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Exact solution of inverse kinematic problem of 6R serial manipulators using Clifford Algebra

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SUMMARY
In this paper, Clifford Algebra is used to model and facilitate solving the inverse kinematic problem for robots with *only* two consecutive parallel axes. It is shown that when a solution exists, it is usually the case that one of the angles of rotation can be arbitrarily chosen from a union of intervals. The remaining angles are then uniquely determined. Of course, there are cases when no solution exists, such as when the object is out of reach. But typically, when solutions exist, there are infinitely many sets of solutions.

KEYWORDS: Clifford Algebra; Inverse kinematic problem; 6R serial robot.

1. Background

1.1. Introduction

The inverse kinematic problem of 6R robots has been a subject of study for several decades. Different approaches have been adopted to solve this problem, which was found to be difficult to solve whether geometrically, algebraically, or numerically.

The common geometrical approach was to divide the robot into two systems and to solve each system independent of the other. Algebraically, the problem can be represented by a six-degree polynomial that is quite impossible to solve, and numerically this problem would require thousands of lines code to replicate the problem and its solution.

Pieper[1] found that for three intersecting pairs of axes there are four different configurations leading to the same position and orientation. He also found that for two intersecting pairs there are eight different configurations and for one intersecting pair there are 16 different configurations. Later, Selig[2,3] used Pieper's theorem along with the Clifford Algebra *Cl(0,3,1)* to formulate a solution for the inverse kinematic problem.

Bayro-Corrochano *et al.*[4–6] worked on the inverse kinematic problem, they formulated a solution for the inverse kinematic problem using *G(3,0,1)*. Bayro-Corrochano *et al.* also used the conformal geometric algebra *G(4,1)*, which makes use of points, lines, planes, and spheres to compute inverse and differential kinematics.[4]

Later, Sariyildiz and Temeltas[7] and Payandeh and Goldenberg[8] used quaternions to represent the screw motion of the robot joints to solve the problem. Vasilyev and Lyashin[9] proposed an analytical solution for the problem, and in their paper they used matrices, a method that has been historically studied and is known to be quite complex. Another analytical solution was also proposed by Pfurner and Husty.[10] Numerical solutions were studied by various researchers.[11,12]

In this paper, we present a new simple method to solve the inverse kinematic problem of 6R serial manipulators. This method uses Clifford Algebra to represent the robot joints[2] in simple equations. These equations are solved together using algebraic operations such as conjugation to relate five of the six unknown angles in terms of one angle that can be arbitrarily chosen.

An important advantage of this method is that all possible solutions can be found by arbitrarily choosing an angle from union of intervals, and then the remaining angles are determined uniquely.

1.2. Clifford Algebra

One type of Clifford Algebra is the $Cl(0,3,1)$ which is an associative algebra of dimension 16 with four anti-commutative generators e_1, e_2, e_3, and e, the first three square to −1 and the fourth to 0. For more on the properties of this algebra, see Ref. [2].

This algebra was used to model points, line, planes, and the group of rigid body motions as follows and as described in ref. [2]. Points are represented by grade three elements

$$p = e_1 e_2 e_3 + x\, e_2 e_3 e - y\, e_1 e_3 e + z e_1 e_2 e, \qquad (1.1)$$

where x, y, z are the coordinates of the position vector of the point.

Lines are represented by grade two elements,

$$l = v_x e_2 e_3 - v_y e_1 e_3 + v_z e_1 e_2 + u_x e_1 e + u_y e_2 e + u_z e_3 e, \qquad (1.2)$$

where $v = (v_x, v_y, v_z)$ is a unit vector in the direction of the line and $u = (u_x, u_y, u_z)$ is the moment vector defined as the cross product of v and a point on the line. So (v, u) are the Plücker coordinates of l.

