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PAPER

Variational inequalities arising from credit rating migration with buffer zone

Xinfu Chen[1,2] and Jin Liang[2]

[1]Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, 15260, USA and [2]School of Mathematical Science, Tongji University, Shanghai, 200092, China
Corresponding author: Jin Liang; Email: liang_jin@tongji.edu.cn

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Abstract
In Chen and Liang previous work, a model, together with its well-posedness, was established for credit rating migrations with different upgrade and downgrade thresholds (i.e. a buffer zone, also called dead band in engineering). When positive dividends are introduced, the model in Chen and Liang (*SIAM Financ. Math.* 12, 941–966, 2021) may not be well-posed. Here, in this paper, a new model is proposed to include the realistic nonzero dividend scenarios. The key feature of the new model is that partial differential equations in Chen and Liang (*SIAM Financ. Math.* 12, 941–966, 2021) are replaced by variational inequalities, thereby creating a new free boundary, besides the original upgrading and downgrading free boundaries. Well-posedness of the new model, together with a few financially meaningful properties, is established. In particular, it is shown that when time to debt paying deadline is long enough, the underlying dividend paying company is always in high grade rating, that is, only when time to debt paying deadline is within a certain range, there can be seen the phenomenon of credit rating migration.

1. Introduction

Following the ever-increasing development and complexity of the financial market, credit rating migration risk is playing a more and more important role. The 2008 financial tsunami and 2010 European debit crisis exemplify the importance of proper credit risk management. It gradually becomes an urgent task both for industries and academics to measure and manage the risk [5, 6]. In theory, there are two main mathematical frameworks for this risk: structure model and intensity one. They are different, one modelled by endogenic causes and the other by exogenous factors.

Structural models involve the reason of the rating objects themselves and usually indicate the rating migration boundaries; therefore, they have some advantages. The works using this kind of model for credit rating migration can be found in [4, 7, 10, 12–17] and the reference therein, where a credit migration boundary is used for both upgrading and downgrading. The stochastic feature of the model will then create a phenomena of infinite many upgrading and downgrading since any Brownian motion path oscillates around any level it reached infinitely many times. Thus, a model equipped with a buffer zone, called dead band or hysteresis, is necessarily needed.

In [3], we establish a model for credit rating migration with different migration upgrading and downgrading thresholds. In the model, credit rating migrations are assumed to depend on the ratio of debt and asset value of the underlying company where debt is assumed to be a zero coupon corporate bond and asset value follows a geometric Brownian motion with volatility depending on credit rating. There is a buffer zone in credit rating migration, so upgrade and downgrade thresholds are different.

Mathematically, this model is a system of partial differential equations (PDEs) with two free boundaries that correspond to the hitting boundaries in state space to upgrade and downgrade credit rating, respectively. Under the condition of zero dividend, the existence, uniqueness, regularity and asymptotic behaviour of the solution and free boundaries are obtained. In addition, some nonzero dividend cases are discussed on the behaviours of the solution. However, the existence and uniqueness of the solution in the nonzero dividend case are still open, since the method for the zero dividend case seems to be no more suitable for the nonzero dividend case. In this paper, we are going to fill this gap.

From the discussion of the asymptotic behaviour of the nonzero dividend case in [3], we see that the migration free boundaries will go to negative infinite time. This adds an interesting phenomena to the model. There exist two times, T^L and T^H with $0 < T^H < T^L$. When time to expiry is larger than T^L, the credit rating is always high. When time to expiry is between T^H and T^L, the credit rating can change at most once and it is from low to high. Only when the time is shorter than T^H, there are credit rating changes that go both directions.

However, the method in [3] cannot be directly applied to the nonzero dividend case, as one PDE with nonzero dividend in the buffer zone is no longer suitable to the maximum principle. That means, the upgrade/downgrade condition might be violated in the buffer zone. Thus, the old model for nonzero dividend needs to be modified. To avoid the ratios run out the threshold inside the buffer zone, obstacle problems are applied to the model instead of pure PDEs.

The paper is organised as follows. In Section 1, a model with nonzero dividend is presented, with the model in [3] being a special case. In Section 3, a priori properties of the solution of our model are discussed; in particular, it is shown that, when dividends are positive, two free boundaries go to $-\infty$ in finite time. In Section 4, some preparations for theoretical proofs are made. In Section 5, we construct two monotonic sequences of super and subsolutions and prove that both sequences approach the unique solution of the problem. Since the free boundaries go to $-\infty$, part of the solutions are limited to finite time to have a financial meaning. Main theorem is stated in Section 6, where uniqueness is also proved. Section 7 gives final conclusion.

2. Model

A credit rating migration model is established in [3], where only Black–Scholes equations are used. When underlying asset pays zero dividend, the model was shown to be well-posed. When positive dividends are added into the model, only short-time existence can be proved and global solutions do not exist in certain cases. Hence, the model in [3] needs to be modified. In this paper, we revise the model by replacing the Black–Scholes equations by variational inequalities. We will show later that the revised model is well-posed.

2.1. Basic assumptions

We assume that the underlying firm issues a corporate bond, which is a contingent claim of its (observable) asset value and its credit rating. We consider the simple case where only two ratings are used: "H" and "L".

Assumption 1 (Firm's asset under different credit ratings). *Let $(\Omega, \{\mathscr{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with filtration. Denoting by S_t the firm's asset value in the risk neutral world,*[1] *we assume that $\{S_t\}$ obeys the stochastic differential equation (SDE)*

[1] One may criticise that the real asset value is not observable; here for mathematical simplicity we assume that the credit rating assign organisation is able enough to learn the real asset value of the underlying company.

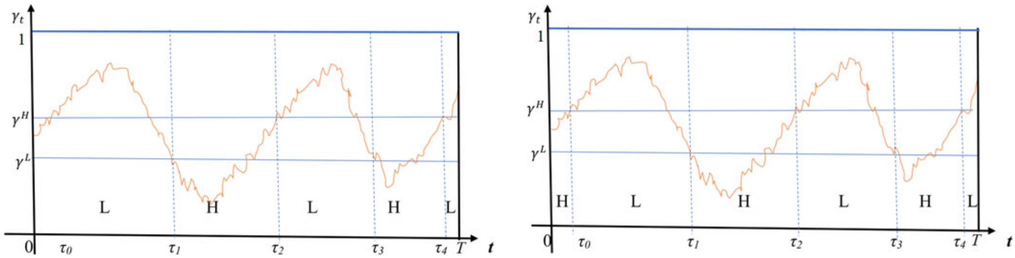

Figure 1. *An illustration of the rating migration. Left: starting from a low rating, the rating will upgrade at τ_1, downgrade at τ_2, upgrade at τ_3 and downgrade at τ_4; Right: with the same sample path as the left, but starting from a high rating, the rating will downgrade at τ_0, the rest is the same as the left figure. Starting from different rating, the bond values are different only up to time τ_0. After τ_0, initial difference of company's rating disappears.*

$$\frac{dS_t}{S_t} = \begin{cases} (r-\delta^H)dt + \sigma^H dW_t & \text{in the high rating region,} \\ (r-\delta^L)dt + \sigma^L dW_t & \text{in the low rating region,} \end{cases} \tag{2.1}$$

where r is the risk free interest rate, and (σ^H, σ^L) and (δ^H, δ^L) are constants that represent volatilities and dividend rates of the firm under the high and low credit grades, respectively. Also $\{W_t\}_{t\geqslant 0}$ is the standard Brownian motion, which generates the filtration $\{\mathscr{F}_t\}$.

