pi)\|_{F^{N/2-1}_t}$ is a continuous function. Therefore, if we assume that $Cc < 1$, the above inequality leads to $(\delta a, \delta u, \nabla\delta\Pi) = 0$ for small time. Standard arguments then yield uniqueness on the whole time-interval $[0,T]$. □

4.2. Case where $N = 2$

The proof of uniqueness is more intricate in the case where $N = 2$. Let us explain the reason why. As emphasized in the previous section, owing to the presence of a transport equation, we loose one derivative when estimating $(\delta a, \delta u, \nabla\delta\Pi)$. This is going to cause problems in the case where $N = 2$ for this loss of regularity forces us to use the endpoint estimate (2.6): the sum of regularity indices of certain nonlinear terms in (4.2) vanishes. As a matter of fact, if we assume that δu belongs to $L^1(0,T;\dot{B}^1_{2,1})$ and $a^1 \in L^\infty(0,T;\dot{B}^1_{2,\infty} \cap L^\infty)$, the term $a^1\Delta\delta u$ *does not* belong to $L^1(0,T;\dot{B}^{-1}_{2,1})$ (as required to close the estimates), but rather to the larger space $L^1(0,T;\dot{B}^{-1}_{2,\infty})$. Therefore, the right-hand side of the second equation in (4.2) belongs only to $L^1(0,T;\dot{B}^{-1}_{2,\infty})$, and proposition 3.2 yields bounds

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for δu in the (larger) space $\tilde{L}^1_T(\dot{B}^1_{2,\infty}) \cap L^\infty(0,T;\dot{B}^{-1}_{2,\infty})$. It turns out that assuming $\delta u \in \tilde{L}^1_T(\dot{B}^1_{2,\infty})$ suffices to get an estimate for $a^1\Delta\delta u$ in $\tilde{L}^1_T(\dot{B}^{-1}_{2,\infty})$, provided that we make a stronger hypothesis on a^1, namely, $a^1 \in \tilde{L}^\infty_T(\dot{B}^1_{2,1})$ (see (2.8)).

However, we still are in trouble since the bound for δu in $\tilde{L}^1_T(\dot{B}^1_{2,\infty})$ does not provide us with estimates for the right-hand side of the first equation in (4.2). An additional bound in $L^1(0,T;L^\infty)$ for δu is needed. The key to that ultimate difficulty is given by the following logarithmic interpolation inequality (see the proof of proposition 2.8):

$$\|\delta u\|_{L^1_T(\dot{B}^1_{2,1})} \lesssim \|\delta u\|_{\tilde{L}^1_T(\dot{B}^1_{2,\infty})} \log\biggl(\mathrm{e} + \frac{\|\delta u\|_{\tilde{L}^1_T(\dot{B}^0_{2,\infty})} + \|\delta u\|_{\tilde{L}^1_T(\dot{B}^2_{2,\infty})}}{\|\delta u\|_{\tilde{L}^1_T(\dot{B}^1_{2,\infty})}}\biggr). \tag{4.6}$$

One can now state our uniqueness result in dimension $N = 2$.

PROPOSITION 4.2. *Let $(a^1, u^1, \nabla\Pi^1)$ and $(a^2, u^2, \nabla\Pi^2)$ solve (1.6) with the same data $a_0 \in \dot{B}^1_{2,1}$, $u_0 \in \dot{B}^0_{2,1}$ with $\operatorname{div} u_0 = 0$, and $f \in L^1_{\mathrm{loc}}([0,T^\star);\dot{B}^0_{2,1})$ with $\mathcal{Q}f \in L^1_{\mathrm{loc}}([0,T^\star);\dot{B}^{-1}_{2,\infty})$. Suppose that, for $i = 1, 2$,*

$$\begin{aligned} a^i &\in C([0,T^\star);\mathcal{S}') \cap L^\infty_{\mathrm{loc}}([0,T^\star);\dot{B}^1_{2,1}),\\ u^i &\in C([0,T^\star);\dot{B}^0_{2,1}) \cap L^1_{\mathrm{loc}}([0,T^\star);\dot{B}^2_{2,1}),\\ \nabla\Pi^i &\in L^1_{\mathrm{loc}}([0,T^\star);\dot{B}^0_{2,1}). \end{aligned}$$

Then there exists a constant $c > 0$ such that, if

$$\|a^1\|_{\tilde{L}^\infty_{T^\star}(\dot{B}^1_{2,1})} \leqslant c, \tag{4.7}$$

then $(a^2, u^2, \nabla\Pi^2) \equiv (a^1, u^1, \nabla\Pi^1)$.

Proof. In order to track the dependence with respect to the data, we assume throughout that the two solutions $(a^1, u^1, \nabla\Pi^1)$ and $(a^2, u^2, \nabla\Pi^2)$ correspond to (possibly) *different* data (a^1_0, u^1_0, f_1) and (a^2_0, u^2_0, f_2). We shall further denote

$$\delta a_0 \stackrel{\text{def}}{=} a^2_0 - a^1_0, \quad \delta u_0 \stackrel{\text{def}}{=} u^2_0 - u^1_0 \quad \text{and} \quad \delta f \stackrel{\text{def}}{=} f_2 - f_1.$$

Of course, if δa_0 is small enough in $\dot{B}^2_{2,1}$, then the smallness condition (4.7) (with $2c$ instead of c) is also fulfilled by a^2 on some non-trivial finite time-interval $[0, T]$.