Planes are represented by grade one elements,

$$\pi = n_x e_1 + n_y e_2 + n_z e_3 + de, \qquad (1.3)$$

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where $n = (n_x, n_y, n_z)$ is a choice of unit normal vector to the plane, and d is the distance to the origin.

Rigid body motions are represented by even grade elements,

$$g = a + \frac{1}{2} t\, a\, e, \quad (1.4)$$

where a is the Clifford Algebra element representing the rotation part of the motion. In fact

$$a = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} l, \quad (1.5)$$

where l is the Clifford Algebra element representing the axis of rotation, and θ is the angle of rotation. Finally, t is the Clifford Algebra element representing the translation part of the motion as follows:

$$t = t_x e_1 + t_y e_2 + t_z e_3, \quad (1.6)$$

where (t_x, t_y, t_z) is the translation vector.

One of the algebraic operations used in this paper is conjugation, which is a linear mapping from the algebra to itself. The conjugate of a generator is its negative. The conjugate of a product of generators is the product of the conjugates of the generators in reverse order. All these elements, p, l, π, g, and a, belong to the group of elements of $Cl(0,3,1)$ satisfying $xx^* = 1$. If x is p, l, or π, then gxg^* is the image of x under the rigid body motion g.

1.3. Inverse kinematic problem

A 6R robot consists of six joints whose axes are represented by the Clifford Algebra element l_i of the form (1.2) with Plücker coordinates (v_i, u_i).

The six equations representing the six joints of the robot form what is called the home position of the joints. It is only necessary to consider the home (initial) and final positions of the joints.

In simple terms, the inverse kinematic problem is about finding all possible sets of the six joint angles in order to obtain a specified gripper location and orientation. This is an important problem in robotics, since whenever we specify the motion of the robot's gripper, we need to know the corresponding joint motions. Mathematically, we proceed as follows: Solving this equation is to find the set of θs for a given g, where g is the Clifford Algebra element representing the rigid body motion that would take the gripper from its home position to the desired final position. The inverse kinematic problem is then to solve for θ_i the following equation:

$$g = a_1\, a_2\, a_3\, a_4\, a_5\, a_6, \quad (1.7)$$

where $a_i = \cos\frac{\theta_i}{2} + \sin\frac{\theta_i}{2} l_i$ is the Clifford Algebra element representing rotation through an angle θ_i about the axis l_i. Trying to solve Eq. (1.7) by expanding both sides will yield a polynomial of degree six in the variables $\sin\frac{\theta_i}{2}$ and $\cos\frac{\theta_i}{2}$, which is difficult to solve. The inverse kinematic problem of a 6R robot has been previously solved under certain restrictions such as three consecutive joints being parallel[13] or intersecting in a point.[1]

Another method for solving the inverse kinematic problem for a 6R robot consists of dividing the joints into two groups and solving each group independently as done in ref. [14]. This method cannot be generalized for all 6R robots. It worked for the well-known robot PUMA because of the nature of its design; its first three joints almost form a planar manipulator and the last three form a 3R wrist, both of these cases are easy to solve.

2. Main Result

We will demonstrate that for a 6R serial robot and for a wide variety of final gripper positions given by g there exist infinite sets of solutions for a robot with two consecutive parallel joints.

2.1. The algorithm

All algebraic operations used in this section can be found in refs. [2, 3]. We solve the case of a 6R robot with two consecutive parallel joint axes starting from Eq. (1.7). Suppose l_2 is parallel to l_3 and let π be the plane perpendicular to both of them passing through the origin.

Using conjugation, we can rewrite Eq. (1.7) as

$$a_1^*\, g\, a_6^*\, a_5^*\, a_4^* = a_2\, a_3. \quad (2.1)$$

Since π is perpendicular to l_2 and l_3, we obtain

$$a_2\, a_3\, \pi\, a_3^*\, a_2^* = \pi = a_1^*\, g\, a_6^*\, a_5^*\, a_4^*\, \pi\, a_4\, a_5\, a_6\, g^*\, a_1. \quad (2.2)$$

Working on right-hand side of Eq. (2.2) we find

$$g^*\, a_1\, \pi\, a_1^*\, g = a_6^*\, a_5^*\, a_4^*\, \pi\, a_4\, a_5\, a_6 = a_6^*\, \pi_5 a_6, \quad (2.3)$$

where here and below $\pi_5 = a_5^*\, \pi_4\, a_5$ and $\pi_4 = a_4^*\, \pi\, a_4$.

