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A switched server system semiconjugate to a minimal interval exchange†

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Switched server systems are mathematical models of manufacturing, traffic and queueing systems that have being studied since the early 1990s. In particular, it is known that typically the dynamics of such systems is asymptotically periodic: each orbit of the system converges to one of its finitely many limit cycles. In this article, we provide an explicit example of a switched server system with exotic behaviour: each orbit of the system converges to the same Cantor attractor. To accomplish this goal, we bring together recent advances in the understanding of the topological dynamics of piecewise contractions and interval exchange transformations (IETs) with flips. The ultimate result is a switched server system whose Poincaré map is semiconjugate to a minimal and uniquely ergodic IET with flips.

Key words: Piecewise contraction, symbolic dynamics, switched server system, pseudo billiard

2010 Mathematics Subject Classification: 37E05 (Primary); 37B10 (Secondary); 37N35 (Secondary)

1 Introduction

Certain aspects of manufacturing, traffic or queueing systems are captured by the mathematical model named *switched server system*, which was introduced by Chase et al. in [4, Section II.B, p. 72]. It is a continuous-time system discretely controlled via a switched state-feedback, also referred to as a hybrid dynamical system (see [21]). It can also be considered a *pseudo-billiard* (see [1]). In this article, we provide an example of a switched server system with atypical non-trivial dynamics. Our approach benefits from recent advances in the understanding of the topological dynamics of piecewise contractions (see [18]).

The switched server system we consider here consists of three buffers (tanks) numbered 1, 2, 3, and a server. It is very convenient to think of each buffer i as a tank partially filled in with

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a fluid (work). At each time $t \geq 0$, a fluid is delivered to each tank i at the constant rate $\rho_i = \frac{1}{3}$ $(i = 1, 2, 3)$ and is removed from a selected tank $i \in \{1, 2, 3\}$ by the server at the constant rate $\rho = 1$. The volume of fluid in the tank i at the time t is denoted by $v_i(t)$. When the tank i is emptied by the server at the time t, the server changes its location to the tank $j \neq i$ with the largest scaled volume $d_{ij} v_j(t)$, where $\{d_{ij} : 1 \leq i, j \leq 3, i \neq j\}$ are the parameters of the system. We assume that $\sum_{i=1}^{3} v_i(0) = 1$. Since the system is closed $(\rho_1 + \rho_2 + \rho_3 = \rho)$, we have that $\sum_{i=1}^{3} v_i(t) = 1$ for every $t \geq 0$. Hence, the state $\mathbf{v}(t) = (v_1(t), v_2(t), v_3(t))$ of the system at the time t is a probability vector and the phase space is the set $\Delta = \{\mathbf{v} = (v_1, v_2, v_3) : v_i \geq 0, \forall i \text{ and } v_1 + v_2 + v_3 = 1\}$. Let $l(t)$ denote the position of the server at the time t. We assume that $t \mapsto l(t)$ is right-continuous. Figure 1(a) shows a switched server system with the server located at the position $l = 1$.

The trajectory $t \in [0, \infty) \mapsto \mathbf{v}(t) \in \Delta$ describes the position of a particle that moves with constant velocity inside Δ and changes its velocity when the particle hits the boundary $\partial\Delta$ according to a non-specular reflection. Hence, the system is a *pseudo-billiard* (see [1]). The times $0 \leq t_1 < t_2 < t_3 \cdots$ at which any of the tanks is empty are called the *switching times*. At the initial time $t = 0$, the server is supposed to be connected to a non-empty tank. Notice that $\mathbf{v}(t) \in \partial\Delta$ (the boundary of the phase space) if and only if $t \in \{t_1, t_2, \ldots\}$ (i.e. if t is a switching time). In other words, at the switching times, the pseudo billiard trajectory hits the boundary $\partial\Delta$. By sampling the system at the switching times, we obtain a map $F : \partial\Delta \to \partial\Delta$ called the *Poincaré map or first-return map induced by the switched server system* (see Figure 1(b)). The *frequency* with which the server is connected to the tank i is defined by

$$freq\,(i) = \lim_{n\to\infty} \frac{1}{n} \#\{1 \leq k \leq n : l(t_k) = i\},$$

whenever the limit exists.

The dynamics of a switched server system with parameters $\{d_{ij} > 0 : 1 \leq i, j \leq 3, i \neq j\}$ depends only on the proportionality between pairs of parameters. More specifically, switched server systems sharing the same ratios d_{13}/d_{12}, d_{21}/d_{23} and d_{32}/d_{31} have the same dynamics. In this way, we assume that if (d_1, d_2, d_3) is a vector with positive entries, then the system parameters d_{ij} are chosen according to the following conditions:

$$\frac{d_{13}}{d_{12}} = d_1, \qquad \frac{d_{21}}{d_{23}} = d_2, \qquad \frac{d_{32}}{d_{31}} = d_3. \tag{1.1}$$

By [15, Theorem 1.4], we have that for Lebesgue almost every vector (d_1, d_2, d_3) with positive entries, any switched server system with parameters d_{ij} satisfying (1.1) is structurally stable and admits finitely many limit cycles that attract all the orbits. The same result was obtained in [4, Theorem 4.1] under the additional restrictions: $d_{21} = d_{31}$, $d_{12} = d_{32}$ and $d_{13} = d_{23}$. Figure 1(b) shows the case in which $d_1 = d_2 = d_3 = 1$ and $d_{ij} = 1$ for all $i \neq j$. In this case, $\left\{\left(0, \frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, 0, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, 0\right)\right\}$ is a limit cycle of the system.

In this article, we are interested in constructing switched server systems with complex dynamics, i.e., with no periodic orbit and therefore with no limit cycle. In the light of what was discussed in the previous paragraph, it necessary to search for the appropriate parameters in a Lebesgue negligible set of parameters (d_1, d_2, d_3). Moreover, the example we provide presents stochastic regularity in the sense that it is possible to compute the frequency with which the server is connected to the tank i at the switching times.

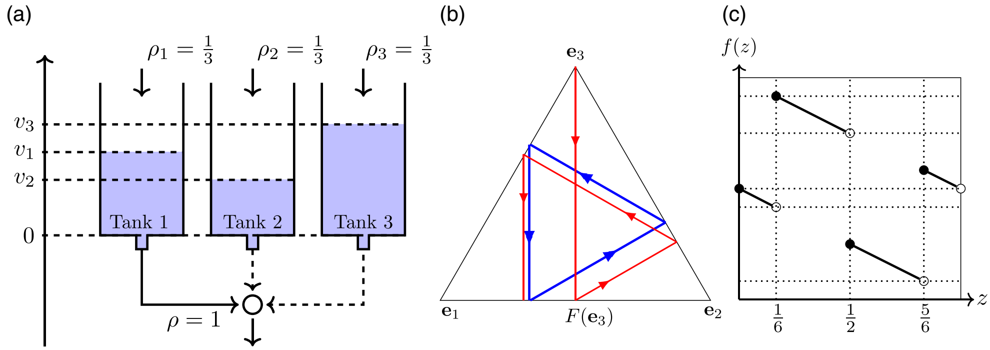

FIGURE 1. The switched server system, the pseudo billiard and the Poincaré map.

The strategy we use to tackle the problem is the following. The dynamics of a switched server system is completely determined by the Poincaré map $F : \partial\Delta \to \partial\Delta$ induced by the system on the boundary $\partial\Delta$ of the phase space. The Poincaré map F is topologically conjugate to the piecewise smooth interval map $f : [0, 1] \to [0, 1]$ defined by $f = \varphi^{-1} \circ F \circ \varphi$, where $\varphi : [0, 1] \to \partial\Delta$ denotes the anticlockwise arc-length parametrisation of $\partial\Delta$ with $\varphi(0) = \mathbf{e}_2 = (0, 1, 0)$. Conversely, the following lemma is provided in this article:

Lemma 1.1 *Given $d_1, d_2, d_3 > 0$, let $f_{d_1,d_2,d_3} : [0, 1] \to [0, 1]$ be the map defined by*

$$f_{d_1,d_2,d_3}(z) = \begin{cases} -\frac{1}{2}z + \frac{1}{2} & \text{if } z \in [z_0, z_1) \\ -\frac{1}{2}z + 1 & \text{if } z \in [z_1, z_2) \\ -\frac{1}{2}z + \frac{1}{2} & \text{if } z \in [z_2, z_3) \\ -\frac{1}{2}z + 1 & \text{if } z \in [z_3, z_4], \end{cases} \tag{1.2}$$

where

$$z_0 = 0, \quad z_1 = \frac{1}{3(1 + d_1)}, \quad z_2 = \frac{1}{3(1 + d_2)} + \frac{1}{3}, \quad z_3 = \frac{1}{3(1 + d_3)} + \frac{2}{3}, \quad z_4 = 1. \tag{1.3}$$

Then the Poincaré map $F : \partial\Delta \to \partial\Delta$ of any switched server system with parameters d_{ij} satisfying (1.1) is topologically conjugate to f_{d_1,d_2,d_3}.

In Figure 1(c), the map $f = f_{1,1,1}$ is plotted considering $d_1 = d_2 = d_3 = 1$. In general, for any $d_1, d_2, d_3 > 0$, the map f_{d_1,d_2,d_3} is a piecewise λ-affine contraction, where $\lambda = \frac{1}{2}$ (see [15]). We say that an infinite word $w = i_0 i_1 \ldots$ over the alphabet $\mathcal{A} = \{1, 2, 3, 4\}$ is a *symbolic itinerary* or *natural coding* of $f = f_{d_1,d_2,d_3}$ if there exists $z \in [0, 1]$ such that, for each $k \geq 0$,

$$f^k(z) \in \begin{cases} [z_{i_k - 1}, z_{i_k}) & \text{if } \quad i_k < 4 \\ [z_3, z_4] & \text{if } \quad i_k = 4. \end{cases}$$

A symbolic itinerary w (i.e. an infinite word over the alphabet $\mathcal{A}$) is *ultimately periodic* if there exist finite words u and v over the alphabet $\mathcal{A}$ such that $w = uvv\ldots$ For instance, the symbolic itinerary $w = 311\,432\,432\,432\ldots$ is ultimately periodic with $u = 311$ and $v = 432$.

The problem we want to solve translates into the following question.

(Q) Does the family of piecewise contractions $\{f_{d_1,d_2,d_3} : d_1 > 0, d_2 > 0, d_3 > 0\}$ contain a map having no ultimately periodic symbolic itinerary (and therefore no periodic orbit and no limit cycle) ?

On the one hand, as already mentioned, recent advances (see [14, 15]) in the understanding of the topological dynamics of piecewise contractions show that generically piecewise contractions have finitely many limit cycles that attract all orbits. Hence, an affirmative answer to (**Q**) is very unlikely. On the other hand, as it was shown very recently (see [18, Theorem 2.2]), there exist piecewise $\frac{1}{2}$-affine contractions with only one gap having no periodic orbit and no ultimately periodic symbolic itinerary. In order to adapt the proof of [18, Theorem 2.2] to our framework, it is necessary to find an isometric model for f_{d_1,d_2,d_3}, that is, a minimal and uniquely ergodic interval exchange transformation (IET) T with four flips and three discontinuities $0 < x_1 < x_2 < x_3$ satisfying $T(x_2) < T(0) < T(x_3) < T(x_1)$ (see Section 2). Surprisingly, as we show in this article, (**Q**) has an affirmative answer.

The use of IETs as isometric models of complex dynamics is quite standard. Lots of piecewise smooth aperiodic interval maps are topologically semiconjugate to IETs (see [3, 5, 6, 17, 18]). Moreover, IETs are the simplest discontinuous interval maps preserving Lebesgue measure (see [9]).

2 Statement of the results

Throughout this article, let P and Q be the integer matrices defined by

$$P = \begin{pmatrix} 3 & 3 & 5 & 4 \\ 1 & 2 & 3 & 3 \\ 1 & 1 & 2 & 1 \\ 2 & 3 & 5 & 5 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

By the Perron–Frobenius Theorem (see [20, Theorem 5.12]), since P has positive entries, the eigenvalue of P of maximum modulus (called *Perron–Frobenius eigenvalue* and denoted by η) is unique, real, greater than zero and simple. Moreover, there is a unique probability eigenvector with positive entries associated with η.

