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A comparison of concepts from computable analysis and effective descriptive set theory

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Computable analysis and effective descriptive set theory are both concerned with complete metric spaces, functions between them and subsets thereof in an effective setting. The precise relationship of the various definitions used in the two disciplines has so far been neglected, a situation this paper is meant to remedy.
As the role of the Cauchy completion is relevant for both effective approaches to Polish spaces, we consider the interplay of effectivity and completion in some more detail.

1. Introduction

Both computable analysis (Weihrauch 1987, 2000) and effective descriptive set theory (Moschovakis 1980) have a notion of computability on (complete, separable) metric spaces as a core concept. Nevertheless, the definitions are prima facie different, and the precise relationship has received little attention so far (contrast e.g. the well-established connections between Weihrauch's and Pour-El & Richard's approach to computable analysis Pour-El and Richards 1989).

The lack of exchange between the two approaches becomes even more regrettable in the light of recent developments that draw on both computable analysis and descriptive set theory:

— The study of Weihrauch reducibility often draws on concepts from descriptive set theory via results that identify various classes of measurable functions as lower cones for Weihrauch reducibility (Brattka 2005; Brattka et al. 2012a; Pauly and de Brecht 2014). The Weihrauch lattice is used as the setting for a metamathematical investigation of the computable content of mathematical theorems (Brattka and Gherardi 2011; Gherardi and Marcone 2009; Pauly 2010).
— In fact, Weihrauch reducibility was introduced partly as an analogue to Wadge reducibility for functions (see the original papers Weihrauch 1992a,b, 2000 and subsequent work Hertling 1996), and as such, can itself be seen as a subfield of (effective) descriptive set (or rather function) theory.
— The Quasi-Polish spaces (de Brecht 2013) allow the generalization of many results from descriptive set theory to a much larger class of spaces (e.g. de Brecht 2014; Motto Ros

et al. 2014), and admit a very natural characterization in terms of computable analysis as those countably based spaces with a total admissible Baire-space representation.

— Even more so, the suggested *synthetic descriptive set theory* Pauly and de Brecht (2013, 2015) would extend some fundamental results from descriptive set theory even further, to general represented spaces (Pauly 2012). This could pave the way to apply some very strong results (Kihara 2015) to the long-outstanding questions regarding generalizations of the Jayne–Rogers theorem (Jayne and Rogers 1982; Motto Ros and Semmes 2009; Kačena et al. 2012; Semmes 2009).

Our goal with the present paper is to facilitate the transfer of results between the two frameworks by pointing out both similarities and differences between definitions. For example, it turns out that the requirements of an effective metric space (as used by Moschovakis) are strictly stronger than those Weihrauch imposes on a computable metric space (CMS) – however, this is only true for specific metrics, by moving to an equivalent metric, the stronger requirements can always be satisfied. Hence, effective Polish spaces and computable Polish spaces are the same concept.

Besides the fundamental layer of metric spaces, we shall also consider the computability structure on hyperspaces such as all Σ_2-measurable subsets of some given Polish spaces. While these spaces do not carry a meaningful topology, they can nevertheless be studied as represented spaces. This was done implicitly in Moschovakis (1980), and more explicitly in Brattka (2005), Pauly and de Brecht (2014), Selivanov (2013) and Pauly and de Brecht (2013, 2015).

As a digression, we will consider a more abstract view point on the Cauchy completion to illuminate the different approaches to metric spaces.

2. Effective Polish spaces and computable Polish spaces

We begin by contrasting the definitions of the fundamental structure on metric spaces used to derive computability notions; Moschovakis defines a *recursively presented metric space* (RPMS) and Weihrauch a CMS. Throughout the text, by $\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q}$ we denote some standard bijection.

Definition 2.1 (Moschovakis 1980). [3B] Suppose $\mathcal{X}$ is a separable, complete metric space with distance function d. A recursive presentation of $\mathcal{X}$ is any function $\mathbf{r} : \mathbb{N} \to \mathcal{X}$ whose image $\mathbf{r}[\mathbb{N}] = r_0, r_1, \ldots$ is dense in $\mathcal{X}$ and such that the relations

$$P^{d,\mathbf{r}}(i, j, k) \iff d(r_i, r_j) \leqslant \nu_{\mathbb{Q}}(k),$$

$$Q^{d,\mathbf{r}}(i, j, k) \iff d(r_i, r_j) < \nu_{\mathbb{Q}}(k),$$

are recursive.

A *RPMS* is a triple $(\mathcal{X}, d, \mathbf{r})$ as above. To every RPMS $(\mathcal{X}, d, \mathbf{r})$ we assign the *nbhd system* $\{N(\mathcal{X}, s) \mid s \in \mathbb{N}\}$, where

$$N(\mathcal{X}, 2^{i+1} \cdot 3^{k+1}) = \{x \in \mathcal{X} \mid d(x, r_i) < \nu_{\mathbb{Q}}(k)\},$$

and $N(\mathcal{X}, s)$ is the empty set if s does not have the form $2^{i+1} \cdot 3^{k+1}$.

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When referring to RPMS, we usually omit the metric and the recursive presentation, and simply write $\mathcal{X}$.

Definition 2.2 (Weihrauch 2000). [cf 8.1.2] We define a CMS with its Cauchy representation as follows:

1. An effective metric space is a tuple $\mathbf{M} = (M, d, (a_n)_{n\in\mathbb{N}})$ such that (M, d) is a metric space and $(a_n)_{n\in\mathbb{N}}$ is a dense sequence in (M, d).
2. The Cauchy representation $\delta_{\mathbf{M}} :\subseteq \mathbb{N}^{\mathbb{N}} M$ associated with the effective metric space $\mathbf{M} = (M, d, (a_n)_{n\in\mathbb{N}})$ is defined by

$$\delta_{\mathbf{M}}(p) = x \ :\Longleftrightarrow \begin{cases} d(a_{p(i)}, a_{p(k)}) \leqslant 2^{-i} \text{ for } i < k \\ \text{and } x = \lim\limits_{i\to\infty} a_{p(i)}. \end{cases}$$

3. Finally, a CMS is an effective metric space such that the following relation (involving a standard numbering $\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q}$)

$$\{(t, u, v, w) \mid \nu_{\mathbb{Q}}(t) < d(a_u, a_v) < \nu_{\mathbb{Q}}(w)\} \text{ is r.e.}$$

Both definitions can only ever apply to separable metric spaces, however, a noticeable difference is Moschovakis' requirement of completeness, which is not demanded by Weihrauch. This is only a superfluous distinction, though:

Observation 2.3. If $\mathbf{M} = (M, d, (a_n)_{n\in\mathbb{N}})$ is a CMS with a Cauchy representation then its completion $\overline{\mathbf{M}} = (\overline{M}, \overline{d}, (a_n)_{n\in\mathbb{N}})$ (where $\overline{d}$ is the expanded distance function for the completion, specifically $\overline{d}|_M = d$) is also a CMS.

A more substantial difference lies in the decidability requirement of distances between basic points and rational numbers. For Weihrauch's definition, being able to semi-decide $q < d(a_u, a_w)$ and $d(a_u, a_w) < q$ is enough, whereas Moschovakis demands these to be decidable. By identifying[†] $(a_n)_{n\in\mathbb{N}}$ and $r : \mathbb{N} \to \mathcal{X}$, we immediately find

Observation 2.4. Every RPMS is a CMS.

The converse fails in general:

Example 2.5. Consider the following CMS: Let the base set be $X = \mathbb{N} \uplus \mathbb{N}$, the dense set also X (with a standard bijection) and the distance function be defined as follows (assuming n_i is the ith element of the first copy of $\mathbb{N}$, n'_i from the second):

$$d(n_i, n_j) := |n_i - n_j|,$$

$$d(n_i, n'_i) := 1 + \frac{1}{s_i}.$$

[†] That this identification actually makes sense follows from the investigation of the class of computable functions between spaces in Section 3.

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Where s_i is the step count of the ith Turing machine started with no arguments if it halts

$$d(n_i, n_i') := 1,$$

if it does not. Then, to ensure the validity of the triangle inequality, we set

$$d(n_j, n_i') := d(n_i, n_i') + d(n_i, n_j),$$

$$d(n_j', n_i') := d(n_j, n_i') + d(n_j', n_j).$$

This space is a CMS but not an RPMS.

Proof. To output the upper bound $d(n_i, n_i') < 1 + \frac{1}{k} \leqslant q_i$ one only has to simulate φ_i, the ith program for k steps, if it did not halt yet, output q_i, if it did halt it will be a lower bound. We can avoid outputting the exact term for the exact step count in case it halts. Similarly, we semidecide the other types of distances.

This will form a CMS (with the representation of eventually constant sequences of points).

Suppose towards a contradiction that (X, d) admits a recursive presentation $\mathbf{r} : \mathbb{N} \to X$. Since the set $\mathbf{r}[\mathbb{N}]$ is dense in (X, d) and the latter space is discrete we have that $\mathbf{r}$ is surjective. It follows easily that there exists a recursive function $f : \mathbb{N} \times \{0, 1\} \to \mathbb{N}$ such that $\mathbf{r}(f(i, 0)) = n_i$ and $\mathbf{r}(f(i, 1)) = n_i'$.

The decidability requirements now imply in particular that $d(\mathbf{r}(f(i, 0)), \mathbf{r}(f(i, 1))) = 1$ is a decidable property – but by the construction of d, this would mean that the Halting problem is decidable, providing the desired contradiction. □

We will proceed to find a weaker counterpart to Observation 2.4. First, note that in a RPMS we can decide whether $r_n = r_m$?, whereas we cannot decide $a_w = a_u$? in a CMS. It is possible, however, to avoid having duplicate points in the dense sequence even in the latter case. First, we shall provide a general criterion for when two dense sequences give rise to homeomorphic CMSs (which presumably is a folklore result):

Lemma 2.6. For two CMSs $\mathbf{X} = (M, d, (a_i)_{i\in\mathbb{N}}), \mathbf{X}' = (M, d, (a_i')_{i\in\mathbb{N}})$ if a_i is uniformly computable in $\mathbf{X}'$ then the

$$id : \mathbf{X} \to \mathbf{X}',$$

identity function is computable.

Proof. The assumption means that given $n, k \in \mathbb{N}$, we can compute some $i_{n,k}$ such that $d(a'_{i_{n,k}}, a_n) < 2^{-k}$. Now, we are given some $x \in \mathbf{X}$ via some sequence $(a_{n_j})_{j\in\mathbb{N}}$ such that $d(a_{n_j}, x) < 2^{-j}$. Consider the sequence $(a'_{i_{n_{j+1},(j+1)}})_{j\in\mathbb{N}}$. We find that $d(a'_{i_{n_{j+1},(j+1)}}, x) \leqslant d(a'_{i_{n_{j+1},(j+1)}}, a_{n_{j+1}}) + d(a_{n_{j+1}}, x) \leqslant 2^{-j-1} + 2^{-j-1} = 2^{-j}$; thus this sequence constitutes a name for $x \in \mathbf{X}'$. □

In general, we shall write $\mathbf{X} \cong \mathbf{Y}$ iff there is a bijection $\lambda : \mathbf{X} \to \mathbf{Y}$ such that λ and λ^{-1} are computable. In this paper, λ will generally be the identity on the underlying sets.

