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DECOMPOSITION PROPERTY FOR MARKOV-MODULATED QUEUES WITH APPLICATIONS TO WARRANTY MANAGEMENT

NAN LIU AND VIDYADHAR G. KULKARNI
Department of Statistics and Operations Research
University of North Carolina
Chapel Hill, NC 27599
E-mail: nliu@email.unc.edu; vkulkarn@email.unc.edu

In this article we study Markov-modulated queues with and without jumps. These queuing models arise naturally in production-inventory systems with and without an external supplier. We show an interesting decomposition property that relates the equilibrium state distributions in these two systems and present an integrated warranty-inventory management model as an application.

1. INTRODUCTION

Queues in a stochastic environment have been extensively studied over decades. Previous articles on this topic (e.g., Eisen and Tainiter [3], Mitrany and Avi-Itzhak [9], Neuts [10], Yechiali and Naor [17] and Yechiali [16]) are dedicated to deriving equilibrium conditions, queue lengths, waiting times, and other system performance measures in steady state. These types of models arise in applications where queues exhibit fluctuations in their arrival and/or service processes (e.g., traffic queues, service operations by different agents, queues subject to server breakdowns, etc.; see, e.g., Neuts [10] and Mitrany and Avi-Itzhak [9]). In this article we study a single-server queue in a stochastic environment in which random batch arrivals ("jumps") occur as soon as the server becomes free; thus, the server is always kept busy. To our best knowledge, no literature has considered such models thus far. Our motivation for studying these queuing models is to provide theoretical insights to the relationships between this model (models with jumps) and the original model (models without jumps). Furthermore,

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queues and production–inventory systems are closely related to each other: customer arrivals and service completions in queues correspond to productions and demands in the production-inventory systems, respectively. In our queuing model, the "jump" behavior is the result of the order placement in inventory setups. Recently, the idea of "stochastic environment" has been viewed as a very useful tool in the inventory management area to model demand fluctuations (see, e.g., Song and Zipkin [14]). Thus, our theoretical results on queues in a stochastic environment can enhance the understanding of inventory systems in a stochastic environment and be helpful for analyzing such inventory control problems.

For the ease of discussion, we use the terminology of inventory to describe our models. We study a production-inventory system where the production and demand are modulated by an external environment process, which is modeled as a continuous time Markov chain (CTMC) with a finite state space. We model the production and demand processes as Poisson processes whose rates are determined by the current state of the environment process. We consider two cases. The first one corresponds to a lost-sale inventory system without an external supplier, for which the inventory level remains zero if its current level is zero until the next production occurs. All of the demands during this interval are lost. This case is actually a Markov-modulated $M/M/1$ queuing model as studied in Yechiali [16]. The second case corresponds to an inventory system with an external supplier, for which the inventory is replenished to a random level immediately after a demand occurs when the inventory level is zero. This system behaves as an inventory system under continuous review where no backlogging is allowed, lead times are zero, an order is placed when the inventory level is zero and a demand occurs, and the order size is random.

We study the limiting distributions of the inventory level for both of the models and show an interesting relationship between them, which is similar to the decomposition results in queues with server vacations (see, e.g., Fuhrmann and Cooper [4] and Shanthikumar [13]). Our result is also a generalization of Lemma 6.3.1 in Zipkin [18, p. 201]: that the inventory position is a uniform random variable in limit if one manages the inventory system in a Markovian demand environment using an (r, q) policy.

Using the theoretical results, we study an interesting warranty–inventory management problem. Warranty models have been studied to a great extent in the past and related literature is scattered across many journals from different disciplines. Blischke and Murthy [1] provided a detailed review. The inventory problem we consider here is of great significance in industries, especially in those that sell durable products with warranties on them (e.g., vehicle batteries, electric appliances, computer components, etc.). In our basic warranty–inventory model, we assume that the demand only arises from replacement requests due to failed products within their warranty periods. Thus, the products with unexpired warranties serve as the external environment process. We also study several extensions to this basic model. In particular, we consider the extension where the demand comes from both new sales and replacement requests and we also relax the Markovian assumption.

Our research is closely related to inventory control problems and fluid processes in a stochastic environment. Song and Zipkin [14] considered the inventory control problem under Markov-modulated Poisson demands and derived the structure of the optimal ordering policy to be an environment-dependent one. A recent article by Schwarz, Sauer, Daduna, Kulik, and Szekli [12] studied an inventory control problem for which the demand is modulated by an $M/M/1$ queue. In the case of a continuous fluid model, Browne and Zipkin [2] studied a model with continuous demand driven by a Markov process. Fluid versions of these models are studied in Kulkarni, Tzenova, and Adan [7], Kulkarni and Yan [8], and Yan and Kulkarni [15]. The closest work to our analysis is [8], in which the authors considered a continuous fluid-flow system with jumps at the boundary. However, as far as we know, no literature thus far has shown the stochastic decomposition property in the systems we study here.