Let $\boldsymbol{\nu}$ be the probability eigenvector with positive entries associated with the Perron–Frobenius eigenvalue η of P. Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be the vector defined by $\boldsymbol{\lambda} = Q\boldsymbol{\nu}$ whose norm is $|\boldsymbol{\lambda}| = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 > 1$. Consider the partition of the interval $[0, |\boldsymbol{\lambda}|]$:

$$I_1 = [0, \lambda_1), \ \ I_2 = [\lambda_1, \lambda_1 + \lambda_2), \ \ I_3 = [\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3),$$
$$I_4 = [\lambda_1 + \lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4].$$

Let $T : [0, |\boldsymbol{\lambda}|] \to [0, |\boldsymbol{\lambda}|]$ be the map (called isometric model) defined by

$$T(x) = \begin{cases} -x + \lambda_1 + \lambda_3 & \text{if} \quad x \in I_1 \\ -x + \lambda_1 + |\boldsymbol{\lambda}| & \text{if} \quad x \in I_2 \\ -x + \lambda_1 + \lambda_2 + \lambda_3 & \text{if} \quad x \in I_3 \\ -x + \lambda_1 + \lambda_3 + |\boldsymbol{\lambda}| & \text{if} \quad x \in I_4. \end{cases} \tag{2.1}$$

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According to the definition given in [7], we have that T is a 4-*IET with flips* (4-IET with flips). In fact, it can be easily verified that T is one-to-one on $(0, |\boldsymbol{\lambda}|)$, $T|_{I_i}$ is an isometry ($i = 1, 2, 3, 4$) and T reverts the orientation of one (in fact, all) of the intervals I_1, I_2, I_3, I_4. We denote by $O_T(x) = \{x, T(x), T^2(x), \ldots\}$ the *T-orbit of* $x \in [0, |\boldsymbol{\lambda}|]$. We say that T is *topologically transitive* if it has a dense orbit; *minimal* if every T-orbit is dense; *uniquely ergodic* if the (normalised) Lebesgue measure on $[0, |\boldsymbol{\lambda}|]$ is the only T-invariant Borel probability measure.

Our first result is the following.

Theorem 2.1 *The map T defined in* (2.1) *is minimal and uniquely ergodic.*

The example given in Theorem 2.1 is rare. Typically, an n-IET with flips has an interval formed by periodic orbits and, therefore, is not minimal (see [13]). This situation is completely different in the case of IETs without flips, also called *standard* IETs. The simplest example is the rotation of the circle $R_\alpha : [0, 1) \to [0, 1)$ defined by $R_\alpha(x) = \{x + \alpha\}$, where $0 < \alpha < 1$. It can be written as the standard 2-IET $T_\alpha : [0, 1] \to [0, 1]$ defined by $T_\alpha(x) = x + 1 - \alpha$ if $x \in [0, \alpha)$ and $T_\alpha(x) = x - \alpha$ if $x \in [\alpha, 1]$. It is widely known that when α is irrational, R_α and T_α are minimal and uniquely ergodic. Concerning standard irreducible n-IETs with $n \geq 2$, Keane's conjecture, answered in the affirmative by many authors (see [2, 10, 12, 19, 22]), states that such maps are typically minimal and uniquely ergodic.

To state our main result, we need some more definitions. Let

$$p_1 = 0, \quad p_2 = T(\lambda_1 + \lambda_2), \quad p_3 = T(\lambda_1 + \lambda_2 + \lambda_3), \quad p_4 = |\boldsymbol{\lambda}|.$$

For $i, j \in \{1, 2, 3, 4\}$, let

$$K_{ij} = \{k \geq 0 : T^k(p_j) \in I_i\}, \quad c_{ij} = \sum_{k \in K_{ij}} \frac{1}{2^k}. \tag{2.2}$$

Let

$$M = \begin{pmatrix} c_{11} - c_{14} & c_{12} - c_{14} & c_{13} - c_{14} \\ c_{21} - c_{24} & c_{22} - c_{24} & c_{23} - c_{24} \\ c_{31} - c_{34} & c_{32} - c_{34} & c_{33} - c_{34} \end{pmatrix}.$$

Let $u_4 = 1 - u_1 - u_2 - u_3 > 0$, where $u_1, u_2, u_3 > 0$ is the unique solution of the linear system

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{2} M \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \frac{1}{2} M \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} c_{14} \\ c_{24} \\ c_{34} \end{pmatrix}.$$

Let

$$z_1 = u_1, \quad z_2 = u_1 + u_2, \quad z_3 = u_1 + u_2 + u_3. \tag{2.3}$$

In what follows, we say that a map $f : [0, 1] \to [0, 1]$ is *topologically semiconjugate* to the isometric model $T : [0, |\boldsymbol{\lambda}|] \to [0, |\boldsymbol{\lambda}|]$ if there exists a continuous, surjective, nondecreasing map $h : [0, 1] \to [0, |\boldsymbol{\lambda}|]$ such that $h \circ f = T \circ h$. We denote by $\overline{S}$ the topological closure of a set S and by $\omega_F(\mathbf{v}) = \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} \{F^k(\mathbf{v})\}}$, the ω-limit set of $\mathbf{v} \in \partial\Delta$ by $F : \partial\Delta \to \partial\Delta$.

Now we state our main result.

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Theorem 2.2 *Let z_1, z_2 and z_3 be the solution of* (2.3) *and let $d_1, d_2, d_3 > 0$ be defined by*

$$d_1 = \frac{1}{3z_1} - 1 = 0.213841\ldots, \quad d_2 = \frac{2-3z_2}{3z_2-1} = 4.036935\ldots, \quad d_3 = \frac{3-3z_3}{3z_3-2} = 1.428826\ldots$$

Then for any switched server system with parameters d_{ij} satisfying (1.1) *the following statements are true:*

(a) *The switched server system has no periodic orbit;*
(b) *The Poincaré map $F : \partial\Delta \to \partial\Delta$ of the system is topologically semiconjugate to T;*
(c) *$\omega_F(\mathbf{v})$ is a Cantor set for every $\mathbf{v} \in \partial\Delta$;*
(d) *The frequency freq* (i) *with which the server is connected to the tank i at the switching times is*

$$\mathit{freq}\,(1) = \frac{\lambda_3}{|\boldsymbol{\lambda}|} \cong 34.44\%, \quad \mathit{freq}\,(2) = \frac{\lambda_1+\lambda_4}{|\boldsymbol{\lambda}|} \cong 41.82\%, \quad \mathit{freq}\,(3) = \frac{\lambda_2}{|\boldsymbol{\lambda}|} \cong 23.72\%.$$

In Theorem 2.2, the item (d) follows from the item (b), from Theorem 2.1 and from the version of Birkhoff's Ergodic Theorem for uniquely ergodic transformations (see [7, Proposition 4.1.13]). The items (a) and (d) are also confirmed by numerical simulations using the *R* programming language. It is also worth mentioning that the matrices P and Q were obtained by using Rauzy induction (see [13]).

A few words are necessary to understand the implications of Theorem 2.2. Let d_1, d_2, d_3 be the real numbers defined in Theorem 2.2 and consider a switched server system with parameters d_{ij} satisfying (1). Then, by Theorem 2.2, the system has a Cantor attractor, no periodic orbit and no limit cycle. Moreover, the system is Li-Yorke chaotic (see [11]). On the other hand, the system presents stochastic regularity in the sense that, regardless the initial state, the server stays 34.44% connected to the tank 1, 41.82% connected to the tank 2 and 23.72% connected to the tank 3. In the real world or in computational simulations, one has to deal with inaccuracies in parameters and rounding errors, then it is natural to wonder whether the theoretical prediction remains valid in such cases. As already discussed, it is known from other previous works that for Lebesgue almost every vector $(\widetilde{d}_1, \widetilde{d}_2, \widetilde{d}_3)$ with positive entries, the system with the parameters $\widetilde{d}_1, \widetilde{d}_2, \widetilde{d}_3$ has finitely many limit cycles that attract all orbits. This shows that the system with the parameters d_1, d_2, d_3 given in Theorem 2.2 is not structurally stable because arbitrarily close to it there are other systems with a completely different qualitative behaviour: instead of Cantor attractors the systems have attractive limit cycles. In spite of that, some properties of the theoretical system still persist when we perturb its parameters. In our computational simulations using rational parameters close to d_1, d_2, d_3, we found no periodic orbit and no limit cycle. Hence, if the system has periodic orbits, their period is extremely high. Besides that, the frequencies that we found in our computational simulations are the same as those approximations given in Theorem 2.2.

3 Poincaré maps of switched server systems and the proof of Lemma 1.1

We keep all the notations given in the previous sections.

Proof of Lemma 1.1 Let $d_1, d_2, d_3 > 0$ be given. Let the switched server system parameters d_{ij} be chosen according to (1.1). Let $0 \le t_1 < t_2 \ldots$ denote the switching times. If at the switching time t_m the server is connected to the tank j, then it keeps connected to the tank j during the

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time-interval $[t_m, t_{m+1})$. Moreover,

$$t_{m+1} - t_m = \frac{v_j(t_m)}{\rho - \rho_j} = \frac{v_j(t_m)}{1 - \frac{1}{3}} = \frac{3}{2} v_j(t_m). \tag{3.1}$$

For every $m \geq 1$ and $t_m \leq t \leq t_{m+1}$, the level $v_k(t)$ of any tank $k \in \{1, 2, 3\}$ is determined by the set of linear equations

$$v_k(t) = \begin{cases} v_k(t_m) + \dfrac{1}{3}(t - t_m) & \text{if} \quad k \neq j \\ v_j(t_m) - \dfrac{2}{3}(t - t_m) & \text{if} \quad k = j, \end{cases} \tag{3.2}$$

where j is the position of the server at the time t_m.

Equation (3.2) shows that the state $\mathbf{v}(t) = \big(v_1(t), v_2(t), v_3(t)\big)$ of the system at any time $t \in [t_m, t_{m+1})$ describes the position of a particle that moves with constant velocity. More precisely, when the particle hits $\partial\Delta$ at the switching time t_m, it takes the velocity $\mathbf{v}'(t_m+) = \lim_{\epsilon \to 0^+} \frac{\mathbf{v}(t_m+\epsilon) - \mathbf{v}(t_m)}{\epsilon}$ and moves with such velocity till it hits the boundary again, at the time t_{m+1}, when then the velocity changes to $\mathbf{v}'(t_{m+1}+)$. In this way, $t \in [0, \infty) \mapsto \mathbf{v}(t) \in \Delta$ is the trajectory of a *pseudo billiard*. By sampling the system at the consecutive switching times t_1 and t_2, we obtain the Poincaré map $F : \partial\Delta \to \partial\Delta$ induced by the flow on the boundary $\partial\Delta$ of Δ. More specifically, considering $m = 1$ in (3.1) and (3.2), $t = t_2$ in (3.2), and $(v_1, v_2, v_3) = (v_1(t_1), v_2(t_1), v_3(t_1)) \in \partial\Delta$ yield

$$\big(F(v_1, v_2, v_3)\big)_k = v_k(t_2) = \begin{cases} v_k + \dfrac{1}{2} v_j & \text{if } k \neq j \\ 0 & \text{if } k = j, \end{cases} \tag{3.3}$$

where j is the position of the server at the time t_1. Notice that if $i \neq j$ denotes the empty tank number at the time t_1, then $d_{ij} v_j = \max\{d_{ik} v_k : 1 \leq k \leq 3\}$, that is, at the time t_1, the server begins emptying the tank j with the largest scaled volume $d_{ij} v_j$. Now we will find a piecewise-defined formula for F. Let

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1).$$

Given $\mathbf{p}, \mathbf{q} \in \mathbb{R}^3$, let the line segments $[\mathbf{p}, \mathbf{q}]$, $(\mathbf{p}, \mathbf{q})$, $[\mathbf{p}, \mathbf{q})$ and $(\mathbf{p}, \mathbf{q}]$ be defined as usual, for instance,

$$[\mathbf{p}, \mathbf{q}] = \{(1 - \alpha)\mathbf{p} + \alpha\mathbf{q} : 0 \leq \alpha \leq 1\}, \quad (\mathbf{p}, \mathbf{q}) = \{(1 - \alpha)\mathbf{p} + \alpha\mathbf{q} : 0 < \alpha < 1\}.$$

Notice that

$$\partial\Delta = [\mathbf{e}_2, \mathbf{e}_3] \cup [\mathbf{e}_3, \mathbf{e}_1] \cup [\mathbf{e}_1, \mathbf{e}_2].$$

Moreover,

$$\begin{cases} (v_1, v_2, v_3) \in [\mathbf{e}_2, \mathbf{e}_3] \iff v_1 = 0 \\ (v_1, v_2, v_3) \in [\mathbf{e}_3, \mathbf{e}_1] \iff v_2 = 0 \\ (v_1, v_2, v_3) \in [\mathbf{e}_1, \mathbf{e}_2] \iff v_3 = 0. \end{cases} \tag{3.4}$$

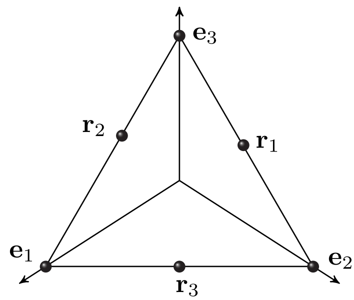

FIGURE 2. Partition of $\partial\Delta$.