Corollary 2.7. For two CMSs $\mathbf{X} = (M, d, (a_i)_{i\in\mathbb{N}}), \mathbf{X}' = (M, d, (a_i')_{i\in\mathbb{N}})$ if a_i is uniformly computable in $\mathbf{X}'$ and a_i' is uniformly computable in $\mathbf{X}$ then $\mathbf{X} \cong \mathbf{X}'$

As a slight detour, we will prove a more general, but ultimately too weak result. We recall from Pauly (2012) that a represented space is called computably Hausdorff, if inequality is recognizable. Inequality is (computably) recognizable in a represented space $\mathbf{X}$ iff the function $\neq : \mathbf{X} \times \mathbf{X} \to \mathbb{S}$ is computable, where $\mathbb{S}$ is the Sierpiński space (with underlying set $\{\bot, \top\}$ and open sets $\emptyset, \{\top\}, \{\bot, \top\}$) and $\neq(x,x) = \bot$ and $\neq(x,y) = \top$ otherwise. Note that every CMS is computably Hausdorff. We define a multivalued map $\text{RemoveDuplicates} :\subseteq \mathcal{C}(\mathbb{N}, \mathbf{X}) \rightrightarrows \mathcal{C}(\mathbb{N}, \mathbf{X})$ by $\text{dom}(\text{RemoveDuplicates}) = \{(x_n)_{n\in\mathbb{N}} \mid \omega = |\{x_n \mid n \in \mathbb{N}\}|\}$ and $(y_n)_{n\in\mathbb{N}} \in \text{RemoveDuplicates}((x_n)_{n\in\mathbb{N}})$ iff $\{y_n \mid n \in \mathbb{N}\} = \{x_n \mid n \in \mathbb{N}\}$ and $\forall n \neq m \in \mathbb{N} \ . \ y_n \neq y_m$. In words, RemoveDuplicates takes a sequence with infinite range, and produces a sequence with the same range but without duplicates.

Proposition 2.8. Let $\mathbf{X}$ be computably Hausdorff. Then, $\text{RemoveDuplicates} :\subseteq \mathcal{C}(\mathbb{N}, \mathbf{X}) \rightrightarrows \mathcal{C}(\mathbb{N}, \mathbf{X})$ is computable.

Proof. Given a sequence $(x_n)_{n\in\mathbb{N}}$ in a computable Hausdorff space, we can compute $\{n \in \mathbb{N} \mid \forall i < n \ x_i \neq x_n\} \in \mathcal{O}(\mathbb{N})$, i.e. as a recursively enumerable set (relative to the sequence). By assumption on the range of the sequence, this set is infinite. It is a basic result from recursion theory that any infinite recursively enumerable set is the range of an injective computable function, and this holds uniformly. Let λ be such a function. Then, $y_n = x_{\lambda(n)}$ satisfies the criteria for the output. □

The combination of Lemma 2.6 and Proposition 2.8 allows us to conclude that for any infinite computable metric space $\mathbf{X}$, there is a computable metric space $\mathbf{X}'$ with the same underlying set and metric, and a repetition-free dense sequence such that $\text{id} : \mathbf{X}' \to \mathbf{X}$ is computable – but we cannot guarantee computability of $\text{id} : \mathbf{X} \to \mathbf{X}'$ thus. Consequently, we shall employ a more complicated construction:

Theorem 2.9. For any infinite CMS $\mathbf{X} = (M, d, (a_i)_{i\in\mathbb{N}})$, there is a repetition-free sequence $(a'_i)_{i\in\mathbb{N}}$ such that $\mathbf{X}' = (M, d, (a'_i)_{i\in\mathbb{N}})$ is a CMS with $\mathbf{X} \cong \mathbf{X}'$.

Proof. We will first describe an algorithm obtaining the sequence $(a'_i)_{i\in\mathbb{N}}$ from the original sequence $(a_i)_{i\in\mathbb{N}}$.

1. At any stage, let A' be the finite prefix sequence of the $(a'_i)_{i\in\mathbb{N}}$ emitted so far. We also keep track of a precision parameter n, starting with $n := 1$.
2. In the first stage, we emit a_0 into A' (i.e. we set $a'_0 := a_0$)
3. Do the following iteration:
   a. Take the next element from $(a_i)_{i\in\mathbb{N}}$ and place it in an auxiliary set B, increment n
   b. For all elements $b \in B$, we can compute the number $\min_b = \min\{d(a,b) \mid a \in A'\} \in \mathbb{R}$ where A' is the finite sequence of a'_is emitted so far.
   c. For each $\min_b$, check (non-deterministically) in parallel: if $\min_b < 2^{-n}$ skip b, if $\min_b > 2^{-n-1}$ emit b, remove b from B, emit b as an a'_i (thus also suffix it on A'), repeat.
   d. If all elements in B were skipped, repeat.

The parallel test in 3(c) is a common trick in computable analysis. The relations by themselves are not decidable, but as at least one of them has to be true, we can wait until

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we recognize a true proposition. If there are multiple b such that $\min_b$ lies between 2^{-n-1} and 2^{-n}, then the choice is non-deterministic in the high level view of real numbers as inputs. If all codings and implementations are fixed, then the choice here is determined, too, though.

First, we shall argue that $(a'_i)_{i\in\mathbb{N}}$ is dense and repetition free. If $(a'_i)_{i\in\mathbb{N}}$ were not dense, then there would be some $m, k \in \mathbb{N}$ such that $\forall i \in \mathbb{N}\ d(a_m, a'_i) > 2^{-k}$. However, then once m has been placed into B and n incremented beyond $k+1$, m would have been chosen for A' – contradiction. The sequence $(a'_i)_{i\in\mathbb{N}}$ cannot have repetitions, because a duplicate element could never satisfy the test in 3(c).

In remains to show that $(a'_i)_{i\in\mathbb{N}}$ is computable in $(a_i)_{i\in\mathbb{N}}$ and vice versa. From Corollary 2.7 we would then know that $(M, d, (a_i)_{i\in\mathbb{N}}) \cong (M, d, (a'_i)_{i\in\mathbb{N}})$.

By construction $(a'_i)_{i\in\mathbb{N}}$ is computable in (M, d, A): Given $i \in \mathbb{N}$, just follow the construction above in order to identify which a_j is the ith element to be put into A', then we have $a'_i = a_j$.

Now to prove that $(a_i)_{i\in\mathbb{N}}$ is computable in $(M, d, (a'_i)_{i\in\mathbb{N}})$; i.e. that given some $i \in \mathbb{N}$ we can compute a sequence $(n_j)_{j\in\mathbb{N}}$ such that $d(a'_{n_j}, a_i) < 2^{-j}$. For this, we inspect the algorithm above beginning from the point when a_i is put into B. If a_i is moved into A' as the kth element to enter A', then $d(a'_k, a_i) = 0$, and we can continue the sequence $(n_j)_{j\in\mathbb{N}}$ as the constant sequence k. If a_i is not moved into A' in the jth round, then this is due to $\min_{a_i} < 2^{-j}$, and there must be some l such that a'_l witnesses this distance, i.e. $d(a_i, a'_l) < 2^{-j}$. Thus, continuing the sequence with $n_j := l$ works. □

In Gregoriades and Moschovakis (in preparation), it is proved that for every RPMS (X, d) there exists a recursive real $0 < \alpha < 1$ such that the metric $\alpha \cdot d$ takes values in $\mathbb{R} \setminus \mathbb{Q} \cup \{0\}$ on the dense sequence. This idea combined with Theorem 2.9 gives the following result.

Theorem 2.10. For every CMS $\mathbf{X} = (M, d, (a_i)_{i\in\mathbb{N}})$ there is a CMS $\mathbf{X}' = (M, \alpha d, (a'_i)_{i\in\mathbb{N}})$ with a computable real $\alpha \leqslant 1$ such that $\mathbf{X} \cong \mathbf{X}'$, and $\mathbf{X}'$ satisfies the criteria for a RPMS.

Proof. If $\mathbf{X}$ is finite, the result is straight-forward. If $\mathbf{X}$ is infinite, we may assume by Theorem 2.9 that $(a_i)_{i\in\mathbb{N}}$ is repetition-free. Let the following be computable bijections:

1. $\langle , \rangle_{-\Delta} : \big(\mathbb{N} \times \mathbb{N} \setminus \{(n, n) \mid n \in \mathbb{N}\}\big) \to \mathbb{N}$
2. $v^+_{\mathbb{Q}} : \mathbb{N} \to \{q \in \mathbb{Q} \mid q > 0\}$
3. $\langle , \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

Then, consider the computable sequence defined via $D_{\langle k, \langle i,j\rangle_{-\Delta}\rangle} = \frac{v^+_{\mathbb{Q}}(k)}{d(a_i, a_j)}$. We can diagonalize to find a computable real number α not in $(D_n)_{n\in\mathbb{N}}$ with $0.5 \leqslant \alpha \leqslant 1$. By choice of α, we find $\alpha d(a_i, a_j) \notin \mathbb{Q}$ for $i \neq j$, hence, the problematic case in the requirements for a RPMS becomes irrelevant. To compute the identity $\mathrm{id} : \mathbf{X} \to \mathbf{X}'$, one just needs to map a fast Cauchy sequence $(x_i)_{i\in\mathbb{N}}$ to $(x_{i+1})_{i\in\mathbb{N}}$ (as $\alpha \geqslant 0.5$), the identity in the other direction does not require any changes at all. □

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3. The induced computability structures

Having proved that computable- and RPMSs are inter-connected through a computable rescaling of the metric, it is natural to compare some of the basic objects derived from them. There are three fundamental types of objects in RPMSs: recursive- sets, functions and points, all of which have the corresponding analogue in CMSs. We will see that these concepts do coincide.

A comparison of the computability structure induced by recursive presentations and CMSs respectively is more illuminating in the framework of represented spaces. We recall some notions from Pauly (2012), and then prove some basic facts about them – these results are known and included here for completeness. A *represented space* is a pair $\mathbf{X} = (X, \delta_X)$ of a set X and a partial surjection $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. A multi-valued function between represented spaces is a multi-valued function between the underlying sets. For $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, we call F a realizer of f (notation $F \vdash f$), iff $\delta_Y(F(p)) \in f(\delta_X(p))$ for all $p \in \mathrm{dom}(f\delta_X)$. A map between represented spaces is called computable (continuous), iff it has a computable (continuous) realizer. Similarly, we call a point $x \in \mathbf{X}$ computable, iff there is some computable $p \in \mathbb{N}^{\mathbb{N}}$ with $\delta_{\mathbf{X}}(p) = x$. Any CMS induces a represented space via its Cauchy representation, and a function between CMSs is called computable, iff it is computable between the induced representations. Note that the realizer-induced notion of continuity coincides with ordinary metric continuity is a basic fact about admissible representation (Weihrauch 2000).