The remainder of the article is organized as follows. In Section 2 we introduce some preliminary results for the system without an external supplier. In Section 3 we study the system with an external supplier and show the stochastic decomposition property. We present the warranty-inventory management model in Section 4. Finally, we mention some possible future research directions in Section 5.

2. PRELIMINARIES: THE MODEL WITHOUT JUMPS

In this section we present some preliminary results on a Markov-modulated $M/M/1$ queuing model. The purpose of this section is to introduce the notation and recapitulate the main results in Yechiali [16] for ready reference in the rest of the article. Consider a production-inventory system in a stochastic environment and without an external supplier. We call this the model without jumps. Let $Y(t)$ be the state of the environment at time t. We assume that $\{Y(t), t \geq 0\}$ is an irreducible CTMC with a finite state space $\Omega = \{1, 2, \ldots, n\}$ and its generator matrix is $Q = [q_{i,j}]$. Let PP(γ) represent a Poisson process with rate parameter γ. During the time intervals when $Y(t) = i$, the demands arise according to PP(λ_i) and the production occurs according to PP(μ_i). The demand process is independent of the production process. Let $\pi = [\pi_i]$ denote the limiting distribution of the process $\{Y(t), t \geq 0)\}$ [i.e., $\lim_{t\to\infty} \mathbf{Pr}(Y(t) = i) = \pi_i$]. It is clear that π is the unique solution to the following system:

$$\pi Q = \vec{0}, \qquad \pi e = 1,$$

where $\vec{0}$ is a row vector with all 0s and e is a column vector with all 1s.

Let $X(t)$ be the inventory level at time t. Since there is no external supplier, unsatisfied demands are lost, so $\{X(t), t \geq 0\}$ has the state space $S = \{0, 1, 2, \ldots\}$. It is easy to see that $\{(X(t), Y(t)), t \geq 0\}$ is a bivariate CTMC on the state space $S \times \Omega$. We assume that it is irreducible. Let $r_i = \lambda_i - \mu_i$ be the net demand rate at state i, $\forall i$, and $r = [r_i]$ be a $1 \times m$ row vector. Let $d = \pi r^T$ denote the expected net demand rate in steady state. It is well known that the system is stable if and only if the average demand rate is strictly larger than the average production rate in equilibrium state (see, e.g., Yechiali [16]); that is,

$$d = \pi r^T > 0. \tag{2.1}$$

Assume that stability condition (2.1) holds. Let

$$p_{i,j} = \lim_{t\to\infty} \mathbf{Pr}((X(t), Y(t)) = (i,j)), \qquad (i,j) \in S \times \Omega.$$

Define the partial generating functions

$$\psi^j(z) = \sum_i z^i p_{i,j}$$

and let

$$\psi(z) = [\psi^1(z), \psi^2(z), \ldots, \psi^n(z)].$$

Define

$$p_0 = [p_{0,1}, p_{0,2}, \ldots, p_{0,n}]$$

and let

$$\Delta(\lambda) = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

represent a diagonal matrix whose (i, i)th entry is λ_i. Denote that

$$\nu = p_0 \Delta(\lambda)$$

and

$$U(z) = \Delta(\mu + \lambda) - z\Delta(\mu) - \frac{1}{z}\Delta(\lambda). \tag{2.2}$$

It is shown in Yechiali [16] that

$$\psi(z) = \frac{z-1}{z}\nu[U(z) - Q]^{-1}. \tag{2.3}$$

We refer the reader to Yechiali [16] for numerical methods to compute ν.

3. THE MODEL WITH JUMPS

3.1. The Model and Stability Conditions

In this section, we consider the production-inventory system of the previous section but with an external supplier who can satisfy orders instantaneously. We call it the model with jumps. To distinguish it from the model without jumps, let $X'(t)$ and $Y'(t)$ denote the inventory level and the state of the environment at time t, respectively. When the inventory level is zero and a demand occurs, we place an order from the external supplier and the order arrives instantaneously. Suppose that the ith order has

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a size of J_i, and J_i's are a sequence of i.i.d. (independent and identically distributed) discrete random variables with support $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. We use J to denote the size of a generic order; thus, the inventory level becomes $J - 1$ right after an order arrives. As we can see, this corresponds to an inventory model under a base-stock policy with random order sizes and with no backlogging and no lead times. Note that the random order sizes include the more common case of constant deterministic order sizes. Random order sizes arise as a model of the situation when the actual amount received from the supplier is a random variable depending on the amount ordered. This may be due to damages in transportation.

