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MAHLER'S AND KOKSMA'S CLASSIFICATIONS IN FIELDS OF POWER SERIES

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Abstract. Let q a prime power and $\mathbb{F}_q$ the finite field of q elements. We study the analogues of Mahler's and Koksma's classifications of complex numbers for power series in $\mathbb{F}_q((T^{-1}))$. Among other results, we establish that both classifications coincide, thereby answering a question of Ooto.

§1. Introduction

Mahler [16], in 1932, and Koksma [15], in 1939, introduced two related measures for the quality of approximation of a complex number ξ by algebraic numbers. For any integer $n \geq 1$, we denote by $w_n(\xi)$ the supremum of the real numbers w for which

$$0 < |P(\xi)| < H(P)^{-w}$$

has infinitely many solutions in integer polynomials $P(X)$ of degree at most n. Here, $H(P)$ stands for the naïve height of the polynomial $P(X)$, that is, the maximum of the absolute values of its coefficients. Furthermore, we set

$$w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n},$$

and, according to Mahler [16], we say that ξ is

$$\begin{gathered}\text{an } A\text{-number, if } w(\xi) = 0;\\ \text{an } S\text{-number, if } 0 < w(\xi) < \infty;\\ \text{a } T\text{-number, if } w(\xi) = \infty \text{ and } w_n(\xi) < \infty, \text{ for any integer } n \geq 1;\\ \text{a } U\text{-number, if } w(\xi) = \infty \text{ and } w_n(\xi) = \infty, \text{ for some integer } n \geq 1.\end{gathered}$$

The set of complex A-numbers is the set of complex algebraic numbers. In the sense of the Lebesgue measure, almost all numbers are S-numbers. Liouville numbers (which, by definition, are the real numbers ξ such that $w_1(\xi)$ is infinite) are examples of U-numbers, while the existence of T-numbers remained an open problem during nearly 40 years, until it was confirmed by Schmidt [22], [23].

Following Koksma [15], for any integer $n \geq 1$, we denote by $w_n^*(\xi)$ the supremum of the real numbers w^* for which

$$0 < |\xi - \alpha| < H(\alpha)^{-w^*-1}$$

has infinitely many solutions in complex algebraic numbers α of degree at most n. Here, $H(\alpha)$ stands for the naïve height of α, that is, the naïve height of its minimal defining polynomial over the integers. Koksma [15] defined A^*-, S^*-, T^*-, and U^*-numbers as above,

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using w_n^* in place of w_n. Namely, setting

$$w^*(\xi) = \limsup_{n\to\infty} \frac{w_n^*(\xi)}{n},$$

we say that ξ is

$$\begin{gathered}\text{an } A^*\text{-number, if } w^*(\xi) = 0;\\ \text{an } S^*\text{-number, if } 0 < w^*(\xi) < \infty;\\ \text{a } T^*\text{-number, if } w^*(\xi) = \infty \text{ and } w_n^*(\xi) < \infty, \text{ for any integer } n \geq 1;\\ \text{a } U^*\text{-number, if } w^*(\xi) = \infty \text{ and } w_n^*(\xi) = \infty, \text{ for some integer } n \geq 1.\end{gathered}$$

Koksma proved that this classification of numbers is equivalent to the Mahler one, in the sense that the classes A, S, T, U coincide with the classes A^*, S^*, T^*, U^*, respectively. For more information on the functions w_n and w_n^*, the reader is directed to [5], [7].

Likewise, we can divide the sets of real numbers and p-adic numbers in classes A, S, T, U and A^*, S^*, T^*, U^*. However, there is a subtle difference with the case of complex numbers, since the field $\mathbb{R}$ of real numbers and the field $\mathbb{Q}_p$ of p-adic numbers are not algebraically closed. This means that, in the definition of the exponent $w_n^*(\xi)$ for a real (resp., p-adic) number ξ, we have to decide whether the algebraic approximants α are to be taken in $\mathbb{C}$ (resp., in an algebraic closure of $\mathbb{Q}_p$) or in $\mathbb{R}$ (resp., in $\mathbb{Q}_p$). Fortunately, in both cases, it makes no difference, as shown in [4], [5]. For instance, it has been proved that, if there is α of degree n in an algebraic closure of $\mathbb{Q}_p$ satisfying $|\xi - \alpha| < H(\alpha)^{-1-w^*}$, then there exists α' in $\mathbb{Q}_p$, algebraic of degree at most n, such that $H(\alpha') \leq cH(\alpha)$ and $|\xi - \alpha'| \leq cH(\alpha')^{-1-w^*}$, where c depends only on ξ and on n.

The analogous question has not yet been clarified for Diophantine approximation in the field $\mathbb{F}_q((T^{-1}))$ of power series over the finite field $\mathbb{F}_q$. Different authors have different practices, some of them define w_n^* by restricting to algebraic elements in $\mathbb{F}_q((T^{-1}))$, while some others allow algebraic elements to lie in an algebraic closure of $\mathbb{F}_q((T^{-1}))$. One of the aims of the present paper is precisely to clarify this point.

Our framework is the following. Let p be a prime number and $q = p^f$ an integer power of p. Any nonzero element ξ in $\mathbb{F}_q((T^{-1}))$ can be written

$$\xi = \sum_{n=N}^{+\infty} a_n T^{-n},$$

where N is in $\mathbb{Z}$, $a_N \neq 0$, and a_n is in $\mathbb{F}_q$ for $n \geq N$. We define a valuation ν and an absolute value $|\cdot|$ on $\mathbb{F}_q((T^{-1}))$ by setting $\nu(\xi) = N$, $|\xi| := q^{-N}$, and $\nu(0) = +\infty$, $|0| := 0$. In particular, if $R(T)$ is a nonzero polynomial in $\mathbb{F}_q[T]$, then we have $|R| = q^{\deg(R)}$. The field $\mathbb{F}_q((T^{-1}))$ is the completion with respect to ν of the quotient field $\mathbb{F}_q(T)$ of the polynomial ring $\mathbb{F}_q[T]$. It is not algebraically closed. Following [26], we denote by C_∞ the completion of its algebraic closure. To describe precisely, the set of algebraic elements in C_∞ is rather complicated. Indeed, Abhyankar [1] pointed out that it contains the element

$$T^{-1/p} + T^{-1/p^2} + T^{-1/p^3} + \cdots,$$

which is a root of the polynomial $TX^p - TX - 1$. Kedlaya [13], [14] constructed an algebraic closure of $K((T^{-1}))$ for any field K of positive characteristic in terms of certain generalized power series.

There should be no confusion between the variable T and the notion of T-number.