The category of represented spaces is cartesian closed, meaning we have access to a general function space construction as follows: Given two represented spaces $\mathbf{X}$, $\mathbf{Y}$ we obtain a third represented space $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ of functions from X to Y by letting $0^n 1p$ be a $[\delta_X \to \delta_Y]$-name for f, if the nth Turing machine equipped with the oracle p computes a realizer for f. As a consequence of the UTM theorem, $\mathcal{C}(-,-)$ is the exponential in the category of continuous maps between represented spaces, and the evaluation map is even computable.

Still drawing from Pauly (2012), we consider the Sierpiński space $\mathbb{S}$, which allows us to formalize semi-decidability. An explicit representation for this space is $\delta_{\mathbb{S}} : \mathbb{N}^{\mathbb{N}} \to \mathbb{S}$ where $\delta_{\mathbb{S}}(0^{\mathbb{N}}) = \bot$ and $\delta_{\mathbb{S}}(p) = \top$ for $p \neq 0^{\mathbb{N}}$. The computable functions $f : \mathbb{N} \to \mathbb{S}$ are exactly those where $f^{-1}(\{\top\})$ is recursively enumerable (and thus $f^{-1}(\{\bot\})$ co-recursively enumerable). In general, for any represented space $\mathbf{X}$ we obtain two spaces of subsets of $\mathbf{X}$; the space of open sets $\mathcal{O}(\mathbf{X})$ by identifying $f \in \mathcal{C}(\mathbf{X}, \mathbb{S})$ with $f^{-1}(\{\top\})$, and the space of closed sets $\mathcal{A}(\mathbf{X})$ by identifying $f \in \mathcal{C}(\mathbf{X}, \mathbb{S})$ with $f^{-1}(\{\bot\})$. In particular, the computable elements of $\mathcal{O}(\mathbb{N})$ are precisely the recursively enumerable sets. An explicit representation for $\mathcal{O}(\mathbb{N})$ is found in $\delta_{rng} : \mathbb{N}^{\mathbb{N}} \to \mathcal{O}(\mathbb{N})$ defined via $n \in \delta_{rng}(p)$ iff $\exists i \in \mathbb{N}\ p(i) = n + 1$.

The focus of computable analysis has traditionally been on the computable admissible spaces. Following Schröder (2002), we call a space $\mathbf{X}$ computably admissible, iff the canonic map $\kappa : \mathbf{X} \to \mathcal{O}(\mathcal{O}(\mathbf{X}))$ has a computable inverse. This essentially means that a point can be effectively recovered from its neighbourhood filter. The computably admissible spaces are those represented spaces that correspond to topological spaces.

We proceed to introduce the notion of an effective countable base. The effectively countably based computable admissible spaces are exactly the computable topological

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spaces studied by Weihrauch (e.g. Rettinger and Weihrauch 2013, Definition 3.1). The countably based admissible spaces that admit a total Baire space representation are the Quasi-Polish spaces (de Brecht 2013).

Definition 3.1. An effective countable base for $\mathbf{X}$ is a computable sequence $(U_i)_{i\in\mathbb{N}} \in \mathcal{C}(\mathbb{N}, \mathcal{O}(\mathbf{X}))$ such that the multivalued partial map Base $:\subseteq \mathbf{X} \times \mathcal{O}(\mathbf{X}) \rightrightarrows \mathbb{N}$ is computable. Here $\text{dom}(\text{Base}) = \{(x, U) \mid x \in U\}$ and $n \in \text{Base}(x, U)$ iff $x \in U_n \subseteq U$. Note that the requirement on Base also gives that $(U_i)_{i\in\mathbb{N}}$ forms a basis of $\mathbf{X}$.

Proposition 3.2 (([†])). Let $\mathbf{X} = (M, d, (a_i)_{i\in\mathbb{N}})$ be a CMS. Then, $B_{\langle i,j\rangle} = \{x \in M \mid d(x, a_i) < 2^{-j}\}$ provides an effective countable base for $\mathbf{X}$.

Proof. We start to prove that this is a computable sequence. By the definition of $\mathcal{O}$, it suffices to show that given x, i, j we can recognize $d(x, a_i) < 2^{-j}$. Let $\delta_M(p) = x$, i.e. $\forall k \ \ d(x, a_{p(k)}) < 2^{-k}$. Now $d(x, a_i) < 2^{-j}$ iff $\exists k \in \mathbb{N} \ \ d(a_{p(k)}, a_i) < 2^{-j} - 2^{-k}$. By the conditions on a CMS, the property is r.e., and existential quantification over an r.e. property still produces an r.e. property.

Next, we need to argue that Base is computable. Given some $x \in M$ and some open set $U \in \mathcal{O}(\mathbf{X})$ with $x \in U$, we do know by definition of $\mathcal{O}(\mathbf{X})$ that $x \in U$ will be recognized at some finite stage. Moreover, we can simulate the computation until this happens. At this point, only some finite prefix of the δ_M-name p of x has been read, say of length N. But then we must have $x \in \bigcap_{k\leqslant N} B_{\langle p(k),k\rangle} \subseteq U$. It is easy to verify that we can identify a particular ball inside the intersection still containing x. □

We now have the ingredients to give a more specific characterization of both $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ and $\mathcal{O}(\mathbf{X})$ for countably based spaces $\mathbf{X}$, $\mathbf{Y}$ and computably admissible $\mathbf{Y}$.

Proposition 3.3. Let $\mathbf{X} = (X, \delta_{\mathbf{X}})$ have an effective countable base $(U_i)_{i\in\mathbb{N}}$ and let $(p_n)_{n\in\mathbb{N}}$ be a computable sequence that is dense in $\text{dom}(\delta_{\mathbf{X}})$. Then, the map $\bigcup : \mathcal{O}(\mathbb{N}) \to \mathcal{O}(\mathbf{X})$ defined via $\bigcup(S) = \bigcup_{i\in S} U_i$ is computable and has a computable multivalued inverse.

Proof. That the map is computable follows from Pauly (2012, Proposition 4.2(4), Proposition 3.3(4)). For the inverse, fix some computable realizer of Base. Given some $U \in \mathcal{O}(\mathbf{X})$, test for any $n \in \mathbb{N}$ if $\delta_{\mathbf{X}}(p_n) \in U$. If this is confirmed, compute $m_n := \text{Base}(\delta_{\mathbf{X}}(p_n), U)$ and list it in $\bigcup^{-1}(U)$.

We will now argue that $\bigcup\bigcup^{-1}(U) = U$ with the algorithm described above. If $m \in \bigcup^{-1}(U)$, then by construction $U_m \subseteq U$, hence $\bigcup\bigcup^{-1}(U) \subseteq U$. On the other hand, let $x = \delta_{\mathbf{X}}(q) \in U$. The realizer for Base will choose some m_q on input q, U. As this happens after some finite time, there is some p_{i_q} so close to x that the realizer works in exactly the same way[†]. This ensures that m_q is listed in $\bigcup^{-1}(U)$, thus $x \in \bigcup\bigcup^{-1}(U)$. □

[†] As mentioned in the introduction to this section, this result is folklore. It has appeared e.g. as (Korovina and Kudinov 2008, Theorem 2.3).

[†] For this, it is important to fix one realizer of Base and to use the same name of U for all calls.

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Proposition 3.4. Let $\mathbf{X} = (X, \delta_{\mathbf{X}})$ be computably admissible and have an effective countable base $(U_i)_{i\in\mathbb{N}}$, and let $(p_n)_{n\in\mathbb{N}}$ be a computable sequence that is dense in $\mathrm{dom}(\delta_{\mathbf{X}})$. Then, $x \mapsto \{i \mid x \in U_i\} : \mathbf{X} \to \mathcal{O}(\mathbb{N})$ is a computable embedding.

Proof. That the map is computable is straight-forward. For the inverse, we shall first argue that $\{i \in \mathbb{N} \mid x \in U_i\} \mapsto \{U \in \mathcal{O}(\mathbf{X}) \mid x \in U\} : \mathcal{O}(\mathbb{N}) \to \mathcal{O}(\mathcal{O}(\mathbf{X}))$ is computable. By type-conversion, this is equivalent to $(\{i \in \mathbb{N} \mid x \in U_i\}, U) \mapsto (x \in U?) : \mathcal{O}(\mathbb{N})\times\mathcal{O}(\mathbf{X}) \to \mathbb{S}$. Here, we understand $(x \in U?) = \top$ if $x \in U$ and $(x \in U?) = \bot$ if $x \notin U$. By employing Proposition 3.3, this follows from $\in : \mathcal{O}(\mathbb{N}) \times \mathcal{O}(\mathcal{O}(\mathbb{N})) \to \mathbb{S}$ being computable.

Finally, we can compute x from $\{U \in \mathcal{O}(\mathbf{X}) \mid x \in U\}$ as $\mathbf{X}$ is computably admissible. □

As a consequence of Proposition 3.3, we see that for countably based spaces $\mathbf{X}$, we may conceive of open sets being given by enumeration of basic open sets exhausting them. For CMSs in particular, an open set is given by an enumeration of open balls with basic points as centers and radii of the form 2^{-i} (or equivalently, rational radii):

Definition 3.5 (Weihrauch 2000). [cf 4.1.2] Given a CMS $\mathbf{X}$, we define a numbering I for the open balls with basic centers and radii of the form 2^{-i} via $\mathrm{I}(\langle n,k\rangle) = B(a_n, 2^{-k})$. For convenience, we shall assume that $\mathrm{I}(0) = \emptyset$.

Definition 3.6 (Weihrauch 2000). [5.1.15.4] Given a CMS $\mathbf{X}$, we define the representation $\theta^{en}_{<} : \mathbb{N}^{\mathbb{N}} \to \mathcal{O}(\mathbf{X})$ by

$$\theta^{en}_{<}(p) := \bigcup_{n\in\mathbb{N}} \mathrm{I}(p(n))$$

which is intuitively a name consisting of the descriptions of open balls that exhaust the particular set (but not necessarily all of them).

An open $V \subseteq \mathbf{X}$ is *computably open* if $V = \theta^{en}_{<}(p)$ for some recursive $p \in \mathbb{N}^{\mathbb{N}}$.

The analogous notion in effective descriptive set theory is the following.

Definition 3.7 (Moschovakis 1980). [1B.1] Given a RPMS $\mathcal{X}$, a pointset $V \subseteq \mathcal{X}$ is semirecursive (or else Σ^0_1) if

$$V = \bigcup_n N(\mathcal{X}, \varepsilon(n)),$$

with some recursive $\varepsilon : \omega \to \omega$.

The following lemma is simple but useful tool.