It is clear that $\{(X'(t), Y'(t)), t \geq 0\}$ is also a bivariate CTMC on the state space $S \times \Omega$. We assume that it is irreducible. Lemma 3.1 states the stability condition for the $\{(X'(t), Y'(t)), t \geq 0\}$ process.

LEMMA 3.1: *The* $\{(X'(t), Y'(t)), t \geq 0\}$ *process is stable if*

$$\begin{aligned} \pi r^T &> 0, \\ \mathbf{Pr}(J \leq M) &= 1 \quad \text{for some } M < \infty. \end{aligned} \tag{3.1}$$

PROOF: Let $S_0 = 0$ and S_k represent the kth jump of the $X'(t)$ process from 0 to $J_k - 1$ [i.e., $X(S_k^-) = 0$ and $X(S_k^+) = J_k - 1$]. Let $Z_k = \{X'(S_k^+), Y'(S_k^+)\}$. It is easy to see that $\{(Z_k, S_k), k \geq 0\}$ is a Markov renewal sequence and $\{(X'(t), Y'(t)), t \geq 0\}$ is the corresponding Markov regenerative process (MRGP) (see, e.g., Kulkarni [6]). From the theory of MRGP, the limiting distribution of $\{(X'(t), Y'(t)), t \geq 0\}$ exists if $\mathbf{E}(S_k) < \infty$. By the stability condition (2.1) as discussed in Section 2, we know that $\mathbf{E}(S_k) < \infty$ if $\pi r^T > 0$ and $\mathbf{Pr}[J \leq M] = 1$. This proves the lemma. ■

3.2. Stochastic Decomposition Property

Let $X' \stackrel{d}{=} \lim_{t\to\infty} X'(t)$ and $X \stackrel{d}{=} \lim_{t\to\infty} X(t)$, where $\stackrel{d}{=}$ represents the equality in distribution. Theorem 3.1 gives the main result of this article—the stochastic decomposition property that relates the distributions of X' and X.

THEOREM 3.1: *Assume that the stability conditions* (3.1) *hold; then*

$$X' \stackrel{d}{=} X_0 + X,$$

where X_0 and X are independent, and the probability mass function of X_0 is given by

$$\mathbf{Pr}(X_0 = i) = \frac{\mathbf{Pr}(J > i)}{\mathbf{E}(J)}, \qquad i \in \mathbb{Z}^+ \cup \{0\}.$$

PROOF: Let $\alpha_i = \mathbf{Pr}(J = i), i \in \mathbb{Z}^+$, and $p'_{i,j} = \lim_{t\to\infty} \mathbf{Pr}[(X'(t), Y'(t)) = (i,j)]$, $(i,j) \in S \times \Omega$. We have the following balance equations corresponding to the system

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with an external supplier:

$$\left(\sum_{k\neq j} q_{j,k} + \mu_j + \lambda_j\right)p'_{0,j} = \sum_{k\neq j} p'_{0,k}q_{k,j} + \alpha_1\lambda_j p'_{0,j} + \lambda_j p'_{1,j}, \quad \forall j, \tag{3.2}$$

$$\left(\sum_{k\neq j} q_{j,k} + \mu_j + \lambda_j\right)p'_{i,j} = \sum_{k\neq j} p'_{i,k}q_{k,j} + \alpha_{(i+1)}\lambda_j p'_{0,j} + \mu_j p'_{i-1,j} + \lambda_j p'_{i+1,j}, \quad \forall i \geq 1, \forall j. \tag{3.3}$$

Let the partial generating functions $\phi^j(z), |z| \leq 1, j = 1, 2, \ldots, n$, be defined by

$$\phi^j(z) = \sum_i z^i p'_{i,j}.$$

Define

$$\phi(z) = [\phi^1(z), \phi^2(z), \ldots, \phi^n(z)] \quad \text{and} \quad \omega = p'_0\Delta(\lambda).$$

Let

$$\phi_J(z) = \mathbf{E}(z^J)$$

be the generating function of J. Through some elementary but tedious algebra, we obtain the following matrix representation from the system (3.2) and (3.3):

$$\phi(z) = \frac{\phi_J(z) - 1}{z}\omega[U(z) - Q]^{-1}, \tag{3.4}$$

where $U(z)$ is defined in (2.2). Now, rewrite (3.4) into the following form:

$$\phi(z) = \frac{1}{\mathbf{E}(J)}\frac{\phi_J(z) - 1}{z - 1}\mathbf{E}(J)\frac{z - 1}{z}\omega[U(z) - Q]^{-1}.$$