As explained above, we are induced to estimate $(\delta a, \delta u, \nabla\delta\Pi)$ in the space

$$G_T \stackrel{\text{def}}{=} C([0,T];\dot{B}^0_{2,\infty}) \times (\tilde{L}^1_T(\dot{B}^1_{2,\infty}) \cap C([0,T];\dot{B}^{-1}_{2,\infty}))^2 \times (\tilde{L}^1_T(\dot{B}^{-1}_{2,\infty}))^2. \tag{4.8}$$

That $(\delta a, \delta u, \nabla\delta\Pi)$ belongs to G_T may be proved by considerations similar to those of the case where $N \geqslant 3$, provided that we make the hypothesis

$$\delta a_0 \in \dot{B}^0_{2,\infty}, \quad \delta u_0 \in \dot{B}^{-1}_{2,\infty} \quad \text{and} \quad \delta f \in \tilde{L}^1_T(\dot{B}^{-1}_{2,\infty}).$$

There is no additional difficulty for δa. The only difference for $(\delta u, \nabla\delta\Pi)$ is that, owing to the product laws in dimension $N = 2$, the right-hand side of equation (4.5) only belongs to $L^2(0,T;\dot{B}^{-2}_{2,\infty})$, and thus to $L^1(0,T;\dot{B}^{-2}_{2,\infty})$, which implies $\nabla\Pi^i \in L^1(0,T;\dot{B}^{-1}_{2,\infty})$, provided that $\|a^i\|_{\tilde{L}^\infty_T(\dot{B}^1_{2,1})} \ll 1$.

Coming back to (4.4) and using once more the product laws, one concludes that

$$(\bar{u}^i, \nabla\bar{\Pi}^i) \in (C([0,T]; \dot{B}^{-1}_{2,\infty}) \cap \tilde{L}^1_T(\dot{B}^1_{2,\infty}))^2 \times (\tilde{L}^1_T(\dot{B}^{-1}_{2,\infty}))^2.$$

From now on, let us assume that $T > 0$ has been chosen so small as to satisfy

$$\exp(C\|\nabla u^2\|_{L^1_T(\dot{B}^1_{2,1})}) \leqslant 2, \tag{4.9}$$

where C is the largest constant appearing in propositions 3.1 and 3.2.

Applying proposition 3.1 and using inequality (2.5) yields

$$\|\delta a\|_{L^\infty_t(\dot{B}^0_{2,\infty})} \lesssim \|\delta a_0\|_{\dot{B}^0_{2,\infty}} + \|\delta u\|_{L^1_t(\dot{B}^1_{2,1})}\|\nabla a^1\|_{L^\infty_t(\dot{B}^0_{2,1})}.$$

Now, inserting the logarithmic inequality (4.6) in the above estimate, we find

$$\begin{aligned}\|\delta a\|_{L^\infty_t(\dot{B}^0_{2,\infty})} &\lesssim \|\delta a_0\|_{\dot{B}^0_{2,\infty}} \\ &\quad + \|a^1\|_{L^\infty_t(\dot{B}^1_{2,1})}\|\delta u\|_{\tilde{L}^1_t(\dot{B}^1_{2,\infty})} \log\biggl(\mathrm{e} + \frac{\|\delta u\|_{\tilde{L}^1_t(\dot{B}^0_{2,\infty})} + \|\delta u\|_{\tilde{L}^1_t(\dot{B}^2_{2,\infty})}}{\|\delta u\|_{\tilde{L}^1_t(\dot{B}^1_{2,\infty})}}\biggr).\end{aligned}$$

Clearly,

$$\begin{aligned}&\|\delta u\|_{\tilde{L}^1_t(\dot{B}^2_{2,\infty})} \leqslant \|u^1\|_{L^1_t(\dot{B}^2_{2,1})} + \|u^2\|_{L^1_t(\dot{B}^2_{2,1})}, \\ &\|\delta u\|_{\tilde{L}^1_t(\dot{B}^0_{2,\infty})} \leqslant \|u^1\|_{L^1_t(\dot{B}^0_{2,1})} + \|u^2\|_{L^1_t(\dot{B}^0_{2,1})} \leqslant t\|u^1\|_{\tilde{L}^\infty_t(\dot{B}^0_{2,1})} + t\|u^2\|_{\tilde{L}^\infty_t(\dot{B}^0_{2,1})}.\end{aligned}$$

Therefore,

$$\|\delta a\|_{L^\infty_t(\dot{B}^0_{2,\infty})} \lesssim \|\delta a_0\|_{\dot{B}^0_{2,\infty}} + c\|\delta u\|_{\tilde{L}^1_t(\dot{B}^1_{2,\infty})} \log\biggl(\mathrm{e} + \frac{V(t)}{\|\delta u\|_{\tilde{L}^1_t(\dot{B}^1_{2,\infty})}}\biggr), \tag{4.10}$$

where V is a non-decreasing locally bounded function on $[0, T^\star)$.

On the other hand, applying proposition 3.2 to the second equation in (4.2) leads to

$$\begin{aligned}\|\delta\|_{L^\infty_t(\dot{B}^{-1}_{2,\infty})}u + \mu\|\delta u\|_{\tilde{L}^1_t(\dot{B}^1_{2,\infty})} + \|\nabla\delta\Pi\|_{\tilde{L}^1_t(\dot{B}^{-1}_{2,\infty})} &\lesssim \|\delta u_0\|_{\dot{B}^{-1}_{2,\infty}} + \|\delta f\|_{\tilde{L}^1_t(\dot{B}^{-1}_{2,\infty})} + \|\delta u \cdot \nabla u^1\|_{\tilde{L}^1_t(\dot{B}^{-1}_{2,\infty})} \\ &\quad + \|\delta a(\mu\Delta u^2 - \nabla\Pi^2)\|_{\tilde{L}^1_t(\dot{B}^{-1}_{2,\infty})} + \|a^1(\mu\Delta\delta u - \nabla\delta\Pi)\|_{\tilde{L}^1_t(\dot{B}^{-1}_{2,\infty})}.\end{aligned} \tag{4.11}$$

