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Wilton ripples generated by a moving pressure distribution

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The two-dimensional steady potential free surface flow due to a pressure distribution moving at a constant velocity at the surface of a fluid of infinite depth is considered. The effects of gravity and surface tension are included in the dynamic boundary condition. The fully nonlinear problem is solved numerically by a boundary integral equation method and the results are compared with those of the linear theory of Rayleigh (1883). It is found that for some values of the capillary number, the nonlinear solutions do not approach the linear solution of Rayleigh as the magnitude of the pressure distribution approaches zero. Appropriate linear and nonlinear solutions are constructed.

1. Introduction

A small object moving at a sufficiently large constant velocity U at the surface of a fluid produces a train of waves behind it and another train in front. Rayleigh (1883) studied this phenomenon by using potential flow theory and by assuming that the disturbance due to the object is small. The equations can then be linearized around a uniform stream with constant velocity U and steady solutions (in a frame of reference moving with the object) are found by Fourier transforms.

Rayleigh's solutions can be described in terms of the critical velocity

$$U_{min} = \left(\frac{4Tg}{\rho}\right)^{1/4}. \tag{1.1}$$

Here T is the surface tension, g the acceleration due to gravity and ρ the density.

There are no periodic linear waves travelling at a phase velocity U smaller than U_{min}. Therefore for $U < U_{min}$, the free surface at large distance from the object is waveless and flat. For $U > U_{min}$, there are two possible trains of periodic waves. The solution of the free-surface flow due to a moving object is thus non-unique for $U > U_{min}$, since multiples of the two trains of waves can always be superimposed on any solution to generate new solutions. However a unique solution is obtained by imposing the radiation condition which requires that there is no supply of energy from infinity. It is then found that a train of waves with the larger wavelength appears behind the object and one with a smaller wavelength in front. Rayleigh showed that a convenient way to impose the radiation condition is to solve the problem with some artificial viscosity μ (known as a Rayleigh viscosity) and then take the limit as $\mu \to 0$. Alternative dissipation mechanisms were used by Spivak, Vanden-Broeck & Miloh (2001) and Vanden-Broeck (2001).

Boundary integral equation methods can be used to solve the fully nonlinear problem numerically (Asavanant & Vanden-Broeck 1994; Dias & Vanden-Broeck

1992; Vanden-Broeck 2001). We should expect these nonlinear solutions to approach Rayleigh's linear solution as the size of the object approaches zero. In this paper we show that there are particular values of the Weber number for which this is not the case. For these values we show that the nonlinear solutions approach linear solutions which differ from Rayleigh's solutions as the size of the object tends to zero. It is found that this discrepancy is related to the fact that there are many families of nonlinear gravity–capillary periodic waves but only one family of linear gravity–capillary periodic waves. The results described in this paper are qualitatively independent of the nature of the object (submerged object, surface piercing object or pressure distribution). Here we present all our results for a prescribed distribution of pressure. This can be viewed as an indirect way to compute the free surface flow due to a moving surface piercing object since the shape of the free surface resulting from the pressure distribution can be replaced at the end of the calculations by a rigid wall.

2. Formulation

We consider the nonlinear free surface flow generated by a pressure distribution moving to the left with a constant velocity U. The fluid is assumed to be inviscid and incompressible and the flow to be irrotational. The depth of the fluid is infinite. We choose a frame of reference moving with the pressure distribution and the y-axis directed vertically upwards. We assume that the flow is steady and that the pressure distribution is symmetric with respect to $x = 0$. As $y \to -\infty$ we have a uniform stream with a constant velocity U in the x-direction. We introduce the potential function ϕ and the streamfunction ψ. We choose $\psi = 0$ on the free surface. We define dimensionless variables by choosing U^2/g as the unit length and U as the unit velocity.

The problem can be formulated in terms of the potential function ϕ as

$$\nabla^2\phi = 0 \quad \text{in} \quad -\infty < y < \eta(x), \tag{2.1}$$

$$\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] + \eta(x) - W\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} + \frac{P_0(x)}{\rho} = \frac{1}{2} \quad \text{on} \quad y = \eta(x), \tag{2.2}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x}\eta_x(x) \quad \text{on} \quad y = \eta(x), \tag{2.3}$$

$$\frac{\partial\phi}{\partial x} \to 1 \quad \text{as} \quad y \to -\infty, \tag{2.4}$$

where

$$W = \frac{Tg}{\rho U^4} \tag{2.5}$$

is the Weber number. Here $y = \eta(x)$ is the unknown shape of the free surface and $P_0(x)$ a prescribed distribution of pressure which tends to zero as $|x| \to \infty$. The acceleration due to gravity is acting in the negative y-direction. Equations (2.2) and (2.3) are the dynamic and kinematic boundary conditions on the free surface. The choice of the Bernoulli constant on the right-hand side of (2.2) fixes the origin of y as the undisturbed free surface level.