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The height $H(P)$ of a polynomial $P(X)$ over $\mathbb{F}_q[T]$ is the maximum of the absolute values of its coefficients. A power series in C_∞ is called algebraic if it is a root of a nonzero polynomial with coefficients in $\mathbb{F}_q[T]$. Its height is then the height of its minimal defining polynomial over $\mathbb{F}_q[T]$. We define the exponents of approximation w_n and w_n^* as follows.

Definition 1.1. Let ξ be in $\mathbb{F}_q((T^{-1}))$. Let $n \geq 1$ be an integer. We denote by $w_n(\xi)$ the supremum of the real numbers w for which

$$0 < |P(\xi)| < H(P)^{-w}$$

has infinitely many solutions in polynomials $P(X)$ over $\mathbb{F}_q[T]$ of degree at most n. We denote by $w_n^*(\xi)$ the supremum of the real numbers w^* for which

$$0 < |\xi - \alpha| < H(\alpha)^{-w^*-1}$$

has infinitely many solutions in algebraic power series α in $\mathbb{F}_q((T^{-1}))$ of degree at most n.

An important point in the definition of w_n^* is that we require that the approximants α lie in $\mathbb{F}_q((T^{-1}))$. In the existing literature, it is not always clearly specified whether the algebraic approximants are taken in C_∞ or in $\mathbb{F}_q((T^{-1}))$. To take this into account, we introduce the following exponents of approximation, where we use the superscript $^@$ to refer to the field C_∞.

Definition 1.2. Let ξ be in $\mathbb{F}_q((T^{-1}))$. Let $n \geq 1$ be an integer. We denote by $w_n^@(\xi)$ the supremum of the real numbers $w^@$ for which

$$0 < |\xi - \alpha| < H(\alpha)^{-w^@-1}$$

has infinitely many solutions in algebraic power series α in C_∞ of degree at most n.

Clearly, we have $w_n^@(\xi) \geq w_n^*(\xi)$ for every $n \geq 1$ and every ξ in $\mathbb{F}_q((T^{-1}))$. The first aim of this paper is to establish that the functions w_n^* and $w_n^@$ coincide.

Theorem 1.3. *For any ξ in $\mathbb{F}_q((T^{-1}))$ and any integer $n \geq 1$, we have*

$$w_n^*(\xi) = w_n^@(\xi).$$

Theorem 1.3 is not surprising, since it seems to be very unlikely that a power series in $\mathbb{F}_q((T^{-1}))$ could be better approximated by algebraic power series in $C_\infty \setminus \mathbb{F}_q((T^{-1}))$ than by algebraic power series in $\mathbb{F}_q((T^{-1}))$. Difficulties arise because of the existence of polynomials over $\mathbb{F}_q[T]$ which are not separable and of the lack of a Rolle lemma, which is a key ingredient for the proof of the analogous result for the classifications of real and p-adic numbers.

Exactly as Mahler and Koksma did, we divide the set of power series in $\mathbb{F}_q((T^{-1}))$ in classes A, S, T, U, A^*, S^*, T^*, and U^*, by using the exponents of approximation w_n and w_n^*. It is convenient to keep the same terminology and to use S-numbers, and so forth, although we are concerned with power series and not with *numbers*. This has been done by Bundschuh [8], who gave some explicit examples of U-numbers. Ooto [19, p. 145] observed that, by the currently known results (with $w_n^@$ used instead of w_n^* in the definitions of the classes), the sets of A-numbers and of A^*-numbers coincide, as do the sets of U-numbers and of U^*-numbers. Furthermore, an S-number is an S^*-number, while a T^*-number is a T-number. However, it is not known whether the sets of S-numbers (resp., T-numbers) and

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of S^*-numbers (resp., T^*-numbers) coincide. The second aim of this paper is to establish that these sets coincide, thereby answering [19, Problem 5.9].

THEOREM 1.4. *In the field $\mathbb{F}_q((T^{-1}))$, the classes A, S, T, U coincide with the classes A^*, S^*, T^*, U^*, respectively.*

In 2019, Ooto [21] proved the existence of T^*-numbers and, consequently, that of T-numbers. His proof is fundamentally different from that of the existence of real T-numbers by Schmidt [22], [23], whose complicated construction rests on a result of Wirsing [29] (alternatively, one can use a consequence of Schmidt's subspace theorem) on the approximation to real algebraic numbers by algebraic numbers of lower degree. In the power series setting, no analogue of Schmidt's subspace theorem, or even to Roth's theorem, holds: Liouville's result is best possible, as was shown by Mahler [18].

Theorem 1.4 is an immediate consequence of the following statement.

THEOREM 1.5. *Let ξ be in $\mathbb{F}_q((T^{-1}))$ and n be a positive integer. Then, we have*

$$w_n(\xi) - n + 1 \leq w_n^*(\xi) \leq w_n(\xi).$$

Theorem 1.5 answers [19, Problem 5.8] and improves [19, Proposition 5.6], which asserts that, for any positive integer n and any ξ in $\mathbb{F}_q((T^{-1}))$, we have

$$\frac{w_n(\xi)}{p^k} - n + \frac{2}{p^k} - 1 \leq w_n^{@}(\xi) \leq w_n(\xi),$$

where k is the integer defined by $p^k \leq n < p^{k+1}$.

Our next result is, in part, a metric statement. It provides a power series analogue to classical statements already established in the real and in the p-adic settings. Throughout this paper, *almost all* always refer to the Haar measure on $\mathbb{F}_q((T^{-1}))$.

THEOREM 1.6. *For any positive integer n and any ξ in $\mathbb{F}_q((T^{-1}))$ not algebraic of degree $\leq n$, the equality $w_n^*(\xi) = n$ holds as soon as $w_n(\xi) = n$. Almost all power series ξ in $\mathbb{F}_q((T^{-1}))$ satisfy $w_n^*(\xi) = n$ for every $n \geq 1$.*

The first assertion of Theorem 1.6 was stated without proof, and with $w_n^{@}$ in place of w_n^*, at the end of [12]. It follows immediately from Theorem 1.5 combined with (2.1) below, which implies that $w_n^*(\xi) \geq n$ holds as soon as we have $w_n(\xi) = n$. By a metric result of Sprindžuk [24], stating that almost all power series ξ in $\mathbb{F}_q((T^{-1}))$ satisfy $w_n(\xi) = n$ for every $n \geq 1$, this gives the second assertion.

Chen [10] established that, for any $n \geq 1$ and any real number $w \geq n$, the set of power series ξ in $\mathbb{F}_q((T^{-1}))$ such that $w_n(\xi) = w$ (resp., $w_n^{@}(\xi) = w$) has Hausdorff dimension $(n+1)/(w+1)$. In view of Theorem 1.3, her result also holds for w_n^* in place of $w_n^{@}$.