Now let us consider the decomposition of $\partial\Delta$ given by (see Figure 2):

$$\partial\Delta = [\mathbf{r}_1, \mathbf{e}_3] \cup [\mathbf{e}_3, \mathbf{r}_2) \cup [\mathbf{r}_2, \mathbf{e}_1] \cup [\mathbf{e}_1, \mathbf{r}_3) \cup [\mathbf{r}_3, \mathbf{e}_2] \cup [\mathbf{e}_2, \mathbf{r}_1),$$

where

$$\mathbf{r}_1 = \frac{d_{13}}{d_{12}+d_{13}}\mathbf{e}_2 + \frac{d_{12}}{d_{12}+d_{13}}\mathbf{e}_3, \quad \mathbf{r}_2 = \frac{d_{21}}{d_{23}+d_{21}}\mathbf{e}_3 + \frac{d_{23}}{d_{23}+d_{21}}\mathbf{e}_1,$$
$$\mathbf{r}_3 = \frac{d_{32}}{d_{31}+d_{32}}\mathbf{e}_1 + \frac{d_{31}}{d_{31}+d_{32}}\mathbf{e}_2.$$

Let $(v_1, v_2, v_3) \in (\mathbf{r}_1, \mathbf{e}_3]$, then $v_1 = 0$, that is, $i = 1$. Moreover,

$$v_3 > \frac{d_{12}}{d_{12}+d_{13}}, \; v_2 < \frac{d_{13}}{d_{12}+d_{13}} \quad \text{and} \quad d_{13}v_3 > \frac{d_{13}d_{12}}{d_{12}+d_{13}} = \frac{d_{12}d_{13}}{d_{12}+d_{13}} > d_{12}v_2,$$

implying that the tank 3 has the largest scaled volume, that is, $j = 3$. Proceeding likewise with respect to $[\mathbf{e}_3, \mathbf{r}_2)$, $[\mathbf{r}_2, \mathbf{e}_1]$, etc., and using the convention that l is right-continuous (see Introduction), we reach the following conclusion:

$$\begin{cases} (v_1, v_2, v_3) \in [\mathbf{r}_1, \mathbf{e}_3] \cup [\mathbf{e}_3, \mathbf{r}_2) \iff j = 3 \\ (v_1, v_2, v_3) \in [\mathbf{r}_2, \mathbf{e}_1] \cup [\mathbf{e}_1, \mathbf{r}_3) \iff j = 1 \\ (v_1, v_2, v_3) \in [\mathbf{r}_3, \mathbf{e}_2] \cup [\mathbf{e}_2, \mathbf{r}_1) \iff j = 2. \end{cases} \tag{3.5}$$

Putting together (3.3)–(3.5), we reach

$$F(v_1, v_2, v_3) = \begin{cases} \left(\frac{1}{2}v_2, 0, v_3 + \frac{1}{2}v_2\right) & \text{if } (v_1, v_2, v_3) \in [\mathbf{e}_2, \mathbf{r}_1) \\ \left(v_1 + \frac{1}{2}v_3, v_2 + \frac{1}{2}v_3, 0\right) & \text{if } (v_1, v_2, v_3) \in [\mathbf{r}_1, \mathbf{e}_3] \cup [\mathbf{e}_3, \mathbf{r}_2) \\ \left(0, v_2 + \frac{1}{2}v_1, v_3 + \frac{1}{2}v_1\right) & \text{if } (v_1, v_2, v_3) \in [\mathbf{r}_2, \mathbf{e}_1] \cup [\mathbf{e}_1, \mathbf{r}_3) \\ \left(v_1 + \frac{1}{2}v_2, 0, \frac{1}{2}v_2\right) & \text{if } (v_1, v_2, v_3) \in [\mathbf{r}_3, \mathbf{e}_2]. \end{cases} \tag{3.6}$$

Let $\varphi : [0, 1] \to \partial\Delta$ be the anticlockwise arc-length parametrisation of $\partial\Delta$ (see Figure 3). More precisely, let

$$\varphi(t) = \begin{cases} (1-3t)\mathbf{e}_2 + 3t\mathbf{e}_3 & \text{if } t \in \left[0, \frac{1}{3}\right) \\ (2-3t)\mathbf{e}_3 + (3t-1)\mathbf{e}_1 & \text{if } t \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ (3-3t)\mathbf{e}_1 + (3t-2)\mathbf{e}_2 & \text{if } t \in \left[\frac{2}{3}, 1\right]. \end{cases} \tag{3.7}$$

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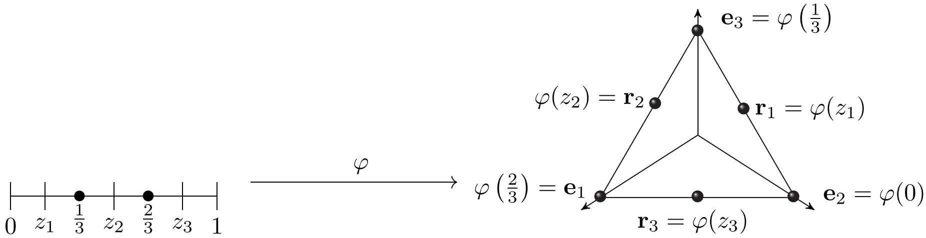

FIGURE 3. The arc-length parametrisation of $\partial\Delta$.

The inverse of φ is defined by

$$\varphi^{-1}(\mathbf{p}) = \begin{cases} \frac{1}{3\sqrt{2}}\|\mathbf{p}-\mathbf{e}_2\| & \text{if} \quad \mathbf{p} \in [\mathbf{e}_2, \mathbf{e}_3] \\ \frac{1}{3\sqrt{2}}\|\mathbf{p}-\mathbf{e}_3\| + \frac{1}{3} & \text{if} \quad \mathbf{p} \in [\mathbf{e}_3, \mathbf{e}_1] \\ \frac{1}{3\sqrt{2}}\|\mathbf{p}-\mathbf{e}_1\| + \frac{2}{3} & \text{if} \quad \mathbf{p} \in [\mathbf{e}_1, \mathbf{e}_2]. \end{cases} \tag{3.8}$$

It follows from (3.6)–(3.8) that the map $f = \varphi^{-1} \circ F \circ \varphi$ is given by

$$f(z) = \begin{cases} -\frac{1}{2}z + \frac{1}{2} & \text{if } z \in [z_0, z_1) \\ -\frac{1}{2}z + 1 & \text{if } z \in [z_1, z_2) \\ -\frac{1}{2}z + \frac{1}{2} & \text{if } z \in [z_2, z_3) \\ -\frac{1}{2}z + 1 & \text{if } z \in [z_3, z_4], \end{cases}$$

where

$$z_0 = 0, \quad z_1 = \frac{d_{12}}{3(d_{12}+d_{13})}, \quad z_2 = \frac{d_{23}}{3(d_{23}+d_{21})} + \frac{1}{3}, \quad z_3 = \frac{d_{31}}{3(d_{31}+d_{32})} + \frac{2}{3}, \quad z_4 = 1. \tag{3.9}$$

By (1.1), we have that (3.9) is equivalent to (1.3), hence $f(z) = f_{d_1,d_2,d_3}(z)$ for every $z \in [0, 1]$. This concludes the proof of Lemma 1.1. □

4 Interval exchange transformations

In this section, we gather some results related to the construction of topologically transitive IETs. We will use them in the next section in the proof of Theorem 2.1. Although Theorem 2.1 is by itself an original and rare example of minimal IET with flips, the arguments used in its proof are known by experts working with standard IETs.

Let $a > 0$ and $I = [0, a]$. Following [7], we say that $T : I \to I$ is an *n-interval exchange transformation* (*n*-IET) if there exist a partition of I into intervals $I_1, I_2, \ldots, I_n$ with endpoints $\{x_0, x_1\}$, $\{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}$ satisfying $0 = x_0 < x_1 < \cdots < x_n = a$ and the following conditions:

(*i*) T is one-to-one on $I \backslash \{x_0, \ldots, x_n\}$;
(*ii*) $T\big(I \backslash \{x_0, \ldots, x_n\}\big) \cap T\big(\{x_0, \ldots, x_n\}\big) = \emptyset$;

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(*iii*) $T|_{(x_{i-1},x_i)}$ is an isometry for all $1 \leq i \leq n$.

Notice that (*ii*) and (*iii*) are automatically satisfied if $T|_{I_i}$ is an isometry ($i = 1, 2, \ldots, n$). The vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_i = x_i - x_{i-1}$ is called the *length vector*. Moreover, there exist $\varepsilon_i \in \{-1, 1\}$ and $b_i \in \mathbb{R}$ ($i = 1, 2, \ldots, n$) such that

$$T(x) = T_i(x) := \varepsilon_i x + b_i \quad \text{for all} \quad x \in (x_{i-1}, x_i) \quad (i = 1, 2, \ldots, n). \tag{4.1}$$

The set $\mathscr{F} = \{1 \leq i \leq n : \varepsilon_i = -1\}$ is denominated the *flip set* of T (see [13]). If $\mathscr{F} = \emptyset$, then $\varepsilon_i = 1$ ($i = 1, 2, \ldots, n$) and T is called *standard* or *without flips*. Otherwise, $\mathscr{F} \neq \emptyset$ and T is said to have *flips*.

We assume that $\mathcal{D}(T) = \{x_1, x_2, \ldots, x_{n-1}\}$ is the set of discontinuities of T, otherwise T would be an m-IET with $m < n$.

4.1 Poincaré maps of IETs

Let $0 = x_0 < x_1 < \cdots < x_n = a$ and let $T : I \to I$ be an n-IET defined on $I = [0, a]$ with set of discontinuities $\mathcal{D}(T) = \{x_1, x_2, \ldots, x_{n-1}\}$.

Definition 4.1 (T-tower) Given $r \geq 1$, we say that $\{J, T(J), \ldots, T^{r-1}(J)\}$ is a *T-tower* if $J, T(J), \ldots, T^{r-1}(J)$ are pairwise disjoint open intervals. Each interval $T^k(J)$, $0 \leq k \leq r - 1$, is called a *floor*.

It is an elementary fact that all the floors in a T-tower have the same length $|J|$. In this way, $r \leq |I|/|J|$. Equivalently, a family $\{J_1, J_2, \ldots, J_r\}$ of pairwise disjoint open intervals is a T-tower if there exists a permutation $\tau : \{1, \ldots, r\} \to \{1, \ldots, r\}$ such that $J_{\tau(i+1)} = T(J_{\tau(i)})$ for every $1 \leq i \leq r - 1$.

The following result is a consequence of the injectivity of T on $(0, a)\backslash\mathcal{D}(T)$.

Lemma 4.2 *If $\{J, T(J), \ldots, T^{r-1}(J)\}$ is a T-tower with $T^{r-1}(J) \cap \mathcal{D}(T) = \emptyset$, then either $T^r(J) \cap J \neq \emptyset$ or $\{J, T(J), \ldots, T^r(J)\}$ is a T-tower.*

Proof Set $U = (0, a)\backslash\mathcal{D}(T)$. Since $J, T(J), \ldots, T^{r-1}(J)$ are open intervals and $T^{r-1}(J) \cap \mathcal{D}(T) = \emptyset$, we have that $J \cup T(J) \cup \ldots \cup T^{r-1}(J) \subset U$ and $T^r(J) = T\big(T^{r-1}(J)\big)$ is an open interval. Without loss of generality, we assume that $r \geq 2$. We claim that $T^r(J) \cap T^k(J) = \emptyset$ for all $1 \leq k \leq r - 1$. In fact,

$$T^r(J) \cap T^k(J) = T(A) \cap T(B), \quad \text{where} \quad A = T^{r-1}(J) \quad \text{and} \quad B = T^{k-1}(J).$$

Because $k - 1 \leq r - 2 < r - 1$ and $J, T(J), \ldots, T^{r-1}(J)$ are pairwise disjoint open intervals, we have that $A \cap B = \emptyset$ and $A \cup B \subset U$. By the injectivity of T on U, we conclude that $T(A) \cap T(B) = \emptyset$, which proves the claim. □

Let $0 < a' < a$ and $I' = [0, a']$. Given $x \in I$, let $N(x) \in \mathbb{N} \cup \{\infty\}$ be defined by

$$N(x) = \inf\{N \geq 1 : T^N(x) \in I'\}, \tag{4.2}$$

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where $\inf \emptyset = \infty$. The map $T' : \mathrm{dom}(T') \to I'$, where $\mathrm{dom}(T') = \{x \in I' : N(x) < \infty\}$ and

$$T'(x) = T^{N(x)}(x) = \underbrace{T \circ T \circ \ldots \circ T}_{N(x)\text{ times}}(x)$$

is called the *Poincaré map or first-return map of T on I'*.

In general, the Poincaré map T' induced by the n-IET T on a subinterval I' of I may have more discontinuities than T. In order to avoid that situation, we introduce the notion of admissible interval (see Corollary 4.7).