Thanks to (2.8),

$$\|a^1(\mu\Delta\delta u - \nabla\delta\Pi)\|_{\tilde{L}^1_t(\dot{B}^{-1}_{2,\infty})} \lesssim \|a^1\|_{\tilde{L}^\infty_t(\dot{B}^1_{2,1})}\|\mu\Delta\delta u - \nabla\delta\Pi\|_{\tilde{L}^1_t(\dot{B}^{-1}_{2,\infty})}.$$

Therefore, if c has been chosen small enough in (4.7), the last term in the right-hand side of (4.11) may be absorbed by the left-hand side. By making use of Minkowski's inequality and (2.6), we eventually find

$$\begin{aligned}\|\delta\|_{L^\infty_t(\dot{B}^{-1}_{2,\infty})}u + \mu\|\delta u\|_{\tilde{L}^1_t(\dot{B}^1_{2,\infty})} + \|\nabla\delta\Pi\|_{\tilde{L}^1_t(\dot{B}^{-1}_{2,\infty})} &\lesssim \|\delta u_0\|_{\dot{B}^{-1}_{2,\infty}} + \|\delta f\|_{\tilde{L}^1_t(\dot{B}^{-1}_{2,\infty})} \\ &\quad + \int_0^t (\|\nabla u^1\|_{\dot{B}^1_{2,1}}\|\delta u\|_{\dot{B}^{-1}_{2,\infty}} + \|\mu\Delta u^2 - \nabla\Pi^2\|_{\dot{B}^0_{2,1}}\|\delta a\|_{\dot{B}^0_{2,\infty}})\,\mathrm{d}\tau.\end{aligned}$$

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Denote

$$\begin{aligned} W(t) &\stackrel{\text{def}}{=} \|\delta\|_{L_t^\infty(\dot B^{-1}_{2,\infty})}u + \|\delta u\|_{\tilde L_t^1(\dot B^1_{2,\infty})} + \|\nabla\delta\Pi\|_{\tilde L_t^1(\dot B^{-1}_{2,\infty})}, \\ W_0 &\stackrel{\text{def}}{=} \|\delta u_0\|_{\dot B^{-1}_{2,\infty}} + \|\delta f\|_{\tilde L_T^1(\dot B^{-1}_{2,\infty})}, \\ Z(t) &\stackrel{\text{def}}{=} \|\nabla u^1(t)\|_{\dot B^1_{2,1}} + \|\mu\Delta u^2(t) - \nabla\Pi^2(t)\|_{\dot B^0_{2,1}}. \end{aligned}$$

Inserting (4.10) in the above inequality, we get, for a constant C depending only on μ,

$$W(t) \leqslant C\biggl(W_0 + \|\delta a_0\|_{\dot B^0_{2,\infty}}\int_0^t Z(\tau)\,\mathrm{d}\tau + \int_0^t Z(\tau)W(\tau)\log\biggl(\mathrm{e} + \frac{V(\tau)}{W(\tau)}\biggr)\mathrm{d}\tau\biggr).$$

Since, for $\alpha \geqslant 0$ and $x \in (0,1]$,

$$\log(\mathrm{e} + \alpha x^{-1}) \leqslant (1 - \log x)\log(\mathrm{e} + \alpha),$$

we eventually find

$$W(t) \leqslant C\biggl(W_0 + \|\delta a_0\|_{\dot B^0_{2,\infty}}\int_0^t Z(\tau)\,\mathrm{d}\tau + \log(\mathrm{e} + V(t))\int_0^t Z(\tau)W(\tau)(1 - \log W(\tau))\,\mathrm{d}\tau\biggr),$$

provided that $W \leqslant 1$ on $[0,t]$.

As V and W are continuous non-decreasing functions and $Z \in L^1(0,T)$, it is easy to check from the above inequality and a Gronwall-type argument (see, for example, [4, lemma 5.2.1]) that, for all $t \in [0,T]$ (with T satisfying condition (4.9)),

$$\frac{W(t)}{\mathrm{e}} \leqslant \biggl(\frac{CW_0 + C\|\delta a_0\|_{\dot B^0_{2,\infty}}\int_0^t Z(\tau)\,\mathrm{d}\tau}{\mathrm{e}}\biggr)^{\exp[-C\log(\mathrm{e}+V(t))\int_0^t Z(\tau)\,\mathrm{d}\tau]}, \tag{4.12}$$

provided that

$$CW_0 + C\|\delta a_0\|_{\dot B^0_{2,\infty}}\int_0^T Z(\tau)\,\mathrm{d}\tau \leqslant \mathrm{e}^{1-\exp[C\log(\mathrm{e}+V(T))\int_0^T Z(\tau)\,\mathrm{d}\tau]}.$$

In particular, if

$$W_0 = \|\delta a_0\|_{\dot B^0_{2,\infty}} = 0,$$

we get $W \equiv 0$ on $[0,T]$, whence also $\delta a \equiv 0$. Standard arguments then yield uniqueness on the whole interval $[0,T^\star)$. ☐

Remark 4.3. What we actually get from (4.12) is Hölder continuity of the solution (seen as a G_T-valued function) with respect to the initial data. Note that the Hölder exponent strongly depends on the solution itself and coarsens when T increases. We recall that a similar phenomenon occurs for the two-dimensional incompressible Euler equations with bounded vorticity (see the work by Yudovitch in [23]). On the other hand, in the case where $N \geqslant 3$, we can obtain (locally) Lipschitz continuity in the space $F_T^{N/2-1}$.