For given $P_0(x)$ and W, we need to solve (2.1) with the conditions (2.2)–(2.4). This is a nonlinear free surface flow problem for which no general exact solution is known. Therefore analytical and numerical approximations are discussed in the next sections.

The solutions were found to be qualitatively independent of the particular choice for the function $P_0(x)$. All the calculations presented in this paper are for

$$P_0(x) = \epsilon\, e^{-5x^2}. \tag{2.6}$$

Here ϵ is a positive parameter. Similar results can be obtained with other distributions of pressure or other disturbances.

3. Rayleigh's linear solution

If ϵ is small, the problem (2.1)–(2.4) can be linearized by assuming a small perturbation around a uniform stream with constant velocity 1. This yields

$$\nabla^2\phi = 0 \quad \text{in} \quad -\infty < y < 0, \tag{3.1}$$

$$\frac{\partial\phi}{\partial x} - 1 + \eta(x) - W\eta_{xx}(x) + \frac{P_0(x)}{\rho} = 0 \quad \text{on} \quad y = 0, \tag{3.2}$$

$$\frac{\partial\phi}{\partial y} = \eta_x(x) \quad \text{on} \quad y = 0, \tag{3.3}$$

$$\frac{\partial\phi}{\partial x} \to 1 \quad \text{as} \quad y \to -\infty. \tag{3.4}$$

The problem (3.1)–(3.4) was solved by Rayleigh (1883) using Fourier transforms. Rayleigh noticed that with $P_0(x) = 0$ (i.e. $\epsilon = 0$ in (2.6)) it has the solution

$$\phi^0 = x - A\,e^{ky}\sin k(x+\delta) + B, \tag{3.5}$$

$$\eta^0 = A\cos k(x+\delta). \tag{3.6}$$

Here A, B and δ are arbitrary constants. Since ϕ is only defined up to an arbitrary additive constant, we can set $B = 0$ without loss of generality. The constant k in (3.5) and (3.6) is a solution of

$$Wk^2 - k + 1 = 0. \tag{3.7}$$

For $W > 1/4$, the roots of (3.7) are complex and we need to set the constant A in (3.5) and (3.6) equal to zero for ϕ and η to be bounded for all x. However for $W < 1/4$, A is arbitrary. There are then two non-trivial solutions of (3.5), (3.6) corresponding to the two roots

$$k_{\pm} = \frac{1 \pm (1-4W)^{1/2}}{2W} \tag{3.8}$$

of (3.7). These two solutions are trains of waves of wavenumber $k_{\pm}$. Because of the linearity of the equations, arbitrary multiples of $\phi^0 - x$ and η^0 can be added to any solution of (3.1)–(3.4) with $P_0(x) \neq 0$. This implies that the solution of (3.1)–(3.4) with $P_0(x) \neq 0$ is not unique. We note that $W = 1/4$ corresponds to $U = U_{min}$ where U_{min} is defined in (1.1). A unique solution of (3.1)–(3.4) can be obtained by applying the radiation condition which requires that there is no supply of energy from infinity. Rayleigh (1883) showed that a convenient way to impose the radiation condition is to introduce an artificial viscosity $\mu > 0$ by rewriting (3.2) as

$$\frac{\partial\phi}{\partial x} - 1 + \eta(x) - W\eta_{xx}(x) + \frac{P_0(x)}{\rho} + \mu(x-\phi) = 0 \quad \text{on} \quad y = 0 \tag{3.9}$$

and to solve the problem defined by (3.1), (3.9), (3.3) and (3.4). The solution is then