As observed by Ooto [19, Lemma 5.5], it follows quite easily from the theory of continued fractions that $w_1(\xi) = w_1(\xi^p)$ for every ξ in $\mathbb{F}_q((T^{-1}))$. This invariance property extends to the exponents w_n.

THEOREM 1.7. *Let ξ be in $\mathbb{F}_q((T^{-1}))$ and n be a positive integer. Then, we have*

$$w_n(\xi) = w_n(\xi^p).$$

Theorem 1.7 is one of the assertions of Theorem 2.2.

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It follows from Liouville's inequality (see, e.g., [19, Theorem 5.2]) that, for any $n \geq 1$ and any algebraic power series ξ in $\mathbb{F}_q((T^{-1}))$ of degree d, we have

$$w_n^*(\xi) \leq w_n(\xi) \leq d-1.$$

Mahler's example [18] of the root $T^{-1}+T^{-p}+T^{-p^2}+\cdots$ of $X^p - X + T^{-1}$ shows that there are algebraic power series ξ in $\mathbb{F}_p((T^{-1}))$ of degree p with $w_1(\xi) = p-1$. For further results on Diophantine exponents of approximation of algebraic power series, the reader is directed to [9], [11], [25], [27] and the references given therein.

The present paper is organized as follows. Further exponents of approximation are defined in Section 2, and (in)equalities between them are stated. Auxiliary results are gathered in Section 3, while the next two sections are devoted to proofs. Several open questions are listed in Section 6.

Throughout this paper, the notation $\ll$, $\gg$ means that there is an implicit, absolute, positive constant.

§2. Uniform exponents and two inequalities between exponents

A difficulty occurring in the proof of the metric statement of Sprindžuk mentioned in the previous section is caused by the fact that the polynomials which are very small at a given power series could be inseparable. Or, said differently, by the possible existence of power series ξ for which $w_n(\xi)$ exceeds $w_n^{\mathrm{sep}}(\xi)$, where w_n^{sep} is defined exactly as w_n, but with the extra requirement that the polynomials $P(X)$ have to be separable. The next result shows that such power series do not exist. Before stating it, we define several exponents of uniform approximation.

Definition 2.1. Let ξ be in $\mathbb{F}_q((T^{-1}))$. Let $n \geq 1$ be an integer. We denote by $\widehat{w}_n(\xi)$ (resp., $\widehat{w}_n^{\mathrm{sep}}(\xi)$) the supremum of the real numbers $\widehat{w}$ for which there exists an integer H_0 such that, for every $H > H_0$, there exists a polynomial $P(X)$ (resp., a separable polynomial $P(X)$) over $\mathbb{F}_q[T]$ of degree at most n and height at most H such that

$$0 < |P(\xi)| < H^{-\widehat{w}}.$$

We denote by $\widehat{w}_n^*(\xi)$ the supremum of the real numbers $\widehat{w}^*$ for which there exists an integer H_0 such that, for every $H > H_0$, there exists an algebraic power series α in $\mathbb{F}_q((T^{-1}))$ of degree at most n and height at most H such that

$$0 < |\xi - \alpha| < H(\alpha)^{-1} H^{-\widehat{w}^*}.$$

We denote by $\widehat{w}_n^{@}(\xi)$ the supremum of the real numbers $\widehat{w}^{@}$ for which there exists an integer H_0 such that, for every $H > H_0$, there exists an algebraic power series α in C_∞ of degree at most n and height at most H such that

$$0 < |\xi - \alpha| < H(\alpha)^{-1} H^{-\widehat{w}^{@}}.$$

For any power series ξ and any $n \geq 1$, we have clearly

$$\widehat{w}_n^{\mathrm{sep}}(\xi) \leq \widehat{w}_n(\xi) \quad \text{and} \quad \widehat{w}_n^*(\xi) \leq \widehat{w}_n^{@}(\xi).$$

The first of these is an equality.

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THEOREM 2.2. *Let ξ be in $\mathbb{F}_q((T^{-1}))$ and n a positive integer. Then, we have*

$$w_n(\xi) = w_n^{sep}(\xi), \quad \widehat{w}_n(\xi) = \widehat{w}_n^{sep}(\xi),$$

and

$$w_n(\xi) = w_n(\xi^p), \quad \widehat{w}_n(\xi) = \widehat{w}_n(\xi^p).$$

To prove Theorem 1.6, we establish the following inequalities.

THEOREM 2.3. *Let $n \geq 1$ be an integer. The lower bounds*

$$w_n^*(\xi) \geq \widehat{w}_n^{@}(\xi) \geq \frac{w_n(\xi)}{w_n(\xi) - n + 1} \tag{2.1}$$

and

$$w_n^*(\xi) \geq \frac{\widehat{w}_n(\xi)}{\widehat{w}_n(\xi) - n + 1}$$

hold for any power series ξ which is not algebraic of degree $\leq n$.

For completeness, we define two exponents of simultaneous approximation and establish that they are invariant under the map $\xi \mapsto \xi^p$.

Below, the *fractional part* $\|\cdot\|$ is defined by

$$\left\| \sum_{n=N}^{+\infty} a_n T^{-n} \right\| = \left| \sum_{n=1}^{+\infty} a_n T^{-n} \right|,$$

for every power series $\xi = \sum_{n=N}^{+\infty} a_n T^{-n}$ in $\mathbb{F}_q((T^{-1}))$.

DEFINITION 2.4. Let ξ be in $\mathbb{F}_q((T^{-1}))$. Let $n \geq 1$ be an integer. We denote by $\lambda_n(\xi)$ the supremum of the real numbers λ for which

$$0 < \max\{\|R(T)\xi\|, \ldots, \|R(T)\xi^n\|\} < q^{-\lambda \deg(R)}$$

has infinitely many solutions in polynomials $R(T)$ in $\mathbb{F}_q[T]$. We denote by $\widehat{\lambda}_n(\xi)$ the supremum of the real numbers $\widehat{\lambda}$ for which there exists an integer d_0 such that, for every $d > d_0$, there exists a polynomial $R(T)$ in $\mathbb{F}_q[T]$ of degree at most d such that

$$0 < \max\{\|R(T)\xi\|, \ldots, \|R(T)\xi^n\|\} < q^{-\widehat{\lambda} d}.$$

Since $\mathbb{F}_q$ is a finite field, requiring *infinitely many solutions in polynomials $R(T)$ in $\mathbb{F}_q[T]$* is equivalent to requiring *solutions in polynomials $R(T)$ in $\mathbb{F}_q[T]$ of arbitrarily large degree.*

PROPOSITION 2.5. *Let ξ be in $\mathbb{F}_q((T^{-1}))$ and n a positive integer. Then, we have*

$$\lambda_n(\xi) = \lambda_n(\xi^p), \quad \widehat{\lambda}_n(\xi) = \widehat{\lambda}_n(\xi^p).$$

§3. Auxiliary results

LEMMA 3.1 (Krasner's lemma). *Let K be a complete, algebraically closed field equipped with a non-archimedean absolute value $|\cdot|$. Let α be an algebraic element of K of degree d at least equal to 2 and separable. Let $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_d$ be the conjugates of α. For any β in K satisfying*

$$|\alpha - \beta| < |\alpha_j - \beta|, \quad 2 \leq j \leq d,$$

we have $K(\beta) \subset K(\alpha)$.