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REMARK 4.4. Whether uniqueness may be proved under the weaker assumption $a_1 \in L^\infty(0,T; L^\infty \cap \dot{B}^1_{2,\infty})$ is unclear. Having $a_1 \in \tilde{L}^\infty_T(\dot{B}^1_{2,1})$ is crucial when estimating $a^1(\mu\Delta\delta u - \nabla\delta\Pi)$.

REMARK 4.5. A similar method has been used for proving uniqueness in dimension $N = 2$ for isentropic compressible fluids, or in dimension $N = 3$ for polytropic fluids (see [11]).

5. Existence

This section is devoted to the proof of solvability for (1.6) in the case where data have critical regularity. The result below clearly entails theorem 1.1.

THEOREM 5.1. *Let $r \in [1, +\infty]$ and $p > 1$. There exists a constant c depending only on N and such that, for any $u_0 \in \dot{B}^{N/2-1}_{2,1}$ with* $\operatorname{div} u_0 = 0$*, $f \in L^1(\mathbb{R}^+; \dot{B}^{N/2-1}_{2,1})$ with $\mathcal{Q}f \in L^p_{\mathrm{loc}}(\mathbb{R}^+; \dot{B}^{N/2-2}_{2,1})$ ($L^p_{\mathrm{loc}}(\mathbb{R}^+; \dot{B}^{-1}_{2,\infty})$ suffices if $N = 2$), and $a_0 \in \dot{B}^{N/2}_{2,r} \cap L^\infty$ with*

$$\|a_0\|_{\dot{B}^{N/2}_{2,\infty} \cap L^\infty} \leqslant c, \tag{5.1}$$

there exists $T \in (0, +\infty]$ such that system (1.6) has a solution $(a, u, \nabla\Pi)$ with

$$a \in C_b([0,T); \dot{B}^{N/2}_{2,r}) \cap L^\infty(0,T; L^\infty), \qquad u \in C_b([0,T); \dot{B}^{N/2-1}_{2,1}) \cap L^1(0,T; \dot{B}^{N/2+1}_{2,1})$$

and $\nabla\Pi \in L^1(0,T; \dot{B}^{N/2-1}_{2,1})$, and, for a constant K depending only on N,

$$\|a\|_{\tilde{L}^\infty_T(\dot{B}^{N/2}_{2,r} \cap L^\infty)} \leqslant K\|a_0\|_{\dot{B}^{N/2}_{2,r} \cap L^\infty} \tag{5.2}$$

and

$$\|u\|_{\tilde{L}^\infty_T(\dot{B}^{N/2-1}_{2,1})} + \mu\|u\|_{L^1_T(\dot{B}^{N/2+1}_{2,1})} + \|\nabla\Pi\|_{L^1_T(\dot{B}^{N/2-1}_{2,1})} \leqslant K(\|u_0\|_{\dot{B}^{N/2-1}_{2,1}} + \|f\|_{L^1_T(\dot{B}^{N/2-1}_{2,1})}). \tag{5.3}$$

Besides, $\nabla\Pi$ belongs to $L^p_{\mathrm{loc}}(0,T; \dot{B}^{N/2-2}_{2,1})$ if $N \geqslant 3$ ($L^p_{\mathrm{loc}}(0,T; \dot{B}^{-1}_{2,\infty})$ if $N \geqslant 2$) and there exists a universal positive constant κ such that T may be bounded by below by

$$\max\left\{ t \geqslant 0 \;\middle|\; \|\mathcal{Q}f\|_{L^1_t(\dot{B}^{N/2-1}_{2,1})} + \sum_{q\in\mathbb{Z}} 2^{q(N/2-1)}(\|\Delta_q u_0\|_{L^2} + \|\Delta_q \mathcal{P}f\|_{L^1_t(L^2)})\left(\frac{1 - \mathrm{e}^{-\kappa\mu 2^{2q} t}}{\kappa}\right) \leqslant \frac{c\mu}{1 + U_0/\mu} \right\}$$

with

$$U_0 \stackrel{\mathrm{def}}{=} \|u_0\|_{\dot{B}^{N/2-1}_{2,1}} + \|\|_{L^1(\dot{B}^{N/2-1}_{2,1})} Pf.$$

If, in addition to (5.1), we have

$$\|u_0\|_{\dot{B}^{N/2-1}_{2,1}} + \|f\|_{L^1(\dot{B}^{N/2-1}_{2,1})} \leqslant c'\mu$$

for some suitably small constant c' depending only on N, we can take $T = +\infty$.

In the case $N \geqslant 3$, uniqueness holds true. In the case where $N = 2$, uniqueness holds if $r = 1$ and $\|a_0\|_{\dot{B}^1_{2,1}} \leqslant c$.

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REMARK 5.2. The assumption on $\mathcal{Q}f$ is used for proving uniqueness and provides some compactness. It may be removed by proving uniqueness in non-homogeneous spaces and using a contraction argument rather than compactness for stating the convergence of approximate solutions. The proof is more technical than the one we present here though.

Proof.

