J. Fluid Mech. (2006), *vol.* 549, *pp.* 191–214. © 2006 Cambridge University Press
doi:10.1017/S0022112005007883 Printed in the United Kingdom

A turbulent constitutive law for the two-dimensional inverse energy cascade

By GREGORY L. EYINK

Department of Applied Mathematics & Statistics, The Johns Hopkins University, Baltimore, MD 21218, USA

(Received 12 April 2005 and in revised form 3 August 2005)

The inverse energy cascade of two-dimensional turbulence is often represented phenomenologically by a Newtonian stress–strain relation with a 'negative eddy viscosity'. Here we develop a fundamental approach to a turbulent constitutive law for the two-dimensional inverse cascade, based upon a convergent multi-scale gradient (MSG) expansion. To first order in gradients, we find that the turbulent stress generated by small-scale eddies is proportional not to strain but instead to 'skew-strain,' i.e. the strain tensor rotated by 45°. The skew-strain from a given scale of motion makes no contribution to energy flux across eddies at that scale, so that the inverse cascade cannot be strongly scale-local. We show that this conclusion extends a result of Kraichnan for spectral transfer and is due to absence of vortex stretching in two dimensions. This 'weakly local' mechanism of inverse cascade requires a relative rotation between the principal directions of strain at different scales and we argue for this using both the dynamical equations of motion and also a heuristic model of 'thinning' of small-scale vortices by an imposed large-scale strain. Carrying out our expansion to second order in gradients, we find two additional terms in the stress that can contribute to the energy cascade. The first is a Newtonian stress with an 'eddy-viscosity' due to differential strain rotation, and the second is a tensile stress exerted along vorticity contour lines. The latter was anticipated by Kraichnan for a very special model situation of small-scale vortex wave-packets in a uniform strain field. We prove a proportionality in two dimensions between the mean rates of differential strain rotation and of vorticity-gradient stretching, analogous to a similar relation of Betchov for three dimensions. According to this result, the second-order stresses will also contribute to inverse cascade when, as is plausible, vorticity contour lines lengthen, on average, by turbulent advection.

1. Introduction

Almost forty years ago, Kraichnan (1967) predicted an inverse cascade of energy in two-dimensional incompressible fluid turbulence. This is perhaps one of the most intriguing turbulent phenomena to occur in classical fluids. Kraichnan proposed an inertial range with a $k^{-5/3}$ power-law energy spectrum, just as in three dimensions, but with a flux of energy from small scales to large scales rather than the reverse. Kraichnan's detailed predictions for steady-state forced two-dimensional turbulence have been confirmed with increasing precision in a series of numerical simulations (Lilly 1971, 1972; Fyfe, Montgomery & Joyce 1977; Siggia & Aref 1981; Hossain, Matthaeus & Montgomery 1983; Frisch & Sulem 1984; Herring & McWilliams 1985; Maltrud & Vallis 1991; Boffetta, Celani & Vergassola 2000) and laboratory experiments (Sommeria 1986; Paret & Tabeling 1998; Rutgers 1998; Rivera 2000). In fact, it can be rigorously proved that an inverse cascade with constant (negative) flux of energy must occur in a forced two-dimensional fluid, if damping at low wavenumbers keeps the energy finite in the high-Reynolds-number limit (Eyink 1996*a*). Kraichnan's seminal concept of an 'inverse cascade' has since been fruitfully extended to other physical situations, such as an inverse cascade of magnetic helicity in three-dimensional magnetohydrodynamic turbulence (Frisch *et al.* 1975), of wave action in weak turbulence (Zakharov & Zaslavskii 1982; see also Zakharov 1967) and of passive scalars in compressible fluid turbulence (Chertkov, Kolokolov & Vergassola 1998).

Attempts have often been made to account for the two-dimensional inverse energy cascade phenomenon by a *negative eddy viscosity*, either within analytical closure theories (Kraichnan 1971*a*, *b*, 1976) or more phenomenologically (Starr 1968). Such a description postulates a constitutive law for the turbulent stress proportional to the strain, $\tau_{ij} = -2\nu_T S_{ij}$, with a viscosity coefficient $\nu_T < 0$. However, an exact elimination of turbulent small scales gives rise to a stress formula which is quite different: non-local in space, history-dependent and stochastic (Lindenberg, West & Kottalam 1987; Eyink 1996*b*). Thus, any local and deterministic parameterization of the stress, such as by an eddy viscosity, can be only an approximate representation at best. Nevertheless, such simplified constitutive relations can be useful for illuminating some of the basic physics of turbulent cascades and they are also important, of course, for use in practical numerical modelling schemes.

In Eyink (2006, hereinafter referred to as I), we developed a general approximation scheme for the turbulent stress, based upon a multi-scale gradient (MSG) expansion. We employed there the *filtering approach* to space-scale resolution in turbulence (Germano 1992), which is also used in large-eddy simulation (LES) modelling schemes (Meneveau & Katz 2000). Within that framework, we developed an expansion of the stress, first in contributions from different scales of motion and then in terms of space gradients of the filtered velocity field. As a concrete application of the general scheme we considered in (I) the forward cascade of energy and helicity in three dimensions. In this paper, we apply the same formalism to the two-dimensional inverse energy cascade. In certain respects, the two-dimensional theory is more difficult than the three-dimensional theory, because of certain peculiarities of the inverse cascade. We find that contributions to the stress from velocity increments at subfilter scales are much more important in two dimensions than in three dimensions. Also, terms second-order in space gradients play a significant role in the two-dimensional inverse cascade, whereas in three dimensions the terms first-order in gradients appear to suffice. Recognizing these facts has proved crucial to unravelling the physics of the two-dimensional inverse energy cascade.

However, two dimensions is simpler than three dimensions in respect of geometry. As we discussed in (I), the local energy flux is given, in general, by a scalar product

$$\Pi = -\overline{\boldsymbol{S}} : \mathring{\boldsymbol{\tau}}, \tag{1.1}$$

where $\overline{\boldsymbol{S}}$ is the filtered strain tensor and $\mathring{\boldsymbol{\tau}}$ is the deviatoric (i.e. traceless part of the) stress tensor $\boldsymbol{\tau}$. The quantity Π defined in (1.1) represents the rate of work done by the large-scale strain against the stress induced by the small scales. In three dimensions, this expression involves three eigenvalues for each tensor, and also three Euler angles which specify the relative orientations of the tensor eigenframes. However, in two dimensions, we have simply

$$\Pi = -\overline{\sigma}(\delta\tau)\cos(2\theta), \tag{1.2}$$

where $\pm\overline{\sigma}$ are the two eigenvalues of $\overline{\boldsymbol{S}}$, $\pm\delta\tau/2$ are the two eigenvalues of $\mathring{\boldsymbol{\tau}}$, and θ is the angle between the eigenframes of these tensors. We have taken $0 \leqslant \theta \leqslant \pi/2$ and $\overline{\sigma}, \delta\tau \geqslant 0$. Thus, the essence of the inverse energy cascade lies exactly in the tendency that $0 \leqslant \theta < \pi/4$. If $\boldsymbol{e}_{\pm}^{(\tau)}$ are the two eigenvectors of the deviatoric stress corresponding to the eigenvalues $\pm\delta\tau/2$, then there is a net tensile or expansive stress $\delta\tau/2$ along the $\boldsymbol{e}_{+}^{(\tau)}$ direction and a net contractile or compressive stress $-\delta\tau/2$ along the $\boldsymbol{e}_{-}^{(\tau)}$ direction. Therefore, when $0 \leqslant \theta < \pi/4$ holds, the stretching direction $\boldsymbol{e}_{+}^{(\sigma)}$ of the strain is aligned primarily along the direction of net tensile stress, whereas the squeezing direction $\boldsymbol{e}_{-}^{(\sigma)}$ of the strain is aligned mainly along the direction of contractile stress. In that case, the stress cooperates with the strain rather than resists it, and negative work is done by the large scales against the small scales.

Our primary objective in this work is to gain some understanding of how this characteristic alignment comes about in two dimensions. In a negative-viscosity model, the stress is directly proportional to the strain or, equivalently, the alignment angle $\theta = 0$. This configuration leads to a maximal inverse cascade, but it is unlikely to occur uniformly throughout the flow. In fact, a main result of our work is that, to first-order in gradients and considering only the contribution to stress from scales of motion near the filter scale, the alignment is instead $\theta = \pi/4$ everywhere (§ 2.1.1). We call such a stress law 'skew-Newtonian' and, from (1.2), it gives zero energy flux. Thus, to first-order in gradients, no energy flux can arise in two dimensions from strongly scale-local interactions, in agreement with a conclusion of Kraichnan (1971*b*). On the other hand, skew-Newtonian stress from smaller subscale modes can give rise to non-vanishing flux, since the stress is oriented at angle $\pi/4$ with respect to the strain at the same scale, not the large-scale strain $\overline{\boldsymbol{S}}$ (§ 2.1.2). We argue that the flux from skew-Newtonian stress produced by more distant subfilter scales is negative, on average, because of a relative rotation of the principal directions of strain at distinct scales. A plausible explanation for this characteristic rotation is advanced based on the exact equation for the rotation angle (Appendix A) and a heuristic model of 'vortex-thinning' (§ 2.1.3). Furthermore, two additional main mechanisms of inverse cascade are predicted by carrying our expansion to second-order in gradients: a Newtonian stress with eddy viscosity owing to differential strain rotation; and a tensile stress directed along vorticity contour lines (§ 2.2.1). The latter effect was anticipated by Kraichnan (1976) (Appendix B) and it produces an inverse cascade when vorticity gradients are stretched by the large-scale strain. We derive an identity (Appendix C) that shows that, under the same condition, the eddy viscosity due to differential strain rotation is negative on average and produces inverse cascade. These mechanisms operate for stress produced by subfilter scales also, but more weakly the more distant in scale (§ 2.2.2).

2. The multi-scale model in two dimensions

In this section, we shall develop for two dimensions the MSG expansion of the turbulent stress that was elaborated in general in (I). To keep our discussion as brief as possible, we shall refer to (I) for most of the technical details and only outline here the main points of the general scheme. We employ the standard 'filtering approach' (Germano 1992), which is reviewed, for example, by Meneveau & Katz (2000). Thus, we filter the velocity field $\boldsymbol{u}$ with a kernel G at a selected length scale ℓ in order to define a 'large-scale' field $\overline{\boldsymbol{u}}$ from scales $> \ell$ and a complementary 'small-scale' field $\boldsymbol{u}' = \boldsymbol{u} - \overline{\boldsymbol{u}}$ from scales $< \ell$. However, we further decompose the velocity field using test kernels $\Gamma_n(\boldsymbol{r}) = \ell_n^{-d}\Gamma(\boldsymbol{r}/\ell_n)$ into contributions $\boldsymbol{u}^{(n)}$ from length scales $> \ell_n = \lambda^{-n}\ell$. The difference $\boldsymbol{u}^{[n]} \equiv \boldsymbol{u}^{(n)} - \boldsymbol{u}^{(n-1)}$ then represents the velocity contribution from length scales between ℓ_{n-1} and ℓ_n and yields a multi-scale decomposition,

$$\boldsymbol{u} = \sum_{n=0}^{\infty} \boldsymbol{u}^{[n]}, \tag{2.1}$$

of the velocity field. In this paper, we assume a scale ratio $\lambda = 2$. We also assume, for simplicity, that the kernels G and Γ are equal. Thus, the two filtered fields $\overline{\boldsymbol{u}}$ and $\widetilde{\boldsymbol{u}} = \boldsymbol{u}^{(0)}$ at length ℓ are equal and we need no longer keep the second as a distinct object.

Since the filtered velocity fields $\boldsymbol{u}^{(n)}$ are smooth, they may be Taylor-expanded into a series of terms from mth-order gradients $\nabla^m \boldsymbol{u}^{(n)}$. Appropriate functionals of the velocity field may be expressed in this manner as a summation over both the gradient index m and the scale index n, which we call a multi-scale gradient (MSG) expansion. Among the most important quantities for which such an MSG representation may be developed is the turbulent stress tensor $\boldsymbol{\tau}$. The latter quantity is defined mathematically as $\boldsymbol{\tau} = \overline{\boldsymbol{u}\boldsymbol{u}} - \overline{\boldsymbol{u}}\,\overline{\boldsymbol{u}}$. Physically, it gives the contribution of the small scales to spatial transport of large-scale momentum and it is the quantity which requires ‘closure’ in the equation for the large-scale velocity $\overline{\boldsymbol{u}}$. It was proved in (I) that there is a convergent MSG expansion for the stress tensor, under realistic conditions for turbulent cascades.