Proof. See, for example, [3, Section 3.4.2]. □

LEMMA 3.2. *Let $P(X)$ be a polynomial in $C_\infty[X]$ of degree $n \geq 1$ and with leading coefficient a_n. Let $\beta_1, \ldots, \beta_n$ be its roots in C_∞. Then, for any $\rho > 0$ and any ξ in C_∞, we have*

$$\prod_{i=1}^{n} \max\{|\xi - \beta_i|, \rho\} \ll\gg \frac{H(P)}{|a_n|}$$

and

$$\prod_{i=1}^{n} \min\{|\xi - \beta_i|, \rho\} \ll\gg \frac{|P(\xi)|}{H(P)}.$$

Proof. See [13]. □

LEMMA 3.3. *Let $P(X) = c_m(T)X^m + \cdots + c_1(T)X + c_0(T)$ be a polynomial in $\mathbb{F}_q[T][X]$ of positive degree. Let $\alpha_1, \ldots, \alpha_m$ be its roots in C_∞. Let ξ be in $\mathbb{F}_q((T^{-1}))$. Then, for any nonempty subset S of $\{1, \ldots, m\}$, we have*

$$|c_m(T) \prod_{i \in S} (\xi - \alpha_i)| \leq (\max(1, |\xi|))^m H(P).$$

Proof. We may assume without loss of generality that $\alpha_1, \ldots, \alpha_s$ are the zeros of $P(X)$ with $|\alpha_i| \geq 1$ and that $|\alpha_j| < 1$ for $j > s$. Let S_0 be the set of i in S with $|\alpha_i| > |\xi|$. Then,

$$|c_m(T) \prod_{i \in S} (\xi - \alpha_i)| \leq |c_m(T)| \prod_{i \in S_0} |\alpha_i||\xi|^{|S| - |S_0|} \leq (\max(1, |\xi|))^m |c_m(T)\alpha_1 \cdots \alpha_s|.$$

Now, by construction, $|\alpha_1 \cdots \alpha_s| > |\alpha_{i_1} \cdots \alpha_{i_s}|$ whenever $i_1 < i_2 < \cdots < i_s$ and $i_s \neq s$. Thus,

$$|c_m(T)\alpha_1 \cdots \alpha_s| = \left| c_m(T) \sum_{i_1 < \cdots < i_s} \alpha_{i_1} \cdots \alpha_{i_s} \right| = |c_{m-s}(T)|,$$

and so the result follows. □

§4. Proof of Theorem 2.2

We establish, in this order, that the exponents w_n and w_n^{sep} coincide, that w_n and $\widehat{w}_n$ are invariant under the map $\xi \mapsto \xi^p$, and, finally, that the exponents $\widehat{w}_n$ and $\widehat{w}_n^{\mathrm{sep}}$ coincide.

A common key tool for the proofs given in this section is the notion of Cartier operator. For a positive integer j, let $\Lambda_0, \ldots \Lambda_{p^j - 1} : \mathbb{F}_q((T^{-1})) \to \mathbb{F}_q((T^{-1}))$ be the operators uniquely defined by

$$G(T^{-1}) = \sum_{i=0}^{p^j - 1} T^i \Lambda_i(G(T^{-1}))^{p^j},$$

for $G(T^{-1})$ in $\mathbb{F}_q((T^{-1}))$. Observe that $\Lambda_i(A + B^{p^j} C) = \Lambda_i(A) + B\Lambda_i(C)$ for A, B, C in $\mathbb{F}_q((T^{-1}))$. Note also that, for $i = 0, \ldots, p^j - 1$, we have

$$\Lambda_i(T^{-p^j + i}) = T^{-1}, \quad \Lambda_i(T^{p^j + i}) = T.$$

• *Proof of the equality* $\widehat{w}_n = \widehat{w}_n^{\mathrm{sep}}$. The following lemma implies that the exponents w_n and w_n^{sep} coincide. □

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LEMMA 4.1. *Let n be a positive integer and ξ in $\mathbb{F}_q((T^{-1}))$. Let $w \geq 1$ be a real number and $P(X)$ be in $\mathbb{F}_q[T][X]$ inseparable, of degree n, and such that $0 < |P(\xi)| \leq H(P)^{-w}$. Then, there exists a separable polynomial $Q(X)$ in $\mathbb{F}_q[T][X]$ of degree at most n/p satisfying $|Q(\xi)| < |P(\xi)|^{1/n}$ and $0 < |Q(\xi)| \leq H(Q)^{-w}$.*

Proof. We start with a polynomial $P(X)$ in $\mathbb{F}_q[T][X]$ of degree at most n with

$$0 < |P(\xi)| = H(P)^{-w}.$$

Write $P(X) = \sum_{i=0}^{n} Q_i(T)X^i$. Let d denote the greatest common divisor of all i such that Q_i is nonzero. Let j be the nonnegative integer such that p^j divides d but p^{j+1} does not. If $j = 0$, then $P(X)$ is separable, so we have $j \geq 1$. Thus, we may write

$$P(X) = \sum_{i \leq n/p^j} Q_{p^j i}(T) X^{p^j i}.$$

Then,

$$P(\xi) = \sum_{i \leq n/p^j} Q_{p^j i}(T) \xi^{p^j i},$$

so, for $0 \leq s < p^j$, we have

$$\Lambda_s(P(\xi)) = \sum_{i \leq n/p^j} \Lambda_s(Q_{p^j i}(T)) \xi^i.$$

Set $n' = \lfloor n/p^j \rfloor$. Notice that

$$G_s(X) := \sum_{i \leq n'} \Lambda_s(Q_{p^j i}(T)) X^i$$

is of degree at most n' and $G_s(\xi) = \Lambda_s(P(\xi))$. Thus,

$$P(\xi) = \sum_{s=0}^{p^j - 1} T^s G_s(\xi)^{p^j}.$$

Since the $\nu(T^s)$ are pairwise distinct mod p^j for $s = 0, \ldots, p^j - 1$, we see that the $\nu(T^s G_s(\xi)^{p^j})$ are pairwise distinct as s ranges from 0 to $p^j - 1$. Let