STEP 1 (smooth approximate solutions). We first smooth out the data. For $n \in \mathbb{N}$, define

$$a_0^n \stackrel{\text{def}}{=} \sum_{|q|\leqslant n} \Delta_q a_0, \quad u_0^n \stackrel{\text{def}}{=} \sum_{|q|\leqslant n} \Delta_q u_0 \quad \text{and} \quad f^n \stackrel{\text{def}}{=} \sum_{|q|\leqslant n} \Delta_q f.$$

Remark that a_0^n and u_0^n belong to H^∞, and that $f \in L^1(\mathbb{R}^+; H^\infty)$. Therefore, there exists a $T_n > 0$ (possibly infinite) such that system (1.6) with data (a_0^n, u_0^n, f^n) has a unique solution $(a^n, u^n, \nabla \Pi^n)$ with (see, for example, [12] and the references therein)

$$a^n \in C([0, T^n); H^\infty), \quad u^n \in C([0, T^n); H^\infty) \quad \text{and} \quad \nabla \Pi^n \in L^1_{\text{loc}}([0, T^n); H^\infty).$$

STEP 2 (uniform estimates). We claim that $\inf_{n\in\mathbb{N}} T^n$ may be bounded by below by some $T \in (0, +\infty]$ for which $(a^n, u^n, \nabla \Pi^n)_{n\in\mathbb{N}}$ is uniformly bounded in

$$(L^\infty(0,T;L^\infty) \cap \tilde{L}^\infty_T(\dot{B}^{N/2}_{2,r})) \times (\tilde{L}^\infty_T(\dot{B}^{N/2-1}_{2,1}) \cap L^1(0,T;\dot{B}^{N/2+1}_{2,1}))^N \times (L^1(0,T;\dot{B}^{N/2-1}_{2,1}))^N.$$

Let $(u_L, \nabla \Pi_L)$ solve the non-stationary Stokes system

$$\left.\begin{aligned} \partial_t u_L - \mu \Delta u_L + \nabla \Pi_L &= f, \\ \operatorname{div} u_L &= 0, \\ u_{L|t=0} &= 0, \end{aligned}\right\} \tag{5.4}$$

and write

$$u_L^n \stackrel{\text{def}}{=} \sum_{|q|\leqslant n} \Delta_q u_L \quad \text{and} \quad \nabla \Pi_L^n \stackrel{\text{def}}{=} \sum_{|q|\leqslant n} \Delta_q \nabla \Pi_L.$$

Decompose $(u^n, \nabla \Pi^n)$ into

$$u^n \stackrel{\text{def}}{=} u_L^n + \bar{u}_n \quad \text{and} \quad \nabla \Pi^n = \nabla \Pi_L^n + \nabla \bar{\Pi}^n.$$

Obviously, $(a^n, \bar{u}^n, \nabla \bar{\Pi}^n)$ satisfies

$$\left.\begin{aligned} \partial_t a^n + u^n \cdot \nabla a^n &= 0, \\ \partial_t \bar{u}^n + u^n \cdot \nabla \bar{u}^n - \mu \Delta \bar{u}^n + \nabla \bar{\Pi}^n &= -u^n \cdot \nabla u_L^n + a^n(\mu \Delta u^n - \nabla \Pi^n), \\ \operatorname{div} \bar{u}^n &= 0, \\ (a^n, \bar{u}^n)_{|t=0} &= (a_0^n, 0). \end{aligned}\right\} \tag{5.5}$$

Let $K \stackrel{\text{def}}{=} \|\mathcal{F}^{-1}\chi\|_{L^1}$. We readily have, $\forall r' \in [r, +\infty]$,

$$\sup_{n\in\mathbb{N}} \|a_0^n\|_{\dot{B}^{N/2}_{2,r'} \cap L^\infty} \leqslant K \|a_0\|_{\dot{B}^{N/2}_{2,r'} \cap L^\infty} \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|u_0^n\|_{\dot{B}^{N/2-1}_{2,1}} \leqslant K \|u_0\|_{\dot{B}^{N/2-1}_{2,1}}.$$

Hence, by virtue of propositions 3.1 and 3.2, and of inequality (2.5),

$$\begin{aligned}\|\bar u^n\|_{L^\infty_T(\dot B^{N/2-1}_{2,1})} + \mu\|\bar u^n\|_{L^1_T(\dot B^{N/2+1}_{2,1})} + \|\nabla\bar\Pi^n\|_{L^1_T(\dot B^{N/2-1}_{2,1})} \\ \leqslant C\mathrm{e}^{C\|\nabla u^n\|_{L^1_T(\dot B^{N/2}_{2,1})}} \\ \times(\|u^n\|_{L^\infty_T(\dot B^{N/2-1}_{2,1})}\|\nabla u^n_L\|_{L^1_T(\dot B^{N/2}_{2,1})} \\ + \|a^n\|_{L^\infty_T(\dot B^{N/2}_{2,\infty}\cap L^\infty)}(\mu\|u^n\|_{L^1_T(\dot B^{N/2+1}_{2,1})} + \|\nabla\Pi^n\|_{L^1_T(\dot B^{N/2-1}_{2,1})}))\end{aligned}$$

and

$$\|a^n\|_{\tilde L^\infty_T(\dot B^{N/2}_{2,r'}\cap L^\infty)} \leqslant K\mathrm{e}^{C\|\nabla u^n\|_{L^1_T(\dot B^{N/2}_{2,1})}}\|a_0\|_{\dot B^{N/2}_{2,r'}\cap L^\infty}.$$

Let $\alpha > 0$ be a small parameter to be fixed hereafter and assume that T has been chosen so small as to satisfy

$$\mu\|u_L\|_{L^1_T(\dot B^{N/2+1}_{2,1})} + \|\nabla\Pi_L\|_{L^1_T(\dot B^{N/2-1}_{2,1})} \leqslant \alpha, \qquad \|u_L\|_{\tilde L^\infty_T(\dot B^{N/2-1}_{2,1})} \leqslant U_0. \qquad (\mathcal{H})$$

From the definition of $(u^n_L, \nabla\Pi^n_L)$, we easily gather that

$$\mu\|u^n_L\|_{L^1_T(\dot B^{N/2+1}_{2,1})} + \|\nabla\Pi^n_L\|_{L^1_T(\dot B^{N/2-1}_{2,1})} \leqslant K\alpha, \qquad \|u^n_L\|_{\tilde L^\infty_T(\dot B^{N/2-1}_{2,1})} \leqslant KU_0. \quad (5.6)$$