We should remark that two related, but distinct, approximations for the subscale stress were developed in (I). The first (I, § 3) was a systematic expansion, which we shall refer to simply as the MSG expansion. This is a doubly-infinite series in orders of space gradients and in scales of the velocity field, which converges to the exact subscale stress. However, as discussed in (I), the rate of convergence of the expansion in order m of space gradients is apt to be slow as the scale index n is increased. To obtain a more rapidly convergent gradient expansion in the small scales, we developed also a more approximate method (I, § 4). In this modified approach, the small-scale stress was estimated from velocity increments for separation vectors in a certain subset for which the gradient expansion is rapidly convergent, at all scales. The hypothesis underlying this approximation is that the stress due to velocity increments for separation vectors from all subregions is similar and can be estimated, to a good approximation, by the stress arising from the distinguished subset. We referred to this modified expansion in (I) as the *coherent-subregions approximation* (CSA), or the CSA-MSG expansion. It is guaranteed to converge rapidly, but its accuracy depends upon the quality of the basic hypothesis. The latter seems plausible, but should be subjected to empirical tests.

As we shall see below, it is more important to consider the contributions of sub-filter scales in the two-dimensional inverse energy cascade than it is in the three-dimensional forward cascade. Therefore, the rapid convergence of the CSA-MSG expansion at small scales makes it more practical than the systematic expansion for two dimensions, and only the former will be considered here. However, given the close formal relation between them, most of our qualitative physical discussion below can be carried over, with some minor changes, to the systematic MSG expansion, and it is only for the purpose of quantitative comparisons that the CSA expansion is to be preferred. To describe this approximation scheme, it is necessary to decompose the turbulent stress as $\boldsymbol{\tau} = \boldsymbol{\varrho} - \boldsymbol{u}'\boldsymbol{u}'$, where we refer to $\boldsymbol{\varrho}$ as the ‘systematic’ contribution to the stress and to $-\boldsymbol{u}'\boldsymbol{u}'$ as the ‘fluctuation’ contribution. For further discussion of these two terms and for mathematical formulae, see (I, § 2.13–2.14). In terms of these two quantities, the general CSA-MSG expression for the stress in any dimension d was given in (I), to nth-order in scale index and mth-order in gradients, as:

$$\boldsymbol{\tau}_*^{(n,m)} = \sum_{k=0}^{n} \boldsymbol{\varrho}_*^{[k],(m)} - \sum_{k,k'=0}^{n} \boldsymbol{u}'^{[k],(m)}_* \boldsymbol{u}'^{[k'],(m)}_*. \tag{2.2}$$

Using the results for $m = 2$ as illustration, as in (I), we have

$$\boldsymbol{\varrho}_*^{[k],(2)} = \frac{\overline{C}_2^{[k]}}{d} \ell_k^2 \frac{\partial \boldsymbol{u}^{[k]}}{\partial x_l} \frac{\partial \boldsymbol{u}^{[k]}}{\partial x_l} + \frac{\overline{C}_4^{[k]}}{2d(d+2)} \ell_k^4 \frac{\partial^2 \boldsymbol{u}^{[k]}}{\partial x_l \partial x_m} \frac{\partial^2 \boldsymbol{u}^{[k]}}{\partial x_l \partial x_m} + \frac{\overline{C}_4^{[k]}}{4d(d+2)} \ell_k^4 \Delta \boldsymbol{u}^{[k]} \Delta \boldsymbol{u}^{[k]}, \tag{2.3}$$

$$\boldsymbol{u}'^{[k],(2)}_* = \frac{-1}{2d\sqrt{N_k}} \overline{C}_2^{[k]} \ell_k^4 \Delta \boldsymbol{u}^{[k]}. \tag{2.4}$$

The coefficients $\overline{C}_p^{[k]}$ in this model for $p = 2, 4, \ldots$ represent the partial pth moments of the filter-kernel G over a spherical shell of separation vectors of length $\approx \ell_k$, corrected by a multiplicative factor of $N_k = 2^{kd}$ to compensate for the decreasing volume of those shells with increasing k. Explicit expressions were given for these coefficients with a Gaussian filter, in (I), Appendix C. (The expressions involve incomplete gamma functions. For the case $d = 2$ relevant here, these become, for $p = 2m$, $\gamma((d+p)/2, x) = \gamma(1+m, x) = m![1-(1+x+x^2/2!+\cdots+x^m/m!)\mathrm{e}^{-x}]$, in terms of elementary functions. See Abramowitz & Stegun (1964, formulae 6.5.2 and 6.5.13).) Notice that, with the volume-corrected coefficients used here, the 'fluctuation' terms in (2.2) are decreased relative to the 'coherent' terms by the factors $1/\sqrt{N_k N_{k'}}$. These were proposed in (I) as a consequence of a central limit theorem argument for the averages over volume that define the 'fluctuation' velocities in (2.4). Because of this, those terms are expected for larger k to be negligible relative to the 'systematic' contributions in (2.2).

This brief synopsis provides enough background on the MSG expansion for our application in this paper to the two-dimensional inverse cascade. For mathematical derivations and more extensive physical discussion, see (I).

2.1. *The first-order model*

To begin our discussion of the two-dimensional energy cascade, we shall consider the CSA-MSG expansion of the stress developed to first order in velocity gradients. According to the general formula in equations (2.2)–(2.4), the expansion of the stress then contains only the 'coherent' part $\boldsymbol{\varrho}$, since the 'fluctuation' velocity $\boldsymbol{u}'$ vanishes to first order. Thus, in any space dimension d, the expansion is given to this order by

$$\boldsymbol{\tau}_*^{(n,1)} = \sum_{k=0}^{n} \boldsymbol{\varrho}_*^{[k],1}, \tag{2.5}$$

with

$$\boldsymbol{\varrho}_*^{[k],(1)} = \frac{\overline{C}_2^{[k]}}{d} \ell_k^2 \frac{\partial \boldsymbol{u}^{[k]}}{\partial x_l} \frac{\partial \boldsymbol{u}^{[k]}}{\partial x_l}, \tag{2.6}$$

consisting of just the first term in (2.3). See also I, § 5.2. Terms for large values of k become negligibly small (UV scale-locality), so that the limit as $n \to \infty$ exists. For a monofractal velocity field with Hölder exponent everywhere 1/3 – as expected in the two-dimensional inverse cascade (Paret & Tabeling 1998; Yakhot 1999; Boffetta *et al.* 2000) – the kth term in (2.6) scales as $\sim \ell_k^{2/3}$ (Eyink 2005; I).

We now specialize the model to two dimensions, using the standard formula for a velocity-gradient (deformation) matrix in two-dimensions,

$$\frac{\partial u_i}{\partial x_j} = S_{ij} - \tfrac{1}{2}\epsilon_{ij}\omega, \tag{2.7}$$

which relates it to the symmetric traceless strain matrix $\boldsymbol{S}$ and (pseudo)scalar vorticity ω. Here, ϵ_{ij} is the antisymmetric Levi–Civita tensor in two dimensions. Substituting (2.7) into (2.6) yields

$$\varrho_{*;ij}^{[k],1} = \tfrac{1}{2}\overline{C}_2^{[k]}\ell_k^2\left\{S_{il}^{[k]}S_{jl}^{[k]} + \omega^{[k]}\widetilde{S}_{ij}^{[k]} + \tfrac{1}{4}\delta_{ij}\left|\omega^{[k]}\right|^2\right\}, \tag{2.8}$$

where we have defined the *skew-strain matrix* as $\widetilde{S}_{ij} = S_{ik}\epsilon_{kj}$. (This differs slightly from the general definition given in (I), which would lead us in two dimensions to term the product $\omega^{[k]}\widetilde{\boldsymbol{S}}^{[k]}$ as 'skew-strain' instead. This slight difference in terminology should cause no difficulty.) In terms of matrix arrays

$$\boldsymbol{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & -S_{11} \end{pmatrix}, \quad \widetilde{\boldsymbol{S}} = \begin{pmatrix} -S_{12} & S_{11} \\ S_{11} & S_{12} \end{pmatrix}. \tag{2.9}$$

Thus, the skew-strain is also symmetric and traceless. It is easy to see that the strain and skew-strain are orthogonal in the standard matrix inner product $\boldsymbol{S}:\widetilde{\boldsymbol{S}} = 0$ (and hence the prefix 'skew'). The various terms that appear in (2.8) are the same as those in equation (5.4) of I for three dimensions and have the same physical interpretations. Note, however, a principal difference with three dimensions is the absence of terms proportional to $\omega_i^{[k]}\omega_j^{[k]}$. Since the only component of vorticity is perpendicular to the plane of motion, no stress can be directed along vortex lines in two dimensions.

2.1.1. *The strong UV-local terms*

It is interesting to consider separately the first term in (2.5), for $k = 0$, since it corresponds to the stress contribution from filter-scale velocity increments. Thus, we refer to this as the strongly UV-local contribution. It is the only summand in (2.5) which is closed in terms of the filtered velocity $\overline{\boldsymbol{u}} = \boldsymbol{u}^{(0)}$. In fact, this term corresponds to the well-known nonlinear model for the turbulent stress (Meneveau & Katz 2000), as discussed at length in (I).

The most important observation about the strongly UV-local term in two dimensions is that it gives zero energy flux, pointwise in space. This is obvious for the term proportional to $|\overline{\omega}|^2$, since it is a pressure contribution. Furthermore, the first term is proportional to $\overline{\boldsymbol{S}}^2 = \overline{\sigma}^2\boldsymbol{I}$ in two dimensions, where $\boldsymbol{I}$ is the identity matrix, and is thus also a pressure contribution. Here, we have used the Cayley–Hamilton theorem and the fact that the strain matrix in two dimensions has two eigenvalues $\pm\overline{\sigma}$ of equal magnitude, but opposite sign. Therefore, the first term contributes also zero flux. The term in (2.8) proportional to the skew-strain is deviatoric, but it does not contribute to energy flux, by the orthogonality mentioned earlier. We can thus conclude that there is no energy flux anywhere in space arising from the strongly UV-local interactions, to first order in velocity gradients.

This conclusion agrees with Kraichnan (1971*b*), who showed that an energy cascade in two dimensions cannot be strongly scale-local. It is worth summarizing his demonstration, which is based on the detailed conservation of energy and enstrophy in

Fourier space. Let $T(k, p, q)$ represent the energy transfer into wavenumber magnitude k from all triads of wavenumbers with magnitudes k, p, q. A measure of the scale-locality of the triad is provided by the parameter

$$\nu = \log_2(k_{med}/k_{min}) \geqslant 0,$$

where k_{min}, k_{med} and k_{max} are the minimum, median and maximum wavenumber magnitudes, respectively, from the triad k, p, q. Intuitively, this quantity represents the 'number of cascade steps' between the minimum and median wavenumber. Note that $k_{max} \leqslant 2k_{med}$ by wavenumber addition, so that $\log_2(k_{max}/k_{min}) \leqslant \nu + 1$. Thus, the parameter ν unambiguously measures the ratio of scales involved in the triadic interaction. In these terms, non-local (note that this definition makes no distinction between ultraviolet (UV) and infrared (IR) non-local interactions, as in Eyink 2005) interactions correspond to those triads with $\nu \gg 1$ and strongly scale-local ones to those with $\nu \ll 1$. Kraichnan (1971*b*) noted that in two dimensions the transfer function satisfies both

$$T(k, p, q) + T(p, q, k) + T(q, k, p) = 0, \tag{2.10}$$

as a consequence of energy conservation, and

$$k^2 T(k, p, q) + p^2 T(p, q, k) + q^2 T(q, k, p) = 0, \tag{2.11}$$

by conservation of enstrophy. Multiplying (2.10) by q^2 and subtracting from (2.11) gives

$$(k^2 - q^2)T(k, p, q) + (p^2 - q^2)T(p, q, k) = 0. \tag{2.12}$$

Thus, if $k \neq p = q$, then $T(k, p, p) = 0$, and substituting back into (2.10) gives also $T(p, p, k) = T(p, k, p) = 0$. Hence, there is zero transfer, if any two wavenumbers have equal magnitudes, and, in particular, if $\nu = 0$. However, it is very plausible to expect that the transfer function will be continuous in the wavenumber magnitude. In that case, transfer will be vanishingly small also in the limit that $\nu \ll 1$. Kraichnan (1971*b*) obtained more quantitative results using his analytical test-field-model (TFM) closure. He found (see his figure 2) that roughly 90 % of the energy flux comes from triads with $\nu \geqslant 1$, 70 % with $\nu \geqslant 2$, and 60 % with $\nu \geqslant 3$. To obtain 90 % of the total energy flux in the TFM closure required including all triads with $\nu \leqslant 5$. Thus, the two-dimensional energy cascade was predicted by Kraichnan to be scale-local (cf. also the exact analysis in Eyink 2005), but only weakly so.