Now π_5 meets l_6 at the point given by a scalar multiple of $l_6 \wedge \pi_5$. The product $x \wedge y$ is called the exterior product or the Grassman product; it is a linear and associative product. On generators, it is given by $e_i \wedge e_j = \frac{1}{2}(e_i e_j - e_j e_i)$. For more on the expansion of this exterior product, see ref. [2]. Hence,

$$a_5^*\, \pi_4\, a_5 l_6^* + l_6 a_5^*\, \pi_4^*\, a_5 \\ = g^*\, a_1\, \pi\, a_1^*\, g\, l_6^* + l_6 g^* a_1\, \pi^*\, a_1^*\, g. \quad (2.4)$$

Equating left-hand side and right-hand side of this equation, we obtain $\cos\theta_4$ and $\sin\theta_5$ each as functions of θ_1, then using Eq. (2.3), we can find $\sin\theta_6$ as a function of θ_4, θ_5, and θ_1, where θ_4 and θ_5 are now known.

Expanding Eq. (2.1) produces eight equations, six of which are redundant. The remaining two equations give θ_2 in terms of θ_1 and consequently θ_3 in terms of θ_1.

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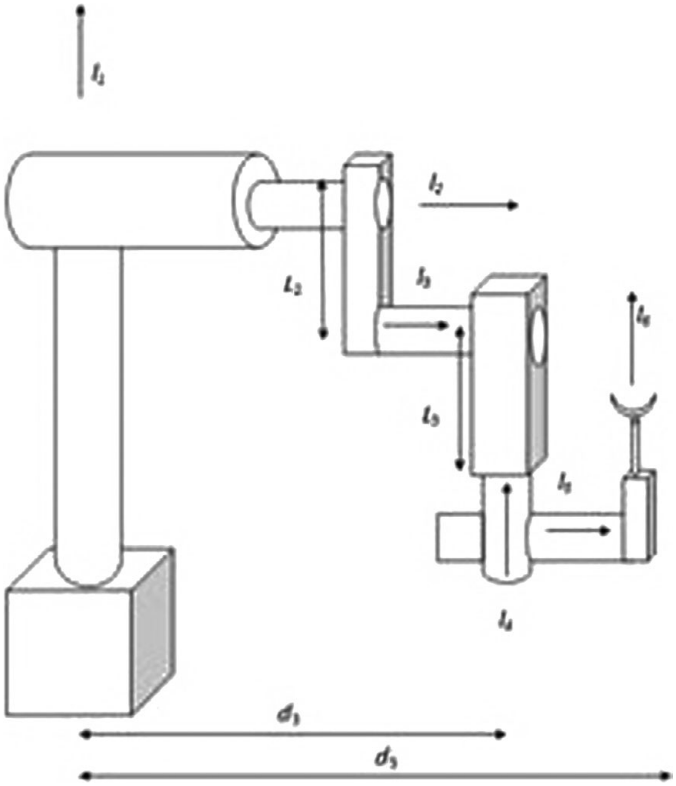

Fig. 1. 6R robot represented by a system of equations.

3. Application

3.1. General setup of home configuration (initial position)

In this section, a system of equations representing initial position of the 6R robot joints using Clifford Algebra and the solution of this system are shown.