Hence

$$\begin{aligned}\|\bar u^n\|_{\tilde L^\infty_T(\dot B^{N/2-1}_{2,1})} + \mu\|\bar u^n\|_{L^1_T(\dot B^{N/2+1}_{2,1})} + \|\nabla\bar\Pi^n\|_{L^1_T(\dot B^{N/2-1}_{2,1})} \\ \leqslant C\mathrm{e}^{(C/\mu)(\alpha+\mu\|\bar u^n\|_{L^1_T(\dot B^{N/2+1}_{2,1})})} \\ \times\biggl(\frac{\alpha}{\mu}(\|\bar u^n\|_{L^\infty_T(\dot B^{N/2-1}_{2,1})} + U_0) \\ + \|a^n\|_{L^\infty_T(\dot B^{N/2}_{2,\infty}\cap L^\infty)}(\alpha + \mu\|\bar u^n\|_{L^1_T(\dot B^{N/2+1}_{2,1})} + \|\nabla\bar\Pi^n\|_{L^1_T(\dot B^{N/2-1}_{2,1})})\biggr)\end{aligned}$$

and

$$\|a^n\|_{\tilde L^\infty_T(\dot B^{N/2}_{2,r'}\cap L^\infty)} \leqslant K\exp((C/\mu)(\alpha + \mu\|\bar u^n\|_{L^1_T(\dot B^{N/2+1}_{2,1})}))\|a_0\|_{\dot B^{N/2}_{2,r'}\cap L^\infty}.$$

From the above inequalities, it is not hard to check by induction that if $(\mathcal{H})$ is fulfilled and, in addition,

$$8C\|a_0\|_{\dot B^{N/2}_{2,\infty}\cap L^\infty} \leqslant 1 \quad\text{and}\quad \frac{C\alpha}{\mu}\biggl(1 + \frac{2CU_0}{\mu}\biggr) \leqslant \frac{1}{8},$$

then, for all $n \in \mathbb{N}$ and $r' \in [r, +\infty]$, we have

$$\|\bar u^n\|_{\tilde L^\infty_T(\dot B^{N/2-1}_{2,1})} + \mu\|\bar u^n\|_{L^1_T(\dot B^{N/2+1}_{2,1})} + \|\nabla\bar\Pi^n\|_{L^1_T(\dot B^{N/2-2}_{2,1})} \leqslant \alpha(2 + 8CU_0\mu^{-1}), \quad (5.7)$$

$$\|a^n\|_{\tilde L^\infty_T(\dot B^{N/2}_{2,r'}\cap L^\infty)} \leqslant 2K\|a_0\|_{\dot B^{N/2}_{2,r'}\cap L^\infty}. \quad (5.8)$$

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Now, according to [7, proposition 2.3], we have

$$\|u_L\|_{\tilde L^\infty_T(\dot B^{N/2-1}_{2,1})} \leqslant \|u_0\|_{\dot B^{N/2-1}_{2,1}} + \|\|_{L^1_T(\dot B^{N/2-1}_{2,1})} Pf,$$

$$\mu\|u_L\|_{L^1_T(\dot B^{N/2+1}_{2,1})} \leqslant C\biggl[\sum_{q\in\mathbb{Z}} 2^{q(N/2-1)}(\|\Delta_q u_0\|_{L^2} + \|\Delta_q \mathcal{P}f\|_{L^1_T(L^2)}) \times \biggl(\frac{1-\mathrm{e}^{-\kappa\mu T 2^{2q}}}{\kappa}\biggr)\biggr]$$

and, obviously,

$$\|\nabla \Pi_L\|_{L^1_T(\dot B^{N/2-1}_{2,1})} \leqslant \|\mathcal{Q}f\|_{L^1_T(\dot B^{N/2-1}_{2,1})}.$$

Therefore, system $(\mathcal{H})$ is satisfied for small positive T, and we conclude that (5.6), (5.7) and (5.8) hold for all $n \in \mathbb{N}$.

Note that

$$\operatorname{div}((1+a^n)\nabla \Pi^n) = \operatorname{div}(\mathcal{Q}f^n - u^n\cdot\nabla u^n + \mu(1+a^n)\Delta u^n).$$

Hence, if $N \geqslant 3$, the assumption

$$\mathcal{Q}f \in L^p_{\mathrm{loc}}(\mathbb{R}^+; \dot B^{N/2-2}_{2,1})$$

entails that $\nabla\Pi^n$ is uniformly bounded in $L^p_{\mathrm{loc}}(0,T;\dot B^{N/2-2}_{2,1})$. If $N = 2$, similar arguments lead to $\nabla\Pi^n$ uniformly bounded in $L^p_{\mathrm{loc}}(0,T;\dot B^{-1}_{2,\infty})$.

Of course, if, for sufficiently small c', we have

$$U_0 + \|\mathcal{Q}f\|_{L^1(\dot B^{N/2-1}_{2,1})} \leqslant c'\mu,$$

then all the estimates above may be made global.