There is a fundamental relationship between our argument and Kraichnan's. This is best understood by recalling the form of the energy flux in three dimensions from the strongly local first-order terms (I, equation (5.11)) (and see also Borue & Orszag 1998):

$$\Pi^{(0,1)} = \tfrac{1}{3} C_2 \ell^2 \{ -\mathrm{Tr}\,(\overline{\boldsymbol{S}}^3) + \tfrac{1}{4} \overline{\boldsymbol{\omega}}^\top \overline{\boldsymbol{S}} \overline{\boldsymbol{\omega}} \}. \tag{2.13}$$

Both of these terms vanish in two dimensions, the second because of absence of vortex stretching. As discussed in (I), the first term can also be related to vortex stretching, at least in a space-average sense, by a relation of Betchov (1956). Of course, the lack of vortex stretching in two dimensions is also what underlies the conservation of enstrophy, used in Kraichnan's argument. The argument that we have given confirms Kraichnan's conclusion and extends it to be also pointwise in space.

2.1.2. *The weakly UV-local terms*

From the preceding discussion, we can see that any energy flux that arises to first order in gradients must be due to subfilter modes, with $k > 0$. Since the contribution from modes with $k \gg 1$ is also small, the flux comes primarily from the weakly local terms with $k \gtrsim 1$. This contribution for each $k \geqslant 1$ can arise solely from the skew-strain term in the stress (2.8), since, by the same reasoning as above, the other two terms are isotropic stresses or pressures. The flux from modes at scale k is thus

$$\Pi_*^{[k],(1)} = \tfrac{1}{2}\overline{C}_2^{[k]}\ell_k^2\omega^{[k]}\left(\boldsymbol{S}^{[0]}\boldsymbol{:}\widetilde{\boldsymbol{S}}^{[k]}\right). \tag{2.14}$$

This can be rewritten in more intuitive fashion using 'polar coordinates' for strain matrices:

$$\boldsymbol{S} = \sigma\begin{pmatrix}\cos(2\alpha) & \sin(2\alpha)\\ \sin(2\alpha) & -\cos(2\alpha)\end{pmatrix}, \quad \widetilde{\boldsymbol{S}} = \sigma\begin{pmatrix}-\sin(2\alpha) & \cos(2\alpha)\\ \cos(2\alpha) & \sin(2\alpha)\end{pmatrix}. \tag{2.15}$$

Here, $\sigma = |\boldsymbol{S}|/\sqrt{2}$ is the (positive) strain eigenvalue and $\alpha = (1/2)\arctan(S_{12}/S_{11})$ is the angle made by the frame of strain eigenvectors $\boldsymbol{e}_+^{(\sigma)}, \boldsymbol{e}_-^{(\sigma)}$ with a fixed orthogonal frame. Note, incidentally, that the skew-strain is obtained by rotating the frame of the strain by $\pi/4$ radians. By choosing appropriately between the two unit eigenvectors $\pm\boldsymbol{e}_+^{(\sigma)}$, we can always ensure that $0 \leqslant |\alpha| < \pi/2$. Thus, from (2.14) and (2.15),

$$\Pi_*^{[k],(1)} = \overline{C}_2^{[k]}\ell_k^2\sigma^{[0]}\sigma^{[k]}\omega^{[k]}\sin\left[2\left(\alpha^{[k]} - \alpha^{[0]}\right)\right], \tag{2.16}$$

a remarkably simple and compact result.

The total flux from all scales $k = 0, 1, \ldots, n$ to first order in gradients is thus

$$\Pi_*^{(n,1)} = \sum_{k=1}^{n}\overline{C}_2^{[k]}\ell_k^2\sigma^{[0]}\sigma^{[k]}\omega^{[k]}\sin\left[2\left(\alpha^{[k]} - \alpha^{[0]}\right)\right]. \tag{2.17}$$

In order to achieve an inverse energy cascade, it must hold that the terms in the sum are negative on average, at least for $k \gtrsim 1$. The sign of (2.16) is determined completely by the factor $\omega^{[k]}\sin[2(\alpha^{[k]} - \alpha^{[0]})]$, which depends upon the relative angle $\alpha^{[k]} - \alpha^{[0]}$. If we choose that $0 \leqslant |\alpha^{[k]} - \alpha^{[0]}| < \pi/2$, then this factor will be negative if the strain-frame at scale k lags the strain-frame at scale 0 ($\alpha^{[k]} < \alpha^{[0]}$) in regions where $\omega^{[k]} > 0$, and leads ($\alpha^{[k]} > \alpha^{[0]}$) in regions where $\omega^{[k]} < 0$. Under these conditions, the small-scale stress will cooperate with the large-scale strain and the latter will do negative work. Note that this is quite different from a 'negative-viscosity' mechanism, with a Newtonian stress proportional to strain $\boldsymbol{S}^{[k]}$. Instead, the crucial deviatoric component of the stress is of the form $\gamma^{[k]}\widetilde{\boldsymbol{S}}^{[k]}$, where $\gamma^{[k]} = \overline{C}_2^{[k]}\ell_k^2\omega^{[k]}/2$ has dimensions of a diffusion constant and may be termed 'skew-viscosity'.

We see that a contribution to inverse energy cascade at scale k requires an anti-correlation in the signs of $\omega^{[k]}$ and $\alpha^{[k]} - \alpha^{[0]}$. A plausible dynamical mechanism for this can be suggested, based upon the exact equation for the strain orientation angle:

$$\left(\sigma^{(k)}\right)^2 D_t^{(k)}\alpha^{(k)} = \tfrac{1}{8}\omega^{(k)}Q^{(k)} + \nabla\boldsymbol{\cdot}\boldsymbol{K}^{(k)} + \cdots, \tag{2.18}$$

with $D_t^{(k)} = \partial_t + \boldsymbol{u}^{(k)}\boldsymbol{\cdot}\nabla$ the advective derivative at scale k,

$$Q^{(k)} = \Delta p^{(k)} = \tfrac{1}{2}\left[\omega^{(k)}\right]^2 - 2\left[\sigma^{(k)}\right]^2 \tag{2.19}$$

the pressure hessian at scale k, and

$$\boldsymbol{K}^{(k)} = \tfrac{1}{4}\left(\nabla p^{(k)}\times\nabla\right)\boldsymbol{u}^{(k)} \tag{2.20}$$

a space transport term due to pressure forces. The $\cdots$ terms in (2.18) represent contributions from the turbulent stress due to modes at length scales $< \ell_k$. See Appendix A for the derivation. According to (2.18), $D_t\alpha^{(k)} \sim \ell_k^{-2/3}$ in a two-dimensional inverse energy cascade range, so that the rotation rate increases with increasing k. Since the pressure contribution $\nabla \cdot \boldsymbol{K}^{(k)}$ is spatially non-local and averages to zero, it can be treated as random noise. We shall likewise disregard the effect of subgrid terms $\cdots$. Thus, the expected correlation will be created in strain-dominated regions with $Q^{(k)} < 0$, since $\alpha^{(k)}$ there rotates against the locality vorticity $\omega^{(k)}$ and faster for larger k. Since the flux (2.16) is proportional also to the strain magnitudes $\sigma^{(0)}, \sigma^{(k)}$, most of the cascade should occur in the strain regions where this counter-rotation occurs.

2.1.3. *A heuristic model*

A simple model problem may help to illuminate the basic mechanism of inverse energy cascade due to skew-strain. We shall consider the effect of a large-scale uniform straining field,

$$\boldsymbol{S}^{(0)} = \begin{pmatrix} \sigma^{(0)} & 0 \\ 0 & -\sigma^{(0)} \end{pmatrix}, \tag{2.21}$$

on a collection of small-scale vortices, each initially circular with support radius ℓ_n. The ith vortex in the assembly will be assumed to have initially a vorticity distribution $\omega_i^{[n]}(|\boldsymbol{r}-\boldsymbol{r}_i|)$ radially symmetric about its centre $\boldsymbol{r}_i$. Let us assume also that the small-scale vortices each have a single sign of vorticity, but with the net circulation of the array equal to zero: $\sum_i \int \mathrm{d}\boldsymbol{r}\, \omega_i^{[n]}(r) = 0$. Kraichnan (1976) considered a very similar model problem of 'vortex blobs' in order to illustrate the mechanism of asymptotic negative viscosities in his test-field model closure. In Appendix B, we review Kraichnan's 'blob model' and compare it with the present one. Suffice it to say here that it was crucial in Kraichnan's calculation to take vortex wave-packets with a very rapid sinusoidal variation in the vorticity. On the contrary, we require no such variation and a particular case of our model is an array of vortex patches with constant vorticity levels, each initially circular.

The effect of the straining field on this set of small-scale vortices will be to deform them into elliptical form, elongated in the x-direction and thinned in the y-direction. Kida (1981) found this behaviour in his exact solution of two-dimensional Euler for an elliptical vortex patch in a uniform shear flow, whenever the strain σ and vorticity level ω satisfy $|\sigma/\omega| \geqslant (3-\sqrt{5})/[2(2+2\sqrt{5})^{1/2}] \doteq 0.15$. More generally, the same phenomenon appears in a rapid distortion limit for the case of a strong strain $\sigma^{(0)} \gg \max_i \|\omega_i^{[n]}\|_\infty$. We can then ignore the self-evolution of the vortices and also their mutual interactions. This permits us to focus on a single vortex centred at $\boldsymbol{r} = \mathbf{o}$ with radial vorticity profile $\omega^{[n]}(r)$. The vorticity level set initially at radius r is distorted into an ellipse whose equation is $x^2/a^2 + y^2/b^2 = 1$ with semi-major axis $a = r\exp(\sigma^{(0)}t)$ and semi-minor axis $b = r\exp(-\sigma^{(0)}t)$ at time t.

The immediate result is that the energy of the small-scale vortex patch is reduced, as a consequence of conservation of circulation. The area inside each elliptical vorticity contour is preserved, but the length of the perimeter is increased. In order to keep the circulation constant, the circumferential velocity must decrease. For example, in the case of a circular vortex patch of constant vorticity-level $\omega^{[n]}$ with initial radius $r = \ell_n$, the patch evolves into an elliptical shape with circulation $\omega^{[n]} \cdot \pi ab \doteq 4u^{[n]}a$, where $u^{[n]}(t)$ is the x-component of the circumferential velocity at time t. The second expression for circulation holds in the limit when $\sigma^{(0)}t \gg 1$ and $a \gg b$, so that the perimeter of the elliptical vortex is approximately $4a$ and is nearly parallel to the x-axis. In that case,

$$u^{[n]}(t) \doteq (\pi/4)\omega^{[n]}b = (\pi/4)\omega^{[n]}\ell_n \exp\left(-\sigma^{(0)}t\right). \tag{2.22}$$

A similar argument can be made for points interior and exterior to the vortex, with the result that the velocity is everywhere reduced by a common factor of $\exp(-\sigma^{(0)}t)$. Thus, the kinetic energy of the vortex is also decreased. (Of course, a single vortex of definite sign would have infinite energy in the unbounded plane, owing to divergence at infinity. Such far-field divergence is absent when considering the array of vortices with zero net circulation.)