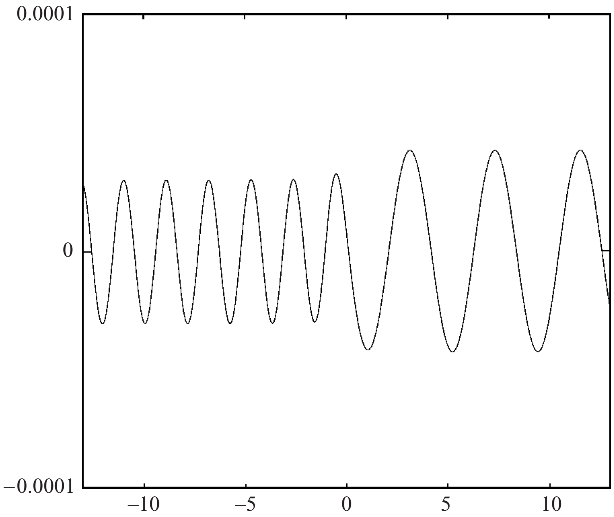

FIGURE 1. Free surface profile of Rayleigh's linear solution for $W = 2/9$ and $\epsilon = 10^{-5}$.

unique and the solution of (3.1)–(3.4) satisfying the radiation condition is obtained by taking the limit $\mu \to 0$.

A typical free surface profile for $W = 2/9$ is shown in figure 1. As $|x| \to \infty$, the solution approaches periodic trains of waves of constant amplitude. These trains of waves are described by (3.5) and (3.6) with $k = k_{\pm}$. Here A and δ are no longer arbitrary and their values are part of the solution. The train of waves with the longer wavelength (i.e. $k = k_-$) appears as $x \to \infty$ and the train with the shorter wavelength (i.e. $k = k_+$) as $x \to -\infty$.

It is usually assumed that linear solutions such as the one shown in figure 1 approximate nonlinear solutions of (2.1)–(2.4) when the parameter ϵ in (2.6) is small. We show in the next sections that there are particular values of W for which it is not the case.

4. Wilton ripples

In the previous section, we showed that Rayleigh's solution for $W < 1/4$ is characterized by linear trains of waves in the far field (i.e. as $|x| \to \infty$). These trains of periodic waves are described by (3.5) and (3.6) with k given by (3.8).

Nonlinear solutions of (2.1)–(2.4) are characterized by nonlinear periodic waves in the far field. In this section, we examine some properties of nonlinear periodic gravity–capillary waves. This will provide useful information on the asymptotic behaviour of the nonlinear solutions of (2.1)–(2.4) with $P_0(x) \neq 0$ as $|x| \to \infty$. Solutions for nonlinear periodic waves can be constructed numerically (see for example Schwartz & Vanden-Broeck 1979; Chen & Saffman 1979; Hogan 1980) or analytically by using perturbation theory. Here we follow the analytical approach and seek solutions of (2.1)–(2.4) with $P_0(x) = 0$ by assuming an expansion in powers of the amplitude of the wave. This is a classical calculation. The pioneering work in this direction was by Stokes (1847) who calculated a one-parameter family of solutions for pure gravity waves. Wilton (1915) generalized Stokes' calculation by including the effect of surface tension. His results and more recent calculations show that in contrast to pure gravity waves, there is an infinite number of families of gravity–capillary solutions.

The origin of this non-uniqueness can be understood by examining conditions under which linear waves of wavenumbers k and nk travel at the same speed U (i.e. are characterized by the same W). Here n is a positive integer. Using (3.7) we obtain

$$Wn^2k^2 - nk + 1 = Wk^2 - k + 1 = 0. \tag{4.1}$$

Relations (4.1) imply

$$W = \frac{n}{(n+1)^2}. \tag{4.2}$$

It then follows that when (4.2) is satisfied, the general solution for linear periodic waves of wavenumber k is not (3.5), (3.6) but

$$\phi^0 = x - A\,\mathrm{e}^{ky}\sin k(x+\delta) - A^*\,\mathrm{e}^{nky}\sin nk(x+\delta) + B, \tag{4.3}$$

$$\eta^0 = A\cos k(x+\delta) + A^*\cos nk(x+\delta), \tag{4.4}$$

where A^* is an arbitrary constant.

When constructing a nonlinear solution as an expansion in powers of the amplitude, (4.3) and (4.4) appear as the first terms in the expansion. It is found that higher-order terms in the expansion can only be calculated if a solvability condition is satisfied. This condition fixes the value of A^*. For example for $n = 2$, it is found that $A^* = \pm A/2$, so that the solution of (3.1)–(3.4) with $P_0(x) = 0$ and

$$W = \tfrac{2}{9} \tag{4.5}$$

is

$$\phi^0 = x - A\,\mathrm{e}^{ky}\sin k(x+\delta) \mp \tfrac{1}{2}A\,\mathrm{e}^{2ky}\sin 2k(x+\delta) + B, \tag{4.6}$$

$$\eta^0 = A\cos k(x+\delta) \pm \tfrac{1}{2}A\cos 2k(x+\delta). \tag{4.7}$$

Both (4.3), (4.4) and (4.6), (4.7) satisfy the linear equations (3.1)–(3.4) with $P_0(x) = 0$. In fact (4.6), (4.7) is a particular case of (4.3), (4.4) where the arbitrary constant A^* is assigned the value $\pm A/2$. However (4.6), (4.7) is an approximation of a nonlinear wave as its amplitude approaches zero, whereas (4.3), (4.4) with $A^* \neq \pm A/2$ is not.