In order to justify the above computations, we still have to check that $T \leqslant T^n$. Writing

$$T^\star \stackrel{\mathrm{def}}{=} \min(T^n, T),$$

we readily have

$$\nabla u^n \in L^1(0,T^\star;\dot B^{N/2}_{2,1}).$$

As $a^n_0 \in H^\infty$, proposition 3.1 ensures that $a^n \in \tilde L^\infty_{T^\star}(\dot H^s)$ whenever $s \in (0, 1+\frac{1}{2}N)$. On the other hand, $a^n_0 \in L^2$, so that $a^n \in L^\infty(0,T^\star;L^2)$. We conclude that

$$a^n \in L^\infty(0,T^\star;H^s) \quad \text{for all } s < 1+\tfrac{1}{2}N.$$

The blow-up criterion derived in [12, proposition 0.5] ensures that no blow-up may occur at time $T^\star$. Hence $T < T^n < +\infty$ or $T^n = T = +\infty$.

Step 3 (convergence). The convergence of $(u^n_L, \nabla\Pi^n_L)$ to $(u_L, \nabla\Pi_L)$ readily stems from the definition of Besov spaces.

As for the convergence of $(a^n, \bar u^n_L, \nabla\bar\Pi^n_L)$, it relies upon compactness properties of the sequence, which are obtained by considering the time derivative of the solution. Indeed, taking advantage of $\mathcal{Q}f \in L^p_{\mathrm{loc}}(0,T;B^{N/2-2}_{2,\infty})$ and using the bounds (5.7) and (5.8), system (5.5) and proposition 2.7, it is not hard to prove that $\nabla\bar\Pi^n$ is uniformly bounded in $L^p_{\mathrm{loc}}(0,T;\dot B^{N/2-2}_{2,\infty})$. Then arguments similar to the ones used in [6] for the compressible Navier–Stokes equations lead to the following lemma.

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LEMMA 5.3.

(i) *The sequence* $(\partial_t a^n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $L^2(0,T;\dot B^{N/2-1}_{2,r})$.

(ii) *The sequence* $(\partial_t \bar u^n)_{n\in\mathbb{N}}$ *is uniformly bounded in* $L^p_{\mathrm{loc}}(0,T;\dot B^{N/2-2}_{2,\infty})$.

As the embeddings

$$B^{N/2}_{2,r,\mathrm{loc}} \hookrightarrow B^{N/2-1}_{2,r,\mathrm{loc}} \quad \text{and} \quad B^{N/2-1}_{2,\infty,\mathrm{loc}} \hookrightarrow B^{N/2-2}_{2,1,\mathrm{loc}}$$

are compact, we get the convergence (up to an extraction) of $(a^n, u^n, \nabla\Pi^n)$ to a limit $(a, u, \nabla\Pi)$ belonging to

$$(L^\infty(0,T;L^\infty)\cap \tilde L^\infty_T(\dot B^{N/2}_{2,r}))\times (L^\infty(0,T;\dot B^{N/2-1}_{2,1}) \cap L^1(0,T;\dot B^{N/2+1}_{2,1}))^N \cap (L^1(0,T;\dot B^{N/2-1}_{2,1}))^N$$

and satisfying (5.2) and (5.3).

Combining with the uniform estimates (5.7) and (5.8), we conclude that $(a^n, u^n, \nabla\Pi^n)$ tends (in G_T) to some limit $(a, u, \nabla\Pi)$ which belongs to

$$(L^\infty(0,T;L^\infty)\cap \tilde L^\infty_T(\dot B^{N/2}_{2,r}))\times (L^\infty(0,T;\dot B^{N/2-1}_{2,1}) \cap L^1(0,T;\dot B^{N/2+1}_{2,1}))^N \cap (L^1(0,T;\dot B^{N/2-1}_{2,1}))^N$$

and satisfies (5.2) and (5.3).

Continuity in time with values in $\dot B^{N/2}_{2,\infty}$ for a stems from proposition 3.1. Moreover, as a is the solution to a transport equation by a solenoidal vector field whose gradient belongs to $L^1(0,T;L^\infty)$, the L^∞ norm of a is a constant. Continuity for u may be proved as in [6].

Uniqueness is a straightforward corollary of propositions 4.1 and 4.2. □

Additional regularity assumptions on the data yield more regular solutions. Slight changes in the proof above would give the following result.

THEOREM 5.4. *Let* (a_0, u_0, f) *satisfy the hypotheses of theorem 5.1 and assume, in addition, that* $a_0 \in \dot B^s_{2,r}$, $u_0 \in \dot B^{s-1}_{2,r}$, $f \in \tilde L^1(\mathbb{R}^+;\dot B^{s-1}_{2,r})$ *for some* $s \in]\frac12 N, \frac12 N+1[$ *and* $r\in[1,+\infty]$. *Then there exists a* $T>0$ (*which may be chosen as in the statement of theorem 5.1*) *such that system (1.6) has a unique solution* $(a,u,\nabla\Pi)$ *that satisfies the same properties as in theorem 5.1 and, in addition, belongs to*

$$C_b([0,T);\dot B^s_{2,r})\times (C_b([0,T);\dot B^{s-1}_{2,r})\cap \tilde L^1_T(\dot B^{s+1}_{2,r}))^N \times (\tilde L^1_T(\dot B^{s-1}_{2,r}))^N.$$

Appendix A.

For the sake of completeness, we prove here the commutation estimate needed in § 3. The statement given below is actually a bit more general.

LEMMA A.1. *Let* $1\leqslant p,\ r\leqslant +\infty$, $\rho<1$, $\eta>-1$ *and* v *be a solenoidal vector field. In addition, assume that*

$$\rho-\eta+N\min(1,2/p)>0 \quad \textit{and} \quad \rho+N/p>0.$$

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Then the following inequality holds true:

$$\bigg(\sum_{q\in\mathbb{Z}} \|[v,\Delta_q]\cdot\nabla a\|_{L^p}^r 2^{rq(N/p+\rho-1-\eta)}\bigg)^{1/r} \lesssim \|\nabla v\|_{\dot B^{N/p+\rho-1}_{p,r}} \|\nabla a\|_{\dot B^{N/p-\eta-1}_{p,r}}.$$

If $\rho=1$*, then* $\|\nabla v\|_{\dot B^{N/p+\rho-1}_{p,r}}$ *has to be replaced by* $\|\nabla v\|_{\dot B^{N/p}_{p,r}\cap L^\infty}$*. If* $\eta=-1$*, then* $\|\nabla a\|_{\dot B^{N/p-\eta-1}_{p,r}}$ *has to be replaced by* $\|\nabla a\|_{\dot B^{N/p}_{p,r}\cap L^\infty}$*.*

Proof. Throughout the proof, the summation convention over repeated indices is used.