The energy lost by the collection of small-scale vortices is transferred to the large scales. To see this, observe that the large-scale straining, in addition to reducing the velocity amplitude of the small-scale vortices, also rectifies the velocity direction. The velocity vector of the elongated vortices points almost entirely in the x-direction and very little in the y-direction. Indeed, the vorticity level curve initially at radius r for the profile $\omega^{[n]}(r)$ now becomes, to leading order, a pair of straight parallel lines $y = \pm b = \pm r\exp(-\sigma^{(0)}t)$. Thus, the vorticity field approximates to $\omega^{[n]}(y,t) = \omega^{[n]}(|y|\exp(\sigma^{(0)}t))$ when $\sigma^{(0)}t \gg 1$. This is merely the vorticity associated to a long narrow shear layer with weakened velocity,

$$u^{[n]}(y,t) = -\exp\left(-\sigma^{(0)}t\right)\operatorname{sign}(y)\int_0^{|y|\exp(\sigma^{(0)}t)} \omega^{[n]}(r)\,\mathrm{d}r, \tag{2.23}$$

directed entirely along the x-axis. If the tensor product $\boldsymbol{u}^{[n]}\boldsymbol{u}^{[n]}$ were integrated over space at the initial time, it would produce only a diagonal stress contribution:

$$T_{ij}(t=0) = \int_{vortex} \mathrm{d}\boldsymbol{r}\; u_i^{[n]}(\boldsymbol{r})u_j^{[n]}(\boldsymbol{r}) = \delta_{ij}\pi\int_0^{\ell_n} \mathrm{d}r\; r\left|u_\theta^{[n]}(r)\right|^2, \tag{2.24}$$

where $u_\theta^{[n]}(r) = (1/r)\int_0^r \rho\omega^{[n]}(\rho)\,\mathrm{d}\rho$ is the tangential velocity around the vortex centre. (Here we have integrated only over the body of the vortex, neglecting the contribution of more distant regions.) However, after 'rectification' there is a net stress component

$$\begin{aligned} T_{11}(t) &\doteq 2a\int_{-b}^{b} \mathrm{d}y\, u^{[n]}(y,t)u^{[n]}(y,t) \\ &= 4\ell_n\int_0^{\ell_n} \mathrm{d}r\,\left\{\int_0^r \omega^{[n]}(\rho)\,\mathrm{d}\rho\right\}^2, \end{aligned} \tag{2.25}$$

with all other components much smaller. This resultant stress reinforces the large-scale strain field, so that $\int \mathrm{d}\boldsymbol{r}\,\Pi(\boldsymbol{r},t) = -S_{ij}T_{ij} < 0$, and negative work is done by the large scales against the small scales.

This simple model of inverse energy cascade illustrates the pattern of relative orientation of strain frames at distinct scales, which was discussed earlier. In fact, within the long narrow shear layer created by thinning of a vortex there is a velocity-gradient (or deformation) tensor of the form

$$\boldsymbol{D}^{[n]}(y,t) = \begin{pmatrix} 0 & -\omega^{[n]}(y,t) \\ 0 & 0 \end{pmatrix}, \tag{2.26}$$

with $(\partial u^{[n]}/\partial y)(y,t) = -\omega^{[n]}(y,t)$. The corresponding strain matrix is

$$\boldsymbol{S}^{[n]}(y,t) = \begin{pmatrix} 0 & -\omega^{[n]}(y,t)/2 \\ -\omega^{[n]}(y,t)/2 & 0 \end{pmatrix}, \tag{2.27}$$

which has eigenvectors

$$\boldsymbol{e}_+^{[n]} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{e}_-^{[n]} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.28}$$

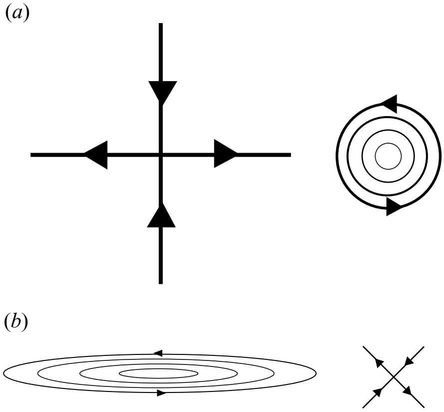

FIGURE 1. Mechanism of vortex-thinning. (*a*) A large scale strain field with stretching direction along the x-axis and shrinking direction along the y-axis, and a small-scale vortex of positive circulation, initially circular. (*b*) The vortex elongated along the x-axis and thinned along the y-axis, and its strain basis, rotated by $-45°$ with respect to the large-scale strain.

for $\omega^{[n]}(y,t) > 0$ and with $\boldsymbol{e}_+^{[n]}, \boldsymbol{e}_-^{[n]}$ reversed for $\omega^{[n]}(y,t) < 0$. See figure 1, which illustrates the case of a vortex patch of positive (counterclockwise) circulation. The small-scale strain basis shown there is rotated relative to the large-scale strain basis by $-\pi/4$ radians. If the vortex patch had had negative (clockwise) circulation, then the rotation would have been by $+\pi/4$ radians instead.

This same model also clarifies the origin of stress proportional to skew-strain in our general scheme. The skew-strain in such an elongated vortex is

$$\widetilde{\boldsymbol{S}}^{[n]}(y,t) = \begin{pmatrix} \omega^{[n]}(y,t)/2 & 0 \\ 0 & -\omega^{[n]}(y,t)/2 \end{pmatrix}. \tag{2.29}$$

Let us introduce a convenient space-average over the vortex of the form

$$\langle \omega^{[n]} \rangle = \frac{2}{b^2}\int_0^b \mathrm{d}y \int_0^y \mathrm{d}y' \omega^{[n]}(y',t) = \frac{2}{\ell_n^2}\int_0^{\ell_n} \mathrm{d}r \int_0^r \mathrm{d}\rho \omega^{[n]}(\rho). \tag{2.30}$$

By the Cauchy–Schwartz inequality,

$$\frac{1}{\ell_n}\int_0^{\ell_n} \mathrm{d}r \left\{ \int_0^r \omega^{[n]}(\rho)\,\mathrm{d}\rho \right\}^2 \geqslant \left[(\ell_n/2)\langle \omega^{[n]} \rangle\right]^2,$$

and, furthermore, these two quantities will generally have a ratio within some specified bounds. It follows that, when $\sigma^{(0)}t \gg 1$,

$$\ell_n^2 \langle \omega^{[n]} \rangle \langle \widetilde{\boldsymbol{S}}^{[n]} \rangle \doteq \begin{pmatrix} \tau_{11}/2 & 0 \\ 0 & -\tau_{11}/2 \end{pmatrix}, \tag{2.31}$$

where we have set $\tau_{11} = T_{11}/\ell_n^2$. Thus, a (deviatoric) stress proportional to skew-strain arises naturally from a narrow shear layer produced by vortex thinning.

It is not completely obvious why small-scale vortices in a two-dimensional inverse cascade range should be elongated and thinned by large-scale strain. After all, in such a range, $\sigma^{(0)} \sim \ell^{-2/3} \ll (\ell_n)^{-2/3} \sim \omega^{[n]}$ for $\ell \gg \ell_n$. Thus, the large-scale strain is weak compared with the vorticity at smaller scales, exactly the opposite as is assumed in the rapid distortion limit above. The vorticity at length scale ℓ_n could be expected to respond more strongly to the larger strains $\sigma^{[n']} \gg \omega^{[n]}$ from length scales $l_{n'} \ll \ell_n$.

However, the large-scale strain, although relatively weak, is coordinated over large distances and is temporally coherent, with a typical lifetime of $t_\ell \sim \ell^{2/3}$. By contrast, the strain from the smaller scales is random and uncoordinated and, furthermore, evolves on a much shorter time scale $t_{\ell_{n'}} \sim (\ell_{n'})^{2/3}$. Thus, the small-scale vorticity can adjust very rapidly to the persistent large-scale strain, whereas it does not have time to adjust to the many, even more rapidly fluctuating strains from the still smaller scales.

Clearly, our simple model calculation does not reflect all of the complexities of the two-dimensional inverse cascade range. However, it gives a simple physical picture for the origin of stress proportional to skew-strain, which, we believe, is essentially the correct one. If the initial profiles of the vorticity, $\omega_i^{[n]}(r)$ for the ith circular vortex, are not constant in the radial distance r from the centre, then vortex thinning also produces large vorticity gradients parallel to the compressing direction of the strain field. This second-order effect will be discussed in detail in the following section.

2.2. *The second-order model*

We have seen that, unlike in three dimensions, the MSG expansion $\boldsymbol{\tau}_*^{(n,m)}$ to lowest order in space gradients, $m = 1$, can only explain energy cascade if subfilter scales $n \geqslant 1$ are considered. However, another possible mechanism may be terms of higher order in space gradients with $m \geqslant 2$. To investigate this possibility, we develop in this section the two-dimensional MSG expansion to second-order in velocity gradients. We can specialize the formulae (2.3) and (2.4) to two dimensions, replacing the velocity derivative with strain and vorticity using (2.7). The result is:

$$\varrho_{*;ij}^{[k],(2)} = \tfrac{1}{2}\overline{C}_2^{[k]}\ell_k^2\left[S_{il}^{[k]}S_{jl}^{[k]} + \omega^{[k]}\widetilde{S}_{ij}^{[k]} + \tfrac{1}{4}\delta_{ij}\left|\omega^{[k]}\right|^2\right] + \tfrac{1}{16}\overline{C}_4^{[k]}\ell_k^4\left[S_{il,m}^{[k]}S_{jl,m}^{[k]} + \left(\partial_l\omega^{[k]}\right)\widetilde{S}_{ij,l}^{[k]} + \tfrac{1}{4}\delta_{ij}\left|\nabla\omega^{[k]}\right|^2\right] + \tfrac{1}{32}\overline{C}_4^{[k]}\ell_k^4\widetilde{\partial}_i\omega^{[k]}\,\widetilde{\partial}_j\omega^{[k]}, \tag{2.32}$$

$$u_{*;i}^{\prime\,[k],(2)} = \frac{1}{4\sqrt{N_k}}\overline{C}_2^{[k]}\ell_k^2\widetilde{\partial}_i\omega^{[k]}. \tag{2.33}$$

In the last term of (2.32) and also in (2.33) we have defined $\widetilde{\partial}_i = \epsilon_{ij}\partial_j$, the *skew-gradient*, which satisfies $\widetilde{\nabla}\boldsymbol{\cdot}\nabla = 0$. This is the same operator that appears in the streamfunction representation of a velocity $u_i = \widetilde{\partial}_i\psi$. Indeed, to derive the last term in (2.32) and the term in (2.33) we used the streamfunction $\psi^{[k]}$ and the Poisson equation $-\Delta\psi^{[k]} = \omega^{[k]}$ in order to write $\Delta u_i^{[k]} = -\widetilde{\partial}_i\omega^{[k]}$.

2.2.1. *The strongly UV-local terms*

As for the first-order expansion, we begin by considering only the strongly UV-local terms with $n = 0$. These give altogether (note that $C_p^{(0)} = \overline{C}_p^{(0)}$ for $n = 0$)

$$\varrho_{*;ij}^{(0,2)} = \tfrac{1}{2}C_2^{(0)}\ell^2\left[\overline{S}_{il}\overline{S}_{jl} + \overline{\omega}\widetilde{\overline{S}}_{ij} + \tfrac{1}{4}\delta_{ij}\overline{\omega}^2\right] + \tfrac{1}{16}C_4^{(0)}\ell^4\left[\overline{S}_{il,m}\overline{S}_{jl,m} + (\partial_k\overline{\omega})\widetilde{\overline{S}}_{ij,k} + \tfrac{1}{4}\delta_{ij}|\nabla\overline{\omega}|^2\right] + \tfrac{1}{32}\left[C_4^{(0)} - 2\left(C_2^{(0)}\right)^2\right]\ell^4(\widetilde{\partial}_i\overline{\omega})\,(\widetilde{\partial}_j\overline{\omega}). \tag{2.34}$$

Let us consider the physical meaning of the various terms that appear.

We have already considered the terms in the initial line of (2.34) that arise from first-order velocity-gradients and have shown that they give no contribution to energy flux. The second line is remarkably similar in appearance to the first. In fact, it is not hard to see that the first term proportional to $\overline{S}_{il,m}\overline{S}_{jl,m}$ is an isotropic (pressure) term, by exactly mimicking the argument we gave earlier for the $\overline{S}_{il}\overline{S}_{jl}$-term, separately for each value of the index m that is summed over. Of course, the final term proportional to $\delta_{ij}|\nabla\overline{\omega}|^2$ is also a pressure. This leaves only the middle deviatoric term as possibly contributing to energy flux. This second-order term,

$$\boldsymbol{\tau}_I^{(0),[2]} = \left(\tfrac{1}{16}\right) C_4^{(0)} \ell^4 (\nabla\overline{\omega}\cdot\nabla)\widetilde{\widetilde{\overline{\mathbf{S}}}}, \tag{2.35}$$

gives rise exactly to an 'eddy viscosity'. To see this, it is easiest to use the 'polar coordinates', (2.15), for the strain and skew-strain. Together with the chain rule, this gives

$$(\nabla\overline{\omega}\cdot\nabla)\widetilde{\widetilde{\overline{\mathbf{S}}}} = -2(\nabla\overline{\omega}\cdot\nabla\overline{\alpha})\overline{\mathbf{S}} + (\nabla\overline{\omega}\cdot\nabla\overline{\lambda})\widetilde{\widetilde{\overline{\mathbf{S}}}}, \tag{2.36}$$

with $\overline{\lambda} = \ln\overline{\sigma}$. Of course, the second term proportional to skew-strain does not contribute to energy flux. Thus, up to such conservative terms, we obtain

$$\boldsymbol{\tau}_I^{(0),[2]} = -\left(\tfrac{1}{8}\right) C_4^{(0)} \ell^4 (\nabla\overline{\omega}\cdot\nabla\overline{\alpha})\overline{\mathbf{S}} + \cdots = -2\nu_T\overline{\mathbf{S}}, \tag{2.37}$$

with $\nu_T = C_4^{(0)}\ell^4(\nabla\overline{\omega}\cdot\nabla\overline{\alpha})/16$. This is a stress of Newtonian form, with an eddy viscosity due to differential rotation of the strain. Indeed, the eddy-viscosity coefficient ν_T is proportional to the rate of rotation of strain along the direction of maximum increase of vorticity.