Lemma 3.8. Suppose that $(\mathcal{X}, d, \mathbf{r})$ is a RPMS, which by Observation 2.4 is a CMS. Then, there exist computable functions

$$\tau : \mathbb{N} \to \mathbb{N}, \quad \sigma : \mathbb{N} \to \mathbb{N}^2,$$

such that

$$\mathrm{I}(w) = N(\mathcal{X}(\tau(w))), \quad \text{and} \quad N(\mathcal{X}, s) = \bigcup_n \mathrm{I}(\sigma(s, n)),$$

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for all w, n.

Proof. The existence of such a function τ follows easily since in the definitions we use computable encoding. Regarding σ, we first claim that

$$N(\mathcal{X}, s) = \cup_{j:\ r_j \in N(\mathcal{X},s)} \cup_{m:\ 2^{-m} < v_{\mathbb{Q}(k)} - d(r_i, r_j)} B(r_j, 2^{-m}),$$

where $s = 2^{i+1} \cdot 3^{k+1}$.

To see the latter, assume first that $x \in N(\mathcal{X}, s)$, where $s = 2^{i+1} \cdot 3^{k+1}$. Then, $d(x, r_i) < v_{\mathbb{Q}}(k)$. We choose m large enough such that $2^{-(m+1)} < v_{\mathbb{Q}}(k) - d(x, r_i)$, from which it follows that $2^{-m} < v_{\mathbb{Q}}(k) - d(x, r_i) - 2^{-m}$. Now, we consider some $r_j \in B(x, 2^{-m})$. Clearly, x belongs to $B(r_j, 2^{-m})$, and

$$d(r_i, r_j) \leqslant d(r_i, x) + d(x, r_j) < d(r_i, x) + 2^{-m} < d(r_i, x) + v_{\mathbb{Q}}(k) - d(x, r_i) - 2^{-m} = v_{\mathbb{Q}}(k) - 2^{-m},$$

hence $2^{-m} < v_{\mathbb{Q}(k)} - d(r_i, r_j)$. This also implies that $d(r_j, r_i) < v_{\mathbb{Q}}(k)$ and so $r_j \in N(\mathcal{X}, s)$. Hence, (j, m) is a suitable pair of naturals such that $x \in B(r_j, 2^{-m})$. Conversely if j, m are such that $2^{-m} < v_{\mathbb{Q}(k)} - d(r_i, r_j)$, and x is a member of $B(r_j, 2^{-m})$, then we have that

$$d(x, r_i) \leqslant d(x, r_j) + d(r_j, r_i) < 2^{-m} + d(r_j, r_i) < v_{\mathbb{Q}}(k),$$

and we have proved the preceding equality. Now, we consider some recursive function $w^* : \mathbb{N}^2 \to \mathbb{N}$ such that $B(r_j, 2^{-m}) = \mathrm{I}(w^*(j, m))$ and we define

$$\sigma(2^{i+1} \cdot 3^{k+1}, 2^{j+1} \cdot 3^{m+1}) = \begin{cases} w^*(j, m), & \text{if } 2^{-m} < v_{\mathbb{Q}(k)} - d(r_i, r_j), \\ 0, & \text{else.} \end{cases}$$

We also let $\sigma(s, n)$ be 0 if the naturals s, n do not have the form above. □

We are now ready to compare the notions of computably-open and semirecursive set.

Theorem 3.9. Suppose that $\mathcal{X}$ is a RPMS, $\mathbf{X} = (M, d, (a_i)_{i \in \mathbb{N}})$ is a CMS, and $\mathbf{X}' \cong \mathbf{X}$ is as in Theorem 2.10. Then,

1. For every $V \subseteq \mathcal{X}$,

$$V \text{ is semirecursive} \iff V \text{ is computably open},$$

(recall from Observation 2.4 that $\mathcal{X}$ is also a CMS).

2. For every $U \subseteq M$,

$$U \text{ is computably open in } \mathbf{X} \text{ (equivalently in } \mathbf{X}') \iff U \text{ is semirecursive in } \mathbf{X}',$$

(recall that $\mathbf{X}'$ is recursively presented).

In particular, the family of all semirecursive subsets of a RPMS is also the family of all computably open subsets of the latter space; and the family of all computably open subsets of a CMS is the family of all semirecursive subsets of a recursive presented metric space, which is $\cong$-equivalent to the original one.

Proof. The second assertion follows from the first one and the fact that the metric space $\mathbf{X}'$ is recursively presented, so let us prove the first assertion. Let $V \subseteq \mathcal{X}$ and assume that V is semirecursive. Then, $V = \cup_m N(\mathcal{X}, \varepsilon(m))$ for some recursive ε. From Lemma 3.8

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we have that

$$V = \bigcup_m N(\mathcal{X}, \varepsilon(m)) = \bigcup_{m,n} \mathrm{I}(\sigma(\varepsilon(m), n)),$$

and V is computably open from the closure properties of the latter class of sets, cf. (Pauly 2012, Proposition 6 (4)). The converse follows again from Lemma 3.8 by using the function τ and the closure properties of semirecursive sets, cf. Moschovakis (1980) 3C.1 (closure under $\exists^\omega$). □

We now shift our attention to computable/recursive functions.

Proposition 3.10. For two represented spaces $\mathbf{X}$, $\mathbf{Y}$ the map $f \mapsto \{(x, U) \mid f(x) \in U\} : \mathcal{C}(\mathbf{X}, \mathbf{Y}) \to \mathcal{O}(\mathbf{X} \times \mathcal{O}(\mathbf{Y}))$ is computable. If $\mathbf{Y}$ is computably admissible, then this map admits a computable inverse.

Proof. That $f \mapsto \{(x, U) \mid f(x) \in U\}$ is computable follows by combining computability of $f \mapsto f^{-1} : \mathcal{C}(\mathbf{X}, \mathbf{Y}) \to \mathcal{C}(\mathcal{O}(\mathbf{Y}), \mathcal{O}(\mathbf{X}))$ and computability of $\in : \mathbf{X} \times \mathcal{O}(\mathbf{X}) \to \mathbb{S}$ and using type conversion.

For the inverse direction, recall that for computably admissible $\mathbf{Y}$ the map $\{U \in \mathcal{O}(\mathbf{Y}) \mid y \in U\} \mapsto y :\subseteq \mathcal{O}(\mathcal{O}(\mathbf{Y})) \to \mathbf{Y}$ is computable. By computability of $\mathrm{Cut} : \mathbf{X} \times \mathcal{O}(\mathbf{X} \times \mathcal{O}(\mathbf{Y})) \to \mathcal{O}(\mathcal{O}(\mathbf{Y}))$ defined via $\mathrm{Cut}(x, V) = \{U \mid (x, U) \in V\}$ and composition, we find that $(x_0, \{(x, U) \mid f(x) \in U\}) \mapsto f(x_0) :\subseteq \mathbf{X} \times \mathcal{O}(\mathbf{X} \times \mathcal{O}(\mathbf{Y})) \to \mathbf{Y}$ is computable. Currying produces the claim. □

Corollary 3.11. Let $\mathbf{Y}$ be computably admissible. Then, $f : \mathbf{X} \to \mathbf{Y}$ is computable iff $\{(x, U) \mid f(x) \in U\} \in \mathcal{O}(\mathbf{X} \times \mathcal{O}(\mathbf{Y}))$ is computable.

Lemma 3.12. $U \mapsto \{(x, n) \in \mathbf{X} \times \mathbb{N} \mid \exists V \in \mathcal{O}(\mathbb{N})\ n \in V \wedge (x, V) \in U\} : \mathcal{O}(\mathbf{X} \times \mathcal{O}(\mathbb{N})) \to \mathcal{O}(\mathbf{X} \times \mathbb{N})$ is computable.

Proof. We start with $U \mapsto \{(V, x, n) \in \mathcal{O}(\mathbb{N} \times \mathbf{X} \times \mathbb{N} \mid n \in V \wedge (x, V) \in U\} : \mathcal{O}(\mathbf{X} \times \mathcal{O}(\mathbb{N})) \to \mathcal{O}(\mathcal{O}(\mathbb{N}) \times \mathbf{X} \times \mathbb{N})$. That this map is computable follows from open sets being effectively closed under products and intersection.

As there is a total representation $\delta_{\mathcal{O}(\mathbb{N})} : \mathbb{N}^{\mathbb{N}} \to \mathcal{O}(\mathbb{N})$, it follows that $\mathcal{O}(\mathbb{N})$ is computably overt (Pauly 2012, Proposition 19). By Pauly (2012, Proposition 40), the existential quantifier over an overt set is a computable map from open sets to open sets, thus the claim follows. □

A similar characterization of computability of functions is used in effective descriptive set theory:

Definition 3.13 (Moschovakis 1980). [3D] A function $f : \mathcal{X} \to \mathcal{Y}$ is recursive if and only if the *neighbourhood diagram* $G^f \subseteq \mathcal{X} \times \mathbb{N}$ of f defined by

$$G^f(x, s) \iff f(x) \in N(\mathcal{Y}, s),$$

is semirecursive.

Theorem 3.14. Suppose that $\mathcal{X}$, $\mathcal{Y}$ are RPMSs, $\mathbf{X} = (M^X, d^X, (a_i^X)_{i\in\mathbb{N}})$, $\mathbf{Y} = (M^Y, d^Y, (a_i^Y)_{i\in\mathbb{N}})$ are CMSs with $\mathbf{Y}$ being admissible, and $\mathbf{X}' \cong \mathbf{X}$, $\mathbf{Y}' \cong \mathbf{Y}$ are as in Theorem 2.10. Then:

1. For every $f : \mathcal{X} \to \mathcal{Y}$, f is recursive exactly when f is computable.
2. For every $g : M^X \to M^Y$, g is $(\mathbf{X},\mathbf{Y})$-computable (equivalently $(\mathbf{X}',\mathbf{Y}')$-computable) exactly when g is $(\mathbf{X}',\mathbf{Y}')$-recursive.

Proof. As before the second assertion follows from the first one and the fact that the metric spaces $\mathbf{X}'$, $\mathbf{Y}'$, are recursively presented. Regarding the first one, if G^f is semirecursive then it is computably open by Theorem 3.9. By Pauly (2012, Proposition 6(7)), the map $\mathrm{Cut} : \mathbf{X} \times \mathcal{O}(\mathbf{X} \times \mathbf{Y}) \to \mathcal{O}(\mathbf{Y})$ is computable. Thus, $x \mapsto \{n \in \mathbb{N} \mid (x, n) \in G^f\} : \mathcal{X} \to \mathcal{O}(\mathbb{N})$ is computable. Lemma 3.8 together with Proposition 3.4 show that we can compute $f(x)$ from $\{n \in \mathbb{N} \mid (x, n) \in G^f\}$. As the composition of computable functions is computable, we conclude that f is computable.