By virtue of Bony's decomposition (see § 3), and as $\operatorname{div} v=0$, the commutator may be decomposed into

$$[v,\Delta_q]\cdot\nabla a = [T_{v^j},\Delta_q]\partial_j a + T'_{\Delta_q\partial_j a} v^j - \Delta_q T_{\partial_j a} v^j - \Delta_q \partial_j R(a,v^j), \qquad \text{(A 1)}$$

where $T'_u v$ stands for $T_u v + R(u,v)$.

We further decompose the first term in the right-hand side of (A 1) into

$$[T_{v^j},\Delta_q]\partial_j a = \sum_{|q'-q|\leqslant 4} [S_{q'-1}v^j,\Delta_q]\Delta_{q'}\partial_j a.$$

(That the summation may be restricted to $|q'-q|\leqslant 4$ is due to (2.2).)

Writing

$$h \stackrel{\text{def}}{=} \mathcal{F}^{-1}\varphi$$

and applying the first-order Taylor's formula, we get, for $x\in\mathbb{R}^N$,

$$\begin{aligned}&[S_{q'-1}v^j,\Delta_q]\Delta_{q'}\partial_j a(x)\\ &\qquad = 2^{-q'}\int_{\mathbb{R}^N}\int_0^1 h(y)(y\cdot\nabla S_{q'-1}v^j(x-2^{-q}\tau y))\Delta_{q'}\partial_j a(x-2^{-q}y)\,\mathrm{d}\tau\mathrm{d}y.\end{aligned}$$

Since, for $\rho<1$,

$$\|\nabla S_{q'-1}v^j\|_{L^\infty} \lesssim 2^{q'(1-\rho)}\|\nabla v\|_{\dot B^{\rho-1}_{\infty,\infty}},$$

we get, for some series $(c_q)_{q\in\mathbb{Z}}$ such that $\sum_{q\in\mathbb{Z}} c_q^r=1$,

$$\begin{aligned}\|[T_{v^j},\Delta_q]\partial_j a]\|_{L^p} &\lesssim c_q 2^{-q(N/p+\rho-\eta-1)}\|\nabla v\|_{\dot B^{\rho-1}_{\infty,\infty}}\|\nabla a\|_{\dot B^{N/p-1-\eta}_{p,r}}\\ &\lesssim c_q 2^{-q(N/p+\rho-\eta-1)}\|\nabla v\|_{\dot B^{N/p+\rho-1}_{p,r}}\|\nabla a\|_{\dot B^{N/p-1-\eta}_{p,r}}.\end{aligned}$$

The above inequality still holds for $\rho=1$, provided we use $\|\nabla v\|_{L^\infty}$ instead of $\|\nabla v\|_{\dot B^{\rho-1}_{\infty,\infty}}$.

According to (2.2), the second term of the right-hand side of (A 1) can be rewritten as

$$T'_{\Delta_q\partial_j a}v^j = \sum_{q'\geqslant q-2} S_{q'+2}\Delta_q\partial_j a\Delta_{q'}v^j,$$

whence

$$\begin{aligned}2^{q(N/p+\rho-\eta-1)}\|T'_{\Delta_q\partial_j a}v^j\|_{L^p} &\lesssim 2^{q(N/p+\rho-\eta-1)}\sum_{q'\geqslant q-2}\|\Delta_{q'}v^j\|_{L^p}\|\Delta_q\partial_j a\|_{L^\infty}\\ &\lesssim \sum_{q'\geqslant q-2} 2^{(q-q')(\rho+N/p)}c_{q'}\|v\|_{\dot B^{N/p+\rho}_{p,r}}\|\nabla a\|_{\dot B^{-\eta-1}_{\infty,\infty}}.\end{aligned}$$

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Convolution inequalities combined with the continuous embedding

$$\dot{B}^{N/p-\eta-1}_{p,r} \hookrightarrow \dot{B}^{-\eta-1}_{\infty,\infty}$$

yield the desired estimate, provided that $\rho + N/p > 0$.

Basic continuity properties for the paraproduct in Besov spaces enable us to bound the last two terms. Indeed, according to [22, § 4.4], we have, for $\eta > -1$,

$$\|T_{\partial_j a} v^j\|_{\dot{B}^{N/p+\rho-1-\eta}_{p,r}} \lesssim \|\nabla a\|_{\dot{B}^{-\eta-1}_{\infty,\infty}} \|v\|_{\dot{B}^{N/p+\rho}_{p,r}},$$

with $\|\nabla a\|_{L^\infty}$ instead of $\|\partial_j a\|_{\dot{B}^{-\eta-1}_{\infty,\infty}}$ if $\eta = -1$, and

$$\|\partial_j R(a, v^j)\|_{\dot{B}^{N/p+\rho-1-\eta}_{p,r}} \lesssim \|v\|_{\dot{B}^{N/p+\rho}_{p,r}} \|a\|_{\dot{B}^{N/p-\eta}_{p,r}}$$

if $\rho - \eta + N \min(1, 2/p) > 0$. □

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