In this paper, we assume that

$$0 < \sigma^H < \sigma^L, \quad 0 \leqslant \delta^L \leqslant \delta^H. \tag{2.2}$$

This means that the volatility in the high rating region should be lower than the one in low rating region, and in high rating, the firm would like to pay more dividend.

When $\delta^L = \delta^H = 0$, as one can see from our analysis, no variational inequalities are needed, so the model reduces to the one in [3]. Hence, in this paper, we consider only $\delta^L > 0$. We will see later that when $\delta^L = 0$ and $\delta^H > 0$ the methods used in [3] still work.

Assumption 2 (The corporate bond). *The firm issues only one zero coupon bond with face value F. Denote by Φ_t the discount value of the bond at time t. Then, on the maturity time T, the bond value is $\Phi_T = g(S_T)$, where $g(S) = S \wedge F$. Throughout this paper, we use notation $a \wedge b = \min\{a, b\}$ and $a \vee b =$* $\max\{a, b\}$.

Assumption 3 (The credit rating migration). *The bond is regarded as debt, and the maturity time T as the debt paying back deadline. We assume that this is the only debt that the underlying company owes. Then the changes of credit ratings are assumed to be determined by the ratio of the debt and asset value. From this view, we define the process of financial situation by*

$$\gamma_t = \frac{\Phi_t}{S_t}.$$

Following [3], we assume that when $\gamma_t \geq \gamma^H$, the company is downgraded; when $\gamma_t \leqslant \gamma^L$, the company is upgraded and when $\gamma^L < \gamma_t < \gamma^H$, the rating will not be changed. See Figure 1 for an illustration. Here, γ^H and γ^L are downgrade and upgrade thresholds, respectively, and satisfy

$$0 < \gamma^L < \gamma^H < 1. \tag{2.3}$$

This assumption describes that the credit rating migrations have a buffer or grace region, where, as in [3], downgrade and upgrade boundaries are different.

2.2. The rating process

We use a stochastic process $\{R_t\}_{t\leqslant T}$ to denote the rating of the underlying company. Assume for simplicity that there are only two ratings, H and L, where H stands for high rating and L stands for low rating. Thus, R_t takes only two values, H and L.

We assume that R_t is observable in the sense that there are closed sets Σ^H and Σ^L, such that

(i) If $(S_t, t) \in \Sigma^H$, the rating remains at or changes to H;
(ii) If $(S_t, t) \in \Sigma^L$, the rating remains at or changes to L;
(iii) If $(S_t, t) \notin \Sigma^H \cup \Sigma^L$, the rating will not change.

It is natural to assume that Σ^H is a graph with a low boundary and Σ^L is a graph with an upper boundary, i.e.

$$\text{If } (S, t) \in \Sigma^H, \text{ then } (S+h, t) \in \Sigma^H \text{ for every } h > 0;$$

$$\text{If } (S, t) \in \Sigma^L, \text{ then } (z, t) \in \Sigma^L \text{ for every } z \in (0, S).$$

That is, there are functions b^L and b^H defined on $(-\infty, T]$ such that

$$\Sigma^L = \{(S, t) | t \leqslant T, S \leqslant b^H(t)\}, \qquad \Sigma^H = \{(S, t) | t \leqslant T, S \geqslant b^L(t)\}.$$

This implies that b^H is the boundary of high credit rating region

$$Q^H = \{(S, t) \mid t \leqslant T, S > b^H(t)\}$$

and b^L is that for low credit rating region

$$Q^L = \{(S, t) \mid t \leqslant T, S < b^L(t)\}.$$

We call the region

$$B = \{(S, t) \mid t \leqslant T, b^H(t) < S < b^L(t)\}$$

the buffer zone in which no credit rating changes.

The purpose of our paper is to find a model to calculate b^H and b^L. Based on the assumption, we have

$$R_t = \begin{cases} H & \text{if } S_t \geqslant b^L(t), \\ L & \text{if } S_t \leqslant b^H(t), \\ R_{t-} & \text{if } b^H(t) < S_t < b^L(t). \end{cases}$$

Then, we have the dynamics of asset value

$$\begin{cases} dS_t = (r - \delta^{R_t})S_t dt + \sigma^{R_t} S_t dW_t, \\ (S_0, R_{0^-}) \text{ is given.} \end{cases} \tag{2.4}$$

2.3. Company debt

Conditioned on $S_t = S$ and $R_t = R$, we use $\Phi^R(S, t)$ to denote the value of the bond (debt) at time t. Note that when $(S, t) \in \Sigma^H$, the rating can only be H; thus, $\Phi^L(S, t)$ is undefined. Similarly, when $(S, t,) \in \Sigma^L$, the rating can only be L; thus, $\Phi^H(S, t)$ is undefined. For technical convenience, we artificially[2] define

$$\Phi^H(S, t) = \gamma^H S \quad \text{if } (S, t) \in \Sigma^L,$$

$$\Phi^L(S, t) = \gamma^L S \quad \text{if } (S, t) \in \Sigma^H.$$

[2]Defining $\Phi^H = \Phi^L$ in $\Sigma^H \cup \Sigma^L$ would be another meaningful extension but not convenient for our presentation.

Suppose that current time is $t \in (-\infty, T)$. We define the upgrade time τ_u^t and downgrade time τ_d^t by

$$\tau_u^t := \inf\{\tau \geqslant t \mid (S_\tau, \tau) \in \Sigma^H\} = \inf\{\tau \geqslant t \mid S_\tau \geqslant b^L(\tau)\};$$
$$\tau_d^t := \inf\{\tau \geqslant t \mid (S_\tau, \tau) \in \Sigma^L\} = \inf\{\tau \geqslant t \mid S_\tau \leqslant b^H(\tau)\}.$$

Here, τ_u^t has double meaning: (1) it is the time to change rating from L and H; (2) the rating remains at H. Similar meaning applies to τ_d^t. Note that if $(S_t, t) \in \Sigma^H$, then $\tau_u^t = t$, and if $(S_t, t) \in \Sigma^L$, then $\tau_d^t = t$. See Figure 1.