Conversely if f is computable then from Corollary 3.11 it follows that $\{(x, U) \mid f(x) \in U\} \in \mathcal{O}(\mathbf{X}\times\mathcal{O}(\mathbf{Y}))$ is computable. Using the characterization of the open sets in Proposition 3.3 together with Lemma 3.8 shows that $\{(x, V) \mid \exists n \in V\ f(x) \in N(\mathcal{Y}, n)\} \in \mathcal{O}(\mathbf{X}, \mathcal{O}(\mathbb{N}))$ is computable. Then Lemma 3.12 implies that G^f is computably open, and so from Theorem 3.9 we have that G^f is semirecursive, i.e., f is a recursive function. □

Finally we deal with points. A point x_0 in a CMS $\mathbf{X}$ is defined to be *computable* if it has a computable name, i.e. if it is the limit of a computable fast Cauchy sequence. On the other hand, point x_1 in a RPMS $\mathcal{X}$ is *recursive* if the set $\{s \in \mathbb{N} \mid x \in N(\mathcal{X}, s)\}$ is semirecursive, cf. the comments preceding 3D.7 (Moschovakis 1980).

Theorem 3.15. Suppose that $\mathcal{X}$ is a RPMS, $\mathbf{X} = (M, d, (a_i)_{i\in\mathbb{N}})$ is a CMS, and $\mathbf{X}' \cong \mathbf{X}$ is as in Theorem 2.10. Then:

1. For every $x \in \mathcal{X}$, x is recursive exactly when it is computable.
2. For every $x \in M$, x is $\mathbf{X}$-computable (equivalently $\mathbf{X}'$)-computable exactly when it is $\mathbf{X}'$-recursive.

Proof. By Proposition 3.4, a point in a CMS is computable iff $\{n \in \mathbb{N} \mid x \in \mathrm{I}(n)\}$ is computably open. Lemma 3.8 and Theorem 3.9 suffice to conclude the claim. □

4. On Cauchy-completions

As a digression, we shall explore the role Cauchy-completion plays in obtained effective versions of metric spaces. An effective version of Cauchy-completion underlies both the definition of CMSs and RPMSs. A crucial distinction, though classically vacuous, lies in the question whether spaces embed into their Cauchy-completion. Our goal in this section is to explore the variations upon effective Cauchy-completion, and to subsequently understand the origin of the discrepancy exhibited in Section 2.

Given a represented space $\mathbf{X}$ and some metric d on X, we define the space $\mathcal{S}_C^d(\mathbf{X}) \subseteq \mathcal{C}(\mathbb{N}, \mathbf{X})$ of fast Cauchy sequences by $(x_n)_{n\in\mathbb{N}} \in \mathcal{S}_C^d(\mathbf{X})$ iff $\forall i, j \geqslant N\ d(x_i, x_j) < 2^{-N}$. If $\mathbf{X}$ is

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complete, the map $\lim_C^d : \mathcal{S}_C^d(\mathbf{X}) \to \mathbf{X}$ is of natural interest (if $\mathbf{X}$ is not complete, we can still study $\lim_C^d$ as a partial map). In fact, it can characterize admissibility as follows:

Proposition 4.1. Let $\mathbf{X}$ admit a computable dense sequence. Let $d : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ be a computable metric, and let $\lim_C^d :\subseteq \mathcal{S}_C^d(\mathbf{X}) \to \mathbf{X}$ be computable. Then, $\mathbf{X}$ is computably admissible.

Proof. To show that $\mathbf{X}$ is computably admissible, we need to show that $\{U \in \mathcal{O}(\mathbf{X}) \mid x \in U\} \mapsto x :\subseteq \mathcal{O}(\mathcal{O}(\mathbf{X})) \to \mathbf{X}$ is computable. We search for some point a_1 such that $B(a_1, 2^{-2}) \in \{U \in \mathcal{O}(\mathbf{X}) \mid x \in U\}$. Then, we search for a_2 with $B(a_2, 2^{-3}) \in \{U \in \mathcal{O}(\mathbf{X}) \mid x \in U\}$ etc. These points form a fast Cauchy sequence converging to x. □

Proposition 4.2. Let $\mathbf{X} = (X, \delta_\mathbf{X})$ be computably admissible and let $\text{dom}(\delta_\mathbf{X})$ contain some computable dense sequence. Let $d : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ be a computable metric, and let $\{B(a_i, 2^{-n}) \mid i, n \in \mathbb{N}\}$ be an effective countable basis for $\mathbf{X}$. Then $\lim_C^d$ is computable.

Proof. We are given some fast Cauchy sequence $(x_i)_{i\in\mathbb{N}}$ converging to some x with $d(x, x_i) < 2^{-i}$ as input. As $x \in B(a_i, 2^{-n}) \Rightarrow x_n \in B(a_i, 2^{-n+1})$ and $x_n \in B(a_i, 2^{-n}) \Rightarrow x \in B(a_i, 2^{-n+1})$, we can compute $\{\langle i, n\rangle \mid x \in B(a_i, 2^{-n})\} \in \mathcal{O}(\mathbb{N})$. Then, we can invoke Proposition 3.4 to extract x. □

This characterization of CMSs in terms of fast Cauchy limits of course presupposes the represented space $\mathbb{R}$ with its canonical structure. In the beginnings of computable analysis, various non-standard representations of $\mathbb{R}$ have been investigated. We will investigate what happens to Cauchy completions, if some other represented space $\mathbf{R}$ (with again the reals as underlying set) is used in place of $\mathbb{R}$.

Definition 4.3. Let $\mathbf{X}$ be a represented space, such that the metric $d : \mathbf{X} \times \mathbf{X} \to \mathbf{R}$ is computable. We obtain its Cauchy-closure $\overline{\mathbf{X}}^{d,\mathbf{R}}$ by taking the usual quotient of $\mathcal{S}_C^d(\mathbf{X})$.

Observation 4.4. Any CMS $\mathbf{X}$ embeds† into its Cauchy-closure $\overline{\mathbf{X}}^{d,\mathbb{R}}$, and d can be extended canonically to $\overline{d} : \overline{\mathbf{X}}^{d,\mathbb{R}} \times \overline{\mathbf{X}}^{d,\mathbb{R}} \to \mathbb{R}$. Definition 2.2 reveals that a complete CMS is the Cauchy-closure of a countable metric space with continuous metric into $\mathbb{R}$.

Proof. The first part of the claim follows from Proposition 4.2 in conjunction with Proposition 3.2. The second part is essentially a reformulation of Observation 2.3. The third part is immediate from Definition 2.2. □

In order to find a contrasting picture of the RPMSs, we first introduce the represented space $\mathbb{R}_{cf}$. Informally, any real number is encoded by its decimal expansion, with infinite repetitions clearly marked†. This just ensures that $x \leqslant q$? and $x \geqslant q$? become both decidable for $x \in \mathbb{R}_{cf}$ and $q \in \mathbb{Q}$.

† A computable embedding $\mathbf{X} \hookrightarrow \mathbf{Y}$ is a computable injection $\iota : \mathbf{X} \to \mathbf{Y}$ such that the partial inverse ι^{-1} is computable, too.

† For example, the unique $\mathbb{R}_{cf}$-name of $\frac{1}{3}$ is $0.\overline{3}$. The number 1 has the names $0.\overline{9}$ and $1.\overline{0}$.

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Observation 4.5. The space $\mathbb{R}_{cf}$ does not embed into $\overline{\mathbb{R}_{cf}}^{d,\mathbb{R}_{cf}}$. Let $d : \mathbf{X} \times \mathbf{X} \to \mathbb{R}_{cf}$ be a computable metric. In general, $\overline{d} : \overline{\mathbf{X}}^{d,\mathbb{R}_{cf}} \times \overline{\mathbf{X}}^{d,\mathbb{R}_{cf}} \to \mathbb{R}_{cf}$ may fail to be computable. Definition 2.1 reveals that a RPMS is (essentially) the Cauchy-closure of a countable metric space with continuous metric into $\mathbb{R}_{cf}$.

Proof. The claims all follow from the observation that $\overline{\mathbb{R}_{cf}}^{d,\mathbb{R}_{cf}} \cong \mathbb{R} \not\cong \mathbb{R}_{cf}$. □

It is not the case, however, that the space $\mathbb{R}$ would be the only space usable in place of **R** when defining the Cauchy-closure to obtain an embedding of a space into its completion. One other example is $\mathbb{R}'$, the jump[‡] of $\mathbb{R}$.

5. Representations of point classes

With a correspondence of the spaces, the continuous/computable functions and the open sets in place, we shall conclude this paper by considering higher-order classes of sets (typically called pointclasses), such as $\underset{\sim}{\mathbf{\Sigma}}^0_n$-sets ($n > 1$), Borel sets or analytic sets. These have traditionally received little attention in the computable analysis community, with the exception of Brattka (2005) and Selivanov (2013). One reason for this presumably was the focus on admissible representations, i.e. spaces carrying a topology – and the natural representations of these classes of sets generally fail to be admissible. The ongoing development of synthetic descriptive set theory does provide representations of all the natural pointclasses.

In descriptive set theory the usual representation of pointclasses is through universal sets and good universal systems. Let $\mathbf{\Lambda}$ be a pointclass, and $\mathbf{Z}$, $\mathbf{X}$ two spaces[†]. For any $P \subseteq \mathbf{Z} \times \mathbf{X}$ and $z \in \mathbf{Z}$, we write $P_z := \{x \in \mathbf{X} \mid (z, x) \in P\}$. We write $\mathbf{\Lambda} \upharpoonright \mathbf{X}$ for all the $\mathbf{\Lambda}$-subsets of $\mathbf{X}$. Now we call $G \in \mathbf{\Lambda} \upharpoonright (\mathbf{Z} \times \mathbf{X})$ a $\mathbf{Z}$-universal set for $\mathbf{\Lambda}$ and $\mathbf{X}$ iff $\{G_z \mid z \in \mathbf{Z}\} = \mathbf{\Lambda} \upharpoonright \mathbf{X}$.

If $\mathbf{Z} = (Z, \delta_{\mathbf{Z}})$ is a represented space and G a $\mathbf{Z}$-universal set for $\mathbf{\Lambda}$ and $\mathbf{X}$, then we obtain a representation γ_G of $\mathbf{\Lambda} \upharpoonright \mathbf{X}$ via $\gamma_G(p) = G_{\delta_{\mathbf{Z}}(p)}$. In this situation, we can safely assume that $\mathbf{Z} \subseteq \mathbb{N}^{\mathbb{N}}$, and replace it by $(\mathrm{dom}(\delta_{\mathbf{Z}}), \mathrm{id}_{\mathrm{dom}(\delta_{\mathbf{Z}})})$ otherwise.

A $\mathbf{Z}$-universal system for $\mathbf{\Lambda}$ is an assignment $(G^{\mathbf{X}})_{\mathbf{X}}$ of a $\mathbf{Z}$-universal set for $\mathbf{\Lambda}$ and $\mathbf{X}$ for each Polish space $\mathbf{X}$. If $\mathbf{Z} = \mathbb{N}^{\mathbb{N}}$, we suppress the explicit reference to $\mathbf{Z}$. A universal system $(G^{\mathbf{X}})_{\mathbf{X}}$ is *good*, if for any space $\mathbf{Y}$ of the form $\mathbf{Y} = \mathbb{N}^l \times (\mathbb{N}^{\mathbb{N}})^k$ with $l, k \geqslant 0$ and any Polish space $\mathbf{X}$ there is a continuous function $S^{\mathbf{Y},\mathbf{X}} : \mathbb{N}^{\mathbb{N}} \times \mathbf{Y} \to \mathbb{N}^{\mathbb{N}}$ such that $(z, y, x) \in G^{\mathbf{Y}\times\mathbf{X}} \Leftrightarrow (S(z, y), x) \in G^{\mathbf{X}}$.