The final term of (2.34) arises from the combination of the last term in (2.32) for $k = 0$ and the product of two terms in (2.33) for $k = k' = 0$. These together give a stress exerted along the direction parallel to the skew-gradient $\widetilde{\nabla}\overline{\omega}$. Equivalently, this stress is directed normal to the vorticity gradient $\nabla\overline{\omega}$, or along the level sets or contour lines of the vorticity. There are two opposing contributions, a tensile stress proportional to $C_4^{(0)}$ from (2.32) and a contractile stress proportional to $(C_2^{(0)})^2$ from (2.33). Which dominates could depend upon the choice of the filter kernel G. However, the concrete calculations in Appendix C of I show that $C = [C_4^{(0)} - 2(C_2^{(0)})]^2/32 > 0$ for a Gaussian kernel. We have also checked this to be true for a few other cases, e.g. an exponential filter $G(\boldsymbol{r}) = \mathrm{e}^{-|\boldsymbol{r}|}/(2\pi)$. At least for these choices we see that there is a tensile stress of strength $C\ell^4|\nabla\overline{\omega}|^2$ exerted by the small scales along vorticity contour lines. As we discuss in Appendix B of the present paper, this effect was anticipated in a calculation of Kraichnan (1976) for a simple model problem of a two-dimensional vorticity wave-packet in a uniform strain field. This tensile stress along vorticity contours should be contrasted with the contractile stress $-C_2^{(0)}\ell^2|\overline{\boldsymbol{\omega}}|^2/2$ exerted along vortex lines in three dimensions, discussed in (I).

The strongly UV-local terms in the stress thus can give a non-vanishing contribution to energy flux, at second-order in gradients. Indeed,

$$\Pi_*^{(0),[2]} = -C\ell^4(\widetilde{\nabla}\overline{\omega})^\top\overline{\mathbf{S}}(\widetilde{\nabla}\overline{\omega}) - C'\ell^4\overline{\mathbf{S}} : (\nabla\overline{\omega}\cdot\nabla)\widetilde{\widetilde{\overline{\mathbf{S}}}}, \tag{2.38}$$

with $C = [C_4^{(0)} - 2(C_2^{(0)})^2]/32$ and $C' = C_4^{(0)}/16$. Using $\boldsymbol{\epsilon}^\top\overline{\mathbf{S}}\boldsymbol{\epsilon} = -\overline{\mathbf{S}}$ and (2.36), this can also be written as

$$\Pi_*^{(0),[2]} = C\ell^4(\nabla\overline{\omega})^\top\overline{\mathbf{S}}(\nabla\overline{\omega}) + 4C'\ell^4\overline{\sigma}^2(\nabla\overline{\omega}\cdot\nabla\overline{\alpha}). \tag{2.39}$$

These are the only UV-local contributions to the energy flux at second order.

It is important to determine the sign of these terms, on average, to see whether they contribute to inverse cascade or direct cascade. In this respect, note that the first term in (2.39) is proportional to the negative of the rate of vorticity-gradient stretching by the large-scale strain. That is, if we consider the equation for the large-scale vorticity gradient, then it has the form

$$\overline{D}_t|\nabla\overline{\omega}|^2 = -2(\nabla\overline{\omega})^\top\overline{\mathbf{S}}(\nabla\overline{\omega}) + \cdots, \tag{2.40}$$

where $\overline{D}_t = \partial_t + \overline{\boldsymbol{u}}\boldsymbol{\cdot}\nabla$ and $\cdots$ denotes neglected terms due to the turbulent stress. Thus, we see that the first term in (2.39) is negative (inverse cascade) precisely when vorticity gradients are magnified, a connection already noted by Kraichnan (1976). Equivalently, inverse cascade requires the stretching direction $\boldsymbol{e}_+^{(\sigma)}$ of the strain field to tend to be parallel to contour lines of the large-scale vorticity. Since we have already seen that the small scales induce a tensile stress along the contour lines, the stress and strain cooperate in this alignment and negative work is done by the large scales against the small scales. Equation (2.40) renders the required alignment plausible, since components of the vorticity gradient parallel to the squeezing direction will tend to grow, according to this equation. Note that this tendency might be moderated somewhat by the small-scale stress terms which we have neglected in (2.40) (cf. Van der Bos *et al.* 2002).

The second term in (2.39) will be negative precisely when $\nabla\overline{\omega}\boldsymbol{\cdot}\nabla\overline{\alpha} < 0$. This means that the strain frame must counter-rotate against vorticity changes, i.e. rotate clockwise moving in the direction of increasing vorticity. We do not have a direct dynamical explanation for this tendency, analogous to the one we gave above for vorticity-gradient stretching. On the other hand, we have found that there is a simple kinematic relation between the rates of differential strain-rotation and vorticity-gradient stretching in two dimensions:

$$\langle(\nabla\overline{\omega})^{\top}\overline{\mathbf{S}}(\nabla\overline{\omega})\rangle = -\langle\overline{\mathbf{S}}\boldsymbol{:}(\nabla\overline{\omega}\boldsymbol{\cdot}\nabla)\widetilde{\overline{\mathbf{S}}}\rangle, \tag{2.41}$$

or, equivalently,

$$\langle(\nabla\overline{\omega})^{\top}\overline{\mathbf{S}}(\nabla\overline{\omega})\rangle = 4\langle\overline{\sigma}^2(\nabla\overline{\omega}\boldsymbol{\cdot}\nabla\overline{\alpha})\rangle. \tag{2.42}$$

Equation (2.41), or (2.42), is an exact two-dimensional analogue of the three-dimensional relation of Betchov (1956), and, like it, depends just on homogeneity and incompressibility of the velocity field. For a proof of the 'two-dimensional Betchov relation' (2.41), see Appendix C. An important immediate consequence is that differential strain counter-rotation and vorticity-gradient stretching must occur together, on average, while differential strain co-rotation is associated with mean shrinking of vorticity gradients. (Because it is purely kinematic, the 'two-dimensional Betchov relation' holds just as well in the enstrophy cascade range. As discussed in Eyink (2001) and Chen *et al.* (2003), forward enstrophy flux is also associated with mean stretching of filtered vorticity gradients. Thus, differential strain counter-rotation must also occur, on average, in the enstrophy cascade.)

The net energy flux from both terms in (2.39) is always negative (inverse cascade) when there is mean stretching of vorticity gradients. Because of the Betchov-like relation (2.42) it follows that $\langle\Pi_*^{(0),[2]}\rangle = (C + C')\ell^4\Gamma$, where Γ is the common average in (2.42) and

$$C + C' = \tfrac{1}{32}C_4^{(0)} + \tfrac{1}{16}\left[C_4^{(0)} - \left(C_2^{(0)}\right)^2\right] \geqslant 0. \tag{2.43}$$

To prove inequality (2.43), note that $C_4^{(0)} \geqslant 0$ by its definition. Furthermore,

$$C_2^{(0)} = \int_{|\boldsymbol{r}|\geqslant 1} \mathrm{d}\boldsymbol{r}\ |\boldsymbol{r}|^2 G(\boldsymbol{r}) \leqslant \sqrt{\int_{|\boldsymbol{r}|\geqslant 1} \mathrm{d}\boldsymbol{r}\ G(\boldsymbol{r})\boldsymbol{\cdot}\int_{|\boldsymbol{r}|\geqslant 1} \mathrm{d}\boldsymbol{r}\ |\boldsymbol{r}|^4 G(\boldsymbol{r})} \leqslant \sqrt{C_4^{(0)}} \tag{2.44}$$

by the Cauchy–Schwartz inequality and normalization of G. This gives (2.43). Thus, for any filter, the net flux is negative when $\Gamma < 0$. The two-dimensional Betchov relation, furthermore, gives the ratio of contribution to inverse cascade of the two terms in (2.39), as C/C'. For a Gaussian filter, this ratio is $C/C' = (1/2) - (9/13)\mathrm{e}^{-1/2} \doteq 0.08$, so that approximately 92.6 % of the mean of (2.39) comes from differential strain-rotation and 7.4 % from vorticity-gradient stretching.

2.2.2. *The weakly UV-local terms*

The terms of the MSG expansion that are second order in gradients contribute to energy flux already from the strongly UV-local modes. However, there are additional contributions at second-order from all the other subscale modes. Here we shall discuss the physical interpretation and significance of those.

In fact, the various terms that appear in the expressions for the two-dimensional model stress, (2.32) and (2.33), can be readily understood. The first term in (2.32), which is first order in gradients, has already been discussed. In the next group of three second-order terms, the first and last are both pressure contributions and do not contribute to energy flux. However, the middle term is deviatoric and can give rise to flux. Using the analogue of (2.36),

$$\left(\nabla\omega^{[k]}\cdot\nabla\right)\widetilde{\boldsymbol{S}}^{[k]} = -2\left(\nabla\omega^{[k]}\cdot\nabla\alpha^{[k]}\right)\boldsymbol{S}^{[k]} + \left(\nabla\omega^{[k]}\cdot\nabla\lambda^{[k]}\right)\widetilde{\boldsymbol{S}}^{[k]}, \tag{2.45}$$

this term can be split into two. The first is a Newtonian stress $-2\nu_T^{[k]}\boldsymbol{S}^{[k]}$ with an eddy-viscosity coefficient,

$$\nu_T^{[k]} = \tfrac{1}{16}\overline{C}_4^{[k]}\ell_k^4\left(\nabla\omega^{[k]}\cdot\nabla\alpha^{[k]}\right), \tag{2.46}$$

arising from differential strain rotation at a length scale ℓ_k. The other term is of the 'skew-Newtonian' form $\gamma_T^{[k]}\widetilde{\boldsymbol{S}}^{[k]}$ with skew-viscosity coefficient

$$\gamma_T^{[k]} = \tfrac{1}{16}\overline{C}_4^{[k]}\ell_k^4\left(\nabla\omega^{[k]}\cdot\nabla\lambda^{[k]}\right) \tag{2.47}$$

arising from differential strain-magnification at the same length-scale ℓ_k. Note that we have defined the logarithm of the strain eigenvalue or magnitude as $\lambda^{[k]} = \ln\sigma^{[k]}$. Since the velocity field in the inverse cascade range is monofractal with Hölder exponent 1/3 (Paret & Tabeling 1998; Yakhot 1999; Boffetta *et al.* 2000), it is not hard to see that both $\nu_T^{[k]}$ and $\gamma_T^{[k]}$ are of order $O(\ell_k^{4/3})$, as expected. (To show this, use the formulae $2\nabla\alpha = (S_{11}\nabla S_{12} - S_{12}\nabla S_{11})/(S_{11}^2 + S_{12}^2)$, $\nabla\lambda = (S_{11}\nabla S_{11} + S_{12}\nabla S_{12})/(S_{11}^2 + S_{12}^2)$, and the general estimates from Eyink (2005) and (I). The last term in (2.32) represents a tensile stress of magnitude $+\overline{C}_4^{[k]}\ell_k^4|\nabla\omega^{[k]}|^2/32$ exerted along contour lines of the vorticity $\omega^{[k]}$ at length scale ℓ_k.