Definition 4.3 (Admissible interval) The interval I' is *admissible* if there exist $0 = x'_0 < x'_1 < \ldots < x'_n = a'$ such that $N(x'_i) < \infty$ for every $1 \le i \le n$ and the set $B = \bigcup_{i=1}^{n} \left\{x'_i, T(x'_i), \ldots, T^{N(x'_i)-1}(x'_i)\right\}$ satisfies

(H1) $B \supset \mathcal{D}(T)$;
(H2) $a' \in T(B)$.

Henceforth, we will assume that I' is an admissible interval.

Lemma 4.4 *Let $K \subset I \backslash B$ be an open interval. Then $K \cap \mathcal{D}(T) = \emptyset$. Moreover, one of the following alternatives happens:*

(i) *$T(K)$ is an open subinterval of I';*
(ii) *$T(K) \cap I' = \emptyset$ and $T(K)$ is an open subinterval of $I \backslash B$.*

Proof By $(H1)$, $\mathcal{D}(T) \subset B$, thus $K \cap \mathcal{D}(T) = \emptyset$ and $T(K)$ is an open interval. Notice that

$$T(K) \cap T(B) \subset \left[T(K) \cap T\big(\{x_0, \ldots, x_n\}\big)\right] \cup \left[T(K) \cap T\big(B \backslash \{x_0, \ldots, x_n\}\big)\right].$$

The first term in the union is empty because of the property (ii) in the definition of n-IET and the fact that $K \subset I \backslash \{x_0, \ldots, x_n\}$. The second term is empty because of property (i) in the definition of n-IET and the fact that K and $B \backslash \{x_0, \ldots, x_n\}$ are pairwise disjoint subsets of $I \backslash \{x_0, \ldots, x_n\}$. Hence, $T(K) \cap T(B) = \emptyset$. Now, by $(H2)$, $a' \notin T(K)$, thus either $T(K) \subset I'$ or $T(K) \cap I' = \emptyset$. In the latter case, $T(K) \cap B \subset B \backslash I' \subset T(B)$, which yields $T(K) \subset I \backslash B$. □

Lemma 4.5 *Let J be an open subinterval of $I' \backslash \{x'_1, \ldots, x'_{n-1}\}$, then there exists $r \ge 1$ such that $\{J, T(J), \ldots, T^{r-1}(J)\}$ is a T-tower, $\bigcup_{k=0}^{r-1} T^k(J) \subset I \backslash \{x_0, \ldots, x_n\}$, $I' \cap \bigcup_{k=1}^{r-1} T^k(J) = \emptyset$ and $T^r(J)$ is a subinterval of I'.*

Proof By the definition of B, we have that $B \cap I' = \{x'_1, \ldots, x'_n\}$, thus $J \subset I \backslash B$ and $T(J)$ is an open interval by (H1). If $T(J) \subset I'$, then we take $r = 1$ and we are done. Otherwise, applying Lemma 4.4 with $K = J$ yields $I' \cap T(J) = \emptyset$ and $T(J) \subset I \backslash B$. Moreover, in this case, we have that the set

$$A = \left\{\alpha \ge 1 : \{J, T(J), \ldots, T^{\alpha-1}(J)\} \text{ is an } \alpha\text{-tower with } I' \cap \bigcup_{k=1}^{\alpha-1} T^k(J) = \emptyset\right\}.$$

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is a non-empty subset of $\left[1, \frac{|I|}{|J|}\right]$. By applying Lemma 4.4 finitely many times, we can prove that $r = \max A$ works. □

Proposition 4.6 *Let $T : I \to I$ be an n-IET and $I' \subset I$ be an admissible interval for T. Then, for each $1 \le i \le n$, there exist $r_i \ge 1$ and a word $i_0 i_1 \ldots i_{r_i-1}$ over the alphabet $\mathcal{A} = \{1, \ldots, n\}$ such that the interval $J_i = (x'_{i-1}, x'_i)$ satisfies*

(A1) $\{J_i, T(J_i), \ldots, T^{r_i-1}(J_i)\}$ is a T-tower with $I' \cap \bigcup_{k=1}^{r_i-1} T^k(J_i) = \emptyset$;
(A2) $T^{r_i}(J_i)$ is an open subinterval of I';
(A3) $T^k(J_i) \subset (x_{i_k-1}, x_{i_k})$ for every $0 \le k \le r_i - 1$;
(A4) $N(x) = r_i$ for all $x \in J_i$.

Moreover, the intervals $T^k(J_i), 0 \le k \le r_i - 1,\ 1 \le i \le n$, are pairwise disjoint.

Proof Applying Lemma 4.5 with $J = J_i$ yields $(A1)$–$(A3)$. The item $(A4)$ follows from $(A1)$ and $(A2)$. We claim that $T^k(J_i)$, $0 \le k \le r_i - 1$, $1 \le i \le n$ are pairwise disjoint. Otherwise, by $(A1)$, there exist $i \ne j$, $0 \le k_i \le r_i - 1$, $0 \le k_j \le r_j - 1$ with $k_i \le k_j$ such that $T^{k_i}(J_i) \cap T^{k_j}(J_j) \ne \emptyset$. By the injectivity of T on $(0, a)$, we obtain that $J_i \cap T^{k_j - k_i}(J_j) \ne \emptyset$, which is a contradiction since $J_i \subset I'$ while $T^{k_j - k_i}(J_j) \cap I' = \emptyset$. □

In Proposition 4.6, the word $i_0 i_1 \ldots i_{r_i-1}$ is the *symbolic itinerary* of the T-tower $\{J_i, T(J_i), \ldots, T^{r_i-1}(J_i)\}$. Concerning the next three corollaries, we let J_i, r_i and $i_0 i_1 \ldots i_{r_i-1}$, be as in the statement of Proposition 4.6.

Corollary 4.7 *Let $T : I \to I$ be an n-IET and $I' \subset I$ be an admissible interval for T. Then the Poincaré map T' of T on I' is the n'-IET, $n' \le n$, defined by*

$$T'(x) = T_{i_{r_i-1}} \circ \cdots \circ T_{i_1} \circ T_{i_0}(x) \quad \text{if} \quad x \in (x'_{i-1}, x'_i),$$

where $T_i : \mathbb{R} \to \mathbb{R}$ is the affine map defined by (4.1). Notice that $\mathcal{D}(T') \subset \{x'_1, \ldots, x'_{n-1}\}$.

Definition 4.8 (Exhaustive family) The family of T-towers $\left\{J_i, T(J_i), \ldots, T^{r_i-1}(J_i)\right\}$, $1 \le i \le n$, is *exhaustive* if all the floors are pairwise disjoint and $I \setminus \bigcup_{i=1}^n \bigcup_{k=0}^{r_i-1} T^k(J_i)$ is a finite set.

Corollary 4.9 *Let $T : I \to I$ be an n-IET and $I' \subset I$ be an admissible interval for T. Suppose that*

(H3) $\sum_{i=1}^n r_i |J_i| = |I|$,

then the family of T-towers $\{J_i, T(J_i), \ldots, T^{r_i-1}(J_i)\}$, $1 \le i \le n$, in Proposition 4.6, is exhaustive.

Proof In fact, in this case, by Proposition 4.6, $S = I \setminus \bigcup_{i=1}^n \bigcup_{k=0}^{r_i-1} T^k(J_i)$ is the union of finitely many compact intervals and has Lebesgue measure zero, which implies that S is a finite set. □

Corollary 4.10 *Let $T : I \to I$ be an n-IET and $I' \subset I$ be an admissible interval for T such that $(H3)$ holds. If T' is topologically transitive, so is T.*

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Proof By Corollary 4.9, $I\backslash \bigcup_{i=1}^{n} \bigcup_{k=0}^{r_i-1} T^k(J_i)$ is a finite set. Moreover, by ($A3$) of Proposition 4.6, $x \in J_i \mapsto T^k(x) \in T^k(J_i)$ is an isometry for every $0 \le k \le r_i - 1$ and $1 \le i \le n$. Since I' is the closure of $\bigcup_{i=1}^{n} J_i$, any T'-orbit dense in I' corresponds to a T-orbit dense in I. □

4.2 Self-similar IETs

Let $I' \subset I$ be an admissible interval for T. By Corollary 4.7, the Poincaré map $T' : I' \to I'$ is an n'-IET with set of discontinuities $\mathcal{D}(T') \subset \{x'_1, \dots, x'_{n-1}\}$.

Definition 4.11 (Self-similar IET) Let $T : I \to I$ be an n-IET and $I' \subset I$ be an admissible interval for T. We say that T *is self-similar on* I' if $T' = L \circ T \circ L^{-1}$ on $I'\backslash\{x'_0, \dots, x'_n\}$, where $L : I \to I'$ is the affine bijection $x \mapsto \frac{a'}{a}x$.

In other words, T is self-similar on I' if $\mathcal{D}(T') = \{x'_1, \dots, x'_{n-1}\}$ and T' is a rescaled copy of T. In particular, we have that $\mathcal{D}(T') = L\big(\mathcal{D}(T)\big)$.

Denote by $\mathcal{A}^*$ the set of (finite) words over the alphabet $\mathcal{A} = \{1, 2, \dots, n\}$. By ($A3$) in Proposition 4.6, to the pair (T, I'), we can associate the map $\sigma : \mathcal{A} \to \mathcal{A}^*$ defined by $\sigma(i) = i_0 i_1 \dots i_{r_i-1}$ called the *substitution associated with* (T, I'). In this way, the substitution σ assigns to each letter $i \in \mathcal{A}$, the symbolic itinerary of the T-tower $\{J_i, T(J_i), \dots, T^{r_i-1}(J_i)\}$. By means of the concatenation operation, we can consider σ as a self-map of $\mathcal{A}^*$. The *matrix associated with* (T, I') is the $n \times n$ matrix M associated with σ, whose j, i-entry is

$$m_{ji} = \#\{k : \sigma(i)_k = j\}, \tag{4.3}$$

where # denotes the cardinality of the set. Notice that m_{ji} is the number of times that the T-orbit of the interval $J_i = (x'_{i-1}, x'_i)$ visits the interval (x_{j-1}, x_j) before return to intersect I'. In particular, we have that

$$r_i = \sum_{j=1}^{n} m_{ji}. \tag{4.4}$$

In what follows, we denote by $m_{ji}^{(k)}$ the j, i-entry of M^k. Moreover, J_i and r_i are as in the statement of Proposition 4.6.

Proposition 4.12 *Let $T : I \to I$ be an n-IET self-similar on some admissible interval $I' \subset I$ in such a way that ($H3$) holds. Given $k \ge 1$, let $J_i^{(k)} = L^{k-1}(J_i)$ for all $1 \le i \le n$. Then*

$$\Big\{J_i^{(k)}, T\big(J_i^{(k)}\big), \dots, T^{(r_i^{(k)}-1)}\big(J_i^{(k)}\big)\Big\}, \quad 1 \le i \le n, \tag{4.5}$$

is an exhaustive family of T-towers, where $r_i^{(k)} = \sum_{j=1}^{n} m_{ji}^{(k)}$.

Proof By Corollary 4.9, we know that $\{J_i, T(J_i), \dots, T^{r_i-1}(J_i)\}$, $1 \le i \le n$, is an exhaustive family of T-towers. Hence, the result is true for $k = 1$ because $J_i^{(1)} = J_i$ and $r_i^{(1)} = r_i$. Since T is self-similar on I', we know that T' is a rescaled copy of T. In particular, by the above, $\big\{L(J_i), T'(L(J_i)), \dots, (T')^{r_i-1}(L(J_i))\big\}$, that is,

$$\big\{J_i^{(2)}, T'(J_i^{(2)}), \dots, (T')^{r_i-1}(J_i^{(2)})\big\}, \quad 1 \le i \le n, \quad \text{is an exhaustive family of } T'\text{-towers}. \tag{4.6}$$

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To obtain (4.5), we will replace each set $(T')^\ell\big(J_i^{(2)}\big)$, $0 \le \ell \le r_i - 1$, in (4.6) by a T-tower. Fix $1 \le i \le n$ and let $i_0 i_1 \ldots i_{r_i-1}$ be the symbolic itinerary of the T-tower $\{J_i, T(J_i), \ldots, T^{r_i-1}(J_i)\}$, which is equal to the symbolic itinerary of the T'-tower $\big\{J_i^{(2)}, T'(J_i^{(2)}), \ldots, (T')^{r_i-1}(J_i^{(2)})\big\}$. Notice that $i_0 = i$ and $(T')^\ell\big(J_i^{(2)}\big) \subset J_{i_\ell}$ for all $0 \le \ell \le r_i - 1$. This implies that the first return time of $(T')^\ell\big(J_i^{(2)}\big)$ to I' under the action of T is r_{i_ℓ}. In particular, we have that

$$\begin{aligned} T'\big(J_i^{(2)}\big) &= T^{r_{i_0}}\big(J_i^{(2)}\big) \\ (T')^2\big(J_i^{(2)}\big) &= T^{r_{i_0}+r_{i_1}}\big(J_i^{(2)}\big) \\ &\;\;\vdots \\ (T')^{r_i-1}\big(J_i^{(2)}\big) &= T^{r_{i_0}+r_{i_1}+\ldots+r_{i_{(r_i-2)}}}\big(J_i^{(2)}\big) \end{aligned}$$

