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PROOF OF TWO CONJECTURES ON SUPERCONGRUENCES INVOLVING CENTRAL BINOMIAL COEFFICIENTS

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Abstract

In this note we use some q-congruences proved by the method of 'creative microscoping' to prove two conjectures on supercongruences involving central binomial coefficients. In particular, we confirm the $m = 5$ case of Conjecture 1.1 of Guo ['Some generalizations of a supercongruence of Van Hamme', *Integral Transforms Spec. Funct.* **28** (2017), 888–899].

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1. Introduction

Let p be an odd prime. A p-adic congruence is called a supercongruence if it holds modulo p^r for some $r \geqslant 2$. In 1997, Van Hamme [13] observed 13 supercongruences on truncated forms of Ramanujan's and Ramanujan-like formulas for $1/\pi$. In particular, the following supercongruence of Van Hamme [13, (B.2)],

$$\sum_{k=0}^{(p-1)/2} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 \equiv p(-1)^{(p-1)/2} \pmod{p^3}, \tag{1.1}$$

was first confirmed by Mortenson [9]. Zudilin [14] reproved (1.1) using the WZ (Wilf–Zeilberger) method. Sun [11] utilised the WZ method again and some properties of the Euler numbers to give a refinement of (1.1) modulo p^4. Swisher [12, (B.3)] proposed an interesting conjecture on a generalisation of (1.1).

Motivated by Zudilin's work [14], the second author [2] investigated other generalisations of (1.1) and also made some related conjectures. For example, the $m = 5$ case of [2, Conjecture 1.1] can be stated as follows: for any odd prime p and

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positive integer r,

$$\sum_{k=0}^{(p^r-1)/2} \frac{(4k+1)^5}{(-64)^k} \binom{2k}{k}^3 \equiv 41p^r(-1)^{(p-1)r/2} \pmod{p^{r+2}}, \tag{1.2}$$

$$\sum_{k=0}^{p^r-1} \frac{(4k+1)^5}{(-64)^k} \binom{2k}{k}^3 \equiv 41p^r(-1)^{(p-1)r/2} \pmod{p^{r+2}}. \tag{1.3}$$

It is clear that the supercongruences (1.2) and (1.3) are equivalent to each other when $r = 1$, since $\binom{2k}{k} \equiv 0 \pmod{p}$ for $(p+1)/2 \leqslant k \leqslant p-1$. The second author [2] proved that (1.2) is true modulo p^3 for $r = 1$ and primes p satisfying some congruence conditions. Liu [8] gave a proof of (1.2) for the complete $r = 1$ case. Hou *et al.* [7] proved [2, Conjecture 1.1] for $r = 1$. However, the supercongruences (1.2) and (1.3) are still open for $r > 1$.

Recently, the second author and Liu [4] proposed the following companion conjecture (the $m = 5$ case of [4, Conjecture 5.1]): for any odd prime p and positive integer r,

$$\sum_{k=0}^{(p^r+1)/2} \frac{(4k-1)^5}{(-64)^k(2k-1)^3} \binom{2k}{k}^3 \equiv -23p^r(-1)^{(p-1)r/2} \pmod{p^{r+2}}, \tag{1.4}$$

$$\sum_{k=0}^{p^r-1} \frac{(4k-1)^5}{(-64)^k(2k-1)^3} \binom{2k}{k}^3 \equiv -23p^r(-1)^{(p-1)r/2} \pmod{p^{r+2}}. \tag{1.5}$$

The aim of this note is to prove the following theorem.

Theorem 1.1. *The supercongruences (1.2)–(1.5) are true.*

We shall prove the theorem by using some q-congruences established by the second author in [3]. Before giving the proof, we recall some standard q-notation: $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ is the *q-shifted factorial*; $[n] = [n]_q = (1-q^n)/(1-q)$ is the *q-integer* and $\Phi_n(q)$ denotes the nth *cyclotomic polynomial* in q,

$$\Phi_n(q) = \prod_{\substack{1\leqslant k\leqslant n \\ \gcd(n,k)=1}} (q - \zeta^k),$$

where ζ stands for an nth primitive root of unity. During the past few years, many authors have proved different q-analogues of supercongruences (see, for example, [1, 3, 5, 6, 10]). This note indicates that q-congruences can do even more than we could ever have expected.

2. Proof of the theorem

Proofs of (1.2) and (1.3). Recently, the second author [3, Theorem 1.2] utilised the method of 'creative microscoping' introduced in [6] to establish the following

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q-congruences: for odd $n>1$, modulo $[n]_{q^2}\Phi_n(q^2)^2$,

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{2k^2-4k}\equiv[n]_{q^2}(-1)^{(n+1)/2}q^{(n-1)^2/2}\frac{2q+1}{q^2}, \tag{2.1}$$

$$\sum_{k=0}^{n-1}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{2k^2-4k}\equiv[n]_{q^2}(-1)^{(n+1)/2}q^{(n-1)^2/2}\frac{2q+1}{q^2}, \tag{2.2}$$

which are q-analogues of the $m=3$ case of [2, Conjecture 1.1]:

$$\sum_{k=0}^{(p^r-1)/2}\frac{(4k+1)^3}{(-64)^k}\binom{2k}{k}^3\equiv-3p^r(-1)^{(p-1)r/2}\pmod{p^{r+2}}, \tag{2.3}$$

$$\sum_{k=0}^{p^r-1}\frac{(4k+1)^3}{(-64)^k}\binom{2k}{k}^3\equiv-3p^r(-1)^{(p-1)r/2}\pmod{p^{r+2}}. \tag{2.4}$$