There remains the 'fluctuation' contribution to the stress from (2.33). This can be best understood by summing over scales, to give $\boldsymbol{u}_*'^{(n,2)} = \widetilde{\nabla}\psi_*^{(n)}$ with a fluctuation streamfunction

$$\psi_*^{(n)} = \tfrac{1}{4}\sum_{k=0}^{n}\frac{\overline{C}_2^{[k]}}{\sqrt{N_k}}\,\ell_k^2\omega^{[k]}. \tag{2.48}$$

Note that the factor $1/\sqrt{N_k}$ reflects the cancellations that are expected to occur in the space integral for the contributions from modes at length scale ℓ_k (I). We see then, finally, that $-\widetilde{\nabla}\psi_*^{(n)}\widetilde{\nabla}\psi_*^{(n)}$ represents a contractile stress along the streamlines of $\psi_*^{(n)}$. This term opposes and, to some degree, cancels against the tensile stress terms in (2.32) exerted along the contour lines of $\omega^{[k]}$ for $k = 1, \ldots, n$.

If the model stress is substituted into (1.1) for the flux, then there results:

$$\begin{aligned}\Pi_*^{(n,2)} = \sum_{k=0}^{n} \Big\{ &\tfrac{1}{2}\overline{C}_2^{[k]} \ell_k^2 \omega^{[k]} \big(\boldsymbol{S}^{(0)} \boldsymbol{:} \widetilde{\boldsymbol{S}}^{[k]}\big) + \tfrac{1}{8}\overline{C}_4^{[k]} \ell_k^4 \big(\nabla\omega^{[k]} \boldsymbol{\cdot} \nabla\alpha^{[k]}\big)\big(\boldsymbol{S}^{(0)} \boldsymbol{:} \boldsymbol{S}^{[k]}\big) \\ &- \tfrac{1}{16}\overline{C}_4^{[k]} \ell_k^4 \big(\nabla\omega^{[k]} \boldsymbol{\cdot} \nabla\lambda^{[k]}\big)\big(\boldsymbol{S}^{(0)} \boldsymbol{:} \widetilde{\boldsymbol{S}}^{[k]}\big) + \tfrac{1}{32}\overline{C}_4^{[k]} \ell_k^4 \big(\nabla\omega^{[k]}\big)^{\top} \boldsymbol{S}^{(0)} \big(\nabla\omega^{[k]}\big) \Big\} \\ &- \big(\nabla\psi_*^{(n)}\big)^{\top} \boldsymbol{S}^{(0)} \big(\nabla\psi_*^{(n)}\big). \end{aligned} \tag{2.49}$$

This is our final CSA expansion result for the energy flux in two dimensions. In addition to the first-order term that appeared in (2.17), there are now second-order contributions arising from differential strain rotation, differential strain magnification, and vorticity-gradient stretching. The final term in (2.49) is expected to be much smaller than the others, because of the cancellations in space averaging discussed above and additional cancellations in the sum over scales in (2.48). We expect that the first four terms contribute to inverse cascade. For small k, $\boldsymbol{S}^{[k]}$ should be correlated to some degree with $\boldsymbol{S}^{(0)}$, so that the differential strain-rotation and vorticity-gradient stretching terms ought to have negative mean values, for similar reasons as the corresponding $k = 0$ terms discussed earlier. Like the first-order 'skew-Newtonian' term, the differential strain-magnification term vanishes for $k = 0$ and can therefore be expected to be relatively smaller than the differential strain-rotation term. The latter has its sign determined by the quantity $\nabla\omega^{[k]} \boldsymbol{\cdot} \nabla\alpha^{[k]} \cos[2(\alpha^{[k]} - \alpha^{(0)})]$, closely related to the signed quantity $\omega^{[k]} \sin[2(\alpha^{[k]} - \alpha^{(0)})]$ that appears in the first-order term. The final term in (2.49) is the only one that we expect to have a positive mean (from vorticity-gradient stretching), but we have already argued that that term will be considerably smaller in magnitude.

Note that the flux term in (2.49) from scale k gives at most a fraction of order $2^{-2k/3}$ to the net energy flux. This agrees with rigorous locality estimates (Eyink 2005). However, the actual contribution is likely to be much smaller, since the correlations which produce the inverse energy cascade must weaken for $k \gg 1$. If the small scales are isotropic, then the mean stress $\boldsymbol{\tau}^{[k]}$ from length scale ℓ_k will satisfy:

$$\big\langle \tau_{ij}^{[k]} \big\rangle = \tfrac{1}{2}\big\langle \mathrm{Tr}\,[\boldsymbol{\tau}^{[k]}]\big\rangle \delta_{ij} \quad \text{for } k \gg 1. \tag{2.50}$$

In that case, if the large-scale strain $\boldsymbol{S}^{(0)}$ and the stress contribution $\boldsymbol{\tau}^{[k]}$ are asymptotically independent for $k \gg 1$, then their mean contribution to the energy flux vanishes, since the deviatoric part of the stress is zero on average. The existence of an energy cascade requires a statistical correlation between the large-scale strain and the small-scale stress contributions from various scales, which becomes progressively weaker for increasing k.

3. Discussion

The theoretical expression that we have developed here for the turbulent stress yields many concrete testable predictions – both qualitative and quantitative – for the two-dimensional inverse energy cascade. Foremost, we predict that strain frames at small scales should lag/lead those at large scales, when the small-scale vorticity is positive/negative. A spatial analogue of this effect is that the strain eigenframes are predicted on average to rotate clockwise in the direction of increasing vorticity (differential counter-rotation). Likewise, we predict that there will be a positive mean rate of stretching of vorticity gradients. More quantitatively, our final CSA-MSG formulae (2.2), (2.32), (2.33) for the stress and (2.49) for the flux may be compared in detail with results obtained from experiment or simulation. If the model survives such tests, then it may be a good point of departure for building a practical LES modelling scheme of the two-dimensional inverse energy cascade.

In our presentation above we have alluded only briefly to the dynamical mechanisms that can produce the various correlations and alignments that are postulated, e.g. based on the evolution equations of strain orientation angles (2.18) and of vorticity gradients (2.40). Many of the mechanisms expected to operate in two dimensions have very close analogues in three dimensions. Notice that vortex stretching in three dimensions is a near relative of the vortex-thinning mechanism in two dimensions, which we discussed in § 2.1.3. However, the result is opposite, because the stretching process in three dimensions 'spins up' the vortices and increases the kinetic energy in the small scales. Vorticity contour lines in two dimensions can also be expected to lengthen on the basis of the same plausible statistical arguments that have been applied to vortex lines or other material lines in three dimensions (Taylor 1938; Batchelor 1952; Cocke 1969). This already argues rather strongly for the stretching of vorticity gradients in two-dimensions incompressible turbulence and, via the Betchov-like relation (2.42), for differential rotation of strain counter to vorticity. On the other hand, in three-dimensions, rather more detailed understanding is available through simple Lagrangian models of the evolution of velocity gradients (Vieillefosse 1982, 1984; Cantwell 1992; Chertkov, Pumir & Shraiman 1999). These phenomenological models have provided plausible dynamical explanations of the key alignments that are observed in DNS (Ashurst *et al.* 1987) and experiment (Tao, Katz & Meneveau 2002). Some of the difficulties in developing such understanding of the inverse energy cascade can be appreciated by considering the exact equations in two dimensions for Lagrangian time derivatives of the velocity gradients:

$$\left.\begin{aligned} \overline{D}_t\overline{\omega} &= 0, \\ \overline{D}_t\overline{S}_{ij} &= \tfrac{1}{2}(\Delta\overline{p})\delta_{ij} - \partial^2_{ij}\overline{p}. \end{aligned}\right\} \qquad (3.1)$$

Here, we have considered separately the evolution of the vorticity and strain. We have also neglected the contribution of turbulent stresses to the evolution of filtered gradients, which may be an important feedback interaction with small scales (Van der Bos *et al.* 2002). The equations (3.1) lack the local self-stretching terms which play the key role in the analogous three-dimensional equations. In fact, the Lagrangian evolution in (3.1) is entirely trivial except for the pressure hessian in the equation for the strain and the latter must play an essential role in the production of strain orientation alignments. More sophistication in the modelling of pressure is therefore likely to be required than in the three-dimensional case (Vieillefosse 1982, 1984; Cantwell 1992; Chertkov *et al.* 1999). Furthermore, we have seen that in the two-dimensional inverse cascade, both higher-order gradient and multi-scale effects are important. Thus, it remains a challenge to develop a detailed dynamical understanding of the two-dimensional inverse energy cascade.

I wish to thank S. Chen, B. Ecke, M. K. Rivera, M.-P. Wang and Z. Xiao for a very fruitful collaboration on two-dimensional turbulence which helped to stimulate the development of the present theory. I would also like to thank C. Meneveau and E. Vishniac for helpful discussions. This work was supported in part by NSF grant ASE-0428325.

Appendix A. Dynamical equation for the strain orientation

It is easy to see from the 'polar' representation (2.15) of the strain $\overline{\boldsymbol{S}}$ that $2\overline{\alpha} = \arctan(\overline{S}_{12}/\overline{S}_{11})$. Since also $\overline{\sigma}^2 = \overline{S}_{12}^2 + \overline{S}_{11}^2$, the Lagrangian derivative may be written as

$$2\overline{\sigma}^2 \overline{D}_t \overline{\alpha} = \overline{S}_{11}(\overline{D}_t \overline{S}_{12}) - \overline{S}_{12}(\overline{D}_t \overline{S}_{11}). \tag{A 1}$$

We can evaluate the time rate of change from the equation (3.1) for the filtered strain, which neglects the contribution from turbulent stress. Substituting into (A 1) we obtain

$$\begin{aligned} 2\overline{\sigma}^2 \overline{D}_t \overline{\alpha} &= \frac{\partial \overline{u}}{\partial x}\left[-\frac{\partial^2 \overline{p}}{\partial x \partial y}\right] - \frac{1}{2}\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right)\left[\tfrac{1}{2}\Delta \overline{p} - \frac{\partial^2 \overline{p}}{\partial x^2}\right] \\ &= \frac{1}{2}\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right)\frac{\partial^2 \overline{p}}{\partial x^2} - \frac{1}{2}\left(\frac{\partial \overline{u}}{\partial x} - \frac{\partial \overline{v}}{\partial y}\right)\frac{\partial^2 \overline{p}}{\partial x \partial y} - \frac{1}{4}\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right)\Delta \overline{p}, \end{aligned} \tag{A 2}$$

where we used incompressibility in the last line and also to derive the next identity:

$$\begin{aligned} \frac{\partial \overline{u}}{\partial y}\frac{\partial^2 \overline{p}}{\partial x^2} - \frac{\partial \overline{v}}{\partial x}\frac{\partial^2 \overline{p}}{\partial y^2} - \left(\frac{\partial \overline{u}}{\partial x} - \frac{\partial \overline{v}}{\partial y}\right)\frac{\partial^2 \overline{p}}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial \overline{p}}{\partial x}\frac{\partial \overline{u}}{\partial y} - \frac{\partial \overline{p}}{\partial y}\frac{\partial \overline{u}}{\partial x}\right) \\ &\quad + \frac{\partial}{\partial y}\left(\frac{\partial \overline{p}}{\partial x}\frac{\partial \overline{v}}{\partial y} - \frac{\partial \overline{p}}{\partial y}\frac{\partial \overline{v}}{\partial x}\right). \end{aligned} \tag{A 3}$$

If (A 3) is used in (A 2) to eliminate the mixed partial derivative of pressure, then we obtain

$$2\overline{\sigma}^2 \overline{D}_t \overline{\alpha} = \frac{1}{4}\left(\frac{\partial \overline{v}}{\partial x} - \frac{\partial \overline{u}}{\partial y}\right)\Delta \overline{p} + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \overline{p}}{\partial x}\frac{\partial \overline{u}}{\partial y} - \frac{\partial \overline{p}}{\partial y}\frac{\partial \overline{u}}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \overline{p}}{\partial x}\frac{\partial \overline{v}}{\partial y} - \frac{\partial \overline{p}}{\partial y}\frac{\partial \overline{v}}{\partial x}\right). \tag{A 4}$$

This last equation is equivalent to (2.18), (2.19), (2.20) in the text.

Appendix B. Vortex-thinning and negative eddy-viscosity

Kraichnan proposed a physical mechanism to explain the origin of negative eddy viscosities in two dimensions (see Kraichnan 1976, § 5.) For this purpose, he employed a simplified model of small-scale vortex wave-packets in a uniform large-scale straining field. His aim was to understand the asymptotic effect of the small scales on much larger scales, and not to give an account of the inverse energy cascade by scale-local interactions. Nevertheless, his ideas turn out to have much in common with our theory of the local cascade interactions. The model proposed by us in § 2.1.3 to explain the stress proportional to skew-strain is just a slight modification of Kraichnan's. Furthermore, his mechanism of 'negative viscosity' is essentially identical with that found in the last term of our model stress, equation (2.34), which corresponds to a tensile stress along vorticity-contour lines. Here we shall review the calculation of Kraichnan (1976), in order to make more clear its relation to the present theory.