Now we will replace each set $(T')^\ell\big(J_i^{(2)}\big)$, $0 \le \ell \le r_i - 1$, in (4.6) by a T-tower as follows:

$$\begin{array}{ccllll} J_i^{(2)} & \mapsto & J_i^{(2)}, & T\big(J_i^{(2)}\big), & \ldots, & T^{r_{i_0}-1}\big(J_i^{(2)}\big), \\ T'\big(J_i^{(2)}\big) & \mapsto & T^{r_{i_0}}\big(J_i^{(2)}\big), & T^{r_{i_0}+1}\big(J_i^{(2)}\big), & \ldots, & T^{r_{i_0}+r_{i_1}-1}\big(J_i^{(2)}\big), \\ \vdots & & & & \vdots & \\ (T')^{r_i-1}\big(J_i^{(2)}\big) & \mapsto & T^{r_{i_0}+r_{i_1}+\cdots+r_{i_{(r_i-2)}}}\big(J_i^{(2)}\big), & T^{r_{i_0}+r_{i_1}+\cdots+r_{i_{(r_i-2)}}+1}\big(J_i^{(2)}\big), & \ldots, & T^{r_{i_0}+r_{i_1}+\cdots+r_{i_{(r_i-1)}}-1}\big(J_i^{(2)}\big). \end{array}$$

This leads us to the exhaustive family of T-towers:

$$\Big\{J_i^{(2)}, T\big(J_i^{(2)}\big), \ldots, T^{r_{i_0}+r_{i_1}+\cdots+r_{i_{(r_i-1)}}-1}\big(J_i^{(2)}\big)\Big\}, \quad 1 \le i \le n.$$

To conclude the proof of the case $k = 2$, we have to show that

$$r_{i_0} + r_{i_1} + \cdots + r_{i_{(r_i-1)}} = r_i^{(2)}.$$

In fact, by (4.4),

$$\begin{aligned} r_{i_0} + r_{i_1} + \cdots + r_{i_{(r_i-1)}} &= \sum_{j=1}^n m_{ji_0} + \sum_{j=1}^n m_{ji_1} + \cdots + \sum_{j=1}^n m_{ji_{(r_i-1)}} \\ &= m_{1i} \sum_{j=1}^n m_{j1} + m_{2i} \sum_{j=1}^n m_{j2} + \cdots + m_{ni} \sum_{j=1}^n m_{jn} \\ &= \sum_{j=1}^n \sum_{\ell=1}^n m_{j\ell} m_{\ell i} = \sum_{j=1}^n m_{ji}^{(2)} = r_i^{(2)}. \end{aligned}$$

Proceeding likewise, we prove that the claim is true for any $k \ge 1$. □

Corollary 4.13 *Let $T : I \to I$ be an n-IET self-similar on some admissible interval $I' \subset I$ in such a way that (H3) holds. If the following conditions are satisfied:*

(H4) *The matrix M associated with (T, I') is positive,*

then T is topologically transitive.

Proof Let $k \ge 1$ be given. For each $1 \le i \le n$, let $J_i^{(k)} = L^{k-1}(J_i)$ be as in (4.5), where $L : I \to I'$ is the affine bijection $x \in I \mapsto \frac{a'}{a} x \in I'$. Let

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$$\mathscr{P}_k = \left\{T^\ell(J_i^{(k)}) : 0 \le \ell \le r_i^{(k)} - 1,\ 1 \le i \le n\right\}.$$

Then, by Proposition 4.12, the union of the intervals in $\mathscr{P}_k$ is equal to I up to finitely many points. Moreover, by (H4), each interval $J_i^{(k+1)}$ visits all the intervals in $\mathscr{P}_k$ before to return to intersect $\bigcup_{i=1}^n J_i^{(k+1)}$. Now let $U, V \subset I$ be open intervals. Since $\max_{J\in\mathscr{P}_k} |J| \to 0$ as $k \to \infty$, by taking k large enough, we may assume that there exist intervals $J_U, J_V \in \mathscr{P}_k$ such that $J_U \subset U$ and $J_V \subset V$. Moreover, by the above, there exist $1 \le i, j \le n$, $1 \le \ell_U \le r_i^{(k+1)}$ and $1 \le \ell_V \le r_j^{(k+1)}$ such that $T^{\ell_U}\left(J_i^{(k+1)}\right) \subset J_U \subset U$, $T^{\ell_V}\left(J_j^{(k+1)}\right) \subset J_V \subset V$ and $T^{r_i^{(k+1)}}\left(J_i^{(k+1)}\right) \cap J_j^{(k+1)}$ is an open interval. In this way, there exists $k \ge 0$ such that $T^k(U) \cap V \ne \emptyset$. By Birkhoff's Transitivity Theorem, we have that T has a dense orbit. □

5 The isometric model and the proof of Theorem 2.1

The aim of this section is to prove Theorem 2.1. The key step required to prove Theorem 2.1 is showing that the map T defined in (2.1) is topologically transitive. Unfortunately, we cannot apply Corollary 4.13 directly to T because T is not self-similar. Thus, instead of T, we consider the Poincaré map $S = T'$ of T on $I' = [0, 1]$. More specifically, we will show that I' is an admissible interval for T and that (H3) holds true. Then, by Corollary 4.10, T will be topologically transitive if S is topologically transitive. This reduction is very convenient because, as we will show, S is self-similar on the subinterval $\left[0, \frac{1}{\eta}\right]$ of $[0, 1]$ and its topological transitivity will follow from Corollary 4.13. To conclude that T is minimal we will prove that T has no periodic orbit. These are the forthcoming steps.

In what follows, let $T : [0, |\boldsymbol{\lambda}|] \to [0, |\boldsymbol{\lambda}|]$ be the map defined in (2.1). Notice that $\mathcal{D}(T) = \{x_1, x_2, x_3\}$, where

$$x_0 = 0, \quad x_1 = \lambda_1, \quad x_2 = \lambda_1 + \lambda_2, \quad x_3 = \lambda_1 + \lambda_2 + \lambda_3, \quad x_4 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = |\boldsymbol{\lambda}|.$$

Some preparatory lemmas are necessary to prove Theorem 2.1.

5.1 Reduction Lemma

Lemma 5.1 *$I' = [0, 1]$ is an admissible interval for T. Moreover, the Poincaré map $T' : I' \to I'$ is given by*

$$T'(x) = \begin{cases} -x + \lambda_1 + \lambda_3 = -x - \nu_2 + 1 & \text{if } x \in [x'_0, x'_1) \\ x + \lambda_3 = x - \nu_1 - \nu_2 + 1 & \text{if } x \in [x'_1, x'_2] \\ x + \lambda_2 + \lambda_3 - |\boldsymbol{\lambda}| = x - \nu_1 - \nu_2 & \text{if } x \in (x'_2, x'_3) \\ -x + \lambda_1 + \lambda_2 + \lambda_3 = -x + \nu_3 + 1 & \text{if } x \in [x'_3, x'_4], \end{cases}$$

where

$$x'_0 = 0, \quad x'_1 = \nu_1, \quad x'_2 = \nu_1 + \nu_2, \quad x'_3 = \nu_1 + \nu_2 + \nu_3, \quad x'_4 = 1,$$

and $\mathcal{D}(T') = \{x'_1, x'_2, x'_3\}$.

Proof See the Appendix. □

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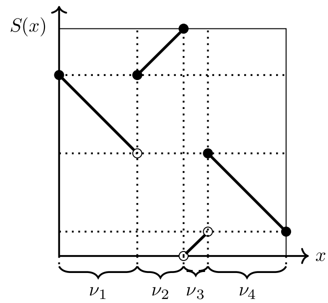
Lemma 5.2 (Reduction Lemma) *If T' is topologically transitive, then so is T.*

Proof See the Appendix. □

5.2 The map S

Let $S : [0, 1] \to [0, 1]$ be the 4-IET defined by

$$S(x) = \begin{cases} -x - \nu_2 + 1 & \text{if} \quad x \in [y_0, y_1) \\ x - \nu_1 - \nu_2 + 1 & \text{if} \quad x \in [y_1, y_2] \\ x - \nu_1 - \nu_2 & \text{if} \quad x \in (y_2, y_3) \\ -x + \nu_3 + 1 & \text{if} \quad x \in [y_3, y_4], \end{cases}$$



where

$$y_0 = x'_0 = 0, \quad y_1 = x'_1 = \nu_1, \quad y_2 = x'_2 = \nu_1 + \nu_2, \quad y_3 = x'_3 = \nu_1 + \nu_2 + u_3, \quad y_4 = x'_4 = 1. \tag{5.1}$$

Then $\mathcal{D}(S) = \{y_1, y_2, y_3\}$. In the previous subsection, we proved that $S = T'$. Let $L : [0, 1] \to \left[0, \frac{1}{\eta}\right]$ be the map $L(y) = \frac{1}{\eta} y$. Set $y'_i = L(y_i)$, $1 \le i \le 4$, then

$$y'_0 = 0, \quad y'_1 = \frac{1}{\eta} y_1, \quad y'_2 = \frac{1}{\eta} y_2, \quad y'_3 = \frac{1}{\eta} y_3, \quad y'_4 = \frac{1}{\eta}.$$

The proofs of the next three lemmas are given in the Appendix.

Lemma 5.3 $\left[0, \frac{1}{\eta}\right]$ *is an admissible interval for S.*

Proof See the Appendix. □

Lemma 5.4 *S is self-similar on* $\left[0, \frac{1}{\eta}\right]$.

Proof See the Appendix. □

Lemma 5.5 *S is topologically transitive.*

Proof See the Appendix. □

Lemma 5.6 *T is topologically transitive.*

Proof By Lemma 5.5, S is topologically transitive. Since $S = T'$, we have that T' is also topologically transitive. The proof is concluded by applying Lemma 5.2. □

Proof of Theorem 2.1 The topological dynamics of n-IETs is well-understood. In particular, it is known that the domain of T splits into the union of periodic components, minimal components and T-connections (see [16, Theorem 3.2] and [8, pp. 470–480]). By Lemma 5.6, T

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is topologically transitive, thus T has no periodic component and has a unique minimal component. Moreover, the minimal component is also a quasi-minimal set in the sense that every non-periodic orbit is dense in it. In this way, T will be minimal if we show that T has no periodic orbit. By way of contradiction, suppose that T has a periodic orbit γ. Then γ contains at least one discontinuity of T, otherwise there would exist a periodic component containing γ. In particular, T has a T-connection, that is, there exist $k \geq 1$ and $x_i, x_j \in \mathcal{D}(T)$ such that $T^k(x_i) = x_j$ and $T^\ell(x_i) \notin \mathcal{D}(T)$ for all $0 < \ell < k$. This contradicts the fact that the Poincaré map of T on I' is a self-similar 4-IET. Therefore, T has no periodic orbit, showing that T is minimal.

Now let us prove that T is uniquely ergodic. Since T has no periodic orbit, all the T-invariant measures are non-atomic and are supported on an uncountable set. Let μ_1, μ_2 be two (non-atomic) T-invariant Borel probability measures, then $\mu_1' = \frac{1}{\mu_1([0,1])}\mu_1$ and $\mu_2' = \frac{1}{\mu_2([0,1])}\mu_2$ are S-invariant Borel probability measures. Moreover, by the proof of Lemma 5.2, T satisfies (H3) on $[0, 1]$, then $\mu_1 = \mu_2$ if and only if $\mu_1' = \mu_2'$. Since S is self-similar on $\left[0, \frac{1}{\eta}\right]$, we have that any S-invariant Borel probability measure μ' is determined by the vector $\mathbf{r} = \Big(\mu'\big((y_0, y_1)\big), \mu'\big((y_1, y_2)\big), \mu'\big((y_2, y_3)\big), \mu'\big((y_3, y_4)\big)\Big)$ which has strictly positive entries, where $y_0, \ldots, y_4$ are as in (5.1). Moreover, since S is self-similar, we have that $\boldsymbol{\nu}$ is the only probability eigenvector of P with strictly positive entries, that is, $\mathbf{r} = \boldsymbol{\nu}$. This means that the only S-invariant measure is the Lebesgue measure, then $\mu_1' = \mu_2'$ and so $\mu_1 = \mu_2$. This proves that T is uniquely ergodic. □

6 Piecewise contractions and the proof of Theorem 2.2

In this section, we will prove Theorem 2.2. By Lemma 1.1 and by Theorem 2.1, all we have to do is to find parameters $d_1, d_2, d_3 > 0$ such that the map f_{d_1,d_2,d_3} defined in (1.2) is topologically semiconjugate to T. The map f_{d_1,d_2,d_3} is a piecewise $\frac{1}{2}$-affine contraction in the following sense.