We note that there are two waves (4.6), (4.7) corresponding to the $+$ and $-$ signs. One has a crest dimple whereas the other has a trough dimple. These two waves are often referred to as the Wilton ripples. This illustrates the non-uniqueness mentioned earlier in this section.

The profile of figure 1 is the solution of the linear system (3.1)–(3.4) satisfying the radiation condition with $P_0(x)$ and W given by (2.6) and (4.5). It is characterized by the train of waves (4.3) and (4.4) with $A^* = 0$ as $|x| \to \infty$. Substituting (4.5) into (3.8), we find $k = k_+ = 3$ as $x \to -\infty$ and $k = k_- = 3/2$ as $x \to \infty$. The above results show that there are no nonlinear periodic waves approaching the train of linear periodic waves with $k = 3/2$ as $x =\to \infty$ in figure 1. This implies that there is no nonlinear solution of (2.1)–(2.4) which approaches the linear solution of figure 1 as $\epsilon \to 0$. In the next Section, we construct numerically linear and nonlinear solutions which approach each other as $\epsilon \to 0$.

5. Nonlinear solutions

We solve the nonlinear problem (3.1)–(3.4) numerically by a boundary integral equation method. The details of the numerical procedure are similar to those described in Asavanant & Vanden-Broeck (1994), so that only the main ideas will be summarized here. We seek $z = x + \mathrm{i}y$ as an analytic function of $f = \phi + \mathrm{i}\psi$ in the lower half-plane $\psi < 0$. We choose $\phi = 0$ at $x = 0$. We rewrite the nonlinear dynamic boundary condition (2.2) on the free surface as

$$\frac{1}{2}\frac{1}{x'^2 + y'^2} + y - W\frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} + \frac{P_0(\phi)}{\rho} = \frac{1}{2}. \tag{5.1}$$

and the linear dynamic boundary condition (3.2) as

$$1 - x' + y - Wy'' + \frac{P_0(\phi)}{\rho} = \frac{1}{2}. \tag{5.2}$$

Here $x(\phi)$ and $y(\phi)$ are the values of x and y on the free surface and x' and y' their derivatives with respect to ϕ.

We choose

$$P_0(\phi) = \epsilon\, \mathrm{e}^{-5\phi^2}. \tag{5.3}$$

The expressions (5.2) and (5.3) are consistent with (3.2) and (2.6) since x can be replaced by ϕ without affecting the order of accuracy of the linear solution.

Next we apply the Cauchy integral formula to the function $x' + \mathrm{i}y' - 1$ in the complex f-plane with a contour consisting of the free surface and a half-circle of arbitrary large radius in the lower half-plane. Since $x' + \mathrm{i}y' - 1 \to 0$ as $\psi \to -\infty$, it can be shown that there is no contribution from the half-circle. Taking the real part we obtain

$$x'(\phi) = 1 - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y'(\varphi)}{\varphi - \phi}\,\mathrm{d}\varphi. \tag{5.4}$$

The integral in (5.4) is a Cauchy principal value.

Eliminating x' between (5.1) and (5.4) we obtain a nonlinear integro–differential equation for the unknown function y'. Similarly eliminating x' between (5.2) and (5.4) we obtain the corresponding linear integro–differential equation for the unknown function y'.

To solve these equations numerically we introduce the mesh points

$$\phi_I = -\frac{(N-1)E}{2} + (I-1)E, \quad I = 1, \ldots, N \tag{5.5}$$

and the unknowns

$$y'_I = y'(\phi_I). \tag{5.6}$$

Here E is the interval of discretization.