3.1.1. Numerical experiment. For a 6R robot we represent the home configuration by $l_1, \ldots, l_6$ as in the following generic case,

$$\begin{aligned} l_1 &= e_1 e_2 \\ l_2 &= e_3 e_1 \\ l_3 &= e_3 e_1 - L_2 e_1 e \\ l_4 &= e_1 e_2 + d_3 e_1 e \\ l_5 &= e_3 e_1 - L e_1 e \\ l_6 &= e_1 e_2 + d_5 e_1 e \end{aligned} \tag{3.1}$$

where $L = L_2 + L_3$, L_2 and L_3 are link lengths, and d_3 and d_5 are joint offsets.

This system of equations can be represented by Fig. 1.

Using these equations and the previously mentioned algorithm, for an arbitrarily chosen θ_1 from a union of intervals, the solution of this system is obtained; solution is given in Table I. We start by substituting $a_i = \cos\frac{\theta_i}{2} + \sin\frac{\theta_i}{2} l_i$ and the set of Eq. (3.1) in Eq. (2.4). Left-hand side of the equation will include two unknowns, θ_4 and θ_5, and right-hand side includes θ_1. By equating left-hand side and right-hand side of Eq. (2.4) we obtain $\cos\theta_4$ in terms of θ_1 as given in Eq. (3.2) and $\sin\theta_5$ in terms of $\sin\theta_4$ (Eq. 3.3), which is known once θ_1 is chosen.

Next we use Eq. (2.3) where now θ_1, θ_4, and θ_5 are known, the only unknown would be θ_6; equating left-hand side and right-hand side of Eq. (2.3), we obtain $\sin\theta_6$ in terms of θ_1, θ_4, and θ_5 as given by Eq. (3.4).

Having θ_1, θ_4, θ_5, and θ_6 known, we substitute in Eq. (2.1) where we equate right-hand side and left-hand side producing eight equations, six of which are redundant. Using one of the remaining two equations, we obtain θ_2 as given by Eq. (3.5) and using the second equation we obtain θ_3 as given by Eq. (3.6).

Equations (3.2)–(3.6) consist of long terms that are fully expanded and shown by sets of Eqs. (3.7)–(3.8),

where

$$\Delta = \cos\theta_1 \left(2 v_x v_y \sin^2\frac{\alpha}{2} + v_z \sin\alpha\right) + \sin\theta_1 \left(-\cos^2\frac{\alpha}{2} + \left(-v_x^2 + v_y^2 + v_z^2\right)\sin^2\frac{\alpha}{2}\right),$$

Table I. System of equations representing the solution.

Angle	Solution	
$\cos\theta_4$	$\frac{1}{(d_5 - d_3)}(\Gamma d_5 - \Omega + \Lambda L - d_3)\ \ d_3 \neq d_5$	(3.2)
$\sin\theta_5$	$\frac{\Lambda}{\sin\theta_4}$	(3.3)
$\sin\theta_6$	$\frac{-\cos\theta_5 \sin\theta_4 \Gamma - \sqrt{\cos^2_{\theta_5}\sin^2_{\theta_4}\Gamma^2 - \left(\cos^2_{\theta_4} + \cos^2_{\theta_5}\sin^2_{\theta_4}\right)\left(\Gamma^2 - \cos^2_{\theta_4}\right)}}{\left(\cos^2_{\theta_4} + \cos^2_{\theta_5}\sin^2_{\theta_4}\right)}$	(3.4)
$\tan\frac{\theta_2}{2}$	$\frac{\cos\frac{\theta_1}{2} K_G + \sin\frac{\theta_1}{2} M_G}{\cos\frac{\theta_1}{2} I_G + \sin\frac{\theta_1}{2} J_G}$	(3.5)
θ_3	$2\cos^{-1}\left[\cos\frac{\theta_1}{2} E_G + \sin\frac{\theta_1}{2} F_G\right] - \theta_2$	(3.6)

Table II. Existing solutions for various design parameters and transformation angles.