We assume that Φ^H and Φ^L are defined by, for $y > 0$ and $t \leqslant T$,

$$\Phi^H(S, t) = (\gamma^H S) \wedge \mathbb{E}\Big[e^{r(t-T)} g(S_T)\mathbf{1}_{\tau_d^t \geqslant T} + \gamma^H e^{r(t-\tau_d^t)} S_{\tau_d^t} \mathbf{1}_{\tau_d^t < T} \Big| S_t = S\Big],$$

$$\Phi^L(S, t) = (\gamma^L S) \vee \mathbb{E}\Big[e^{r(t-T)} g(S_T)\mathbf{1}_{\tau_u^t \geqslant T} + \gamma^L e^{r(t-\tau_u^t)} S_{\tau_u^t} \mathbf{1}_{\tau_u^t < T} \Big| S_t = S\Big].$$

To explain the meaning, let us consider $\Phi^L(S, t)$. Conditioned on $S_t = S$ and $R_{t^-} = L$, there are two cases

(i) $(S, t) \in \Sigma^H$. Then $\tau_u^t = t$; hence $\Phi^L(S, t) = \gamma^L S$;

(ii) $(S, t) \notin \Sigma^H$. Then $\tau_u^t > t$ and $R_t = L$ for $\tau \in [t, \tau_u^t)$, so that

$$dS_\tau = (r - \delta^L) S_\tau d\tau + \sigma^L S_\tau dW_\tau \qquad \text{for } \tau \in [t, \tau_u^t).$$

Thus, by a standard theory of stochastic process (e.g. [9]), we can drive that Φ^L is a solution of the variational inequality:

$$\begin{cases} \mathscr{F}^L \Phi^L(S, t) = 0 & \text{for } t < T,\ S < b^L(t), \\ \Phi^L(S, t) = \gamma^L S & \text{for } t < T,\ S \geqslant b^L(t), \\ \Phi^L(S, T) = (\gamma^L S) \vee g(S) & \text{for } S > 0, \end{cases} \tag{2.5}$$

where

$$\mathscr{F}^L \Phi(S, t) = \min\left\{\Phi - \gamma^L S,\ -\frac{\partial \Phi}{\partial t} - \frac{(\sigma^L)^2}{2} S^2 \frac{\partial^2 \Phi}{\partial S^2} - (r - \delta^L) S \frac{\partial \Phi}{\partial S} + r\Phi\right\}.$$

Similarly, Φ^H is the solution of the variational inequality

$$\begin{cases} \mathscr{F}^H \Phi^H(S, t) = 0 & \text{for } t < T, S > b^H(t), \\ \Phi^H(S, t) = \gamma^H S & \text{for } t < T, S \leqslant b^H(t), \\ \Phi^H(S, T) = (\gamma^H S) \wedge g(S) & \text{for } S > 0, \end{cases} \tag{2.6}$$

where

$$\mathscr{F}^H \Phi(S, t) = \max\left\{\Phi - \gamma^H S, -\frac{\partial \Phi}{\partial t} - \frac{(\sigma^H)^2}{2} S^2 \frac{\partial^2 \Phi}{\partial S^2} - (r - \delta^H) S \frac{\partial \Phi}{\partial S} + r\Phi\right\}.$$

2.4. Rating changes

We assume that when the credit rate changes, the bond price does not change. This produces the free boundary condition

$$\begin{cases} \Phi^H(b^L(t), t) = \gamma^L b^L(t) & \text{if } b^L(t) > 0, \\ \Phi^L(b^H(t), t) = \gamma^H b^H(t) & \text{if } b^H(t) > 0. \end{cases} \tag{2.7}$$

In conclusion, we model the credit rating change by $(\Phi^H, \Phi^L, b^H, b^L)$ which is a solution of (2.5), (2.6), (2.7).

Remark 2.1. Define times

$$T^H = \frac{1}{\delta^L}\ln\frac{1}{\gamma^H}, \quad \hat{T}^L = \frac{1}{\delta^L}\ln\frac{1}{\gamma^L}, \quad T^L = T^H + \frac{1}{\delta^H}\ln\frac{\gamma^H}{\gamma^L}. \tag{2.8}$$

Due to the dividend paying, we shall show that

$$\begin{cases} b^H(t) > 0 & \text{for } t \in (T - T^H, T], \\ b^H(t) = 0 & \text{for } t \leqslant T - T^H, \end{cases}$$

$$\begin{cases} b^L(t) > b^H(t) \geqslant 0 & \text{for } t \in (T - T^L, T], \\ b^L(t) = 0, & \text{for } t \leqslant T - T^L. \end{cases}$$

In particular, if $t \leqslant T - T^L$, the company is always in high rating and will remain in the high rating at least up to time $T - T^H$.

2.5. *The free boundary problem*

Denote by, for $t > 0$ and $x \in \mathbb{R}$,

$$s^H(t) = b^H(T-t), \qquad v^H(x,t) = \frac{\Phi^H(Fe^{x-rt}, T-t)}{Fe^{x-rt}},$$

$$s^L(t) = b^L(T-t), \qquad v^L(x,t) = \frac{\Phi^L(Fe^{x-rt}, T-t)}{Fe^{x-rt}}.$$

Note that here t represents the amount of time left from maturity date of the bond. From (2.5)–(2.7), we then derive the following free boundary problem.

(FBP): Find lower semicontinuous functions $s^L,\ s^H : [0,\infty) \mapsto [-\infty, \infty)$ and continuous functions $v^L,\ v^H : \mathbb{R} \times [0,\infty) \mapsto [0,1]$ such that

$$\begin{cases} \min\{\mathscr{L}^L v^L, v^L - \gamma^L\} = 0 & \text{in } \ Q^L := \{(x,t) | t > 0, x < s^L(t)\}, \\ \max\{\mathscr{L}^H v^H, v^H - \gamma^H\} = 0 & \text{in } \ Q^H := \{(x,t) | t > 0, x > s^H(t)\}, \\ v^L = \gamma^L & \text{in } \ \Sigma^H := \mathbb{R} \times (0,\infty) \setminus Q^L, \\ v^H = \gamma^H & \text{in } \ \Sigma^L := \mathbb{R} \times (0,\infty) \setminus Q^H, \\ v^L(s^H(t), t) = \gamma^H & \text{if } \ t \geqslant 0 \text{ and } s^H(t) > -\infty, \\ v^H(s^L(t), t) = \gamma^L & \text{if } \ t \geqslant 0 \text{ and } s^L(t) > -\infty, \\ v^L(\cdot, 0) = v_0^L(\cdot) & \text{on } \ \mathbb{R}, \\ v^H(\cdot, 0) = v_0^H(\cdot) & \text{on } \ \mathbb{R}, \end{cases} \tag{2.9}$$

where for $l = L, H$,

$$\mathscr{L}^l = \frac{\partial}{\partial t} - \Big(\frac{(\sigma^l)^2}{2}\frac{\partial}{\partial x} - \delta^l\Big)\Big(\frac{\partial}{\partial x} + 1\Big),$$