$$v = \min_{0 \leq s \leq p^j - 1} \{-s + p^j \nu(G_s(\xi))\}.$$

Then, $\nu(G_i(\xi)) \geq v/p^j$ for $i = 0, \ldots, p^j - 1$, and there exists one s such that $\nu(G_s(\xi)) = v/p^j + s/p^j$. For this particular s, we must have

$$0 < |G_s(\xi)| = q^{-v/p^j - s/p^j} \leq q^{-v/p^j} = |P(\xi)|^{1/p^j}.$$

In addition, by construction, if $B(T)$ is a polynomial of degree ℓ, then $\Lambda_s(B)$ has degree at most ℓ/p^j, and so $H(G_s) \leq H(P)^{1/p^j}$. Thus,

$$0 < |G_s(\xi)| = |P(\xi)|^{1/p^j} \leq H(P)^{-w/p^j} \leq H(G_s)^{-w}.$$

If $G_s(X)$ is inseparable, then one repeats the argument to obtain a nonzero polynomial of lesser degree small at ξ. After at most $\log_p n$ steps, we will get a separable polynomial. This proves the lemma. ∎

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• *Proof that* w_n *is invariant under the map* $\xi \mapsto \xi^p$. Let ξ and n be as in the theorem. Let $P(X)$ be a polynomial of degree at most n in $\mathbb{F}_q[T][X]$, and define w by $|P(\xi)| = H(P)^{-w}$. Write

$$P(X) = a_n(T)X^n + \cdots + a_1(T)X + a_0(T)$$

and

$$Q(X) = a_n(T^p)X^n + \cdots + a_1(T^p)X + a_0(T^p).$$

Then, we have $P(\xi)^p = Q(\xi^p)$, $H(Q) = H(P)^p$, and

$$|Q(\xi^p)| = H(P)^{-pw} = H(Q)^{-w}.$$

This shows that $w_n(\xi^p) \geq w_n(\xi)$.

The reverse inequality is more difficult and rests on Lemma 4.1. Let $P(X)$ be a polynomial of degree at most n in $\mathbb{F}_q[T][X]$, which does not vanish at ξ^p, and define w by $|P(\xi^p)| = H(P)^{-w}$. Write

$$P(X) = a_n(T)X^n + \cdots + a_1(T)X + a_0(T)$$

and

$$Q(X) = a_n(T)X^{pn} + \cdots + a_1(T)X^p + a_0(T).$$

Then, we have $P(\xi^p) = Q(\xi)$ and $|Q(\xi)| = H(Q)^{-w}$. Obviously, the polynomial $Q(X)$ is not separable and of degree at most pn. It follows from Lemma 4.1 that there exists a polynomial $R(X)$, of degree at most n, such that $|R(\xi)| \leq H(P)^{-w}$ and

$$|R(\xi)| \leq H(R)^{-w}.$$

Consequently, if, for some w, there are polynomials $P(X)$ of degree at most n and with arbitrarily large height such that $|P(\xi^p)| \leq H(P)^{-w}$, then there are polynomials $R(X)$ of degree at most n and with arbitrarily large height such that $|R(\xi)| \leq H(R)^{-w}$. This shows that $w_n(\xi) \geq w_n(\xi^p)$ and completes the proof of the theorem. □

In the next proofs, we make use of the following convention. Given a nonzero polynomial $P(X) = a_0 + a_1X + \cdots + a_mX^m$ in $\mathbb{F}_q[T][X]$ and $i = 0, \ldots, p-1$, we let $\Lambda_i(P)(X)$ denote the polynomial

$$\sum_{j=0}^{m} \Lambda_i(a_j)X^j.$$

• *Proof that* $\widehat{w}_n$ *is invariant under the map* $\xi \mapsto \xi^p$. Let $\varepsilon > 0$. By assumption, for any sufficiently large H, there is some polynomial $P(X) = a_0 + a_1X + \cdots + a_mX^m$ of degree m at most n and height at most $H^{1/p}$ such that

$$0 < |P(\xi)| < H^{-\widehat{w}_n(\xi)/p + \varepsilon/p}.$$

Set $Q(X) = a_0^p + a_1^pX + \cdots + a_m^pX^m$. Then, $Q(X)$ has degree at most n and height at most H, and, by construction, it satisfies

$$0 < |Q(\xi^p)| < H^{-\widehat{w}_n(\xi)+\varepsilon}.$$

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It follows that $\widehat{w}_n(\xi^p) \geq \widehat{w}_n(\xi) - \varepsilon$ for every $\varepsilon > 0$, and so we get the inequality

$$\widehat{w}_n(\xi^p) \geq \widehat{w}_n(\xi).$$

We now show the reverse inequality. By assumption, for any sufficiently large H, there is some polynomial $Q(X) = a_0 + a_1 X + \cdots + a_m X^m$ of degree m at most n and height at most H^p such that

$$0 < |Q(\xi^p)| < (H^p)^{-\widehat{w}_n(\xi^p)+\varepsilon}.$$

Then, for each i in $\{0,\ldots,p-1\}$, we define

$$Q_i(X) = \sum_{j=0}^{m} \Lambda_i(a_j) X^j.$$

By construction, we have $H(Q_i) \leq H$ for $i = 0,\ldots,p-1$. In addition, we have $Q_i(\xi) = \Lambda_i(Q)(\xi)$, for $i = 0,\ldots,p-1$, and so $Q(\xi^p) = \sum_{j=0}^{p-1} T^j Q_i(\xi)^p$. Since the valuations are distinct, we have

$$|Q_i(\xi)| < H^{-\widehat{w}_n(\xi^p)+\varepsilon},$$

for $i = 0,\ldots,p-1$. Since $Q(X)$ is nonzero, there is some k in $\{0,\ldots,p-1\}$ such that $Q_k(\xi) \neq 0$, and we see

$$0 < |Q_k(\xi)| < H^{-\widehat{w}_n(\xi^p)+\varepsilon}.$$

It follows that $\widehat{w}_n(\xi) \geq \widehat{w}_n(\xi^p)$, giving us the reverse inequality. □

The next lemma is used in the proof that the uniform exponents $\widehat{w}_n$ and $\widehat{w}_n^{\mathrm{sep}}$ coincide. We let $\log_p$ denote the logarithm in base p.