Case	d_3	d_5	L_2	L_3	α (degrees)	t_x	t_y	t_z	v_x	v_y	v_z	Range of θ_1 (in degrees)
1	1	7	4.5	4.5	31–114	0.15	0.15	0.1	0.4	0.3	0.86603	(0–360)
2	1	7	4.5	4.5	120	0.15	0.15	0.1	0.4	0.3	0.86603	(0–54) U (100–360)
3	1	7	4.5	4.5	150	0.15	0.15	0.1	0.4	0.3	0.86603	(0–31) U (134–237) U (297–360)
4	1	4	4.5	4.5	34–66	0.15	0.15	0.1	0.4	0.3	0.86603	(0–360)
5	1	4	4.5	4.5	30	0.15	0.15	0.1	0.4	0.3	0.86603	(0–149) U (203–360)
6	1	4	4.5	4.5	120	0.15	0.15	0.1	0.4	0.3	0.86603	(0–12) U (150–217) U (308–360)
7	1	4	4.5	4.5	30	1	1	1	0.4	0.3	0.86603	(0–107) U (214–360)
8	1	4	4.5	4.5	100	1	1	1	0.4	0.3	0.86603	(0–22) U (154–229) U (307–360)
9	1	4	4.5	4.5	80	1	1	1	0.4	0.3	0.86603	(0–36) U (150–360)
10	1	4	4.5	4.5	150	1	1	1	0.4	0.3	0.86603	(0–21) U (173–217) U (338–360)
11	2	3	4.5	4.5	100	1	1	1	0.4	0.3	0.86603	(192–213) U (326–348)
12	2	3	4.5	4.5	150	1	1	1	0.4	0.3	0.86603	(0–3) U (199–213) U (349–360)
13	1	4	2	4	80	1	1	1	0.4	0.3	0.86603	(0–86) U (121–360)
14	1	4	2	4	100	1	1	1	0.4	0.3	0.86603	(0–43) U (138–360)
15	1	4	2	4	150	1	1	1	0.4	0.3	0.86603	(0–22) U (151–228) U (321–360)
16	1	4	2	4	45–92	0.15	0.15	0.1	0.4	0.3	0.86603	(0–360)
17	1	4	2	4	100	0.15	0.15	0.1	0.4	0.3	0.86603	(0–43) U (106–360)
18	1	4	2	4	150	0.15	0.15	0.1	0.4	0.3	0.86603	(0–15) U (150–229) U (307–360)

$$\Gamma = \cos\theta_1 \left(\cos^2\frac{\alpha}{2} - \left(v_x^2 - v_y^2 + v_z^2\right)\sin^2\frac{\alpha}{2}\right) + \sin\theta_1\left(-2v_xv_y\sin^2\frac{\alpha}{2} + v_z\sin\alpha\right),$$

$$\Lambda = \cos\theta_1\left(2v_yv_z\sin^2\frac{\alpha}{2} - v_x\sin\alpha\right) + \sin\theta_1\left(-2v_xv_z\sin^2\frac{\alpha}{2} - v_y\sin\alpha\right),$$

$$\Omega = -2\cos\theta_1\left(B\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\left(-Cv_x + Dv_y + Av_z\right)\right) + 2\sin\theta_1\left(A\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\left(Dv_x + Cv_y - Bv_z\right)\right) \tag{3.7}$$

$$\begin{aligned} A &= \frac{1}{2}\left(t_x\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\left(t_yv_z - t_zv_y\right)\right),\\ B &= \frac{1}{2}\left(t_y\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\left(t_zv_x - t_xv_z\right)\right),\\ C &= \frac{1}{2}\left(t_z\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\left(t_xv_y - t_yv_x\right)\right),\\ D &= \frac{1}{2}\sin\frac{\alpha}{2}(t_xv_x + t_yv_y + t_zv_z). \end{aligned} \tag{3.8}$$