The nonlinear and linear integro–differential equations are discretized and satisfied at the intermediate mesh points

$$\phi_{I+1/2} = \frac{\phi_I + \phi_{I+1}}{2}, \quad I = 1, \ldots, N-1. \tag{5.7}$$

This leads to $N-1$ nonlinear algebraic equations. The Cauchy principal value in (5.4) is approximated by the trapezoidal rule with a summation over the mesh points (5.5). This yields

$$x'(\phi_{I+1/2}) = 1 - \frac{1}{\pi}\sum_{J=1}^{N} w_J \frac{y'_J}{\phi_J - \phi_{I+1/2}}, \tag{5.8}$$

where $w_1 = w_N = E/2$ and $w_J = E$ otherwise. The symmetry of the quadrature enables us to evaluate the Cauchy principal value integral in (5.4) as if it were an ordinary integral. More details can be found in Asavanant & Vanden-Broeck (1994).

The approximation (5.8) replaces the integral from $-\infty$ to ∞ in (5.4) by an integral from $-A$ to A where $A = (N-1)E/2$. To obtain the results presented here, we improve this truncation by first rewriting (5.4) as

$$x'(\phi) = 1 - \frac{1}{\pi}\int_{-A}^{A} \frac{y'(\varphi)}{\varphi - \phi}\,\mathrm{d}\varphi - \frac{1}{\pi}\int_{-B}^{-A} \frac{y_l'(\varphi)}{\varphi - \phi}\,\mathrm{d}\varphi - \frac{1}{\pi}\int_{A}^{B} \frac{y_r'(\varphi)}{\varphi - \phi}\,\mathrm{d}\varphi \tag{5.9}$$

before applying the trapezoidal rule. Here $B \gg A$. For A sufficiently large, y_l' and y_r' are the periodic trains of waves of §4. We present results for $W = 2/9$. For this value of the Weber number, we showed that the Rayleigh's solution of figure 1 fails in the nonlinear regime because there are no nonlinear periodic waves approaching the linear wave on the far right of figure 1 as $\epsilon \to 0$. The discussion at the end of §4 suggests that the correct nonlinear solutions for $W = 2/9$ should have a train of Wilton ripples on the far right. As we shall see this is confirmed by our numerical calculations. We restrict our calculations to $\epsilon \ll 1$, so that formulae for y_l' and y_r' can be derived from (4.6) and (4.7). Replacing x by ϕ in (4.7) and differentiating with respect to ϕ yields

$$y_r' = -Ak_- \sin k_-(\phi + \delta) \mp Ak_- \sin 2k_-(\phi + \delta). \tag{5.10}$$

Similarly (3.6) yields

$$y_l' = -A^* k_+ \sin k_+(\phi + \delta^*). \tag{5.11}$$

The constants A, A^*, δ, δ^* are found by imposing the continuity conditions

$$y_r'(\phi_N) = y_N', \quad y_r'(\phi_{N-1}) = y_{N-1}', \quad y_r'(\phi_{N-2}) = y_{N-2}', \tag{5.12}$$

$$y_l'(\phi_1) = y_1', \quad y_l'(\phi_2) = y_2'. \tag{5.13}$$

Relation (5.11) imposes a train of waves of short wavelength on the far left similar to the one on the far right of figure 1. Relation (5.10) replaces the train of waves on the far left of figure 1 by Wilton ripples in accordance with the above discussion.

Relations (5.12) and (5.13) together with the $N-1$ equations obtained by discretizing the nonlinear or linear integro–differential equations define a system of $N+4$ nonlinear algebraic equations for the $N+4$ unknowns A, A^*, δ, δ^* and y_I', $I = 1, \ldots, N$. This system is solved by Newton's method.

The non-uniqueness of the linear problem discussed in §3 implies that numerical solutions of the linear integral equation can be obtained by replacing (5.10) by

$$y_r' = -Ak_- \sin k_-(\phi + \delta) \mp \bar{A}k_- \sin 2k_-(\phi + \delta) \tag{5.14}$$

where $\bar{A}$ is arbitrary. The linear solution satisfying the radiation condition (i.e. Rayleigh's solution) is obtained by setting $\bar{A} = 0$. It is shown in figure 1.

Numerical solutions of the nonlinear problem can only be obtained when $\bar{A} = \pm A$ in (5.14) (i.e. by using (5.10)). For other values of $\bar{A}$, the scheme does not converge. There are two nonlinear solutions corresponding to the $\pm$ signs in (5.10). These are shown in figures 2(*a*) and 2(*b*). As $x \to \infty$, these solutions approach Wilton ripples. Since ϵ is small, the nonlinear solutions of figure 2 approximately satisfy the linear equations (3.1)–(3.4). However they differ from Rayleigh's solution of figure 1. As expected the difference between the nonlinear solutions (figure 2) and Rayleigh's solution (figure 1) is a linear train of wave of wavenumber $k = 3$. This is illustrated in figure 3 where we show the difference between the profiles of figure 2(*a*) and figure 1.