Notice that $\phi(z)$ is a valid generating function of a random vector. Let X_0 be a random variable with probability mass function given by

$$\mathbf{Pr}(X_0 = i) = \frac{\mathbf{Pr}(J > i)}{\mathbf{E}(J)}, \qquad i \in \mathbb{Z}^+ \cup \{0\}.$$

It can be shown that the generating function of X_0 is

$$\mathbf{E}(z^{X_0}) = \frac{1}{\mathbf{E}(J)}\frac{\phi_J(z) - 1}{z - 1}.$$

Thus,

$$\mathbf{E}(J)\frac{z - 1}{z}\omega[U(z) - Q]^{-1}$$

must constitute a valid generating function of a random vector. Recall the form of (2.3). Since ν is the unique vector that makes $\psi(z)$ of (2.3) a valid generating function of a random vector, we must have $\nu = \mathbf{E}(J)\omega$. Hence, the result follows. ■

A sample path argument similar to that in Kulkarni and Yan [8] can also be used to prove this theorem.

3.3. Special Cases

Theorem 3.1 implies that in steady state, the inventory level in the model with jumps is the sum of two independent random variables: one only depends on the distribution of random jump size J and the other is the equilibrium inventory level in the model without jumps. Consider the following two cases, which are of special interest in the inventory control context.

Case 1: If J is deterministic [i.e., $\mathbf{Pr}(J = q) = 1$ for some $q \in \mathbb{Z}^+$], then it is easy to verify that X_0 is a discrete uniform random variable on $\{0, 1, \ldots, q - 1\}$.

Case 2: Suppose that there is no production (i.e., $\mu_i = 0, \forall i$ and $\lambda_i > 0$ for at least one i); then the system without an external supplier (as discussed in Section 2) is stable but reducible and $p_{0,j} = \pi_j$, $\forall j$, and $p_{i,j} = 0$, $\forall i \geq 1, \forall j$. It is easy to show that Theorem 3.1 still holds for this special case. Notice that, in this situation, $X = 0$ with probability 1 so $X' \stackrel{\mathrm{d}}{=} X_0$. Since X_0 depends only on J and is independent in the $\{Y'(t), t \geq 0\}$ process, the limiting distribution of inventory level is also independent in the external environment process. This is a very useful result, and it simplifies the analysis of an inventory system in which the demand process is modulated by a CTMC.

3.4. Extensions

The stochastic decomposition property shown in Theorem 3.1 also holds in the following extensions.

3.4.1. Backlogging. We had assumed no backlogging so far. Thus, an order is placed with an external supplier when the inventory level is zero and a demand occurs. We can extend this to a general inventory system with backlogging where the reorder point is $r < 0$ rather than -1. To prove that the stochastic decomposition property still holds in this situation, one only needs to study $X''(t) = r + 1 + X'(t)$. Notice that the $\{X''(t), t \geq 0\}$ process has the state space $\{0, 1, 2, \ldots\}$ and behaves as if the inventory level process in a system without backlogging.

It is worth pointing out that Theorem 3.1 is a generalization of the well-known results stated in Zipkin [18, Lemma 6.3.1, p. 201]: If one manages an inventory system with positive lead times in a Markovian demand environment using an (r, q) policy, then the inventory position is uniformly distributed on $\{r + 1, r + 2, \ldots, r + q\}$ in the limiting state. To see this, notice that J is deterministic and there is no production in the model of Zipkin [18, p. 201].

3.4.2. Interacting environment process. Suppose that the occurrence of a demand (or a production) triggers an instantaneous probabilistic change in the state of the external environment process as follows. Let $\{Y(t), t \geq 0\}$ be the external environment process. Let $U_0 = 0$ and U_k be the kth demand or production event.