Letting $q\mapsto q^{-1}$ in (2.1) and noticing that $\Phi_n(q^{-1})=\Phi_n(q)q^{-\varphi(n)}$ for $n>1$, where $\varphi(n)$ is Euler's totient function,

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-2k^2-6k}$$
$$\equiv[n]_{q^2}(-1)^{(n+1)/2}q^{3-(n-1)^2/2-2n}(2+q)\pmod{[n]_{q^2}\Phi_n(q^2)^2}. \tag{2.5}$$

Subtracting (2.1) from (2.5) and then dividing both sides by $1-q$,

$$\sum_{k=0}^{(n-1)/2}(-1)^k[4k+1]_{q^2}[4k+1]^2\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}[4k^2+2k]q^{-2k^2-6k}$$
$$\equiv[n]_{q^2}(-1)^{(n+1)/2}q^{3-(n-1)^2/2-2n}\frac{2+q-(2q+1)q^{n^2-4}}{1-q}\pmod{[n]_{q^2}\Phi_n(q^2)^2}. \tag{2.6}$$

It is easy to see that, for $k\geqslant 0$ and any prime power p^r,

$$\lim_{q\to1}\frac{(q^2;q^4)_k}{(q^4;q^4)_k}=\frac{1}{4^k}\binom{2k}{k},$$

$$\lim_{q\to1}\frac{2+q-(2q+1)q^{n^2-4}}{1-q}=3n^2-11$$

and $\Phi_{p^r}(1)=p$. Thus, taking $n=p^r$ and $q\to1$ in (2.6),

$$\sum_{k=0}^{(p^r-1)/2}\frac{(4k+1)^3}{(-64)^k}\binom{2k}{k}^3(4k^2+2k)\equiv11p^r(-1)^{(p-1)r/2}\pmod{p^{r+2}}. \tag{2.7}$$

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Since $(4k+1)^2 = 4(4k^2+2k)+1$, combining (2.3) and (2.7), we are led to (1.2). Similarly, applying (2.2) and (2.4), we can prove (1.3). □

Proofs of (1.4) and (1.5). The second author [3] also proved the following q-congruences: for odd $n > 1$, modulo $[n]_{q^2}\Phi_n(q^2)^2$,

$$\sum_{k=0}^{M}(-1)^k[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}q^{2k^2} \equiv [n]_{q^2}(-1)^{(n-1)/2}q^{(n-1)^2/2}\frac{2+q}{q^3}, \tag{2.8}$$

where $M = (n+1)/2$ or $n-1$. Letting $q \mapsto q^{-1}$ in (2.8) and multiplying both sides by q^{-8},

$$\sum_{k=0}^{M}(-1)^k[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}q^{2k-2k^2}$$
$$\equiv [n]_{q^2}(-1)^{(n-1)/2}q^{-4-2n-(n-1)^2/2}(2q+1) \pmod{[n]_{q^2}\Phi_n(q^2)^2}. \tag{2.9}$$

Subtracting (2.8) from (2.9) and dividing both sides by $1-q$,

$$\sum_{k=0}^{M}(-1)^k[4k-1]_{q^2}[4k-1]^2\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}[4k^2-2k]q^{2k-2k^2}$$
$$\equiv [n]_{q^2}(-1)^{(n-1)/2}q^{-4-2n-(n-1)^2/2}\frac{2q+1-(2+q)q^{n^2+2}}{1-q} \pmod{[n]_{q^2}\Phi_n(q^2)^2}. \tag{2.10}$$

Letting $n = p^r$ and $q \to 1$ in (2.8) and (2.10) and noticing that

$$\lim_{q\to 1}\frac{(q^{-2};q^4)_k}{(q^4;q^4)_k} = \frac{-1}{4^k(2k-1)}\binom{2k}{k},$$
$$\lim_{q\to 1}\frac{2q+1-(2+q)q^{p^{2r}+2}}{1-q} = 3p^{2r}+5$$

yields

$$\sum_{k=0}^{N}\frac{(4k-1)^3}{(-64)^k(2k-1)^3}\binom{2k}{k}^3 \equiv -3p^r(-1)^{(p-1)r/2} \pmod{p^{r+2}}, \tag{2.11}$$

$$\sum_{k=0}^{N}\frac{(4k-1)^3}{(-64)^k(2k-1)^3}\binom{2k}{k}^3(4k^2-2k) \equiv -5p^r(-1)^{(p-1)r/2} \pmod{p^{r+2}}, \tag{2.12}$$

where $N = (p^r+1)/2$ or p^r-1. Since $(4k-1)^2 = 4(4k^2-2k)+1$, combining (2.11) and (2.12), we immediately obtain (1.4) and (1.5). □

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