Definition 6.1 (Piecewise $\frac{1}{2}$-affine contraction) A map $f : [0, 1] \to [0, 1]$ is a *piecewise $\frac{1}{2}$-affine contraction* if there exist a partition of $[0, 1]$ into intervals $J_1, \ldots, J_n$, numbers $a_1, \ldots, a_n \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$ and $b_1, \ldots b_n \in \mathbb{R}$ such that $f(x) = a_i x + b_i$ for all $x \in J_i$ $(i = 1, 2, \ldots, n)$.

Our strategy is the following: first we construct a class $\mathscr{C}$ of piecewise $\frac{1}{2}$-affine contractions topologically semiconjugate to T (Proposition 6.4). Then we prove that there exist $d_1, d_2, d_3 > 0$ such that $f_{d_1,d_2,d_3} \in \mathscr{C}$ (Proposition 6.5).

Definition 6.2 (The map $g_{\boldsymbol{u},\ell}$) Given vectors $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ and $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3, \ell_4)$ with positive entries satisfying $|\boldsymbol{u}| = u_1 + u_2 + u_3 + u_4 = 1$ and $|\boldsymbol{\ell}| = \ell_1 + \ell_2 + \ell_3 + \ell_4 = \frac{1}{2}$, let $g_{\boldsymbol{u},\boldsymbol{\ell}} : [0, 1] \to [0, 1]$ be the piecewise $\frac{1}{2}$-affine contraction defined by

$$g_{\boldsymbol{u},\boldsymbol{\ell}}(x) = \begin{cases} -\dfrac{x}{2} + \dfrac{u_1}{2} + \dfrac{u_3}{2} + \ell_1 + \ell_2 & \text{if} \quad x \in J_1, \\[2mm] -\dfrac{x}{2} + \dfrac{u_1}{2} + \dfrac{1}{2} + \ell_1 + \ell_2 + \ell_3 & \text{if} \quad x \in J_2, \\[2mm] -\dfrac{x}{2} + \dfrac{u_1}{2} + \dfrac{u_2}{2} + \dfrac{u_3}{2} + \ell_1 & \text{if} \quad x \in J_3, \\[2mm] -\dfrac{x}{2} + \dfrac{u_1}{2} + \dfrac{u_3}{2} + \dfrac{1}{2} + \ell_1 + \ell_2 & \text{if} \quad x \in J_4, \end{cases} \tag{6.1}$$

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where J_1, J_2, J_3, J_4 is the partition of $[0, 1]$ given by

$$J_1 = [0, u_1), \quad J_2 = [u_1, u_1 + u_2), \quad J_3 = [u_1 + u_2, u_1 + u_2 + u_3), \quad J_4 = [u_1 + u_2 + u_3, 1].$$

Also let

$$\mathscr{C} = \left\{ g_{\boldsymbol{u},\boldsymbol{\ell}} : u_i, \ell_i > 0, \forall i, \sum_{i=1}^{4} u_i = 1 \text{ and } \sum_{i=1}^{4} \ell_i = \frac{1}{2} \right\}.$$

In what follows, let $T : [0, |\boldsymbol{\lambda}|] \to [0, |\boldsymbol{\lambda}|]$ be the isometric model and let I_1, I_2, I_3, I_4 be the partition of $[0, |\boldsymbol{\lambda}|]$ associated with T (see (2.1)). We will also keep all the notations and values given in Sections 1 and 2. Let

$$p_1 = 0, \quad p_2 = T(\lambda_1 + \lambda_2), \quad p_3 = T(\lambda_1 + \lambda_2 + \lambda_3) \quad \text{and} \quad p_4 = |\boldsymbol{\lambda}|.$$

Lemma 6.3 *The T-orbits of p_1, p_2 and p_3 are pairwise disjoint.*

Proof Denote by $O(x) = \{x, T(x), \ldots\}$ the T-orbit of $x \in [0, |\boldsymbol{\lambda}|]$. By (2.1), $T(\lambda_1) = |\boldsymbol{\lambda}|$ and $T(0) = T(|\boldsymbol{\lambda}|)$. Hence,

$$O(p_1) \subset \{0\} \cup O(\lambda_1), \quad O(p_2) \subset O(\lambda_1 + \lambda_2), \quad O(p_3) \subset O(\lambda_1 + \lambda_2 + \lambda_3).$$

In the proof of Theorem 2.1, we showed that T has no T-connection, thus there exists no T-orbit that passes through two discontinuities of T. This together with the injectivity of T on $(0, |\boldsymbol{\lambda}|]$ implies that $O(\lambda_1)$, $O(\lambda_1 + \lambda_2)$ and $O(\lambda_1 + \lambda_2 + \lambda_3)$ are pairwise disjoint. Moreover, we have that 0 has no preimage, which concludes the proof. □

Proposition 6.4 *Let $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ and $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3, \ell_4)$ be vectors with positive entries satisfying $\sum_{i=1}^{4} u_i = 1$, $\sum_{i=1}^{4} \ell_i = \frac{1}{2}$, and*

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = M \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} c_{14} \\ c_{24} \\ c_{34} \end{pmatrix}, \tag{6.2}$$

then $g = g_{\boldsymbol{u},\boldsymbol{\ell}}$ is topologically semiconjugate to T.

Proof Let $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3, \ell_4)$ be a vector with positive entries such that $\sum_{i=1}^{4} \ell_i = \frac{1}{2}$. Let

$$\mathcal{P} = \left\{ T^k(p_i) : k \geq 0 \text{ and } 1 \leq i \leq 4 \right\}.$$

By Theorem 2.1, $\mathcal{P}$ is a denumerable dense subset of $[0, |\boldsymbol{\lambda}|]$. Since $T(p_1) = T(p_4)$, we may write $\mathcal{P} = \left\{ T^k(p_i) : k \geq 0 \text{ and } 1 \leq i \leq 3 \right\} \cup \{p_4\}$. Let $\phi : \mathcal{P} \to (0, 1)$ be the map defined by $\phi(p_i) = \ell_i$, $1 \leq i \leq 4$, and, for all $k \geq 1$,

$$\phi\left(T^k(p_1)\right) = \frac{\ell_1 + \ell_4}{2^k}, \quad \phi\left(T^k(p_2)\right) = \frac{\ell_2}{2^k}, \quad \phi\left(T^k(p_3)\right) = \frac{\ell_3}{2^k}.$$

By Lemma 6.3, ϕ is well-defined. To each $p \in \mathcal{P}$, let $G_p \subset [0, 1]$ be the compact interval defined by $G_{p_1} = [0, \ell_1]$, $G_{p_4} = [1 - \ell_4, 1]$ and

$$G_p = \left[\sum_{\substack{q<p \\ q\in\mathcal{P}}} \phi(q),\ \ \phi(p) + \sum_{\substack{q<p \\ q\in\mathcal{P}}} \phi(q) \right] \quad \text{if} \quad p \notin \{p_1, p_4\}. \tag{6.3}$$

Notice that G_p has length $|G_p| = \phi(p)$ for all $p \in \mathcal{P}$. Hence,

$$\sum_{p\in\mathcal{P}} |G_p| = \sum_{i=1}^{4} \ell_i + \sum_{k\geq 1} \frac{\ell_1 + \ell_4 + \ell_2 + \ell_3}{2^k} = \frac{1}{2}\left(1 + \sum_{k\geq 1} \frac{1}{2^k}\right) = 1. \tag{6.4}$$

By (6.3) and by the density of $\mathcal{P}$ in $[0, |\boldsymbol{\lambda}|]$, we have that $\mathcal{P}$ and $\{G_p\}_{p\in\mathcal{P}}$ share the same ordering meaning that if $p, q \in \mathcal{P}$, then

$$p < q \iff \sup G_p < \inf G_q. \tag{6.5}$$

In particular, we have that the intervals $G_p, p \in \mathcal{P}$, are pairwise disjoint and, by (6.4), their union is dense in [0, 1].

Let $\widehat{h}: \bigcup_{p\in\mathcal{P}} G_p \to [0, |\boldsymbol{\lambda}|]$ be the function that on G_p takes the constant value p. By (6.4) and (6.5), we have that $\widehat{h}$ is nondecreasing and has dense domain and dense range. Thus, $\widehat{h}$ admits a unique nondecreasing continuous surjective extension $h: [0, 1] \to [0, |\boldsymbol{\lambda}|]$. It is elementary to see that $h^{-1}(\{p\}) = G_p$ for every $p \in \mathcal{P}$. Denote by J_1, J_2, J_3, J_4 the partition of [0, 1] defined by $J_i = h^{-1}(I_i)$, where I_1, I_2, I_3, I_4 are as in the definition of the isometric model T.

Let $\widehat{g}: \bigcup_{p\in\mathcal{P}} G_p \to \bigcup_{p\in\mathcal{P}} G_{T(p)}$ be such that $\widehat{g}|_{G_p}: G_p \to G_{T(p)}$ is an affine bijection with slope $-\frac{1}{2}$ for every $p \in \mathcal{P}$. We claim that for each $1 \leq i \leq 4$, there exist a dense subset $\widehat{J_i}$ of J_i and $b_i \in \mathbb{R}$ such that

$$\widehat{g}(x) = -\frac{1}{2}x + b_i \quad \text{for all} \quad x \in \widehat{J_i}\,. \tag{6.6}$$

Let $1 \leq i \leq 4$, $\widehat{I_i} = I_i \cap \mathcal{P}$, and $\widehat{J_i} = \bigcup_{p\in\widehat{I_i}} G_p$, then, by (6.4) and (6.5), we have that

(*i*) $J_i \cap \bigcup_{p\in\mathcal{P}} G_p$ is dense in J_i;
(*ii*) $J_i \cap \bigcup_{p\in\mathcal{P}} G_p = h^{-1}(I_i) \cap \bigcup_{p\in\mathcal{P}} h^{-1}(\{p\}) = \bigcup_{p\in\mathcal{P}} h^{-1}(\{p\} \cap I_i) = \bigcup_{p\in\widehat{I_i}} G_p = \widehat{J_i}$,

showing that $\widehat{J_i}$ is a dense subset of J_i.

Moreover, by definition, $\widehat{g}|_{G_p}: G_p \to G_{T(p)}$ is an affine bijection with slope $-\frac{1}{2}$ for all $p \in \mathcal{P}$, thus there exists $c_p \in \mathbb{R}$ such that

$$\widehat{g}(x) = -\frac{1}{2}x + c_p \quad \text{for all} \quad x \in G_p \text{ and } p \in \mathcal{P}. \tag{6.7}$$

Let us prove that $\widehat{g}$ is strictly decreasing on $\widehat{J_i} = \bigcup_{p\in\widehat{I_i}} G_p$. Let $x < y$ be two points in $\widehat{J_i}$. Since $\widehat{g}$ is already strictly decreasing on each interval G_p, we may assume that $x \in G_p$ and $y \in G_q$, where $p, q \in \widehat{I_i}$ are such that $\sup G_p < \inf G_q$. By (6.5), we have that $p < q$ and $\{p, q\} \subset I_i$. Then, since $T'(z) = -1$ for all $z \in I_i$, we have that $T|_{I_i}$ is decreasing, thus $T(p) > T(q)$. By (6.5) once more, we get $\sup G_{T(q)} < \inf G_{T(p)}$. By definition, $\widehat{g}(p) \in G_{T(p)}$ and $\widehat{g}(q) \in G_{T(q)}$, thus $\widehat{g}(p) > \widehat{g}(q)$. This proves that $\widehat{g}$ is decreasing on $\widehat{J_i}$. It remains to prove that c_p in (6.7) is the same for all $p \in \widehat{I_i}$. Let $p, q \in \widehat{I_i}$ with $p \neq q$. We may assume that $a = \sup G_p < \inf G_q = b$. Notice that since $\widehat{g}$ is decreasing on $\widehat{J_i}$,

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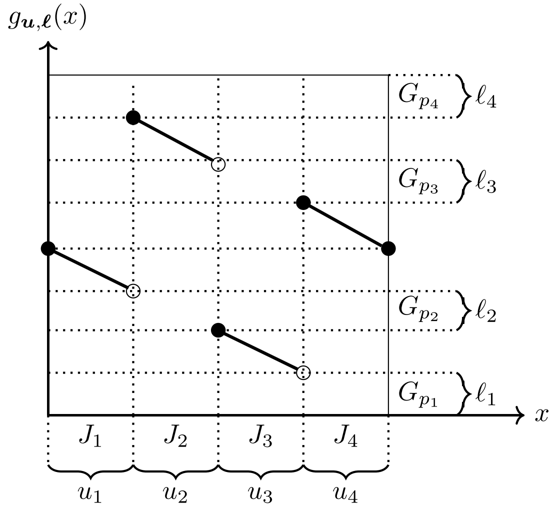

FIGURE 4. The plot of $g = g_{\boldsymbol{u},\boldsymbol{\ell}}$.

$$\frac{1}{2}(b-a) - (c_q - c_p) = -\big(\widehat{g}(b) - \widehat{g}(a)\big) = \sum_{G_r \subset [a,b]} \big|\widehat{g}(G_r)\big|$$
$$= \frac{1}{2} \sum_{G_r \subset [a,b]} |G_r| = \frac{1}{2}(b-a),$$

yielding $c_p = c_q$. Thus, (6.6) is true.