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Assume that over (U_k, U_{k+1}), the $\{Y(t), t \in (U_k, U_{k+1})\}$ process behaves as an irreducible CTMC with generator matrix $Q = [q_{i,j}]$. Furthermore, when a demand (or a production) occurs at time U_k, the state of $\{Y(t), t \geq 0\}$ changes instantaneously with transition probabilities given by

$$a_{i,j} = \mathbf{Pr}(Y(U_k^+) = j | Y(U_k^-) = i \qquad \text{the } k\text{th event is a demand,}$$
$$b_{i,j} = \mathbf{Pr}(Y(U_k^+) = j | Y(U_k^-) = i \qquad \text{the } k\text{th event is a production.}$$

Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be $n \times n$ matrixes. Notice that the A and B matrixes do not depend on the inventory level itself. It is not difficult to see that $\{Y(t), t \geq 0\}$ is a CTMC with the state space $\Omega = \{1, 2, \ldots, n\}$ and its generator matrix is given by

$$Q_Y = Q - \Delta(\mu + \lambda) + \Delta(\mu)B + \Delta(\lambda)A.$$

Note that, in general, $Q_Y \neq Q$ unless $A = B = I$, where I is an identity matrix. Since Q is the generator matrix of an irreducible CTMC, so is Q_Y. Now it is straightforward to verify that Theorem 3.1 holds with with $U(z)$ in (2.2) replaced by

$$U(z) = \Delta(\mu + \lambda) - z\Delta(\mu)B - \frac{1}{z}\Delta(\lambda)A.$$

3.4.3. Compound Poisson production process. Suppose that the production process is a compound Poisson process with rate (vector) μ and the ith production generates D_i items, where the D_i's are a sequence of i.i.d. positive integer-valued random variables. We use D to represent the number of items generated in a generic production batch. One can show that the decomposition property still holds by using arguments similar to those in Sections 2 and 3.2 with $U(z)$ in (2.2) replaced by

$$U(z) = \Delta(\mu + \lambda) - \mathbf{E}(z^D)\Delta(\mu) - \frac{1}{z}\Delta(\lambda).$$

However, if the demand process is assumed to be a compound Poisson process, the decomposition property breaks down due to the boundary effect.

3.4.4. Semi-Markov environment process. It is clear that if the sojourn time of $Y(t)$ in an arbitrary state i is a phase-type random variable, the external environment process can still be modeled as a CTMC (but with more states) and, thus, Theorem 3.1 holds as well. Since the set of phase-type distributions is dense in the set of all distributions with support in $[0, +\infty)$ (see Neuts [11] for further details), the stochastic decomposition property holds even if $\{Y(t), t \geq 0\}$ is a semi-Markov process rather than a CTMC.

4. APPLICATIONS TO WARRANTY-INVENTORY MANAGEMENT

In this section we apply the results obtained above to an integrated warranty-inventory management model under continuous review. In our model, we assume that each product is sold with a warranty. If a product fails within its warranty period, it gets a free replacement from the inventory. Thus, the demands come from two sources: (1) external demands from new sales and (2) replacement demands from failed products within their warranty periods. We study two types of warranties, namely renewing and nonrenewing (see Blischke and Murthy [1]). Under a renewing warranty, each replaced product has the same warranty period as a brand new product; under a nonrenewing warranty, the replacement gets the residual warranty period. To start with, we concentrate on the case in which new sales are handled by a separate source and the inventory is maintained only to to cover the demands arising from product replacement requests. We subsequently present several extensions to this basic model.

4.1. The Basic Model

In this subsection we consider the inventory system that is dedicated to replacement demands. We will study the inventory system facing demands from both new sales and product replacements in Section 4.2. Suppose that the sales process follows a PP(θ) and each sale is satisfied by an external supplier. The inventory is maintained to service demands arising from the failure of the sold items during their warranty periods. Product lifetimes are i.i.d phase-type random variables with m phases and parameters (α, T), denoted as PH(α, T). Here, the notation of a phase-type random variable follows Kulkarni [6, p. 299]: α is a $1 \times m$ row vector representing a probability mass function and T is an $m \times m$ matrix that is a submatrix of a generator matrix of an irreducible CTMC. If L is a PH(α, T) random variable, its cumulative distribution function is given by

$$\mathbf{Pr}(L \leq x) = 1 - \alpha \exp(Tx)e, \qquad x \geq 0.$$

Warranty periods of each item are i.i.d. PH(β, W) with n phases. The sales process, product lifetimes, and warranty periods are mutually independent. Each product that fails during its warranty is replaced instantaneously with a new one from the inventory. We study the renewing and nonrenewing warranties separately. We assume that an order of size q is placed from an external source when the inventory is zero and a demand occurs. The order is delivered instantaneously. Thus, each failed item is replaced instantaneously.