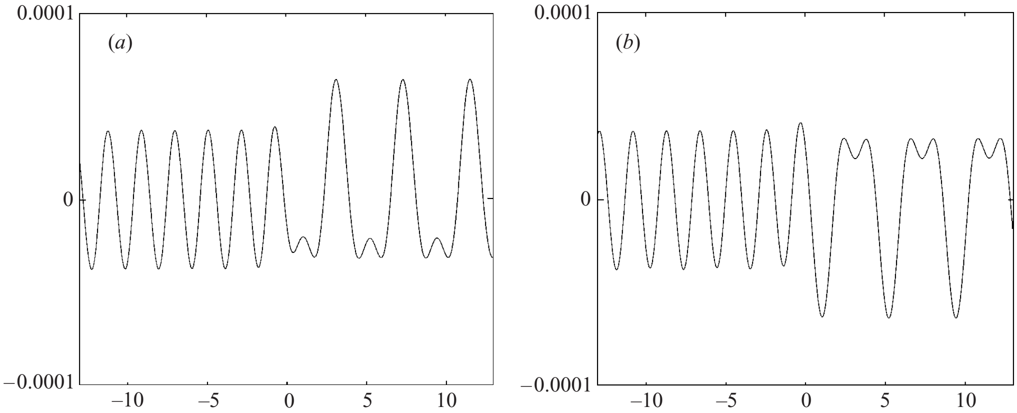

FIGURE 2. Free surface profile of a nonlinear solution for $W = 2/9$ and $\epsilon = 10^{-5}$, (a) corresponding to the + sign in (5.10) and (b) to the − sign.

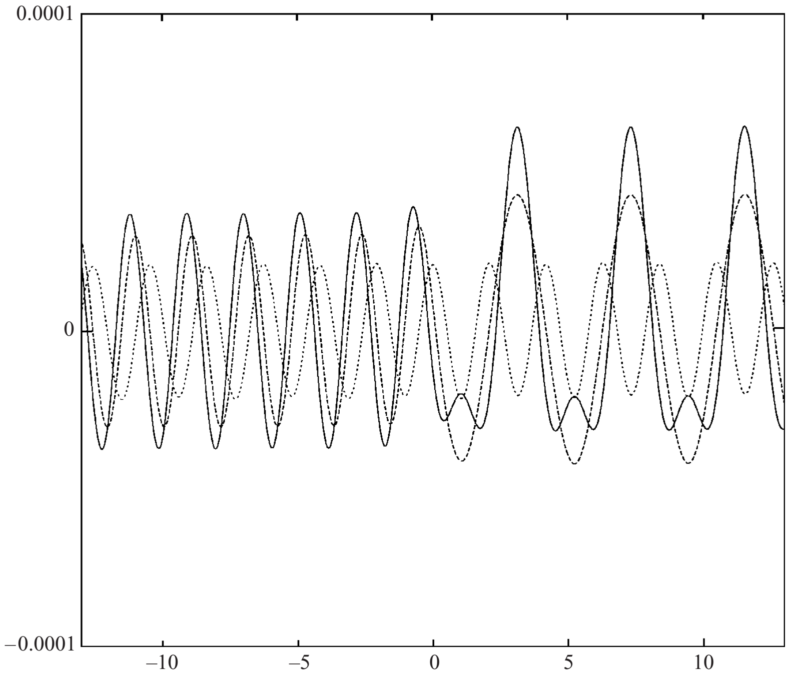

FIGURE 3. The dotted line is the difference between the profiles of figures 1 and 2(a), represented by dashed and solid lines respectively.

6. Conclusions

We have revisited the problem of gravity–capillary waves generated by a moving pressure distribution. We have shown that for some values of the Weber number the classical linear solution fails to approximate nonlinear solutions. Appropriate solutions containing Wilton ripples were derived for a particular value of the Weber number. Our calculations were restricted to steady flows in water of infinite depth. The corresponding time-dependent problem in shallow water was considered by Milewski & Vanden-Broeck (1999). These authors model the flow by a non-homogeneous fifthorder Korteweg–de Vries equation. Their results show that in addition to the steady solutions there are many unsteady solutions. Interestingly some of their solutions exhibit Wilton ripples (see figure 5 in Milewski & Vanden-Broeck 1999). This suggests that solutions similar to those in figure 2 could also be obtained by solving the complete nonlinear time-dependent problem.

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