Lemma 4.2. *Let $\xi \in \mathbb{F}_q((T^{-1}))$ and let $P(X) = c_0 + c_1 X + \cdots + c_n X^n \in \mathbb{F}_q[T][X]$ be a nonconstant polynomial, that is, a product of irreducible inseparable polynomials such that $P(\xi)$ is nonzero. Then, there exist an integer r with $0 \leq r \leq \log_p(n)$ and a polynomial $P_0(X)$ such that the following hold:*

(1) *$P_0(X)$ has a nontrivial separable factor.*
(2) *$p^r \deg(P_0) \leq \deg(P)$.*
(3) *$0 < |P_0(\xi)|^{p^r} < q^{p^r-1}|P(\xi)|$.*
(4) *$H(P_0)^{p^r} \leq H(P)$.*

Proof. Suppose that this is not the case. Then, there must be some smallest n for which it is not true. Then, since $P(X)$ is a product of irreducible inseparable polynomials, $n = pm$ for some m. Then, $P(X) = Q(X^p)$ for some polynomial Q of degree m. Observe that

$$P(\xi) = \sum_{j=0}^{p-1} T^j (\Lambda_j(Q)(\xi))^p.$$

For $j = 0,\ldots,p-1$ such that $\Lambda_j(Q)(\xi)$ is nonzero, write

$$\Lambda_j(Q)(\xi) = c_j T^{-a_j} + \text{ larger powers of } T^{-1},$$

with $c_j \neq 0$. Then, we have

$$T^j(\Lambda_j(Q)(\xi))^p = c_j^p T^{-pa_j+j} + \text{larger powers of } T^{-1}.$$

Now, there must be some unique j_0 such that $pa_{j_0} - j_0$ is minimal among all $pa_j - j$ (and it must be finite), thus

$$|P(\xi)| = q^{-(pa_{j_0} - j_0)}.$$

Then, the polynomial $A(X) := \Lambda_{j_0}(Q)$ has the property that

$$|A(\xi)| = q^{-a_{j_0}}.$$

To summarize, we have:

(a) $p\deg(A) \leq \deg(P)$.
(b) $0 < |A(\xi)|^p \leq |P(\xi)|$.
(c) $H(A)^p \leq H(P)$.

By construction, $\deg(A) < n$, and so, by minimality, there is some $r \leq \log_p(m)$ and a polynomial $Q(X)$ such that:

(d) $p^r \deg(Q) \leq \deg(A)$.
(e) $0 < |Q(\xi)|^{p^r} \leq |A(\xi)|$.
(f) $H(Q)^{p^r} \leq H(A)$.
(g) Q has a nontrivial separable factor.

Then, by construction, $p^{r+1}\deg \mathrm{Q} \leq \deg(P)$, $0 < |Q(\xi)|^{p^{r+1}} < q^{p^{r+1}-1}|P(\xi)|$, and $H(Q)^{p^{r+1}} \leq H(P)$. Furthermore, $r+1 \leq 1+\log_p(m) \leq \log_p(n)$, and so we get the desired result. ∎

• *Proof of the equality* $\widehat{w}_n = \widehat{w}_n^{\mathrm{sep}}$. It is clear that $\widehat{w}_n(\xi) \geq \widehat{w}_n^{\mathrm{sep}}(\xi)$. We now show the reverse inequality. Let $\varepsilon > 0$. Then, there is some H_0 such that, for every $H > H_0$, there is a polynomial $P(X)$ of degree at most n and height at most H such that

$$0 < |P(\xi)| < H^{-\widehat{w}_n(\xi)+\varepsilon}.$$

We take the infimum over all $d \leq n$ for which there is some positive constant C such that, for every $H > H_0$, there is a polynomial $A(X)B(X)$ with $A(X)$ separable and $B(X)$ a polynomial of degree at most d, that is, a product of irreducible inseparable polynomials with

$$0 < |A(\xi)B(\xi)| < C \cdot H^{-\widehat{w}_n(\xi)+\varepsilon}.$$

Then, by assumption, d must be positive, and since the polynomial $B(X)$ is a product of inseparable irreducible polynomials, we see that p divides d.

Let $H > H_0$. Then, there is a fixed constant $C > 0$ that does not depend on H such that there are polynomials $A(X)$ and $B(X)$ with A separable and B a polynomial of degree at most d, that is, a product of irreducible separable polynomials with

$$0 < |A(\xi)B(\xi)| < C \cdot H^{-\widehat{w}_n(\xi)+\varepsilon}.$$

Then, by Lemma 4.2, there is some $r \leq \log_p(n)$ and a polynomial $B_0(X)$ with a nontrivial separable factor such that $\deg(B) = p^r \deg(B_0)$ and

$$0 < |B_0(\xi)| \leq |B(\xi)|$$

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and $H(B_0^{p^r}) < H(B)$. Thus, the polynomial

$$A(X)B_0(X)^{p^r}$$

has degree at most n and height at most H and

$$0 < |A(\xi)B_0(\xi)^{p^r}| < Cq^{p^r-1}H^{-\widehat{w}_n(\xi)+\varepsilon}.$$

By assumption, we can write $B_0(X) = C(X)D(X)$ with $C(X)$ nonconstant and separable and $D(X)$ a product of irreducible inseparable polynomials. Then, we have

$$\deg(D(X)^{p^r}) \le \deg(B) - p^r < d.$$

But this contradicts the minimality of d, and so we see that d must be zero, and so we get $\widehat{w}_n^{\mathrm{sep}}(\xi) \ge \widehat{w}_n(\xi) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get the desired result. □

Proof of Proposition 2.5. Observe that, for $j \ge 1$, the equality $(R(T)\xi^j)^p = R(T^p)\xi^{pj}$ immediately yields $\lambda_n(\xi^p) \ge \lambda_n(\xi)$ and $\widehat{\lambda}_n(\xi^p) \ge \widehat{\lambda}_n(\xi)$, for $n \ge 1$.

Take λ with $0 < \lambda < \lambda_n(\xi^p)$. Then, there is an infinite set $\mathcal{S}$ of polynomials $R(T)$ such that

$$0 < \max\{\|R(T)\xi^p\|, \ldots, \|R(T)\xi^{pn}\|\} < q^{-\lambda \deg(R)}.$$

By replacing $\mathcal{S}$ with a well-chosen infinite subset, we may assume that there is a fixed j in $\{0, 1, \ldots, p-1\}$ such that the degree of every polynomial in $\mathcal{S}$ is congruent to j modulo p. Then, for $R(T)$ in $\mathcal{S}$ and i in $\{1, \ldots, n\}$, we apply the jth Cartier operator Λ_j to $R(T)\xi^{pi}$, and we have $\|\Lambda_j(R(T))\xi^i\| < q^{-\lambda \deg(R)/p}$. We let $Q(T) = \Lambda_j(R(T))$. Then, the degree of Q is $(\deg(R) - j)/p$, and so we see

$$\|Q(T)\xi^i\| < q^{-\lambda(p\deg(Q)+j)/p} \le q^{-\lambda \deg(Q)},$$

for $i = 1, \ldots, n$. Since the degrees of the elements of $\Lambda_j(\mathcal{S})$ are arbitrarily large, we deduce that $\lambda_n(\xi) \ge \lambda$. Consequently, we have established that $\lambda_n(\xi) \ge \lambda_n(\xi^p)$.