Let $Y_{ij}(t)$ be the number of products with lifetime in phase i and warranty period in phase j at time t and let $Y(t) = [Y_{ij}(t)]$ be the $m \times n$ matrix. One can easily see that $\{Y(t), t \geq 0\}$ is a CTMC. This inventory model is a special case of the model with jumps as discussed in Section 3 with $\{Y(t), t \geq 0\}$ serving the role of the external environment process. In particular, the production rate here is zero for all states of $Y(t)$ (see Case 2 in Section 3.3). Let $X(t)$ represent the inventory level at time t and

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$(X, Y) \stackrel{\mathrm{d}}{=} \lim_{t\to\infty}(X(t), Y(t))$. By Theorem 3.1, we know that X and Y are independent and X is a discrete uniform random variable on $\{0, 1, \ldots, q-1\}$.

We also study the limiting distribution of $Y(t)$ and use it to derive formulas for computing the expected net demand rate in steady state, which is an important measure in evaluating the system performance and optimizing the control policy, as will be discussed at the end of Section 4. The next lemma gives the limiting distribution of $Y(t)$. Let $P(\gamma)$ represent a Poisson random variable with mean γ. Define $\mu_{ij} = -T_{ii} - W_{jj}$, $\forall i,j$, and let $T^0 = -Te$ be the $m \times 1$ column vector whose ith entry is T_i^0. We distinguish two cases: renewing and nonrenewing warranties.

LEMMA 4.1: *For a renewing warranty, Y has the following properties:*

1. *Y_{ij} is independent of Y_{st} if $(i,j) \neq (s,t)$.*
2. *$Y_{ij} \stackrel{\mathrm{d}}{=} P(a_{ij}/\mu_{ij})$, where the $\{a_{ij}\}$ solve the following linear system:*

$$a_{ij} = \alpha_i\beta_j\theta + \sum_{k\neq j}\frac{w_{kj}}{\mu_{ik}}a_{ik} + \sum_{k\neq i}\frac{t_{ki}}{\mu_{kj}}a_{kj} + \sum_{x,y}\frac{T_x^0}{\mu_{xy}}\alpha_i\beta_j a_{xy}, \quad \forall i,j. \tag{4.1}$$

For a nonrenewing warranty, Y has the following properties:

1. *Y_{ij} is independent of Y_{st} if $(i,j) \neq (s,t)$.*
2. *$Y_{ij} \stackrel{\mathrm{d}}{=} P(a_{ij}/\mu_{ij})$, where the $\{a_{ij}\}$ solve the following linear system:*

$$a_{ij} = \alpha_i\beta_j\theta + \sum_{k\neq j}\frac{w_{kj}}{\mu_{ik}}a_{ik} + \sum_{k\neq i}\frac{t_{ki}}{\mu_{kj}}a_{kj} + \sum_{k}\frac{T_k^0}{\mu_{kj}}\alpha_i a_{kj}, \quad \forall i,j. \tag{4.2}$$

PROOF: We prove the results for the renewing warranty case; the nonrenewing warranty case follows similarly. For the renewing case, it is not difficult to see that the process $\{Y(t) = [Y_{ij}(t)], t \geq 0\}$ behaves as a Jackson network with $m \times n$ stations and infinite servers at each station. In terms of Jackson network settings, $Y_{ij}(t)$ represents the queue length of station (i,j) at time t. The service rate of a server at station (i,j) is μ_{ij}. The external arrival rate to station (i,j) is $\alpha_i\beta_j\theta$. When a customer completes service at station (i,j), he joins the queue at station (i,k) with probability $w_{jk}/\mu_{ij} + (T_i^0/\mu_{ij})\alpha_i\beta_k$, joins the queue at station (k,j) with probability $t_{ik}/\mu_{ij} + (T_i^0/\mu_{ij})\alpha_k\beta_j$, joins the queue at station (x,y), where $x \neq i$ and $y \neq j$ with probability $(T_i^0/\mu_{ij})\alpha_x\beta_y$, and departs the system with probability W_j^0/μ_{ij}, where W_j^0 is the jth entry of the $n \times 1$ column vector $W^0 = -We$. So a_{ij} represents the total arrival rate to station (i,j). Then the results directly follow from the limiting distribution of a Jackson network. See Kulkarni [6, Thm. 7.5] for more details. ■

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Notice that T_i^0 is the failure rate of a product with lifetime in phase i; thus at time t, the demand rate in state $Y(t)$ can be computed as

$$\mu_{Y(t)} = \sum_{i,j} Y_{ij}(t)T_i^0, \quad \forall t.$$

Note that a_{ij}/μ_{ij} is the long-run average number of products with lifetime in phase i and warranty period in phase j. Recall that d denotes the expected net demand rate in steady state; thus, it can be computed as

$$d = \sum_{i,j,k} k\,\mathbf{Pr}(Y_{ij} = k)T_i^0 = \sum_{i,j} \frac{a_{ij}}{\mu_{ij}} T_i^0. \tag{4.3}$$

One may observe from (4.1) and (4.2) that a_{ij} is directly proportional to the sales rate θ, and so should be d. Lemma 4.2 clarifies this point and provides an alternative way to compute d.