σ^l, δ^l and γ^l are positive constants satisfying (2.2) and (2.3) and v_0^l is defined as, for $x \in \mathbb{R}$,

$$v_0(x) = 1 \wedge e^{-x}, \ \ v_0^L(x) = \gamma^L \vee v_0(x), \ \ v_0^H(x) = \gamma^H \wedge v_0(x). \tag{2.10}$$

It is clear that

$$s^H(0) = -\ln\gamma^H, \quad s^L(0) = -\ln\gamma^L. \tag{2.11}$$

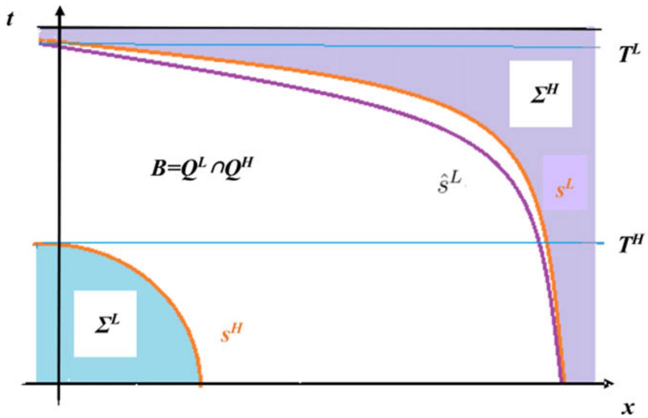

Figure 2. *Sketch of three free boundaries in the scaled variable (x, t): upgrading boundary $x = s^H(t)$, downgrading boundary $x = s^L(t)$ and obstacle boundary $x = \hat{s}^L(t)$; and three regions: low rating Σ^L (blue), high rating Σ^H (purple) and buffer zone $B = Q^H \cap Q^L$ (white), where $Q^H = B \cup \Sigma^H$ and $Q^L = B \cup \Sigma^L$.*

This is a new model and in mathematics, it is a new problem. Thus, first, we need to establish the well-posed of the problem, then analyse the properties of the solution and finally verify financial implications.

The rest of the paper is to study the free boundary problem (2.9). Thus, in the sequel, t is not the current time; instead, it is time to maturity. In addition, subscripts denote partial derivatives with respect to the subscripts.

2.6. Idea of the proof of the existence of (2.9)

The free boundary problem (2.9) contains two unknown functions, v^L and v^H, and two unknown boundaries, s^H and s^L. The unknown v^L can be solved from a standard variational inequality in Q^L, provided that we know its boundary s^L. Similarly, v^H can be solved from a standard variational inequality in Q^H (indeed an equation $\mathscr{L}^H v^H = 0$) if we know its boundary s^H. See Figure 2 for an illustration.

In Section 3, we shall establish certain properties of the solution, in particular, the free boundaries. We shall use a fixed point theorem to show the existence. The properties established in Section 3 will be used to define function spaces for the mapping.

For the existence, first by assuming that $s^L = h_1$ is a given known function in certain function class we solve the variational inequality

$$\min\{\mathscr{L}^L u_1, u_1 - \gamma^L\} = 0 \quad \text{in } \{x < h_1(t)\}. \quad ①$$

Supplied with initial and boundary conditions, this falls into a category of standard free boundary problems [8]. We shall show in Section 4 that there exists a free boundary $\hat{h}$ such that

$$\mathscr{L}^L u_1 = 0 \;\&\; u_1 > \gamma^L \text{ if } x < \hat{h}(t), \quad \mathscr{L}^L u_1 \geqslant 0 \;\&\; u_1 \equiv \gamma^L \text{ if } \hat{h}(t) \leqslant x \leqslant h_1(t).$$

We extend naturally by $u_1 \equiv \gamma$ for $x > h(t)$.

From the solution, we study the level set function h_2 defined implicitly by

$$u_1(h_2, t) = \gamma^H \quad (\text{i.e. } h_2(t) = \inf\{x | u_1(x, t) < \gamma^H\}). \quad ②$$

The map from h_1 to u_1 and then to h_2, together with the function $\hat{h}$, is studied in Section 4.

Next, we assume that $s^H = h_2$ is a known function. We solve u_2 from the variational inequality

$$\max\{\mathscr{L}^H u_2, u_2 - \gamma^H\} = 0 \quad \text{in } \{x > h_2(t)\}. \quad ③$$

Indeed, we can show by a maximum principle that $u_2 < \gamma^H$ so that the variational inequality ③ is equivalent to the PDE $\mathscr{L}^H u_2 = 0$ in $\{x > h_2(t)\}$.

From u_2, we study the level set function $\tilde{h}_1$ defined by

$$u_2(\tilde{h}_1, t) = \gamma^L \quad (\text{i.e. } \hat{h}_1(t) = \inf\{x | u_2(x, t) < \gamma^L\}). \quad ④$$

For the existence, we shall show in Section 5 that map from h_1 to $\tilde{h}_1$ via

$$h_1 \overset{①}{\to} u_1 \overset{②}{\to} h_2 \overset{③}{\to} u_2 \overset{④}{\to} \tilde{h}_1$$

admits a fixed point. Indeed, we can show by comparison that the iteration of the map provides a monotonic sequence if we start from $h_1 \equiv s^L(0)$. The limit of the sequence is the fixed point.

The iteration presented in Section 5.1 uses the map

$$h_2 \overset{③}{\to} u_2 \overset{④}{\to} h_1 \overset{①}{\to} u_1 \overset{②}{\to} \tilde{h}_2$$

starting from $h_2 \equiv -\infty$.

One complication of our analysis arises from the phenomenon that $s^H(t) = -\infty$ when $t \geqslant T^H$ and $s^L(t) = -\infty$ when $t \geqslant T^L$.

Remark 2.2. We will show that $v^H < \gamma^H$ in Q^H; thus, the variational inequality for v^H is indeed the original Black–Scholes equation $\mathscr{L}^H v^H = 0$ in Q^H, the one that is used in the model in [3]. When $\delta^L = 0$, one can shown that $v^L > \gamma^L$ in Q^L, thus, again the variational inequality for v^L is the equation $\mathscr{L}^L v^L = 0$, used in [3]. In conclusion, when $\delta^L = 0 \leq \delta^H$, the current model produces a unique solution that is also the solution of the old model in [3].

For this reason, in the sequel, we assume that $0 < \delta^L \leqslant \delta^H$.

3. A priori property of the solution

The construction of a solution of (2.9), to be given in the next section, is a little bit awkward, partially due to the fact that upgrading and downgrading boundaries go to $-\infty$ in finite time. Hence, in this section, we establish a few properties of the solution, thereby shedding light towards our construction of the solution.

Thus, suppose we have a solution. We investigate certain properties of the solution. Let T^H, $\hat{T}^L$ and T^L be defined in (2.8). Note that $\hat{T}^L \geqslant T^L$ because of $\delta^L \leqslant \delta^H$.