Take $\widehat{\lambda}$ with $0 < \widehat{\lambda} < \widehat{\lambda}_n(\xi^p)$. For any sufficiently large integer d, there is a polynomial $R(T)$ of degree at most pd such that

$$0 < \max\{\|R(T)\xi^p\|, \ldots, \|R(T)\xi^{pn}\|\} < q^{-\widehat{\lambda} pd}.$$

Let j be in $\{0, 1, \ldots, p-1\}$ such that the degree of $R(T)$ is congruent to j modulo p. Apply the jth Cartier operator to $R(T)\xi^{pi}$, and let $Q(T) = \Lambda_j(R(T))$. Then, the degree of Q is at most equal to d, and so we see

$$\|Q(T)\xi^i\| < q^{-\widehat{\lambda} pd/p} \le q^{-\widehat{\lambda} d},$$

for $i = 1, \ldots, n$. This shows that $\widehat{\lambda}_n(\xi) \ge \widehat{\lambda}$. Thus, we obtain $\widehat{\lambda}_n(\xi) \ge \widehat{\lambda}_n(\xi^p)$. □

§5. Proofs of Theorems 1.3, 1.5, 1.7, and 2.3

By adapting the proof of Wirsing [28] to the power series setting, Guntermann [13, Satz 1] established that, for every $n \ge 1$ and every ξ in $\mathbb{F}_q((T^{-1}))$ not algebraic of degree $\le n$, we have

$$w_n^{@}(\xi) \ge \frac{n+1}{2}.$$

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Actually, it is easily seen that instead of starting her proof with polynomials given by Mahler's analogue [17], [24] of Minkowski's theorem, she could have, like Wirsing, started with polynomials $P[X]$ satisfying

$$0 < |P(\xi)| < H(P)^{-w_n(\xi)+\varepsilon},$$

where ε is an arbitrarily small positive real number. By doing this, one gets the stronger assertion

$$w_n^{@}(\xi) \geq \frac{w_n(\xi)+1}{2}, \tag{5.1}$$

which is crucial for proving Theorem 1.3. Note that Guntermann [13] did not obtain any lower bound for $w_n^*(\xi)$, except when $n = 2$.

Proof of Theorem 1.3. Set $w = w_n(\xi)$, $w^{@} = w_n^{@}(\xi)$, and $w^* = w_n^*(\xi)$. Suppose that $w^{@} > w^*$ and pick ε in $(0, 1/3)$ such that $w^{@} > w^* + 2\varepsilon$. Then, there are infinitely many α in C_∞ algebraic of degree at most n such that

$$|\xi - \alpha| < H(\alpha)^{-1-w^{@}+\varepsilon}.$$

Let $P_\alpha(X)$ denote the minimal polynomial of α over $\mathbb{F}_q[T]$. Then, $H(P_\alpha) = H(\alpha)$. We let $\alpha = \alpha_1, \ldots, \alpha_m$ denote the roots of $P_\alpha(X)$ (with multiplicities), where $m = \deg(P_\alpha) \leq n$. We may assume that $|\xi - \alpha_1| \leq \cdots \leq |\xi - \alpha_m|$. Let r be the largest integer such that

$$|\xi - \alpha_1| = \cdots = |\xi - \alpha_r|.$$

If $r = 1$ for infinitely many α as above, then $P_\alpha(X)$ is separable over $\mathbb{F}_q(T)$, and we conclude from Krasner's Lemma 3.1 that α_1 lies in $\mathbb{F}_q((T^{-1}))$. For $H(\alpha)$ large enough, we then get

$$H(\alpha)^{-1-w_n^*-\varepsilon} < |\xi - \alpha| < H(\alpha)^{-1-w^{@}+\varepsilon},$$

thus $w^{@} \leq w^* + 2\varepsilon$, a contradiction.

Thus, we have $r \geq 2$. Observe that $|P_\alpha(\xi)| > H(\alpha)^{-w-\varepsilon}$ if $H(P_\alpha)$ is large enough. On the other hand, with $c_\alpha(T)$ being the leading coefficient of $P_\alpha(X)$, we get

$$\begin{aligned} |P_\alpha(\xi)| &= \left| c_\alpha(T)(\xi - \alpha_1)\cdots(\xi - \alpha_r) \prod_{i=r+1}^{m} (\xi - \alpha_i) \right| \\ &= |\xi - \alpha|^r \cdot \left| c_\alpha(T) \prod_{i=r+1}^{m} (\xi - \alpha_i) \right| \\ &< (\max\{1, |\xi|\})^n \cdot H(\alpha)^{1-r(1+w^{@}-\varepsilon)}, \end{aligned}$$

where the last step follows from Lemma 3.3.

By (5.1), we have $w^{@} \geq (w+1)/2$, thus we get

$$H(\alpha)^{-w-\varepsilon} \ll |P_\alpha(\xi)| \ll H(\alpha)^{1-r(1+w^{@}-\varepsilon)} \ll H(\alpha)^{1-r(1+(w+1)/2)+r\varepsilon}.$$

This then gives

$$w + \varepsilon \geq -1 + r + \frac{r(w+1)}{2} - r\varepsilon,$$

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and since $r \geq 2$, we deduce

$$w+\varepsilon \geq -1+w+1+r-r\varepsilon,$$

which is absurd. Since ε can be taken arbitrarily small, we deduce that $w_n^{@}(\xi) \leq w_n^*(\xi)$. As the reverse inequality immediately follows from the definitions of $w_n^{@}$ and w_n^*, the proof is complete. □

We are ready to complete the proof of Theorem 1.5.