LEMMA 4.2: *For a renewing warranty, let* $\{f_{ij}\}$ *solve the following linear system:*

$$f_{ij} = \frac{T_i^0}{\mu_{ij}} + \sum_{j\neq k} \frac{w_{jk}}{\mu_{ij}} f_{ik} + \sum_{i\neq k} \frac{t_{ik}}{\mu_{ij}} f_{kj} + \frac{T_i^0}{\mu_{ij}} \sum_{x,y} \alpha_x \beta_y f_{xy} \quad \forall i,j. \tag{4.4}$$

For a nonrenewing warranty, let $\{f_{ij}\}$ *solve the following linear system:*

$$f_{ij} = \frac{T_i^0}{\mu_{ij}} + \sum_{j\neq k} \frac{w_{jk}}{\mu_{ij}} f_{ik} + \sum_{i\neq k} \frac{t_{ik}}{\mu_{ij}} f_{kj} + \frac{T_i^0}{\mu_{ij}} \sum_{k} \alpha_k f_{kj} \quad \forall i,j.$$

In both cases, the expected demand rate is given by

$$d = \theta \sum_{i,j} \alpha_i \beta_j f_{ij}. \tag{4.5}$$

PROOF: Use $\otimes$ to represent the Kronecker product operator. Define $a_i = [a_{i1}, a_{i2}, \ldots, a_{in}]$ and $f_i = [f_{i1}, f_{i2}, \ldots, f_{in}]$ for $i = 1, 2, \ldots, m$. Let $a = [a_1, a_2, \ldots, a_m]^T$, $f = [f_1, f_2, \ldots, f_m]^T$, $u = \theta\alpha^T \otimes \beta^T$, and $v = [v_1, v_2, \ldots, v_{m\times n}]^T$, where $v_{(i-1)n+j} = T_i^0/\mu_{ij}$. Note that a, f, u, and v are all collum vectors with dimension $m \times n$. For the renewing warranty case, we can write systems (4.1) and (4.4) into matrix form as follows:

$$Ga = u,$$
$$Hf = v,$$

where G is the coefficient matrix of a in system (4.1) and H is the coefficient matrix of f in system (4.4). It is tedious but straightforward to verify that $G^T = H$; hence, $u^T f = a^T G^T f = a^T H f = a^T v$. This proves the results for the renewing warranty case. Similarly, we can prove the nonrenewing warranty case. ■

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Notice that f_{ij} has a physical interpretation: It represents the expected replacements for a new product starting with lifetime in phase i and warranty period in phase j before its warranty expires. Thus, $\sum_{i,j} \alpha_i \beta_j f_{ij}$ is the expected replacements for a new product during its warranty period. In the long run, sales per unit of time is θ and each product requires $\sum_{i,j} \alpha_i \beta_j f_{ij}$ replacements on average. So, on average, we require $d = \theta \sum_{i,j} \alpha_i \beta_j f_{ij}$ replacements per unit of time. This is the intuition behind Lemma 4.2.

4.2. Extensions

In this subsection we discuss some extensions to the warranty-inventory model discussed in Section 4.1.

4.2.1. General sales process. Instead of assuming that the sales process follows a PP(θ), we generalize it to be a renewal process with interarrival time distributed as a phase-type random variable, say PH(γ, D) with l phases. Let $S(t)$ denote the current phase of an interarrival time during the sales process, $Y(t)$ be defined in the same way as in Section 4.1, and $Z(t) = (S(t), Y(t)), t \geq 0$. One can easily see that this inventory model is also a special case of the model with jumps as discussed in Section 3, where $\{Z(t), t \geq 0\}$ is a CTMC and serves the role of the external environment. Let $X(t)$ represent the inventory level at time t and $(X, Z) \stackrel{d}{=} \lim_{t\to\infty}(X(t), Z(t))$. By Theorem 3.1 we immediately acknowledge that X and Z are independent and X is a discrete uniform random variable on $\{0, 1, \ldots, q-1\}$.

However, it does not seem to be an easy task to analyze the $\{Z(t), t \geq 0\}$ process as we do in Lemma 4.1 to obtain an explicit solution for the steady-state distribution. Instead, similar to Lemma 4.2, Lemma 4.3 gives a method to compute the expected net demand rate in steady state without analyzing the $\{Z(t), t \geq 0\}$ process explicitly. Let $\tau_s = -\gamma D^{-1} e$ be the mean intersale time and let $\theta_s = 1/\tau_s$.