Lemma 3.1. *Let (v^L, v^H, s^L, s^H) be a solution of (2.9). Then $v^L(\cdot, t) \leqslant e^{-\delta^L t}$ for $t \in [0, \hat{T}^L]$, $v^L \equiv \gamma^L$ on $\mathbb{R} \times [\hat{T}^L, \infty)$, and $s^L \equiv -\infty$ in $[\hat{T}^L, \infty)$.*

Proof. Set $\bar{v}^L(x, t) = e^{-\delta_L t}$. By comparison, we have

$$\bar{v}^L \geqslant v^L \quad \text{on } \mathbb{R} \times [0, \hat{T}^L].$$

This implies $v^L(\cdot, \hat{T}^L) \equiv \gamma^L$. Consequently, $v^L \equiv \gamma^L$ in $\mathbb{R} \times [\hat{T}^L, \infty)$ and $s^L = -\infty$ in $[\hat{T}^L, \infty)$. □

Lemma 3.2. *Let (v^L, v^H, s^L, s^H) be a solution of (2.9). The upgrading boundary $x = s^H(t)$ has the following properties:*

(i) $s^H \in C^\infty([0, T^H))$; *in particular* $s^H(t) > -\infty$, *for* $t \in [0, T^H)$;

(ii) $\lim_{t \nearrow T^H} s^H(t) = -\infty$;

(iii) $s^H \equiv -\infty$ *in* $[T^H, \infty)$.

Proof. 1. Let

$$T^* = \inf\{t > 0 | s^H(t) = -\infty\}. \tag{3.1}$$

Since $v^L(s^H(t), t) = \gamma^H$ if $s^H(t) > -\infty$ and by Lemma 3.1, $v^L(\cdot, t) \leqslant e^{-\delta^L t} \vee \gamma^L$, we see that $T^* \leqslant T^H$.

2. Fix a small $\epsilon > 0$ and define v^ϵ by

$$\begin{cases} \mathscr{L}^H v^\epsilon = 0 \quad \text{in } \mathbb{R} \times (T^* - \epsilon, \infty), \\ v^\epsilon(x, T^* - \epsilon) = \begin{cases} \gamma^H & \text{if } x < s^H(T^* - \epsilon), \\ 0 & \text{if } x \geqslant s^H(T^* - \epsilon). \end{cases} \end{cases}$$

Then $v^\epsilon < \gamma^H$ in $\mathbb{R} \times (T^* - \epsilon, \infty)$. By comparison, $v^H \geqslant v^\epsilon$ in $\mathbb{R} \times [T^* - \epsilon, \infty)$. This implies

$$v^H(-\infty, t) \geqslant v^\epsilon(-\infty, t) = \gamma^H e^{-\delta^H(t-(T^*-\epsilon))} > \gamma^L,$$

when $t < T^* - \epsilon + \frac{1}{\delta^H} \ln \frac{\gamma^H}{\gamma^L}$. Thus, $s^L(t) > -\infty$ for $t \in \left[0, T^* - \epsilon + \frac{1}{\delta^H} \ln \frac{\gamma^H}{\gamma^L}\right)$. Sending $\epsilon \to 0$, we see that

$$s^L(t) > -\infty \text{ for } t \in \left[0, T^* + \frac{1}{\delta^H} \ln \frac{\gamma^H}{\gamma^L}\right).$$

3. Next, by comparing $v^L(\cdot + \epsilon, \cdot)$ and $v^L(\cdot, \cdot)$, we find that

$$v^L(\cdot + \epsilon, \cdot) \leqslant v^L(\cdot, \cdot).$$

Thus, $v^L_x(x, t) \leqslant 0$ for $(x, t) \in \mathbb{R} \times [0, \infty)$. In addition, we can show with $\eta = \frac{2\delta^L}{(\sigma^L)^2}$ and $h(t) = \min\{s^L(\tau) \mid 0 \leqslant \tau \leqslant t\}$ that $e^{-\delta^L t} - e^{\eta(x-h(t)+h(0))} \leqslant v^L(x, t) \leqslant e^{-\delta^L t} \vee \gamma^L$. Thus

$$v^L(-\infty, t) = e^{-\delta^L t} \vee \gamma^L \quad \text{for } t \in \left[0, T^* + \frac{1}{\delta^H} \ln \frac{\gamma^H}{\gamma^L}\right). \tag{3.2}$$

This implies $v^L(-\infty, t) > \gamma^H$ when $t \in [0, \min\{T^H, T^* + \frac{1}{\delta^H} \ln \frac{\gamma^H}{\gamma^L}\})$. It then follows that $s^H(t) > -\infty$, when $0 \leqslant t < \min\{T^H, T^* + \frac{1}{\delta^H} \ln \frac{\gamma^H}{\gamma^L}\}$. Thus, $T^* \geqslant \min\{T^H, T^* + \frac{1}{\delta^H} \ln \frac{\gamma^H}{\gamma^L}\}$. Hence $T^* = T^H$ and $T^* + \frac{1}{\delta^H} \ln \frac{\gamma^H}{\gamma^L} = T^L$.

Finally, define $\hat{h}(t) := \inf\{x | v^L(x, t) = \gamma^L\}$ $\forall t \in [0, \infty)$. One can show that $v^L \in C^\infty(Q_{\hat{h}})$ and $v_x < 0$ in $Q_{\hat{h}}$. Thus, by the implicit function theorem for $v^L(s^H(t), t) = \gamma^H$, we derive that $s^H \in C^\infty([0, T^H))$ and $\lim_{t \nearrow T^H} s^H(t) = -\infty$. □

Lemma 3.3. *The downgrading boundary $x = s^L(t)$ has the following properties:*

$$s^L(t) > -\infty \quad \textit{for } t \in [0, T^L); \quad s^L \in C^\infty([0, T^L));$$
$$\lim_{t \nearrow T^L} s^L(t) = -\infty; \quad s^L \equiv -\infty \quad \textit{in } [T^L, \infty).$$

Proof. Upon knowing that $s^H \equiv -\infty$ in $[T^H, \infty)$, we have $\mathscr{L}^H v^H = 0$ in $\mathbb{R} \times [T^H, \infty)$. Hence, $v^H(-\infty, t) = \gamma^H e^{-\delta_H(t-T^H)}$, for $t \geqslant T^H$. Also we can show that $v^H_x(x, t) < 0$, for $t \geqslant 0$, $x > s^H(t)$. Hence, $v^H(s^L(t), t) = \gamma^L$ is uniquely solvable for $t \in [0, T^L)$. Thus, the claim of the proposition comes. □

Remark 3.1. As shown in Figure 3, when $\delta^H > \delta^L$, $\lim_{t \nearrow T^L} v^L(-\infty, t) = e^{-\delta^L T^L} > \gamma^L$ and $v^L(\cdot, T^L) \equiv \gamma^L$. It follows that in the original variable, Φ^L is discontinuous at $y = 0$ and $t = T - T^L$.