Proof of Theorem 1.5. Let ξ be in $\mathbb{F}_q((T^{-1}))$ and n be a positive integer. The inequality $w_n^*(\xi) \leq w_n(\xi)$ is clear. Let ε be a positive real number. By Lemma 4.1, there exist separable polynomials $P(X)$ in $\mathbb{F}_q[T][X]$ of arbitrarily large height such that

$$0 < |P(\xi)| < H(P)^{-w_n(\xi)+\varepsilon}.$$

Then, the (classical) argument given at the beginning of the proof of [19, Lemma 5.4] yields the existence of a root α of $P(X)$ such that

$$0 < |\xi-\alpha| \leq |P(\xi)|\, H(P)^{n-2}.$$

Thus, we get the inequality

$$w_n^{@}(\xi) \geq w_n(\xi)-n+1,$$

and we conclude by applying Theorem 1.3 which asserts that $w_n^{@}(\xi) = w_n^*(\xi)$. □

Proof of Theorem 2.3. We obtain (2.1) by taking Wirsing's argumentation [28]. Let $n \geq 2$ be an integer, and let ξ be a power series which is either transcendental, or algebraic of degree $> n$. Let $\varepsilon > 0$, and set $w = w_n(\xi)(1+\varepsilon)^2$. Let $i_1,\ldots,i_n$ be distinct integers in $\{0,\ldots,n\}$ such that $\nu(\xi) \neq i_j$ for $j = 1,\ldots,n$. By Mahler's analogue [17], [24] of Minkowski's theorem, there exist a constant c and, for any positive real number H, a nonzero polynomial $P(X)$ of degree at most n such that

$$|P(\xi)| \leq H^{-w}, \quad |P(T^{i_1})|,\ldots,|P(T^{i_{n-1}})| \leq H, \quad \text{and} \quad |P(T^{i_n})| \leq cH^{w-n+1}.$$

The definitions of $w_n(\xi)$ and w show that $H(P) \gg H^{1+\varepsilon}$. It follows from Lemma 3.2 that $P(X)$ has some root in a small neighborhood of each of the points $\xi, T^{i_1},\ldots,T^{i_{n-1}}$. Denoting by α the closest root to ξ, we get

$$|\xi-\alpha| \gg\ll \frac{|P(\xi)|}{H(P)} \ll H(P)^{-1}\,(H^{w-n+1})^{-w/(w-n+1)}$$

and

$$H(P) \ll H^{w-n+1}.$$

Since all of this holds for any sufficiently large H, we deduce that $\widehat{w}_n^{@}(\xi) \geq w/(w-n+1)$. Selecting now ε arbitrarily close to 0, we obtain the first assertion.

Now, we establish the second assertion. Since $\widehat{w}_n(\xi) \geq n$, there is nothing to prove if $w_n^*(\xi) \geq n$. Otherwise, let $A > 2$ be a real number with $w_n^*(\xi) < A-1 < n$. Thus, we have $|\xi-\alpha| \geq H(\alpha)^{-A}$ for all algebraic power series α of degree $\leq n$ and sufficiently large height. We make use of an idea of Bernik and Tishchenko (see also [5, Section 3.4]). Let $\varepsilon > 0$ be given. We may assume that $|\xi| \leq 1$. Again, by Mahler's analogue [17], [24] of Minkowski's theorem, there exist a constant c and, for any positive real number H, a nonzero polynomial $P(X) = a_n X^n + \cdots + a_1 X + a_0$ of degree at most n such that

$$|a_1| \leq H^{1+\varepsilon}, \quad |a_2|, \ldots, |a_n| \leq H, \text{and} \quad |P(\xi)| \leq cH^{-n-\varepsilon}.$$

If $P(X)$ is a product of irreducible inseparable factors, then $a_1 = 0$ and $H(P) \ll H$. Assume now that $P(X)$ has a separable factor. Let α in C_∞ be the closest root of $P(X)$ to ξ. If $|a_1| > H$, then we deduce from $|na_n\xi^{n-1} + \cdots + 2a_2\xi| \leq H$ that $|P'(\xi)| = |a_1| \gg H(P)$. Thus, we get

$$|P(\xi)| \geq |\xi - \alpha| \cdot |P'(\xi)| \gg H(\alpha)^{1-A}$$

and

$$|P(\xi)| \leq H(P)^{-(n+\varepsilon)/(1+\varepsilon)},$$

which also holds if $a_1 = 0$. Consequently, if $A - 1 \leq (n+\varepsilon)/(1+\varepsilon)$, that is, if

$$\varepsilon < \frac{n+1-A}{A-2},$$

then we get a contradiction if H is large enough. We conclude that, for any $\varepsilon < (n+1-A)/(A-2)$ and any sufficiently large H, there exists a polynomial $P(X)$ of height $\leq H$ and degree $\leq n$ satisfying $|P(\xi)| \leq H^{-n-\varepsilon}$. Consequently, we have $\widehat{w}_n(\xi) \geq n + \varepsilon$, and thus $\widehat{w}_n(\xi) \geq n + (n+1-A)/(A-2)$. We obtain the desired inequality by letting A tend to $1 + w_n^*(\xi)$. ∎

§6. Further problems

Despite some effort, we did not succeed to solve the following problem.

PROBLEM 6.1. *Let n be a positive integer and ξ in $\mathbb{F}_q((T^{-1}))$. Prove that*

$$\widehat{w}_n^*(\xi) = \widehat{w}_n^{@}(\xi) = \widehat{w}_n^*(\xi^p), \quad w_n^*(\xi) = w_n^*(\xi^p).$$

For $n \geq 2$, Ooto [19] proved the existence of ξ in $\mathbb{F}_q((T^{-1}))$ for which $w_n^*(\xi) < w_n(\xi)$. His strategy, inspired by [6], was to use continued fractions to construct power series ξ with $w_2^*(\xi) < w_2(\xi)$ and $w_2^*(\xi)$ sufficiently large to ensure that, for small (in terms of $w_2^*(\xi)$) values of n, we have

$$w_2^*(\xi) = w_3^*(\xi) = \cdots = w_n^*(\xi), \quad w_2(\xi) = w_3(\xi) = \cdots = w_n(\xi).$$

Very recently, Ayadi and Ooto [2] answered a question of Ooto [20, Problem 2.2] by proving, for given $n \geq 2$ and $q \geq 4$, the existence of algebraic power series ξ in $\mathbb{F}_q((T^{-1}))$ for which $w_n^*(\xi) < w_n(\xi)$.

PROBLEM 6.2. *Do there exist power series ξ in $\mathbb{F}_q((T^{-1}))$ such that*

$$w_n^*(\xi) < w_n(\xi), \quad \textit{for infinitely many } n?$$

The formulation of the next problem is close to that of [20, Problem 2.4].

PROBLEM 6.3. *Let ξ be an algebraic power series in $\mathbb{F}_q((T^{-1}))$ and n a positive integer. Is $w_1(\xi)$ always rational? Are $w_n(\xi), w_n^*(\xi)$, and $\lambda_n(\xi)$ always algebraic numbers?*

No results are known on uniform exponents of algebraic power series in $\mathbb{F}_q((T^{-1}))$.

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PROBLEM 6.4. *Let ξ be an algebraic power series in $\mathbb{F}_q((T^{-1}))$ and n a positive integer. Do we have*

$$\widehat{w}_n(\xi) = \widehat{w}_n^*(\xi) = n?$$

In the real case, there are many of relations between the six exponents w_n, w_n^*, λ_n, $\widehat{w}_n$, $\widehat{w}_n^*$, $\widehat{\lambda}_n$ (see, e.g., the survey [7]). We believe that most of the proofs can be adapted to the power series setting.

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