LEMMA 4.3: *The expected net demand rate in steady state d can be computed as*

$$d = \theta_s \sum_{i,j} \alpha_i \beta_j f_{ij}.$$

PROOF: Let $R(t)$ represent the total number of replacements up to time t. By the definition of d, we have

$$\lim_{t\to\infty} \frac{R(t)}{t} = d.$$

It remains to show that

$$\lim_{t\to\infty} \frac{R(t)}{t} = \theta_s \sum_{i,j} \alpha_i \beta_j f_{ij}.$$

Let $N'(t)$ represent the total sales up to time t. Denote R'_i as the number of replacements for the ith sale and T'_i as the ith intersale time. Let $R'(t) = \sum_{i=1}^{N'(t)} R'_i$. By definition of

$R(t)$ and $R'(t)$, we know

$$\lim_{t\to\infty} \frac{R(t)}{t} = \lim_{t\to\infty} \frac{R'(t)}{t}.$$

It is not difficult to see that $\{R'(t), t \geq 0\}$ is a renewal reward process and, thus,

$$d = \lim_{t\to\infty} \frac{R'(t)}{t} = \frac{\mathbf{E}(R'_1)}{\mathbf{E}(T'_1)} = \frac{\sum_{i,j} \alpha_i \beta_j f_{ij}}{-\gamma D^{-1} e} = \theta_s \sum_{i,j} \alpha_i \beta_j f_{ij}.$$

This proves our results. ■

4.2.2. Combined demand. In this subsection we discuss the warranty-inventory model where the inventory is used to satisfy the demand for new sales as well the demand from product replacement requests. We call such a system a combined demand system and the systems discussed in Sections 4.1 and 4.2.1 the pure demand systems. First, notice that the decomposition property still holds for combined demand systems since one can easily model a combined demand system as a pure demand system with certain changes to the demand rates of some, if not all, states of the environment process. For example, if the sales process is PP(θ), one can use a pure demand model where the demand rates of each state is increased by θ to model a combined demand system. Similar adaptation can be used to model combined demand systems with general sales processes as discussed in Section 4.2.1. To compute the expected net demand rate in steady state, one can still use Lemma 4.3. Instead of R'_i units of demands, the ith sale incurs $R'_i + 1$ units of demands to the inventory. So, one just need to replace f_{ij} by $f_{ij} + 1$ in Lemma 4.3 for the combined demand model.

4.2.3. Non-Markovian models. Since the set of phase-type distributions is dense in the set of all distributions with support in $[0, +\infty)$, the above discussion implies that if the sales process is a renewal process, the product lifetime and warranty period follow two independent general distributions over $[0, +\infty)$, the stochastic decomposition property still holds. Although a full analysis of such a general system is difficult, we can compute the expected net demand rate in steady state d_g for a pure demand model as

$$d_g = \theta_g f_g, \tag{4.6}$$

where θ_g is the sales rate and f_g represents the expected replacements for a newly sold product within its warranty period. Ja [5] provided methods to compute f_g for both renewing and nonrenewing warranty cases under very general setups. To deal with a combined demand system, one just needs to replace f_g by $f_g + 1$.

Using the analysis of the expected net demand rate in steady state and the decomposition property, we can show that the optimal order quantity that minimizes the long-run average cost of the inventory system is given by the classic EOQ (economic order quantity) formula

$$q^* = \sqrt{2Kd/h},$$

where K is the fixed ordering cost, h is the holding cost of one item for one unit of time, and d is the expected net demand rate in steady state as explained earlier.

5. CONCLUSION

In this article we consider a single-server queue in a stochastic environment with random jumps when the server becomes idle. This queuing model can be interpreted as a production-inventory system with an external supplier, in which the productions and demands are modulated by an external stochastic process. We study such a system as well as the same one but without an external supplier, and we show an interesting decomposition relationship between the limiting distributions of the inventory level in these two systems. We present an integrated warranty-inventory model as an interesting application and discuss several extensions.

Although the jump size in our model is random, it is independent of the state of the environment process prior to the jump. It will be interesting to study the problem in which the jump size depends on the environment process. Another possible extension to the warranty inventory model will be to consider a positive lead time between the placement and arrival of an order and/or allow backlogging in the inventory management. However, the decomposition property breaks down under these two extensions, and this makes the problem substantially more difficult.

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