and

$$E_G = \cos\frac{\alpha}{2}\cos\frac{\theta_5}{2}\cos\left(\frac{\theta_4+\theta_6}{2}\right) + \sin\frac{\alpha}{2}\left[\cos\frac{\theta_5}{2}\sin\left(\frac{\theta_4+\theta_6}{2}\right)v_z - \sin\frac{\theta_5}{2}\sin\left(\frac{\theta_4-\theta_6}{2}\right)v_x + \sin\frac{\theta_5}{2}\cos\left(\frac{\theta_4-\theta_6}{2}\right)v_y\right],$$

$$F_G = -\cos\frac{\alpha}{2}\cos\frac{\theta_5}{2}\sin\left(\frac{\theta_4+\theta_6}{2}\right) + \sin\frac{\alpha}{2}\left[\cos\frac{\theta_5}{2}\cos\left(\frac{\theta_4+\theta_6}{2}\right)v_z - \sin\frac{\theta_5}{2}\sin\left(\frac{\theta_4-\theta_6}{2}\right)v_y - \sin\frac{\theta_5}{2}\cos\left(\frac{\theta_4-\theta_6}{2}\right)v_x\right],$$

$$G_G = \cos\frac{\alpha}{2}\sin\frac{\theta_5}{2}\sin\left(\frac{\theta_4-\theta_6}{2}\right) + \sin\frac{\alpha}{2}\left[\cos\frac{\theta_5}{2}\cos\left(\frac{\theta_4+\theta_6}{2}\right)v_x - \cos\frac{\theta_5}{2}\sin\left(\frac{\theta_4+\theta_6}{2}\right)v_y + \sin\frac{\theta_5}{2}\cos\left(\frac{\theta_4-\theta_6}{2}\right)v_z\right],$$

$$H_G = -\cos\frac{\alpha}{2}\sin\frac{\theta_5}{2}\cos\left(\frac{\theta_4-\theta_6}{2}\right) + \sin\frac{\alpha}{2}\left[\cos\frac{\theta_5}{2}\cos\left(\frac{\theta_4+\theta_6}{2}\right)v_y + \cos\frac{\theta_5}{2}\sin\left(\frac{\theta_4+\theta_6}{2}\right)v_x + \sin\frac{\theta_5}{2}\sin\left(\frac{\theta_4-\theta_6}{2}\right)v_z\right],$$

$$I_G = \cos\frac{\alpha}{2}X_G + \sin\frac{\alpha}{2}\left[-Y_Gv_z + Z_Gv_y - W_Gv_x\right] + A\cos\frac{\theta_5}{2}\cos\left(\frac{\theta_4+\theta_6}{2}\right)$$

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$$- B\cos\frac{\theta_5}{2}\sin\left(\frac{\theta_4+\theta_6}{2}\right)$$
$$+ C\sin\frac{\theta_5}{2}\cos\left(\frac{\theta_4-\theta_6}{2}\right)$$
$$- D\sin\frac{\theta_5}{2}\sin\left(\frac{\theta_4-\theta_6}{2}\right),$$

$$J_G = \cos\frac{\alpha}{2}Y_G + \sin\frac{\alpha}{2}\left[X_G v_z - Z_G v_x - W_G v_y\right] + A\cos\frac{\theta_5}{2}\sin\left(\frac{\theta_4+\theta_6}{2}\right) + B\cos\frac{\theta_5}{2}\cos\left(\frac{\theta_4+\theta_6}{2}\right) + C\sin\frac{\theta_5}{2}\sin\left(\frac{\theta_4-\theta_6}{2}\right) + D\sin\frac{\theta_5}{2}\cos\left(\frac{\theta_4-\theta_6}{2}\right),$$

$$K_G = \cos\frac{\alpha}{2}Z_G + \sin\frac{\alpha}{2}\left[-X_G v_y + Y_G v_x - W_G v_z\right] - A\sin\frac{\theta_5}{2}\cos\left(\frac{\theta_4-\theta_6}{2}\right) - B\sin\frac{\theta_5}{2}\sin\left(\frac{\theta_4-\theta_6}{2}\right) + C\cos\frac{\theta_5}{2}\cos\left(\frac{\theta_4+\theta_6}{2}\right) + D\cos\frac{\theta_5}{2}\sin\left(\frac{\theta_4+\theta_6}{2}\right),$$