4. Preparation

In this section, we work on the variational inequalities in (2.9) with s^H and s^L replaced by a known function h, which admits certain characteristic features collected in the following definition:

$$\mathfrak{X}^L_T := \{h \in C^\infty([0, T)) \mid h(0) = s^L(0),\ h' < 0 \text{ in } (0, T),\ h(T^-) = -\infty\}, \tag{4.1}$$

$$\mathfrak{X}^H_T := \{h \in C^\infty([0, T)) \mid h(0) = s^H(0),\ h' < 0 \text{ in } (0, T),\ h(T^-) = -\infty\}. \tag{4.2}$$

As we have seen in Section 3, we know that the solution of (2.9) satisfies $s^L \in \mathfrak{X}^L_{T^L}$ and $s^H \in \mathfrak{X}^L_{T^H}$. Since the variational inequality for v^H is indeed trivially a PDE, we focus on v^L. Thus, for a given constant

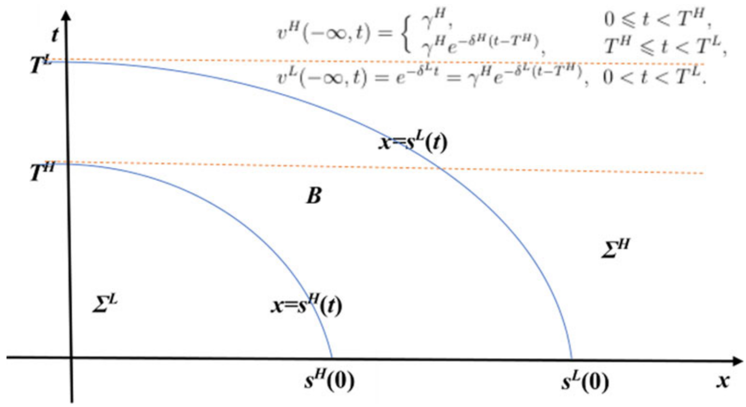

Figure 3. *Sketch of the rating boundaries and solutions' asymptotic behaviour.*

$T > 0$ and a given function $h \in \mathfrak{X}_T^L$, here in this section, we consider the variational problem (known as an obstacle problem):

$$\begin{cases} \min\{\mathscr{L}^L u, u - \gamma^L\} = 0 & \text{in } Q_h := \{(x,t) \mid 0 < t < T, x < h(t)\}, \\ u(x,0) = v_0^L(x) & \text{for } x \in \mathbb{R}, \\ u = \gamma^L & \text{on } \Sigma_h := \mathbb{R} \times (0,\infty) \setminus Q_h. \end{cases} \tag{4.3}$$

If h is a constant function, (4.3) is a standard free boundary problem, see, for example [8]. Similar to the idea of [8], we shall show the following:

Theorem 4.1 (Existence and uniqueness of obstacle problem (4.3)). *Given $T > 0$ and $h \in \mathfrak{X}_T^L$, problem (4.3) admits a unique solution. It satisfies $u_x \leqslant 0$ and $u_t \leqslant 0$ in Q_h and*

$$\lim_{x \to -\infty} u(x,t) = e^{-\delta^L t} \vee \gamma^L \quad \forall t \in [0,T). \tag{4.4}$$

Consequently, we can define

$$\hat{h}(t) := \inf\{x | u(x,t) = \gamma^L\} \quad \forall t \in [0,\infty), \tag{4.5}$$

$$s(t) := \inf\{x | u(x,t) < \gamma^H\} \quad \forall t \in [0,\infty). \tag{4.6}$$

In addition, letting T^L and T^H be defined in (2.8), we have the following:

1. *$\hat{h}$ is a decreasing function in $[0,T)$;*
2. *$\hat{h}(t) > -\infty$ if $t \in [0, T^L \wedge T)$; $\hat{h}(t) = -\infty$ and $u(\cdot,t) \equiv \gamma^L$ if $t \in [T^L, T)$;*
3. *$u \in C^\infty(Q_{\hat{h}})$; $u_x < 0$ and $u_t < 0$ in $Q_{\hat{h}}$;*
4. *$s \in \mathfrak{X}^H_{T^H \wedge T}$; $s \equiv -\infty$ in $[T^H, T)$.*

We call $x = \hat{h}(t)$ the free boundary of the obstacle problem (4.3).

Proof. Since $u \equiv \gamma^L$ on $\mathbb{R} \times [0,\infty) \setminus Q_h$, we need only to solve it in Q_h. Using the penalty method (e.g. [8]), a solution of (4.3) can be obtained by taking the limit, as $\epsilon \searrow 0$, of the solution of the approximation problem

$$\begin{cases} \mathscr{L}^L u^\epsilon = \beta_\epsilon(u^\epsilon) & \text{in } Q_h, \\ u^\epsilon(h(t),t) = \gamma^L & \text{for } t \in (0,T), \\ u^\epsilon(x,0) = v_0^\epsilon(x) & \text{for } x \leqslant h(0), \end{cases} \tag{4.7}$$

where $\beta_\epsilon(u) = \delta^L\gamma^L(\gamma^L + \epsilon - u^\epsilon)^+/\epsilon$, $v_0^\epsilon = v_0^L + w^\epsilon\, \mathbf{1}_{[0,h(0)]}$ and

$$w^\epsilon(x) = \begin{cases} x(1 - x/(4\epsilon)) & \text{if } 0 \leqslant x \leqslant 2\epsilon, \\ \epsilon & \text{if } 2\epsilon \leqslant x \leqslant h(0) - 2\epsilon, \\ (h(0) - x)[1 - (h(0) - x)/(4\epsilon)] & \text{if } h(0) - 2\epsilon \leqslant x \leqslant h(0). \end{cases}$$

One can verify that $v_0^\epsilon \in C^1((-\infty, h(0)])$, $(v_0^\epsilon)' \leqslant 0$, $v_0^L \leqslant v_0^\epsilon \leqslant v_0^L + \epsilon$ and

$$\left(\frac{(\sigma^L)^2}{2}\frac{d}{dx} - \delta^L\right)\left(\frac{d}{dx} + 1\right)v_0^\epsilon + \beta_\epsilon(v_0^\epsilon) \leqslant 0 \quad \text{in } (-\infty, h(0)], \tag{4.8}$$

when $0 < \epsilon < \epsilon_0$ with a sufficiently small positive ϵ_0.