It follows from (6.6) that $\widehat{g}|_{\widehat{J}_i}$ admits a unique monotone continuous extension to the interval $J_i = h^{-1}(I_i)$. This extension is also an affine map with slope equal to $-\frac{1}{2}$. Since i is arbitrary, we obtain an injective piecewise $\frac{1}{2}$-affine extension g of $\widehat{g}$ to the whole interval $[0, 1] = \bigcup_{i=1}^4 J_i$.

We claim that $h \circ g = T \circ h$. In fact, for every $y \in G_p$, we have that

$$h\big(g(y)\big) = \widehat{h}\big(\widehat{g}(y)\big) = T(p) = T\big(\widehat{h}(y)\big) = T\big(h(y)\big). \tag{6.8}$$

Hence, (6.8) holds for a dense set of $y \in [0, 1]$. By continuity, (6.8) holds for every $y \in [0, 1]$. In this way, g is topologically semiconjugate to T.

Figure 4 gives a geometrical picture of the map g. All the slopes equal $-\frac{1}{2}$. It is elementary to verify that $g = g_{\boldsymbol{u},\boldsymbol{\ell}}$, where $u_i = |J_i|$. Thus, the formula of g is the one provided in Definition 6.2.

It remains to prove that $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ satisfies (6.2). In fact, $\sum_{i=1}^4 u_i = \sum_{i=1}^4 |J_i| = 1$. Moreover, we have that

$$u_i = |J_i| = \sum_{G_p \subset J_i} \big|G_p\big| = \sum_{p \in \mathcal{P} \cap I_i} \phi(p) = \sum_{j=1}^4 \sum_{k \in K_{ij}} \frac{\ell_j}{2^k} = \sum_{j=1}^4 c_{ij}\ell_j.$$

Replacing ℓ_4 by $\frac{1}{2} - \ell_1 - \ell_2 - \ell_3$ yields, for all $1 \le i \le 3$,

$$u_i = \sum_{j=1}^3 (c_{ij} - c_{i4})\ell_j + \frac{1}{2}c_{i4},$$

which concludes the proof. □

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Proposition 6.5 *Let $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ be such that $u_1, u_2, u_3 > 0$, $u_4 = 1 - u_1 - u_2 - u_3$,*

$$0 < u_1 < \frac{1}{3}, \quad \frac{1}{3} < u_1 + u_2 < \frac{2}{3}, \quad \frac{2}{3} < u_1 + u_2 + u_3 < 1, \tag{6.9}$$

and let $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3, \ell_4)$ be a vector with positive entries satisfying $\sum_{i=1}^{4} \ell_i = \frac{1}{2}$. If

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 2 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{6.10}$$

$$z_1 = u_1, \quad z_2 = u_1 + u_2, \quad z_3 = u_1 + u_2 + u_3, \tag{6.11}$$

and

$$d_1 = \frac{1}{3z_1} - 1, \quad d_2 = \frac{2 - 3z_2}{3z_2 - 1}, \quad d_3 = \frac{3 - 3z_3}{3z_3 - 2}, \tag{6.12}$$

then $g_{\boldsymbol{u},\boldsymbol{\ell}} = f_{d_1,d_2,d_3}$, that is, $g_{\boldsymbol{u},\boldsymbol{\ell}}$ is the Poincaré map of a switched server system.

Proof By replacing (6.10) in (6.1), and (6.11) and (6.12) in (1.2), it can be easily verified that $g_{\boldsymbol{u},\boldsymbol{\ell}} = f_{d_1,d_2,d_3}$. □

Proof of Theorem 2.2 The items (a) and (b) of Theorem 2.2 follow immediately from Propositions 6.4, 6.5 and Theorem 2.1. Let $\mathbf{v} \in \partial\Delta$. It is clear that $\omega_F(\mathbf{v})$ is a closed, therefore compact, set for every $\mathbf{v} \in \partial\Delta$. Since $F : \partial\Delta \to \partial\Delta$ is topologically conjugate to a piecewise contraction $f : [0, 1] \to [0, 1]$ injective on $(0, 1]$, by [18, Lemma 4.1], there are finitely many pairwise disjoint open connected subsets $F_1, \ldots, F_r$ of $\partial\Delta$ such that $\bigcup_{j=1}^{r} \bigcup_{k\geq 0} F^k(F_j)$ is dense in $\partial\Delta$ and $F^k(\mathbf{v}) \in \partial\Delta \setminus \bigcup_{j=1}^{r} \bigcup_{\ell=0}^{k-1} F^{\ell}(F_j)$ for all $k \geq 1$. In this way, because $\left\{\partial\Delta \setminus \bigcup_{j=1}^{r} \bigcup_{\ell=0}^{k-1} F^{\ell}(F_j)\right\}_{k\geq 1}$ is a nested sequence of compact sets, we have that $\omega_F(\mathbf{v})$ belongs to the set $\partial\Delta \setminus \bigcup_{j=1}^{r} \bigcup_{k\geq 0} F^k(F_j)$, which has empty interior. Hence, $\omega_F(\mathbf{v})$ has empty interior, thus $\omega_F(\mathbf{v})$ is totally disconnected. Since, by the item (b), F is topologically semiconjugate to T, we have that $\omega_F(\mathbf{v})$ is a perfect set. In this way, $\omega_F(\mathbf{v})$ is a Cantor set. This proves the item (c). Let us prove the item (d). Let $0 \leq t_1 < t_2 \cdots$ be the switching times. Let $\mathbf{v}(t_k) = \big(v_1(t_k), v_2(t_k), v_3(t_k)\big)$ be the state of the server at the time t_k. By (3.5), we have that $l(t_k) = 1$ (i.e. the server is connected to the tank 1) if and only if $\mathbf{v}(t_k) \in [\mathbf{r}_2, \mathbf{e}_1] \cup [\mathbf{e}_1, \mathbf{r}_3)$. Since F is topologically semiconjugate to T, this translates into interval dynamics as follows: $l(t_k) = 1$ if and only if $w_k \in [\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3)$, where $w_k = h(\mathbf{v}(t_k))$ is the projection of $\mathbf{v}(t_k)$ by the topological semiconjugacy h. In this way, since T is uniquely ergodic, the normalised Lebesgue measure μ in the only T-invariant Borel probability measure, then by the version of Birkhoff's Ergodic Theorem for uniquely ergodic transformations (see [7, Proposition 4.1.13]), we reach

$$\begin{aligned} \operatorname{freq}(1) &= \lim_{n\to\infty} \frac{1}{n} \#\{1 \leq k \leq n : l(t_k) = 1\} \\ &= \lim_{n\to\infty} \frac{1}{n} \#\{1 \leq k \leq n : w_k \in [\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3)\} \\ &= \lim_{n\to\infty} \frac{1}{n} \#\{1 \leq k \leq n : T^{k-1}(w_1) \in [\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3)\} \\ &= \mu\big([\lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3)\big) = \frac{\lambda_3}{|\boldsymbol{\lambda}|}. \end{aligned}$$

Likewise, $l(t_k) = 2$ if and only if $\mathbf{v}(t_k) \in [\mathbf{r}_3, \mathbf{e}_2] \cup [\mathbf{e}_2, \mathbf{r}_1)$. In terms of interval dynamics, this means that $l(t_k) = 2$ if and only if $w_k \in [\lambda_1 + \lambda_2 + \lambda_3, |\boldsymbol{\lambda}|] \cup [0, \lambda_1)$. Therefore,

$$\begin{aligned}\text{freq}\,(2) &= \lim_{n\to\infty} \frac{1}{n}\#\{1 \le k \le n : l(t_k) = 2\}\\ &= \lim_{n\to\infty} \frac{1}{n}\#\{1 \le k \le n : w_k \in [\lambda_1 + \lambda_2 + \lambda_3, |\boldsymbol{\lambda}|] \cup [0, \lambda_1)\}\\ &= \lim_{n\to\infty} \frac{1}{n}\#\{1 \le k \le n : T^{k-1}(w_1) \in [\lambda_1 + \lambda_2 + \lambda_3, |\boldsymbol{\lambda}|] \cup [0, \lambda_1)\}\\ &= \mu\big([\lambda_1 + \lambda_2 + \lambda_3, |\boldsymbol{\lambda}|]\big) + \mu\big([0, \lambda_1)\big) = \frac{\lambda_1 + \lambda_4}{|\boldsymbol{\lambda}|}.\end{aligned}$$

Finally, $l(t_k) = 3$ if and only if $\mathbf{v}(t_k) \in [\mathbf{r}_1, \mathbf{e}_3] \cup [\mathbf{e}_3, \mathbf{r}_2)$. In terms of interval dynamics, this means that $l(t_k) = 3$ if and only if $w_k \in [\lambda_1, \lambda_1 + \lambda_2)$. Therefore,

$$\begin{aligned}\text{freq}\,(3) &= \lim_{n\to\infty} \frac{1}{n}\#\{1 \le k \le n : l(t_k) = 2\}\\ &= \lim_{n\to\infty} \frac{1}{n}\#\{1 \le k \le n : w_k \in [\lambda_1, \lambda_1 + \lambda_2)\}\\ &= \lim_{n\to\infty} \frac{1}{n}\#\{1 \le k \le n : T^{k-1}(w_1) \in [\lambda_1, \lambda_1 + \lambda_2)\}\\ &= \mu\big(\,[\lambda_1, \lambda_1 + \lambda_2)\,\big) = \frac{\lambda_2}{|\boldsymbol{\lambda}|}.\end{aligned}$$

□

Conflicts of interest

The authors declare that they have no conflicts of interest.

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Appendix. Numerical analysis

The proofs of the results that demand numerical analysis are provided in this section. In order not to overstretch the discussion, we skip some details. Since the isometric model $T : I \to I$ is a piecewise-defined map, in order to compute $T^k(x)$, it is necessary to know which of the intervals I_1, I_2, I_3, I_4 the point $T^{k-1}(x)$ belongs to. In other words, we need to know the address i_{k-1} determined by the equation $T^{k-1}(x) \in I_{i_{k-1}}$. By recursion, if we know the word $i_0 i_1 \ldots i_{k-1}$, then we can compute $T^k(x)$ exactly by means of Corollary 4.7. All we need is to compute $\{x, T(x), \ldots, T^k(x)\}$ for finitely many x's and finitely many k's.

A.1 Spectral analysis of the matrix *P*

The characteristic polynomial p of P is the product of polynomials:

$$p(t) = (t-1)(t^3 - 11t^2 + 7t - 1).$$

Hence, the Perron–Frobenius eigenvalue η of P is a root of the irreducible polynomial over $\mathbb{Q}$: $t^3 - 11t^2 + 7t - 1$. In particular, 1, η and η^2 are rationally independent. Namely, η is equal to

$$\eta = \frac{1}{3}\left(11 + \frac{50 \cdot 2^{2/3}}{\sqrt[3]{499 + 3i\sqrt{111}}} + \sqrt[3]{998 + 6i\sqrt{111}}\right) \cong 10.331851$$

and the associated probability eigenvector $\boldsymbol{v} = (v_1, v_2, v_3, v_4)$ is given by

$$\boldsymbol{v} = \left(\frac{-3\eta^2 + 32\eta - 9}{4}, \frac{5\eta^2 - 54\eta + 25}{4}, \frac{\eta^2 - 10\eta - 3}{4}, \frac{-3\eta^2 + 32\eta - 9}{4}\right), \qquad \text{(A1)}$$

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Table A.1. *Symbolic itineraries of T-orbits of some points*

i	x'_i	$\left\{T^k(x'_i): 0 \le k \le N(x'_i)-1\right\}$		$T^{N(x'_i)}(x'_i)$	$N(x'_i)$
0	0	0		0.796052 . . .	1
1	ν_1	x_1	1.311107 . . .	0.796052 . . .	2
2	$\nu_1+\nu_2$	0.548394 . . .	x_3	x'_4	2
3	$\nu_1+\nu_2+\nu_3$	x_2		0.451606 . . .	1
4	1	1		0.107159 . . .	1

which is, approximately, equal to

$$\boldsymbol{\nu} \cong (0.344446,\ 0.203947,\ 0.107159,\ 0.344446).$$

The vector $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = Q\boldsymbol{\nu}$ is given by

$$\boldsymbol{\lambda} = \left(\frac{-3\eta^2+32\eta-9}{4}, \frac{6\eta^2-64\eta+22}{4}, \frac{-2\eta^2+22\eta-12}{4}, \frac{5\eta^2-54\eta+25}{4}\right), \tag{A2}$$

which is, approximately, equal to

$$\boldsymbol{\lambda} \cong (0.344446,\ 0.3111078,\ 0.4516059,\ 0.203947).$$

Notice that $|\boldsymbol{\lambda}| \cong 1.311107$.