$$M_G = \cos\frac{\alpha}{2}W_G + \sin\frac{\alpha}{2}\left[X_G v_x + Y_G v_y + Z_G v_z\right] + A\sin\frac{\theta_5}{2}\sin\left(\frac{\theta_4-\theta_6}{2}\right) - \mathrm{B}\sin\frac{\theta_5}{2}\cos\left(\frac{\theta_4-\theta_6}{2}\right) - C\cos\frac{\theta_5}{2}\sin\left(\frac{\theta_4+\theta_6}{2}\right) + D\cos\frac{\theta_5}{2}\cos\left(\frac{\theta_4+\theta_6}{2}\right). \quad (3.9)$$

$$X_G = -\cos\frac{\theta_5}{2}\left(\sin\frac{\theta_4}{2}\cos\frac{\theta_6}{2}d_3 + \cos\frac{\theta_4}{2}\sin\frac{\theta_6}{2}d_5\right) + \sin\frac{\theta_5}{2}\cos\left(\frac{\theta_4-\theta_6}{2}\right)L,$$

$$Y_G = -\sin\frac{\theta_6}{2}\cos\frac{\theta_5}{2}\sin\frac{\theta_4}{2}(d_5-d_3) + \sin\frac{\theta_5}{2}\sin\left(\frac{\theta_4-\theta_6}{2}\right)L,$$

$$Z_G = \sin\frac{\theta_5}{2}\left(-\cos\frac{\theta_6}{2}\sin\frac{\theta_4}{2}d_3 + \sin\frac{\theta_6}{2}\cos\frac{\theta_4}{2}d_5\right),$$

$$W_G = -\sin\frac{\theta_6}{2}\sin\frac{\theta_5}{2}\sin\frac{\theta_4}{2}(d_5-d_3). \quad (3.10)$$

The trigonometric nature of the equations given in Table I shows that there should be two solutions for each of $\theta_4, \theta_5, \theta_6$, and θ_3. However, numerical evidence shows that not all 16 possibilities are solutions. Depending on the transformation angle and design parameters, the results vary from one robot to the other, but it was not found in any case that there are 16 solutions.

We can find infinite sets of θs representing the solution of the inverse kinematic problem of a 6R robot. It is found that there are ranges of values of θ_1 for which the remaining θ_i are uniquely determined by θ_1 and also there are ranges of values of θ_1 for which there is no solution.

To implement the algorithm we arbitrarily take a range of trial values for θ_1 and for each of them determine whether the values of $\theta_4, \theta_5, \theta_6, \theta_2$ and θ_3 can be obtained.

Table II represents some of the numerical solutions for the robot given by Fig. 1 in Section 3.1.1. It shows that for various design parameters and transformation angles α, the values of θ_1 where a solution exists is given.

Take an example case 6, for which a solution exists for θ_1 varying from 0° to 12°, from 150° to 217° and from 308° to 360°, while for the remaining values of θ_1 there is no solution. In case 16, for all αs varying from 45° to 92°, there exists a solution for all values of θ_1.

It is shown in Table II that for various design parameters, a unique solution or no solution was always obtained for a given θ_1. Of course, there are values for those parameters where no solution exists at all, such as if the length link is too short or too long, and for the design parameter that is too small or too big.

4. Summary

This work shows that for an arbitrary choice of one of the angles of rotation within a union of intervals, there exists a unique solution for the inverse kinematic problem for a 6R serial manipulator with two consecutive joint axes being parallel. Clifford Algebra formed an important tool in simplifying the representation of the robot's home configuration and in solving the system.

Acknowledgments

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