Comment. In this section, we consider the classical pointclasses e.g. $\underset{\sim}{\mathbf{\Sigma}}^0_n$ rather than the corresponding ones Σ^0_n in effective descriptive set theory. The classical pointclasses are also known as boldface pointclasses, because they typically arise from the effective (or else lightface) pointclasses through the process of 'boldification,' (see comments preceding

‡ The jump of a represented space is discussed in Ziegler (2007); Brattka et al. (2012b); Pauly and de Brecht (2013, 2015).

† Usually the spaces involved would be restricted to Polish spaces. However, the formalism is useful for us in a more general setting.

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3H.1 in Moschovakis (1980)). To be more precise for every pointclass Γ of sets in Polish spaces one defines the corresponding *boldface* pointclass $\underset{\sim}{\Gamma}$ as follows: a set $P \subseteq \mathcal{X}$, where $\mathcal{X}$ is Polish, belongs to $\underset{\sim}{\Gamma}$ if there exists some $Q \subseteq \mathbb{N}^{\mathbb{N}} \times \mathcal{X}$ in Γ and some $\varepsilon \in \mathbb{N}^{\mathbb{N}}$ such that P is the ε-section of Q,

$$P = \{x \in \mathcal{X} \mid (\varepsilon, x) \in Q\}.$$

It is well-known that the boldface pointclasses constructed by the lightface Σ^0_n, Σ^1_n are the classical Borel pointclasses $\underset{\sim}{\mathbf{\Sigma}}^0_n$ and $\underset{\sim}{\mathbf{\Sigma}}^1_n$ respectively. In fact, the sets belonging to the effective pointclasses Γ are all ε-sections of a set in Γ for some recursive $\varepsilon \in \mathbb{N}^{\mathbb{N}}$ (see 3H.1 in Moschovakis 1980), hence the effective notion can always be recovered by the classical one[‡].

Observation 5.1. Let γ_H be a representation of $\mathbf{\Lambda} \upharpoonright \mathbf{X}$ obtained from the universal set H. Further let $(G^{\mathbf{X}})_{\mathbf{X}}$ be a good universal system, and let γ_G be the induced representation of $\mathbf{\Lambda} \upharpoonright \mathbf{X}$. Then, $\mathrm{id} : (\mathbf{\Lambda} \upharpoonright \mathbf{X}, \gamma_H) \to (\mathbf{\Lambda} \upharpoonright \mathbf{X}, \gamma_G)$ is continuous[§].

Proof. By assumption, $H \in \mathbf{\Lambda} \upharpoonright (\mathbb{N}^{\mathbb{N}} \times \mathbf{X})$. Hence, there is some $h \in \mathbb{N}^{\mathbb{N}}$ such that $G_h^{\mathbb{N}^{\mathbb{N}} \times \mathbf{X}} = H$. Now $p \mapsto S^{\mathbb{N}^{\mathbb{N}}, \mathbf{X}}(h, p)$ is a continuous realizer of id. □

As such, we see that the representations obtained from good universal system for some fixed pointclass are the weakest one (w.r.t. continuous reducibilities) among those obtained from universal systems in general. Consequently, the particular choice of a good universal system can only ever matter for computability considerations, but not for continuity.

We can now contrast the approach to representations of pointclasses via good universal system with the approach via function spaces and Sierpiński -like spaces underlying (Pauly and de Brecht 2013, 2015). A Sierpiński -like space is a represented space $\mathbf{S}$ with underlying set $\{\top, \bot\}$ – no assumptions on the representation are made. Any such space $\mathbf{S}$ induces a pointclass $\mathcal{S}$ over the represented spaces via $U \in \mathcal{S} \upharpoonright \mathbf{X}$ iff $\chi_U : \mathbf{X} \to \mathbf{S}$ is continuous (computable), where $\chi_U(x) = \top$ iff $x \in U$. Note that this approach simultaneously provides for the effective and the classical version of $\mathcal{S}$. This pointclass comes with a represented space $\mathcal{S}(\mathbf{X})$ via the function space constructor $\mathcal{C}(-,-)$ and identification of a set and its characteristic function.

By the properties of the function space construction, we see that $\ni : \mathcal{S}(\mathbf{X}) \times \mathbf{X} \to \mathbf{S}$ is computable, which immediately implies that we may interpret $\ni$ as a $\mathcal{S}$-subset of $\mathcal{S}(\mathbf{X}) \times \mathbf{X}$. Thus, any representation of a fixed slice $\mathcal{S} \upharpoonright \mathbf{X}$ arises from some $\mathcal{S}(\mathbf{X})$-universal set. By moving along the representation, we may replace $\mathcal{S}(\mathbf{X})$ with some suitable $\mathbf{Z} \subseteq \mathbb{N}^{\mathbb{N}}$ here.

Next, we may relax the requirements for $\mathbf{Z}$-universal systems for $\mathbf{\Lambda}$ to allow $\mathbf{Z}$ to vary as $\mathbf{Z}_{\mathbf{X}}$ with the space $\mathbf{X}$, and will also let $\mathbf{X}$ range over all represented spaces, rather than just Polish spaces. The resulting notion shall be called a generalized universal system.

[‡] It is also worth pointing out that the effective hierarchy of lightface Σ^0_n pointclasses can be extended transfinitely to recursive ordinals ξ, but it is still not known if the corresponding boldface pointclass of Σ^0_ξ is actually the classical Borel pointclass $\underset{\sim}{\mathbf{\Sigma}}^0_\xi$. We nevertheless keep the latter notation for the classical Borel pointclasses with the danger of abusing the notation.

[§] This is continuity in the sense of represented spaces, generally not continuity in a topological setting.

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Such a system $(\mathbf{Z_X}, G^{\mathbf{X}})_{\mathbf{X}}$ is good, if for any represented spaces $\mathbf{Y}$, $\mathbf{X}$ there is a continuous function $S^{\mathbf{Y},\mathbf{X}} : \mathbf{Z}_{\mathbf{X}\times\mathbf{Y}} \times \mathbf{Y} \to \mathbf{Z_X}$ such that $(z, y, x) \in G^{\mathbf{Y}\times\mathbf{X}} \Leftrightarrow (S(z, y), x) \in G^{\mathbf{X}}$.

Observation 5.2. Let the generalized universal system $(\mathbf{Z_X}, G^{\mathbf{X}})_{\mathbf{X}}$ be obtained from the Sierpiński -like space $\mathbf{S}$. Then, it is good.

Proof. $S^{\mathbf{Y},\mathbf{X}} : \mathcal{C}(\mathbf{X} \times \mathbf{Y}, \mathbf{S}) \times \mathbf{Y} \to \mathcal{C}(\mathbf{X}, \mathbf{S})$ is realized via partial function application. □

For most natural choices of a Sierpiński -like space $\mathbf{S}$, we may actually replace the occurrence of $\mathcal{C}(\mathbf{X}, \mathbf{S})$ in the induced generalized universal system by $\mathbb{N}^{\mathbb{N}}$ again, thus closing the distance between the two approaches. We recall from Kreitz and Weihrauch (1985) that a representation $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbf{X}$ is called precomplete, if for any computable partial $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ there is a computable total $\overline{F} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\delta \circ F(p) = \delta \circ \overline{F}(p)$ for all $p \in \operatorname{dom}(\delta \circ F(p))$. Now note that if $\mathbf{S}$ admits a precomplete representation, then $\mathcal{C}(\mathbf{X}, \mathbf{S})$ admits a total representation for any $\mathbf{X}$. Subsequently, we note:

Observation 5.3. Let the Sierpiński -like space $\mathbf{S}$ admit a precomplete representation. Then, it induces a pointclass $\mathcal{S}$ together with a good universal system.

We will now explore which pointclasses on Polish spaces are obtainable from some Sierpiński -like space. First, note that any such class $\mathcal{S}$ is closed under taking preimages under continuous functions. Then, for any Polish space $\mathbf{X}$ and total representation $\delta : \mathbb{N}^{\mathbb{N}} \to \mathbf{X}$, we observe that $A \in \mathcal{S} \upharpoonright \mathbf{X}$ iff $\delta^{-1}(A) \in \mathcal{S} \upharpoonright \mathbb{N}^{\mathbb{N}}$. Generally, we shall call any pointclass satisfying this property for all total admissible representations of Polish spaces to be $\mathbb{N}^{\mathbb{N}}$-*determined*.

Proposition 5.4. Let $\mathbf{\Lambda}$ be $\mathbb{N}^{\mathbb{N}}$-determined, closed under continuous preimages and admit a good universal system. Then, there is some Sierpiński -like space $\mathbf{S}$ with $\mathbf{\Lambda} = \mathcal{S}$.

Proof. Let $G \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ be a universal set for $\mathbf{\Lambda}$ and $\mathbb{N}^{\mathbb{N}}$. We define a representation $\delta_G : \mathbb{N}^{\mathbb{N}} \to \{\top, \bot\}$ via $\delta_G(\langle p, q\rangle) = \top$ iff $(p, q) \in G$. Let the resulting space be $\mathbf{S}$. We claim that the pointclass induced by $\mathbf{S}$ coincides with $\mathbf{\Lambda}$.

Let $A \in \mathbf{\Lambda} \upharpoonright \mathbf{X}$. Then $\delta_{\mathbf{X}}^{-1}(A) \in \mathbf{\Lambda} \upharpoonright \mathbb{N}^{\mathbb{N}}$. Thus, there is some $a \in \mathbb{N}^{\mathbb{N}}$ with $q \in A \Leftrightarrow (a, q) \in G$. Now $q \mapsto \langle a, q\rangle$ is a continuous realizer of $\chi_A : \mathbf{X} \to \mathbf{S}$.

Conversely, assume $\chi_A \in \mathcal{C}(\mathbf{X}, \mathbf{S})$. Let $c_A : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a continuous realizer of χ_A. Note $((\pi_1, \pi_2) \circ c_A)^{-1}(G) = \delta_{\mathbf{X}}^{-1}(A)$. The left hand side of this equation shows that the set is in $\mathbf{\Lambda}$, as $\mathbf{\Lambda}$ is closed under continuous preimages. The right hand side then implies that $A \in \mathbf{\Lambda} \upharpoonright \mathbf{X}$, as $\mathbf{\Lambda}$ is $\mathbb{N}^{\mathbb{N}}$-determined. □

Proposition 5.5. Let $\mathbf{\Lambda}$ be a pointclass.