By a standard theory for parabolic equations [11], problem (4.7) admits a unique solution. We have the following observation:

1. $\underline{u} \equiv \gamma^L$ is a subsolution. Thus, $u^\epsilon \geq \gamma^L$. Consequently $0 \leqslant \mathscr{L}^L u^\epsilon \leqslant \delta^L\gamma^L$ in Q_h.
2. Since $u^\epsilon(h(t), t) = \gamma^L \leqslant u^\epsilon(x, t)$ for $x < h(t)$, we have $u_x^\epsilon(h(t), t) \leqslant 0$ for $t \in [0, T)$. Also $u_x^\epsilon(x, 0) \leqslant 0$ for $x \leqslant h(0)$. Applying the maximum principle to u_x^ϵ, we find that $u_x^\epsilon < 0$ in Q_h.
3. Note that from (4.8), we have $u_t^\epsilon(x, 0) \leqslant 0$ for $x \leqslant h(0)$. Differentiating $u^\epsilon(h(t), t) = \gamma^L$ with respect to t, we find that $u_t^\epsilon(h(t), t) = -h'(t)u_x^\epsilon(h(t), t) \leqslant 0$ for $t \in [0, T)$. By the maximum principle, $u_t^\epsilon < 0$ in Q_h.
4. Set $\bar{u}(x, t) = \max\{e^{-\delta^L t}, \epsilon + \gamma^L\}$. One can verify that $\bar{u}$ is a supersolution. Set $\underline{u}(x, t) = e^{-\delta^L t} - e^{\eta(x - h(t) + h(0))}$, where $\eta = \frac{2\delta^L}{(\sigma^L)^2}$. One can verify that $\underline{u}$ is a subsolution. Thus, for $(x, t) \in \overline{Q}_h$,
$$e^{-\delta^L t} - e^{\eta(x - h(t) + h(0))} \leqslant u^\epsilon(x, t) \leq \max\{e^{-\delta^L t}, \epsilon + \gamma^L\}.$$
5. Note that $(\gamma^L + \epsilon - u)^+/\epsilon = [1 - (u - \gamma^L)/\epsilon]^+$ is an increasing function of ϵ, so does v_0^ϵ. Thus, $\{u^\epsilon\}_{0<\epsilon<\epsilon_0}$ is a family that is increasing in ϵ. Sending $\epsilon \searrow 0$, we obtain a limit u, which is a solution of (4.3) (c.f. [8]). The solution satisfies the estimate $u_x \leqslant 0$, $u_t \leqslant 0$ in Q_h and
$$e^{-\delta^L t} - e^{\eta(x - h(t) + h(0))} \leqslant u(x, t) \leqslant e^{-\delta^L t} \vee \gamma^L \quad \forall (x, t) \in \overline{Q}_h. \tag{4.9}$$

Hence, we can define $\hat{h}$ and s as in (4.5) and (4.6). Note that (4.4) follows from (4.9). We see from (4.4) that $-\infty < \hat{h}(t) \leqslant h(t)$ for $t \in [0, T \wedge T^L)$, and $-\infty < s(t) < \hat{h}(t)$ for $t \in [0, T \wedge T^H)$. By strong maximum principle, we have $v_t < 0$, and $v_x < 0$ in $Q_{\hat{h}}$. The implicit function theorem shows that $s \in \mathfrak{X}^H_{T \wedge T^H}$. This establishes the existence of a solution to (4.3) that has the properties stated in the theorem. The following comparison principle implies the uniqueness of the solution, completing the proof of Theorem 3.1. □

Theorem 4.2 (Comparison principle). *Let $T > 0$ and $h \in C^1([0, T))$ be given. Assume that $\underline{u}$ and $\bar{u}$ are bounded and continuous and satisfy*

$$\begin{cases} \mathscr{F}^L\underline{u} \leqslant \mathscr{F}^L\bar{u} & \text{in } Q_h, \\ \underline{u}(h(t), t) \leq \bar{u}(h(t), t) & \text{for } t \in (0, T), \\ \underline{u}(x, 0) \leqslant \bar{u}(x, 0) & \text{for } x \leqslant h(0). \end{cases} \tag{4.10}$$

Then $\bar{u} \geqslant \underline{u}$ in Q_h.

Proof. We use a contradiction argument. Assume that the assertion is not true. Then there exist $t_0 \in (0, T)$ and $x_0 < h(t)$ such that $\epsilon := \underline{u}(x_0, t_0) - \bar{u}(x_0, t_0) > 0$. Set

$$\phi(x) = e^{-x} + e^{\frac{\delta^L}{2(\sigma^L)^2}x}, \quad w(x, t) = \underline{u}(x, t) - \bar{u}(x, t) - \frac{\epsilon}{2\phi(x_0)}\phi(x).$$

Note that $w(-\infty, t) = -\infty$, $w(h(t), t) < 0$ for $t \in (0, t_0]$ and $w(x, 0) < 0$ for $x \leqslant h(0)$. Since $w(x_0, t_0) > 0$, there exist $t_1 \in (0, t_0]$ and $x_1 < h(t)$ such that

$$w(x_1, t_1) = \max_{0 \leqslant t \leqslant t_0, x \leqslant h(t)} w(x, t) > 0.$$

Then, $w_x(x_1,t_1)=0$, $w_t(x_1,t_1)\geqslant 0$, and $w_{xx}(x_1,t_1)\leqslant 0$. By noting that $\mathscr{L}^L\phi\equiv 0$, we then obtain

$$\delta^L w(x_1,t_1)\leqslant \mathscr{L}^L w=\mathscr{L}^L\underline{u}(x_1,t_1)-\mathscr{L}^L\bar{u}(x_1,t_1).$$

This implies

$$\begin{aligned}\mathscr{F}^L\bar{u}(x_1,t_1)&=\min\{\mathscr{L}^L\bar{u}(x_1,t_1),\bar{u}(x_1,t_1)-\gamma^L\}\\&\leqslant \min\{\mathscr{L}^L\underline{u}(x_1,t_1)-\delta^L w(x_1,t_1),\underline{u}(x_1,t_1)-\gamma^L-w(x_1,t_1)\}\\&<\mathscr{F}^L\underline{u}(x_1,t_1).\end{aligned}$$

This contradicts the assumption. Thus, we have $\underline{u}\leqslant\bar{u}$ in Q_h. □

Remark 4.1. The proof given above is still in the formal level, since we did not specify the regularity. It can be made rigorous if viscosity solutions are introduced. Here, we omitted the technicality. For more details, see, for example, [1, 2].

5. Existence via sub-super solution iterations

Here, we establish the existence of a solution to the free boundary problem (2.9). We use a monotonic iteration method. More precisely, starting from $\underline{s}_0^H(\cdot)\equiv-\infty$, we construct successively a sequence of subsolutions $\{\underline{v}_k^H,\underline{s}_k^L,\underline{v}_k^L,\underline{s}_{k+1}^H\}_{k=0}^\infty$ that is increasing in k. A solution is then obtained by sending $k\to\infty$.

We can also start from $\bar{s}_0^L(\cdot)\equiv s^L(0)$ to construct successively a sequence of supersolutions $\{\bar{v}_k^L,\bar{s}_k^H,\bar{v}_k^H,\bar{s}_{k+1}^L\}_{k=0}^\infty$ that is decreasing in k. The limit, as $k\to\infty$, is also a solution.

Upon establishing the uniqueness of solution, we see that the limits obtained from the two monotonic iterations are the same. These sub-supersolutions can be used to estimate the solution and derive asymptotic behaviours of the solution as $t\to T^H$ and T^L. In the case of zero dividend, the analysis is presented in [3, 7].