Kraichnan's model of the small scales was a Gaussian wave-packet of vorticity – called a 'blob' – or an 'assembly of uncorrelated blobs' (Kraichnan 1976). The streamfunction of each blob was taken to have the form

$$\left.\begin{aligned} \psi(\boldsymbol{x}) &= k^{-2} f(\boldsymbol{x}) \cos(k x_2), \\ f(\boldsymbol{x}) &= \exp\left(-\tfrac{1}{2}\left(x_1^2 + x_2^2\right)/D^2\right), \end{aligned}\right\} \tag{B 1}$$

where f is a Gaussian envelope function with a standard deviation $\sim D$ that is modulated by an oscillating cosine with wavevector $\boldsymbol{k}$ pointing in the vertical $\boldsymbol{e}_2$-direction. A basic assumption is that $kD \gg 1$, so that the wavenumber of the packet can be regarded as nearly sharp. Calculating the small-scale velocity field from $\boldsymbol{u} = -\widetilde{\nabla}\psi$, it is not hard to show that the leading component of the velocity is

$$u_1 \sim k^{-1} f(\boldsymbol{x}) \sin(kx_2), \tag{B 2}$$

and of the vorticity-gradient is

$$(\nabla\omega)_2 \sim -kf(\boldsymbol{x}) \sin(kx_2), \tag{B 3}$$

asymptotically for $kD \gg 1$ (cf. (5.4) in Kraichnan 1976). Thus, the dominant component of the total stress $\boldsymbol{T} = \int \boldsymbol{\tau} = \int \boldsymbol{uu}$ is

$$T_{11} = k^{-2} \int \mathrm{d}x_1 \int \mathrm{d}x_2 \exp\left(-\left(x_1^2 + x_2^2\right)/D^2\right) \sin^2(kx_2) \sim \pi D^2/2k^2 \tag{B 4}$$

for $kD \gg 1$. That is, the dominant stress is positive, or tensile, and exerted along the horizontal direction $\boldsymbol{e}_1$. This is perpendicular to the direction of the vorticity gradient $\boldsymbol{e}_2$, or along the direction of the vorticity contours. Thus, Kraichnan's 'blob model' leads to a result in agreement with our general conclusion.

As a model of the large scales, Kraichnan took a uniform strain field,

$$\boldsymbol{S} = a \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix}, \tag{B 5}$$

with eigenvalues $\pm a$ and eigenframe oriented at an angle ϕ with respect to the fixed coordinate frame. The streamfunction corresponding to this large-scale field is just $V(\boldsymbol{x}) = \boldsymbol{x}^\top \widetilde{\boldsymbol{S}} \boldsymbol{x}/2$. Actually, Kraichnan kept the strain fixed with frame axes along the coordinate directions and instead rotated the wavenumber of the small-scale blob, as $\boldsymbol{k} = k(\boldsymbol{e}_1 \sin\phi + \boldsymbol{e}_2 \cos\phi)$, ((5.14) in Kraichnan 1976). This is physically more natural, if we think of the small scales as isotropic and the large scales as having fixed anisotropy. However, it is mathematically equivalent to rotate the strain, and it relates more easily to our analysis in the text.

Kraichnan (1976) worked out in detail the energy balance for his simple two-scale model of the velocity field. The initial energy in the small-scales is

$$E = \tfrac{1}{2} \int |\boldsymbol{u}|^2 \sim (1/2)T_{11} \sim \pi D^2/4k^2. \tag{B 6}$$

The effect of the straining field on the small-scale wavevector is to change its magnitude by

$$\mathrm{d}k^2/\mathrm{d}t = -2\boldsymbol{k}^\top \boldsymbol{Sk} = 2a\cos(2\phi)k^2. \tag{B 7}$$

Thus, Kraichnan concluded that, to leading order,

$$(\mathrm{d}E/\mathrm{d}t)_{t=0} = -\pi a D^2 \cos(2\phi)/2k^2. \tag{B 8}$$

Cf. (5.8) in Kraichnan (1976) for the case where $k = 1$ and $\phi = 0$. This reduction in energy of the small-scale blob is a consequence of the transfer of its enstrophy to higher wavenumber.

Kraichnan showed further that the energy budget was maintained by a deposit into the 'interaction energy' $\int \boldsymbol{v} \cdot \boldsymbol{u}$ between the large-scale and small-scale velocity fields. In his calculation, he rewrote the interaction energy as $\int V\omega$, in terms of the large-scale streamfunction V and small-scale vorticity ω, and considered the nonlinear self-interaction of the latter. He found that the small-scale vorticity field set up a secondary flow of four equal-strength vortices with alternating signs of circulation which, for $\phi = 0$, reinforced the large-scale strain. In his own words:

If a small-scale motion has the form of a compact blob of vorticity, or an assembly of uncorrelated blobs, a steady straining will eventually draw a typical blob out into an elongated shape, with corresponding thinning and increase of typical wavenumber. The typical result will be a decrease of the kinetic energy of the small-scale motion and a corresponding reinforcement of the straining field.

In this way, the energy loss from the small scales that is observed in (B 8) can be traced to a transfer of equal size into the interaction energy between large scales and small scales.

This transfer can be shown to be equivalent to the scale-to-scale energy flux that we defined in (1.1). Indeed, using the fact that the large-scale velocity $\boldsymbol{v}$ is stationary and its velocity gradient $\nabla\boldsymbol{v}$ is uniform, we find that

$$(\mathrm{d}E/\mathrm{d}t)_{t=0} = -\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{v}\cdot\boldsymbol{u} = \int \boldsymbol{v}\cdot[\nabla\cdot(\boldsymbol{u}\boldsymbol{u})] = -\int(\nabla\boldsymbol{v})\boldsymbol{:}\boldsymbol{u}\boldsymbol{u} = -\boldsymbol{S}\boldsymbol{:}\boldsymbol{T}. \tag{B 9}$$

This is the area-integral of the quantity that appears in (1.1). We can use this expression to verify easily the energy balance result from Kraichnan (1976). Substituting the stress from (B 4) and the strain from (B 5), we obtain $(\mathrm{d}E/\mathrm{d}t)_{t=0} = -\pi a D^2\cos(2\phi)/2k^2$, in agreement with (B 8). Note that the flux is negative and the small scales lose energy only if $|\phi| < \pi/4$, whereas the flux is positive for $\pi/4 < |\phi| < \pi/2$. If we assume that the angle ϕ is random with an isotropic distribution and $\boldsymbol{k} = k\boldsymbol{e}_2$ is fixed, then the average flux is $\langle(\mathrm{d}E/\mathrm{d}t)_{t=0}\rangle_{\mathrm{ang}} = 0$. Kraichnan (1976) had already noted this result and established its consistency with the mean growth of small-scale wavenumber magnitude or, equivalently, the mean stretching of small-scale vorticity gradients. As we discussed around our equation (2.50), a mean energy flux under isotropic conditions requires statistical correlations between disparate scales. In Kraichnan's case where he assumed a very wide separation between the two scales of motion, it was realistic to assume negligible correlations and thus zero net transfer. However, this is an unrealistic assumption in the context of a local energy cascade, where the stress and strain in (1.1) obtain most of their contributions from adjacent scales (Eyink 2005) and are highly correlated.

The mechanism that Kraichnan identified as acting between distant scales can also be identified with several of the mechanisms that we have found in our analysis of local cascade interactions. Note that in Kraichnan's vortex-blob model

$$(\nabla\omega)^{\top}\boldsymbol{S}(\nabla\omega) = -ak^2 f^2(\boldsymbol{x})\sin^2(kx_2)\cos(2\phi), \tag{B 10}$$

using (B 3) and (B 5). Integrated over space, this yields

$$\int(\nabla\omega)^{\top}\boldsymbol{S}(\nabla\omega) = -\pi a(Dk)^2\cos(2\phi)/2, \tag{B 11}$$

to leading order for $Dk \gg 1$. Thus, we have agreement of (B 8) with the fourth term in our formula $\Pi_*^{(n,2)}$ for the energy flux, equation (2.49), by taking $\ell_k = 1/k$ there. The second term in (2.49) corresponding to differential strain rotation is zero in the vortex-blob model because the orientations of the strain fields (both large-scale and small-scale) are uniform in space. However, we can equally well understand the energy flux in the blob model based upon the first term in (2.49) (the same as (2.14)) that corresponds to relative rotation of strain at disparate scales. Indeed, in the blob model, the vorticity is

$$\omega(\boldsymbol{x}) \sim -f(\boldsymbol{x})\cos(kx_2) \tag{B 12}$$

and the small-scale strain of the blob is

$$\boldsymbol{S}'(\boldsymbol{x}) = \tfrac{1}{2}\omega(\boldsymbol{x})\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{B 13}$$

to leading order. Thus, it is not hard to calculate that

$$\omega \boldsymbol{S} : \widetilde{\boldsymbol{S}}' = -af^2(\boldsymbol{x})\cos^2(kx_2)\cos(2\phi), \tag{B 14}$$

and integrated over space this gives also

$$\int \omega \boldsymbol{S} : \widetilde{\boldsymbol{S}}' = -\pi a D^2 \cos(2\phi)/2 \tag{B 15}$$

to leading order for $Dk \gg 1$. Multiplying (B 15) by $\ell_k^2 = 1/k^2$, we also find agreement of (B 8) with (2.14). Before averaging over space, the two contributions from (B 10) and (B 14) are exactly out of phase. It is another simple exercise to verify that the third term in (2.49), from differential strain magnification, is also non-zero in the blob model and gives a contribution of the same sort.

Thus, it is clear that most of the terms in our CSA-MSG formula, (2.49), are represented in Kraichnan's blob model, in particular, the flux from skew-strain, from differential strain magnification, and from vorticity-gradient stretching. All of these can be produced by a single mechanism of 'vortex thinning'. Our somewhat simpler model of vortex patches in § 2.1.3 also illustrates these same flux terms, except in the case of constant-vorticity patches, for which only the flux from skew-strain survives. The increase in wavenumber that was considered by Kraichnan in his blob model and the asymptotics $Dk \gg 1$ play no essential role in the skew-strain mechanism. Indeed, note that (B 12)–(B 15) for the blob model all have non-vanishing values at $k = 0$, whereas (B 10)–(B 11) tend to zero as $k \to 0$.