1. If $\mathbf{\Lambda}$ is $\mathbb{N}^{\mathbb{N}}$-determined, then so are $\mathbf{\Lambda}^{\mathfrak{C}}$ and $\mathbf{\Lambda} \cap \mathbf{\Lambda}^{\mathfrak{C}}$; where $\mathbf{\Lambda}^{\mathfrak{C}} := \{A^C \mid A \in \mathbf{\Lambda}\}$.
2. For countable ordinals α, $\utilde{\mathbf{\Sigma}}^0_\alpha$ is $\mathbb{N}^{\mathbb{N}}$-determined.
3. $\utilde{\mathbf{\Sigma}}^1_n$ is $\mathbb{N}^{\mathbb{N}}$-determined, $n \geqslant 1$.

Proof.

1. Just observe that $\delta^{-1}(A^C) = (\delta^{-1}(A))^C$.
2. This is a result FROM Saint Raymond (2007) (cf. de Brecht and Yamamoto 2009)

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3. Let $\mathbf{X}$ be a Polish space and $A \subseteq \mathbf{X}$ be $\undertilde{\mathbf{\Sigma}}^1_n$. Since the continuous preimage of a $\undertilde{\mathbf{\Sigma}}^1_n$ set is $\undertilde{\mathbf{\Sigma}}^1_n$, and δ is continuous it follows that $\delta^{-1}[A] \in \undertilde{\mathbf{\Sigma}}^1_n \upharpoonright \mathbb{N}^{\mathbb{N}}$. Conversely using that as a representation, δ is surjective, we have that $A = \delta[\delta^{-1}[A]]$. So if $\delta^{-1}[A]$ is a $\undertilde{\mathbf{\Sigma}}^1_n$ subset of $\mathbb{N}^{\mathbb{N}}$ it follows from the closure of $\undertilde{\mathbf{\Sigma}}^1_n$ under continuous images that $A \in \undertilde{\mathbf{\Sigma}}^1_n \upharpoonright \mathbf{X}$.

□

Conclusion: The approaches to continuity and computability for $\undertilde{\mathbf{\Sigma}}^0_\alpha$ and $\undertilde{\mathbf{\Sigma}}^1_1$ from effective descriptive set theory and synthetic descriptive set theory coincide.

A very important pointclass not yet proven to receive equivalent treatment are the Borel sets $\mathcal{B}$, alternatively $\undertilde{\mathbf{\Delta}}^1_1$ by Suslin's theorem (e.g. Moschovakis 2010). There cannot be any $\mathbb{N}^{\mathbb{N}}$-universal Borel sets† – however, there are $\mathbf{B}$-universal sets for $\undertilde{\mathbf{\Delta}}^1_1$, with non-Polish $\mathbf{B}$. Such a set can be obtained from the Borel codes used in effective descriptive set theory. We currently cannot prove uniform equivalence of the two approaches for Borel sets on arbitrary Polish spaces, as this would require a uniform version of Saint Raymond's result in Saint Raymond (2007). For our purposes, this result is that if δ is a standard representation of a computable Polish space $\mathbf{X}$, and A is a Borel subset of $\mathbb{N}^{\mathbb{N}}$, then $\delta[A]$ is a Borel subset of $\mathbf{X}$. A uniform version would allow us to compute a Borel code for $\delta[A]$ from a Borel code for A. Thus, we first provide a non-uniform treatment of Borel sets on arbitrary Polish spaces, and then a uniform treatment of Borel subsets of $\mathbb{N}^{\mathbb{N}}$.

Definition 5.6 (Moschovakis 1980, 3H). The set of *Borel codes* $\mathrm{BC} \subseteq \mathbb{N}^{\mathbb{N}}$ is defined by recursion as follows:

$$p \in \mathrm{BC}_0 \iff p(0) = 0$$
$$p \in \mathrm{BC}_\alpha \iff p = 1\langle p_0, p_1, \ldots, \rangle \;\&\; (\forall n)(\exists \beta < \alpha)[p_n \in \mathrm{BC}_\beta]$$
$$\mathrm{BC} = \cup_\alpha \mathrm{BC}_\alpha \quad \text{for all countable ordinals } \alpha.$$

With an easy induction one can see that $\mathrm{BC}_\alpha \subseteq \mathrm{BC}_\beta$ for all $\alpha < \beta$ and that BC_α is a Borel set.

For all $p \in \mathrm{BC}$ we denote by $|p|$ the least ordinal α such that $p \in \mathrm{BC}_\alpha$ (This essentially provides a representation of the space of all countable ordinals. This idea is investigated in some detail in Pauly 2015a,b). It is not hard to verify that

$$|1\langle p_0, p_1, \ldots \rangle| = \sup_{n \in \mathbb{N}} |p_n| + 1,$$

Let $\mathbf{X}$ be a Polish space, and $\delta_{\mathcal{O}} : \mathbb{N}^{\mathbb{N}} \to \mathcal{O}(\mathbf{X})$ a standard representation of its open sets. For some subset $A \subseteq \mathbf{X}$, let A^C denote its complement $\mathbf{X} \setminus A$. For all countable ordinals α

† Any such set would fall into $\undertilde{\mathbf{\Sigma}}^0_\alpha$ for some countable ordinal α, but then cannot have any set $A \in \undertilde{\mathbf{\Sigma}}^0_{\alpha+1} \setminus \undertilde{\mathbf{\Sigma}}^0_\alpha$ as a section.

we define the function $\pi^{\mathbf{X}}_\alpha : \mathrm{BC}_\alpha \to \mathcal{B} \upharpoonright \mathbf{X}$ recursively by

$$\pi^{\mathbf{X}}_0(0p) = \delta_{\mathcal{O}}(p),$$
$$\pi^{\mathbf{X}}_\alpha(1\langle p_0, p_1, \ldots\rangle) = \pi^{\mathbf{X}}_{|\bigcup_n p_n|}\left(\bigcup_n p_n\right)^C.$$

An easy induction shows that the function $\pi^{\mathbf{X}}_\alpha$ is onto $\mathbf{\Sigma}^0_\alpha \upharpoonright \mathbf{X}$, and that $\pi^{\mathbf{X}}_\beta \upharpoonright \mathrm{BC}_\alpha = \pi^{\mathbf{X}}_\alpha$ for all $\alpha < \beta$. So one can define the *Borel coding* $\pi^{\mathbf{X}} : \mathrm{BC} \to \mathcal{B} \upharpoonright \mathbf{X}$ by

$$\pi^{\mathbf{X}}(p) = \pi^{\mathbf{X}}_{|p|}(p).$$

so that the family $\mathbf{\Sigma}^0_\alpha \upharpoonright \mathbf{X}$ is exactly the family of all $\pi^{\mathbf{X}}(p)$ for $p \in \mathrm{BC}_\alpha$, in particular a set $A \subseteq \mathbf{X}$ is Borel exactly when $A = \pi^{\mathbf{X}}(p)$ for some $p \in \mathrm{BC}$.

The following are more or less well-known facts in descriptive set theory:

Lemma 5.7.

1. For all countable ordinals α the set $\{p \in \mathrm{BC} \mid |p| \leqslant \alpha\}$ is Borel.

   Proof. This is because $\{p \in \mathrm{BC} \mid |p| \leqslant \alpha\} = \mathrm{BC}_\alpha$. □

2. The set BC is a Π^1_1 subset of $\mathbb{N}^{\mathbb{N}}$ and so in particular it is a $\mathbf{\Pi}^1_1$ set.

   Proof. The latter is a consequence of 7C.8 in Moschovakis (1980), since one can see that the set BC is the least fixed point of a suitably chosen monotone operation. □

3. There exists a $\mathbf{\Sigma}^1_1$ relation $\leqslant_\Sigma \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that for all $p \in \mathrm{BC}$ and all $q \in \mathbb{N}^{\mathbb{N}}$ we have that

   $$[q \in \mathrm{BC} \;\&\; |q| \leqslant |p|] \iff q \leqslant_\Sigma p.$$

   Proof. Note that $|1\langle q_0, q_1, \ldots\rangle| \leqslant |1\langle p_0, p_1, \ldots\rangle|$ iff $\exists t \in \mathbb{N}^{\mathbb{N}}$ s.t. $\forall n \in \mathbb{N}\ |q_n| \leqslant |p_{t(n)}|$, assuming $q_i, p_i \in \mathrm{BC}$. Building upon this idea, consider the closed relation R defined as the least fixed point of:

   $$R(p, q, \langle t', \langle t_0, t_1, \ldots\rangle\rangle) :\Leftrightarrow q(0) = 0 \vee \big(p = 1\langle p_0, p_1, \ldots\rangle \wedge q = 1\langle q_0, q_1, \ldots\rangle \wedge \forall n \in \mathbb{N}\ R(p_n, q_{t'(n)}, t_n)\big)$$

   Now $q \leqslant_\Sigma p :\Leftrightarrow \exists t \in \mathbb{N}^{\mathbb{N}}\ R(p, q, t)$ is a $\mathbf{\Sigma}^1_1$ relation, and satisfies our criterion. □

4. The set BC is not a Borel subset of $\mathbb{N}^{\mathbb{N}}$.

   Proof. We will show that if BC were Borel, then the set of well-founded trees would be analytic, which is a contradiction (as shown e.g. in Bruckner et al. 1997, Section 11.8).
   Note that a tree T is well-founded iff there exists an assignment $P : T \to \mathrm{BC}$ such that for all $u, v \in T$ if v extends u then $|P(v)| < |P(u)|$.
   This is easy to see: If T is well-founded then we use bar recursion to get P such that $|P(u)| = \sup |(un)| + 1$. Conversely if P is such an assignment and T contained an infinite branch then we would get a strictly decreasing sequence of ordinals, a contradiction.

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Now condition $|P(v)| < |P(u)|$ can be replaced by $S(P(v)) \leqslant_\Sigma P(u)$, with $\leqslant_\Sigma$ as above, and S is a continuous function such that $|S(q)| = |q| + 1$. Thus, we have

$$\begin{aligned} &T \text{ is well-founded } \Leftrightarrow \\ &\quad \exists P \quad \forall u, v \in \mathbb{N}^*. u \in T \Rightarrow P(u) \in \mathrm{BC} \text{ and } v \text{ extends } u \Rightarrow \ S(P(v)) \leqslant_\Sigma P(u). \end{aligned}$$

The preceding P varies through the set of all functions from $\mathbb{N}^*$ to $\mathbb{N}^\mathbb{N}$, and the latter set is homeomorphic to $\mathbb{N}^\mathbb{N}$. If the set BC were Borel, then the right-hand side of the preceding equivalence would define a $\underset{\sim}{\Sigma}^1_1$ set, and hence the set of all well-founded trees would be $\underset{\sim}{\Sigma}^1_1$, a contradiction. □

5. Let $f : \mathbb{N}^\mathbb{N} \to \mathrm{BC}$ be Borel measurable. Then, there is a countable ordinal α such that $\forall p \in \mathbb{N}^\mathbb{N}\ |f(p)| \leqslant \alpha$.