Proof of Lemma 5.1 Let I_1, I_2, I_3, I_4 be the partition of $[0, |\boldsymbol{\lambda}|]$ defined by

$$I_1 = [x_0, x_1), \quad I_2 = [x_1, x_2), \quad I_3 = [x_2, x_3), \quad I_4 = [x_3, x_4],$$

where

$$x_0 = 0, \quad x_1 = \lambda_1, \quad x_2 = \lambda_1 + \lambda_2, \quad x_3 = \lambda_1 + \lambda_2 + \lambda_3, \quad x_4 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = |\boldsymbol{\lambda}|.$$

Then

$$\begin{cases} I_1 \cong [0,\ 0.344446) \\ I_2 \cong [0.344446\,, 0.655553) \\ I_3 \cong [0.655553,\ 1.107159) \\ I_4 \cong [1.107159,\ 1.311107]. \end{cases}$$

Let

$$x'_0 = 0, \quad x'_1 = \nu_1, \quad x'_2 = \nu_1 + \nu_2, \quad x'_3 = \nu_1 + \nu_2 + \nu_3, \quad x'_4 = 1.$$

By using the equality $\boldsymbol{\lambda} = Q\boldsymbol{\nu}$, by (2.1) and some numerical analysis, we reach Table A.1.

Table A.1 shows that (H1) and (H2) in Definition 4.3 are satisfied for $B = \bigcup_{i=1}^4 \left\{x'_i, T(x'_i), \ldots, T^{N(x'_i)-1}(x'_i)\right\}$ and $a' = x'_4 = 1$. In fact, $\mathcal{D}(T) = \{x_1, x_2, x_3\} \subset B$ and $a' \in T(B)$. Hence, I' is an admissible interval for T. By Proposition 4.6, for each $1 \le i \le 4$, there exist $r_i \ge 1$ and a word $i_0 i_1 \ldots i_{r_i-1}$ over the alphabet $\mathcal{A} = \{1, 2, 3, 4\}$ such that $(A1)$–$(A4)$ are true. In particular, we have that $r_i = N(c_i)$, where $c_i = (x'_{i-1} + x'_i)/2$. The values of r_i and $i_0 i_1 \ldots i_{r_i-1}$ are given

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Table A.2. *Symbolic itineraries of T-orbits of some intervals*

i	$c_i = (x'_{i-1} + x'_i)/2$	$\left\{T^k(c_i) : 0 \le k \le r_i - 1\right\}$		$T^{r_i}(c_i)$	$N(c_i)$	$i_0 i_1 \ldots i_{r_i - 1}$
1	0.172223 ...	0.172223 ...		0.623829 ...	1	1
2	0.4464201 ...	0.446420 ...	1.209134 ...	0.898026 ...	2	24
3	0.601974 ...	0.601974 ...	1.053579 ...	0.053579 ...	2	23
4	0.827777 ...	0.827777 ...		0.2793829 ...	1	3

in Table A.2. By Corollary 4.7, Table A.2 and the equality $\boldsymbol{\lambda} = Q\boldsymbol{\nu}$, we have that the Poincaré map T' of T on $I' = [0, 1]$ is given by

$$T'(x) = \begin{cases} -x + \lambda_1 + \lambda_3 = -x + \nu_1 + \nu_3 + \nu_4 = -x - \nu_2 + 1 & \text{if} \quad x \in [x'_0, x'_1) \\ x + \lambda_3 = x + \nu_3 + \nu_4 = x - \nu_1 - \nu_2 + 1 & \text{if} \quad x \in [x'_1, x'_2] \\ x + \lambda_2 + \lambda_3 - |\boldsymbol{\lambda}| = x - \lambda_1 - \lambda_4 = x - \nu_1 - \nu_2 & \text{if} \quad x \in (x'_2, x'_3) \\ -x + \lambda_1 + \lambda_2 + \lambda_3 = -x + \nu_1 + \nu_2 + 2\nu_3 + \nu_4 = -x + \nu_3 + 1 & \text{if} \quad x \in [x'_3, x'_4]. \end{cases}$$

This concludes the proof of Lemma 5.1. □

Proof of Lemma 5.2 It suffices to verify the hypotheses of Corollary 4.10. By Lemma 5.1, I' is an admissible interval for T. Moreover, by the $N(c_i)$-column in Table A.2 and by the equality $\boldsymbol{\lambda} = Q\boldsymbol{\nu}$, we reach for $J_i = (x'_{i-1}, x'_i)$,

$$\sum_{i=1}^{4} r_i |J_i| = \sum_{i=1}^{4} r_i (x'_i - x'_{i-1}) = \sum_{i=1}^{4} r_i \nu_i = \nu_1 + 2\nu_2 + 2\nu_3 + \nu_4 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = |\boldsymbol{\lambda}|,$$

which shows that (H3) is true. □

Proof of Lemma 5.3 The proof consists in verifying the hypotheses (H1) and (H2) in Definition 4.3 considering the map $S : [0, 1] \to [0, 1]$, defined in Subsection 5.2, and the interval $I' = \left[0, \frac{1}{\eta}\right] \cong [0,\ 0.096788]$. Notice that $\mathcal{D}(S) = \{y_1, y_2, y_3\}$, where

$$y_0 = 0,\ \ y_1 = \nu_1 \cong 0.344446,\ \ y_2 = \nu_1 + \nu_2 \cong 0.548394,\ \ y_3 = \nu_1 + \nu_2 + \nu_3 = 0.655553,\ \ y_4 = 1.$$

Let

$$y'_0 = 0, \quad y'_1 = \frac{1}{\eta} y_1, \quad y'_2 = \frac{1}{\eta} y_2, \quad y'_3 = \frac{1}{\eta} y_3, \quad y'_4 = \frac{1}{\eta}.$$

By using the equality $P\boldsymbol{\nu} = \eta\boldsymbol{\nu}$ and some numerical analysis, we reach Table A.3. Table A.3 shows that (H1) and (H2) in Definition 4.3 are satisfied for $B = \bigcup_{i=1}^{4} \left\{y'_i, S(y'_i), \ldots, S^{N(y'_i)-1}(y'_i)\right\}$ and $a' = y'_4 = \frac{1}{\eta}$. In fact, $\mathcal{D}(T) = \{y_1, y_2, y_3\} \subset B$ and $a' \in S(B)$. Hence, $I' = \left[0, \frac{1}{\eta}\right]$ is an admissible interval for S, which concludes the proof. □

Proof of Lemma 5.4 By Lemma 5.3 and Proposition 4.6, for each $1 \le i \le 4$, there exist $r_i \ge 1$ and a word $i_0 i_1 \ldots i_{r_i - 1}$ over the alphabet $\mathcal{A} = \{1, 2, 3, 4\}$ such that $(A1)$–$(A4)$ are true. In particular, we have that $r_i = N(c_i)$, where $c_i = (y'_{i-1} + y'_i)/2$. The iterates $S^k(c_i)$ are shown in Table A.4.

Table A.3. *Symbolic itineraries of S-orbits of some points*

i	y'_i	$\{S^k(y'_i) : 0 \le k \le N(y'_i) - 1\}$	$S^{N(y'_i)}(y'_i)$	$N(y'_i)$
0	0	0 0.796052 ... 0.311107 ... 0.484944 ... 0.936550 ... 0.170609 ... 0.625442 ...	0.077048 ...	7
1	$\frac{\nu_1}{\eta}$	0.033338 ... 0.762713 ... y_1 0.796052 ... 0.311107 ... 0.484944 ... 0.936550 ... 0.170609 ... 0.625442 ...	0.077048 ...	9
2	$\frac{\nu_1 + \nu_2}{\eta}$	0.053078 ... 0.742974 ... 0.364185 ... 0.815791 ... 0.291368 ... 0.504683 ... 0.956289 ... 0.150869 ... 0.645182 ...	y'_4	9
3	$\frac{\nu_1 + \nu_2 + \nu_3}{\eta}$	0.063449 ... 0.732602 ... 0.374557 ... 0.826163 ... 0.280996 ... 0.515055 ... 0.966661 ... 0.140498 ... y_3 0.451605 ... 0.903211 ... 0.203947 0.592104 ...	0.043710 ...	13
4	$\frac{1}{\eta}$	0.096788 ... 0.699263 ... 0.407895 ... 0.859501 ... 0.247658 ... y_2 1 0.107159 ... 0.688892 ... 0.418267 ... 0.869873 ... 0.237286 ... 0.558765 ...	0.010371 ...	13

Table A.4. *Symbolic itineraries of S-orbits of some intervals*

i	$c_i = \frac{(y'_{i-1} + y'_i)}{2}$	$\{S^k(c_i) : 0 \le k \le r_i - 1\}$	$S^{r_i}(c_i)$	$N(c_i)$
1	0.016669 ...	0.016669 ... 0.779382 ... 0.327776 ... 0.468275 ... 0.919881 ... 0.187278 ... 0.608773 ...	0.060379 ...	7
2	0.043208 ...	0.043208 ... 0.752843 ... 0.354315 ... 0.805921 ... 0.301237 ... 0.494814 ... 0.946420 ... 0.160739 ... 0.635312 ...	0.086918 ...	9
3	0.0582639 ...	0.058263 ... 0.737788 ... 0.369371 ... 0.820977 ... 0.286182 ... 0.509869 ... 0.961475 ... 0.145684 ... 0.650368 ... 0.101973 ... 0.694078 ... 0.413081 ... 0.864687 ... 0.242472 ... 0.553579 ...	0.005185 ...	15
4	0.08011894 ...	0.080118 ... 0.715933 ... 0.391226 ... 0.842832 ... 0.264327 ... 0.531724 ... 0.983330 ... 0.123829 ... 0.672223 ... 0.434936 ... 0.886542 ... 0.220617 ... 0.575434 ...	0.027040 ...	13

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Table A.5. *Symbolic itineraries of the tower*

i	$r_i = N(c_i)$	$i_0 i_1 \ldots i_{r_i-1}$
1	7	1 4 1 2 4 1 3
2	9	1 4 2 4 1 2 4 1 3
3	15	1 4 2 4 1 2 4 1 3 1 4 2 4 1 3
4	13	1 4 2 4 1 2 4 1 4 2 4 1 3

The values of r_i and $i_0 i_1 \ldots i_{r_i-1}$ are given in Table A.5. By Corollary 4.7, Table A.5 and the equality $\boldsymbol{\lambda} = Q\boldsymbol{\nu}$, we have that the Poincaré map S' of S on $\left[0, \frac{1}{\eta}\right]$ is given by

$$S'(x) = \begin{cases} -x + 2 - 2\nu_1 - 5\nu_2 - 2\nu_3 & \text{if} \quad x \in (y'_0, y'_1) \\ \quad x + 2 - 3\nu_1 - 4\nu_2 - \nu_3 & \text{if} \quad x \in (y'_1, y'_2) \\ \quad x + 3 - 5\nu_1 - 6\nu_2 - \nu_3 & \text{if} \quad x \in (y'_2, y'_3) \\ -x + \nu_3 & \text{if} \quad x \in (y'_3, y'_4). \end{cases} \tag{A3}$$

By (A3) and by the equality $\frac{1}{\eta}\boldsymbol{\nu} = P^{-1}\boldsymbol{\nu}$, it follows that $S' = L \circ S \circ L^{-1}$ on $I' \backslash \{y'_0, \ldots, y'_4\}$, proving that S in fact self-similar on $\left[0, \frac{1}{\eta}\right]$. This concludes the proof of Lemma 5.4.

□

Proof of Lemma 5.5 It suffices to verify the hypotheses of Corollary 4.13. By Lemma 5.3, $\left[0, \frac{1}{\eta}\right]$ is an admissible interval for S. By Lemma 5.4, S is self-similar on $\left[0, \frac{1}{\eta}\right]$. Let p_{ij} denote the i, j-entry of the matrix P. By the $N(c_i)$-column in Table A.5 and by the equality $P\boldsymbol{\nu} = \eta\boldsymbol{\nu}$, we reach for $J_i = (y'_{i-1}, y'_i)$,

$$\sum_{i=1}^{4} r_i |J_i| = \sum_{i=1}^{4} r_i (y'_i - y'_{i-1}) = \sum_{i=1}^{4} r_i \frac{\nu_i}{\eta} = \frac{1}{\eta}(7\nu_1 + 9\nu_2 + 15\nu_3 + 13\nu_4)$$
$$= \frac{1}{\eta} \sum_{j=1}^{4} \sum_{i=1}^{4} p_{ij}\nu_i = \frac{1}{\eta} \sum_{i=1}^{n} \eta\nu_i = 1,$$

which shows that (H3) is true. Applying (4.3) to the third column in Table A.5 yields $M = P$, where M is the matrix associated with $\left(S, \left[0, \frac{1}{\eta}\right]\right)$. Hence, M is positive and (H4) holds. By Corollary 4.13, S is topologically transitive. By Lemma 5.2, T is topologically transitive. □