Appendix C. Two-dimensional Betchov relation

For any incompressible or solenoidal field $\boldsymbol{u}$ in two dimensions, we can define a corresponding 'strain' $S_{ij}^{(u)}$ and 'vorticity' $\omega^{(u)}$ via

$$\frac{\partial u_i}{\partial x_j} = u_{i,j} = S_{ij}^{(u)} - \tfrac{1}{2}\epsilon_{ij}\omega^{(u)}. \tag{C 1}$$

Observe our notation for partial derivative with respect to x_j, indicated by subscript j preceded by a comma. Likewise, we write $\partial^2 u_i/\partial x_j \partial x_k = u_{i,jk}$, etc. Using these notations and definitions, the first step in the derivation of the two-dimensional Betchov relation is the following identity:

$$\begin{aligned} \partial_l\left[S_{ij}^{(u)} v_{i,k} w_{j,kl}\right] - \partial_k\left[S_{ij}^{(u)} v_{i,k} w_{j,ll}\right] &= \underbrace{S_{ij}^{(u)} v_{i,kl} w_{j,kl}}_{①} - \underbrace{S_{ij}^{(u)} v_{i,kk} w_{j,ll}}_{②} \\ &\quad + \underbrace{S_{ij,l}^{(u)} v_{i,k} w_{j,kl}}_{③} - \underbrace{S_{ij,k}^{(u)} v_{i,k} w_{j,ll}}_{④}. \end{aligned} \tag{C 2}$$

Here $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are all incompressible fields. The identity (C 3) follows straightforwardly from the product of the rule of differentiation.

https://doi.org/10.1017/S0022112005007883 Published online by Cambridge University Press

The terms labelled ①, ②, and ③ are easily calculated by the substitutions $v_{i,j} = S_{ij}^{(v)} - \epsilon_{ij}\omega^{(v)}/2$, $v_{i,jk} = S_{ij,k}^{(v)} - \epsilon_{ij}\partial_k\omega^{(v)}/2$, and $v_{i,kk} = -\epsilon_{il}\partial_l\omega^{(v)}$, and similar substitutions for the field $\boldsymbol{w}$. The term ① becomes

$$S_{ij}^{(u)} v_{i,kl} w_{j,kl} = \tfrac{1}{2} S_{ij}^{(u)} \left(\widetilde{S}_{ij,l}^{(v)} \partial_l \omega^{(w)} + \widetilde{S}_{ij,l}^{(w)} \partial_l \omega^{(v)} \right), \tag{C 3}$$

② becomes

$$-S_{ij}^{(u)} v_{i,kk} w_{j,ll} = S_{ij}^{(u)} \partial_i \omega^{(v)} \partial_j \omega^{(w)}, \tag{C 4}$$

and ③ becomes

$$S_{ij,l}^{(u)} v_{i,k} w_{j,kl} = -\tfrac{1}{2} S_{ik}^{(v)} \widetilde{S}_{ik,l}^{(u)} \partial_l \omega^{(w)} + \tfrac{1}{2} S_{ij,l}^{(u)} \widetilde{S}_{ij,l}^{(w)} \omega^{(v)}. \tag{C 5}$$

Term ④ requires as an additional step the use of the identity

$$S_{ij,k}^{(u)} - \tfrac{1}{2}\epsilon_{ij}\partial_k\omega^{(u)} = u_{i,jk} = u_{i,kj} = S_{ik,j}^{(u)} - \tfrac{1}{2}\epsilon_{ik}\partial_j\omega^{(u)} \tag{C 6}$$

to replace $S_{ij,k}^{(u)}$ by $S_{ik,j}^{(u)}$. Then using the same substitutions as for the other three terms, ④ becomes

$$-S_{ij,k}^{(u)} v_{i,k} w_{j,ll} = \widetilde{S}_{ik,j}^{(u)} S_{ik}^{(v)} \partial_j \omega^{(w)} + \tfrac{1}{2} S_{jk}^{(v)} \partial_k \omega^{(u)} \partial_j \omega^{(w)} + \tfrac{1}{4} \epsilon_{kj} \partial_k \omega^{(u)} \partial_j \omega^{(w)} \omega^{(v)}. \tag{C 7}$$

We are now able to sum the contributions from all four terms, ① ② ③ and ④. In order to simplify the result, it is helpful to define the quantity

$$T_{ij}^{(u,v)} = \partial_i \omega^{(u)} \partial_j \omega^{(v)} + \widetilde{S}_{ij,k}^{(u)} \partial_k \omega^{(v)}. \tag{C 8}$$

Then the sum of the four terms yields, after some elementary algebra,

$$S_{ij}^{(v)} T_{ij}^{(u,w)} + S_{ij}^{(u)} T_{ij}^{(v,w)} + S_{ij}^{(u)} T_{ij}^{(w,v)} = \epsilon_{ij} \left[S_{ik,l}^{(u)} S_{jk,l}^{(w)} - \partial_i \omega^{(u)} \partial_j \omega^{(w)} \right] \omega^{(v)} \\ - 2\partial_k \left[S_{ij}^{(u)} v_{i,k} w_{j,ll} \right] + 2\partial_l \left[S_{ij}^{(u)} v_{i,k} w_{j,kl} \right]. \tag{C 9}$$

Observe that the first term on the right-hand side of (C 9) is antisymmetric under the interchange $\boldsymbol{u} \leftrightarrow \boldsymbol{w}$. Thus, if we symmetrize (C 9) in $\boldsymbol{u}$ and $\boldsymbol{w}$, we obtain

$$S_{ij}^{(u)} T_{ij}^{(v,w)} + \text{perm.} = \text{div}\,[\cdots], \tag{C 10}$$

where the sum on the left-hand side is over all six permutations of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and div $[\cdots]$ on the right-hand side indicates a total divergence. It therefore follows that

$$\left\langle S_{ij}^{(u)} T_{ij}^{(v,w)} \right\rangle + \text{perm.} = 0, \tag{C 11}$$

where $\langle\cdots\rangle$ denotes either an average over a homogeneous ensemble or a space average with boundary conditions that permit integrations by parts (e.g. periodic). We call the relation (C 11) the *generalized Betchov identity in two dimensions*. Setting $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w}$ gives

$$\left\langle S_{ij}^{(u)} T_{ij}^{(u,u)} \right\rangle = 0, \tag{C 12}$$

where $S_{ij}^{(u)}$ and $T_{ij}^{(u,u)}$ are now constructed from the field $\boldsymbol{u}$ alone. Equation (C 12) is equivalent to the two-dimensional Betchov relation (2.41) or (2.42) stated in the text.

REFERENCES

Abramowitz, M. & Stegun, I. A. 1964 *Handbook of Mathematical Functions*. National Bureau of Standards.

Ashurst, W. T., Kerstein, A. R., Kerr, R. M. & Gibson, C. H. 1987 Alignment of vorticity and scalar gradient with strain rate in simulated Navier–Stokes turbulence. *Phys. Fluids* **30**, 2343–2353.

Batchelor, G. K. 1952 The effect of turbulence on material lines and surfaces. *Proc. R. Soc. Lond.* A **213**, 349–366.

Betchov, R. 1956 An inequality concerning the production of vorticity in isotropic turbulence. *J. Fluid Mech.* **1**, 497–504.

Boffetta, G., Celani, A. & Vergassola, M. 2000 Inverse energy cascade in two-dimensional turbulence: deviations from Gaussian behavior. *Phys. Rev.* E **61**, R29–R32.

Borue, V. & Orszag, S. A. 1998 Local energy flux and subgrid-scale statistics in three-dimensional turbulence. *J. Fluid Mech.* **366**, 1–31.

Cantwell, B. 1992 Exact solution of a restricted Euler equation for the velocity gradient tensor. *Phys. Fluids* A **4**, 782–793.

Chen, S., Ecke, R. E., Eyink, G. L., Wang, X. & Xiao, Z. 2003 Physical mechanism of the two-dimensional enstrophy cascade. *Phys. Rev. Lett.* **91**, 214501.

Chertkov, M., Kolokolov, I. & Vergassola, M. 1998 Inverse versus direct cascades in turbulent advection. *Phys. Rev. Lett.* **80**, 512–515.

Chertkov, M., Pumir, A. & Shraiman, B. I. 1999 Lagrangian tetrad dynamics and the phenomenology of turbulence. *Phys. Fluids* **11**, 2394–2410.

Cocke, W. J. 1969 Turbulent hydrodynamic line stretching: consequences of isotropy. *Phys. Fluids* **12**, 2488–2492.

Eyink, G. L. 1996*a* Exact results for stationary turbulence in 2D: consequences of vorticity conservation. *Physica* D **91**, 97–142.

Eyink, G. L. 1996*b* Turbulence noise. *J. Stat. Phys.* **83**, 955–1019.

Eyink, G. L. 2001 Dissipation in turbulent solutions of two-dimensional Euler equations. *Nonlinearity* **14**, 787–802.

Eyink, G. L. 2005 Locality of turbulent cascades. *Physica* D **207**, 91–116.

Eyink, G. L. 2006 Multi-scale gradient expansion of the turbulent stress tensor. *J. Fluid Mech.* **549**, 159–190.

Frisch, U., Pouquet, A., Leorat, J. & Mazure, A. 1975 Possibility of an inverse cascade of magnetic helicity in magnetohydrodynamic turbulence. *J. Fluid Mech.* **68**, 769–778.

Frisch, U. & Sulem, P. L. 1984 Numerical simulation of the inverse cascade in two-dimensional turbulence. *Phys. Fluids* **27**, 1921–1923.

Fyfe, D., Montgomery, D. & Joyce, G. 1977 Dissipative, forced turbulence in 2-dimensional magnetohydrodynamics. *J. Plasma Phys.* **17**, 369–398.

Germano, M. 1992 Turbulence: the filtering approach. *J. Fluid Mech.* **238**, 325–336.

Herring, J. R. & McWilliams, J. C. 1985 Comparison of direct numerical-simulation of two-dimensional turbulence with 2-point closure – the effects of intermittency. *J. Fluid Mech.* **153**, 229–242.

Hossain, M., Matthaeus, W. H. & Montgomery, D. 1983 Long-time states of inverse cascades in the presence of a maximal length scale. *J. Plasma Phys.* **30**, 479–493.

Kida, S. 1981 Motion of an elliptic vortex in a uniform shear flow. *J. Phys. Soc. Japan* **50**, 3517–3520.

Kraichnan, R. H. 1967 Inertial ranges in two-dimensional turbulence. *Phys. Fluids* **10**, 1417–1423.

Kraichnan, R. H. 1971*a* An almost-Markovian Galilean-invariant turbulence model. *J. Fluid Mech.* **47**, 513–524.

Kraichnan, R. H. 1971*b* Inertial-range transfer in two- and three-dimensional turbulence. *J. Fluid Mech.* **47**, 525–535.

Kraichnan, R. H. 1976 Eddy viscosity in two and three dimensions. *J. Atmos. Sci.* **33**, 1521–1536.

Lilly, D. K. 1971 Numerical simulation of two-dimensional turbulence. *Phys. Fluids Suppl.* **12**, 240–249.

Lilly, D. K. 1972 Numerical simulation studies of two-dimensional turbulence: I. models of statistically steady turbulence. *Geophys. Fluid Dyn.* **3**, 289–319.

Lindenberg, K., West, B. J. & Kottalam, J. 1987 Fluctuations and dissipation in problems of geophysical fluid dynamics. In *Irreversible Phenomena and Dynamical Systems Analysis in Geosciences, NATO ASI Ser. C* (ed. C. Nicolis & G. Nicolis). Reidel.

Maltrud, M E. & Vallis, G. K. 1991 Energy-spectra and coherent structures in forced 2-dimensional and beta-plane turbulence. *J. Fluid Mech.* **228**, 321–342.

Meneveau, C. & Katz, J. 2000 Scale-invariance and turbulence models for large-eddy simulation. *Annu. Rev. Fluid Mech.* **32**, 1–32.

Paret, J. & Tabeling, P. 1998 Intermittency in the two-dimensional inverse cascade of energy: experimental observations. *Phys. Fluids* **10**, 3126–3136.

Rivera, M. K. 2000 The inverse energy cascade of two-dimensional turbulence. PhD thesis, University of Pittsburgh.

Rutgers, M. A. 1998 Forced two-dimensional turbulence: experimental evidence of simultaneous inverse energy and forward enstrophy cascades. *Phys. Rev. Lett.* **81**, 2244–2247.

Siggia, E. D. & Aref, H. 1981 Point-vortex simulation of the inverse energy cascade in 2-dimension turbulence. *Phys. Fluids* **24**, 171–173.

Sommeria, J. 1986 Experimental study of the two-dimensional inverse energy cascade in a square box. *J. Fluid Mech.* **170**, 139–168.

Starr, V. P. 1968 *Physics of Negative Viscosity Phenomena*. McGraw–Hill.

Tao, B., Katz, J. & Meneveau, C. 2002 Statistical geometry of subgrid-scale stresses determined from holographic particle image velocimetry measurements. *J. Fluid Mech.* **467**, 35–78.

Taylor, G. I. 1938 Production and dissipation of vorticity in a turbulent fluid. *Proc. R. Soc. Lond.* A **164**, 15–23.

Van der Bos, F., Tao, B., Meneveau, C. & Katz, J. 2002 Effects of small-scale turbulent motions on the filtered velocity gradient tensor as deduced from holographic PIV measurements. *Phys. Fluids* **14**, 2456–2474.

Vieillefosse, P. 1982 Local interaction between vorticity and shear in a perfect incompressible fluid. *J. Phys. Paris* **43**, 837–842.

Vieillefosse, P. 1984 Internal motion of a small element of fluid in an inviscid flow. *Physica* A **125**, 150–162.

Yakhot, V. 1999 Two-dimensional turbulence in the inverse cascade range. *Phys. Rev.* E **60**, 5544–5551.

Zakharov, V. E. 1967 On the theory of surface waves. PhD thesis, Budker Institute for Nuclear Physics, Novosibirsk, USSR.

Zakharov, V. E. & Zaslavskii, M. M. 1982 Kinetic equation and Kolmogorov's spectra in a weak turbulent theory of wind waves. *Izv. Akad. Nauk SSSR, Atmos. Ocean Phys.* **18**, 747–753.