Proof. If this were not the case we would have that

$$q \in \mathrm{BC} \iff (\exists p)[q \leqslant_\Sigma f(p)],$$

where $\leqslant_\Sigma$ is as above. Since f is Borel measurable the preceding equivalence would imply that the set BC is a $\underset{\sim}{\Sigma}^1_1$ subset of $\mathbb{N}^\mathbb{N}$. Hence from the Suslin Theorem, it would follow that BC is a Borel set, a contradiction and our claim is proved. □

Definition 5.8. We define the Sierpiński -like space $\mathbf{S}_\mathcal{B} = (\{\bot, \top\}, \delta_\mathcal{B})$ recursively via

$$\begin{aligned} \delta_\mathcal{B}(p) \text{ is defined } &\iff p \in \mathrm{BC} \\ \delta_\mathcal{B}(0p) &= \delta_\mathbb{S}(p) \\ \delta_\mathcal{B}(1\langle p_0, p_1, \dots \rangle) &= \bigvee_{i \in \mathbb{N}} \neg \delta_\mathcal{B}(p_i). \end{aligned}$$

Note that by construction of $\mathbf{S}_\mathcal{B}$, we find that $\in : \mathbb{N}^\mathbb{N} \times \mathcal{B} \to \mathbf{S}_\mathcal{B}$ is computable.

Proposition 5.9. Fix a Polish space $\mathbf{X}$. For $A \subseteq \mathbf{X}$ we find the following to be equivalent:

1. $A \in \mathcal{B} \upharpoonright \mathbf{X}$
2. $\chi_A : \mathbf{X} \to \mathbb{S}_\mathcal{B}$ is continuous.
3. $\chi_A : \mathbf{X} \to \mathbb{S}_\mathcal{B}$ is Borel measurable.

Proof.

1. $\Rightarrow$ 2.Fix a total admissible representation $\delta_\mathbf{X} : \mathbb{N}^\mathbb{N} \to \mathbf{X}$. Let us assume that $A \in \mathcal{B} \upharpoonright \mathbf{X}$. Then $\delta_\mathbf{X}^{-1}(A) \in \mathcal{B} \upharpoonright \mathbb{N}^\mathbb{N}$. If a is a Borel code for $\delta_\mathbf{X}^{-1}(A)$, then $q \mapsto \in(q, a)$ is a continuous realizer for $\chi_A : \mathbf{X} \to \mathbb{S}_\mathcal{B}$.

2. $\Rightarrow$ 3.Trivial.

3. $\Rightarrow$ 1.Now, let us assume that $\chi_A : \mathbf{X} \to \mathbf{S}_\mathcal{B}$ is Borel measurable. Let $c_A : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be a Borel measurable realizer of χ_A. We remark that $c_A(p) \in \mathrm{BC}$ for all $p \in \mathbb{N}^\mathbb{N}$. Consider now some countable ordinal α_A such that $|c_A(p)| < \alpha_A$ for all $p \in \mathbb{N}^\mathbb{N}$, which we may obtain from Lemma 5.7 (5). The set $S_{\alpha_A} := \{p \in \mathbb{N}^\mathbb{N} \mid \delta_\mathcal{B}(p) = \top \wedge |p| \leqslant \alpha_A\}$ is a Borel subset of $\mathbb{N}^\mathbb{N}$. Then, $c_A^{-1}(S_{\alpha_A}) = \delta_\mathbf{X}^{-1}(A)$ is Borel as well and hence it is $\underset{\sim}{\Sigma}^0_{\beta_A}$ for some countable ordinal β_A. By Proposition 5.5 (2), we find that $A \in \underset{\sim}{\Sigma}^0_{\beta_A} \upharpoonright \mathbf{X}$, in particular, A is Borel. □

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As announced above, we will proceed to show that for Baire space the representation of $\mathcal{B}$ via Borel codes is computably equivalent to the representation via the function space into $\mathbf{S}_{\mathcal{B}}$. In this, we will consider the Borel codes to be the default representation of $\mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}}$. A new ingredient of the proof will be:

Lemma 5.10. The operation $r :\subseteq \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}} \to \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}}$ with $\mathrm{dom}(r) = \{A \in \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}} \mid A \subseteq \mathrm{BC}\}$ and $r(A) = \{p \in A \mid \delta_{\mathcal{B}}(p) = \top\}$ is well-defined and computable.

Proof. We start by providing Σ^1_1-sets T and B, such that $\delta_{\mathcal{B}}(p) = \top \Leftrightarrow p \in \mathrm{BC} \cap T$ and $\delta_{\mathcal{B}}(p) = \bot \Leftrightarrow p \in \mathrm{BC} \cap B$. This is done by constructing two Π^0_1-sets $P, Q \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ via an interleaving fixed point construction. (The reader coming from a computable analysis background may prefer to see the following as instructions for a dove-tailing programme trying to disprove $(p,q) \in P$ or $(p,q) \in Q$ by unravelling the instructions. If ever one of the first two cases is reached and yields a negative answer, this is propagated back and disproves the original membership query. It is perfectly fine to have ill-founded computation paths, these can never yield contradictions and thus may cause queries to fall in P or Q where the first parameter is not a Borel code.)

$$\begin{aligned}(0p, q) \in P &:\Leftrightarrow p = 0^{\mathbb{N}}\\ (0p, nq) \in Q &:\Leftrightarrow p(n) = 1\\ (1\langle p_0, p_1, \ldots\rangle, \langle q_0, q_1, \ldots\rangle) \in P &:\Leftrightarrow \forall n \in \mathbb{N}\ (p_n, q_n) \in Q\\ (1\langle p_0, p_1, \ldots\rangle, nq) \in Q &:\Leftrightarrow (p_n, q) \in P,\end{aligned}$$

Now $p \in T :\Leftrightarrow \exists q \in \mathbb{N}^{\mathbb{N}}\ (p,q) \in Q$ and $p \in B :\Leftrightarrow \exists q \in \mathbb{N}^{\mathbb{N}}\ (p,q) \in P$ are our desired sets.

Given $A \in \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}}$ we can compute $A \cap T \in \underset{\sim}{\mathbf{\Sigma}}^1_1 \upharpoonright \mathbb{N}^{\mathbb{N}}$ and $A^C \cup B \in \underset{\sim}{\mathbf{\Sigma}}^1_1 \upharpoonright \mathbb{N}^{\mathbb{N}}$, and note that $A \subseteq \mathrm{BC}$ implies $(A \cap T)^C = A^C \cup B$, so by applying the effective Suslin theorem (Moschovakis 2010) we can obtain $r(A) = A \cap T \in \mathcal{B}$. □

Theorem 5.11. The map $A \mapsto \chi_A : \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}} \to \mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbf{S}_{\mathcal{B}})$ is a computable isomorphism.

Proof. That this map is computable follows by currying from the computability of $\in\ : \mathbb{N}^{\mathbb{N}} \times \mathcal{B} \to \mathbf{S}_{\mathcal{B}}$; that it is a bijection from Proposition 5.9. It only remains to prove that its inverse is computable, too.

Given $\chi_A \in \mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbf{S}_{\mathcal{B}})$, we can compute the $\underset{\sim}{\mathbf{\Sigma}}^1_1$ set $\chi_A[\mathbb{N}^{\mathbb{N}}]$. Then, we use the effective Suslin theorem (Moschovakis 2010) on $\chi_A[\mathbb{N}^{\mathbb{N}}]$ and the Σ^1_1-set BC^C to obtain some $B \in \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}}$ with $\chi_A[\mathbb{N}^{\mathbb{N}}] \subseteq B \subsetneq \mathrm{BC}$. Using the computable map r from Lemma 5.10 we can then obtain $A \in \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}}$ as $A = \chi_A^{-1}(r[B])$. □

Theorem 5.11 allows us to conclude some effective closure properties of $\mathcal{B}$ either directly, or using some basic properties of $\mathbf{S}_{\mathcal{B}}$. We start with the latter:

Proposition 5.12. The following maps are computable:

1. $\neg : \mathbf{S}_{\mathcal{B}} \to \mathbf{S}_{\mathcal{B}}$
2. $\wedge, \vee : \mathbf{S}_{\mathcal{B}} \times \mathbf{S}_{\mathcal{B}} C \to \mathbf{S}_{\mathcal{B}}$
3. $\bigwedge, \bigvee : \mathcal{C}(\mathbb{N}, \mathbf{S}_{\mathcal{B}}) \to \mathbf{S}_{\mathcal{B}}$.

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Proof.

1. This is realized by $p \mapsto \langle 1, p, p, p, \ldots \rangle$.
2. This follows from (3.).
3. $(p_i)_{i\in\mathbb{N}} \mapsto \langle 1, \langle 1, p_0, p_1, \ldots \rangle, \langle 1, p_0, p_1, \ldots \rangle, \ldots \rangle$ realizes $\bigwedge$. Computability of $\bigvee$ follows using de Morgan's law and (1.).

□

Corollary 5.13. The following maps are computable:

1. $(f, U) \mapsto f^{-1}(U) : \mathcal{C}(\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}) \times \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}} \to \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}}$
2. $U \mapsto U^C : \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}} \to \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}}$
3. $\cap, \cup : \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}} \times \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}} \to \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}}$
4. $\bigcap, \bigcup : \mathcal{C}(\mathbb{N}, \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}}) \to \mathcal{B} \upharpoonright \mathbb{N}^{\mathbb{N}}$.

Proof. Use the characterization of $\mathcal{B}$ given by Theorem 5.11. The first map is realized by function composition, the remaining by composing with the appropriate map from Proposition 5.12. □

While proving the results of Corollary 5.13 directly would not have been particularly cumbersome either, the present approach immediate generalizes to all represented spaces. In analogy to Theorem 5.11, we could *define* the space $\mathcal{B}_{\mathbf{X}}$ of Borel subsets of some represented space $\mathbf{X}$ by identifying $U \subseteq \mathbf{X}$ with (continuous) $\chi_U : \mathbf{X} \to \mathbf{S}_{\mathcal{B}}$. Corollary 5.13 then immediately shows that $\mathcal{B}_{\mathbf{X}}$ has the expected effective closure properties.

6. Conclusions

We have demonstrated that the computability notions used in computable analysis (and synthetic descriptive set theory) and effective descriptive set theory respectively coincide for objects in the scope of both. When it comes to metric spaces, the scope of effective descriptive set theory is more restrictive, however, the difference disappears modulo a rescaling of the metric. While the requirements for pointclasses to be treatable in the two frameworks differ significantly, the computability notions for $\utilde{\mathbf{\Sigma}}^0_\alpha$, $\utilde{\mathbf{\Pi}}^0_\alpha$, $\utilde{\mathbf{\Sigma}}^1_1$ and $\utilde{\mathbf{\Pi}}^1_1$ coincide for Polish spaces, and $\mathcal{B}$ ($\utilde{\mathbf{\Delta}}^1_1$) is the same in both frameworks for Baire space.

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