5.1. Subsolutions

Set $\underline{s}_0^H(\cdot)\equiv-\infty$ and $T_0=0$. We shall inductively define $\{\underline{v}_k^H,\underline{s}_k^L,\underline{v}_k^L,\underline{s}_{k+1}^L\}_{k=0}^\infty$. For this, we make an induction assumption that there exists $T_k\geqslant 0$ such that $\underline{s}_k^H\equiv-\infty$ in $[T_k,\infty)$ and $\underline{s}_k^H\in\mathfrak{X}_{T_k}^H$, where $\mathfrak{X}_T^H$ is defined in (4.2).

Suppose that $k\geqslant 0$ is an integer and $\underline{s}_k^H$ is known, we construct $\underline{v}_k^H$, $\underline{s}_k^L$, $\underline{v}_k^L$, $\underline{s}_{k+1}^L$ and T_{k+1} as follows.

Step 1. Define $\underline{v}_k^H(t)$ as the solution of the initial boundary value problem

$$\begin{cases}\mathscr{L}^H\underline{v}_k^H=0, & \text{for } x>\underline{s}_k^H(t)\ t>0,\\ \underline{v}_k^H(x,t)=\gamma^H & \text{for } x\leqslant\underline{s}_k^H(t),\ \ t>0,\\ \underline{v}_k^H(x,0)=\gamma^H\wedge e^{-x} & \text{for } x\in\mathbb{R},\ \ t=0.\end{cases}\tag{5.1}$$

We consider $k=0$ and $k\geqslant 1$ separately.

When $k=0$, this is a Cauchy problem. There exists a unique solution $\underline{v}_0^H\in C^\infty(\mathbb{R}\times[0,\infty)\setminus(s^H(0),0))$. Since γ^H is a supersolution and e^{-x} is a steady state, $\gamma^H\wedge e^{-x}$ is a supersolution so $\underline{v}_0^H(x,\tau)<\underline{v}_0^H(x,0)$ for $\tau>0$ and $x\in\mathbb{R}$. This implies by comparison that $\underline{v}_0^H(\cdot,t+\tau)<\underline{v}_0^H(\cdot,t)$ for all $t\geqslant 0$ and $\tau>0$. Thus, $\underline{v}_{0t}^H\leqslant 0$ in $\mathbb{R}\times(0,\infty)$. By the strong maximum principle, $\underline{v}_{0t}^H<0$ for $t>0$. Similarly, we derive that with $k=0$,

$$\underline{v}_{kx}^H(x,t)<0,\quad \underline{v}_{kt}^H(x,t)<0,\qquad \forall\, t>0,\ x>\underline{s}_k^H(t).\tag{5.2}$$

When $k\geqslant 1$, (5.1) is an initial boundary value problem for a diffusion equation in $\{(x,t)\mid t>0,x>\underline{s}_k^H(t)\}$ and there exists a C^∞ solution in $\{(x,t)|t>0,x>\underline{s}_k^H(t)\}$. The maximum principle shows that $\max\underline{v}_k^H=\gamma^H$. Hence, by Hopf's lemma, $\underline{v}_{kx}^H(\underline{s}_k^H(t)^+,t)<0$ for $0\leqslant t<T_k$. Also the inductive assumption

$\dot{\underline{s}}_k^H < 0$ implies that $\underline{v}_{kt}^H(\underline{s}_k^H(t)^+, t) = -v_x(\underline{s}_k^H(t)^+, t)\dot{\underline{s}}_k^H(t) < 0$ for each $t \in (0, T_k)$. The maximum principle for $\underline{v}_{kx}^H$ and $\underline{v}_{kt}^H$ implies that (5.2) holds.

Notice that, since $0 < v_k^H < \gamma^H$ in $\{(x,t) \mid t > 0, x > \underline{s}_k^H(t)\}$, problem (5.1) is equivalent to the obstacle problem as follows:

$$\begin{cases} \max\{\mathscr{L}^H \underline{v}_k^H, \underline{v}_k^H - \gamma^H\} = 0 & \text{for } x > \underline{s}_k^H(t),\ t > 0, \\ \underline{v}_k^H(x, 0) = \min\{\gamma^H, e^{-x}\} & \text{for } x \in \mathbb{R}, t = 0, \\ \underline{v}_k^H(x,t) = \gamma^H & \text{for } x \leqslant \underline{s}_k^H(t),\ t > 0. \end{cases} \tag{5.3}$$

Step 2. Define $\underline{s}_k^L(t)$ by

$$\underline{s}_k^L(t) := \inf\{x \mid \underline{v}_k^H(x,t) < \gamma^L\} \in [-\infty, \infty),\ \forall t \geqslant 0. \tag{5.4}$$

For constant $A > 1$, $\pm Ae^{-x}$ is a super/subsolution. It follows that $\underline{v}_k^H(\infty, t) = 0$. Similarly, we can show that

$$\lim_{x\to-\infty} \bar{v}_k^H(x,t) = \begin{cases} \gamma^H & \text{if } t \in [0, T_k), \\ \gamma^H e^{\delta^H(t-T_k)} & \text{if } t \geqslant T_k. \end{cases}$$

In view of (5.2), we see that $\underline{s}_k^L$ is well-defined. In addition, by the implicit function theorem, we derive that $\bar{s}_k^L \in \mathfrak{X}_{T_k^L}^L$ and $\underline{s}_k^L \equiv -\infty$ in $[T_k^L, \infty)$, where $\mathfrak{X}_T^L$ is defined in (4.1) and

$$T_k^L := T_k + \frac{1}{\delta^H} \ln \frac{\gamma^L}{\gamma^H}.$$

Using $(\underline{s}_k^H)' < 0$ in $(0, T_k)$ one can check that $\gamma^H e^{-x+\underline{s}_k^H(t)}$ is a subsolution so $\underline{v}_k^H(x,t) > \gamma^H e^{-x+\underline{s}_k^H(t)}$ for $x > \underline{s}_k^H(t)$ and $t \in [0, T_k)$. Thus,

$$\underline{s}_k^L(t) - \underline{s}_k^H(t) > \ln \frac{\gamma^H}{\gamma^L} \qquad \forall\, t \in [0, T_k). \tag{5.5}$$

Step 3. Next we define $\underline{v}_k^L$ as the solution of the following obstacle solution:

$$\begin{cases} \min\{\mathscr{L}^L \underline{v}_k^L, \underline{v}_k^L - \gamma^L\} = 0 & \text{for } x < \underline{s}_k^L(t), t \in (0, T_k^L), \\ \underline{v}_k^L(x, 0) = \max\{\gamma^L, v_0(x)\} & \text{for } x \in \mathbb{R}, t = 0, \\ \underline{v}_k^L(x,t) = \gamma^L & \text{for } x \geqslant \underline{s}_k^L(t),\ t > 0. \end{cases} \tag{5.6}$$

This is an obstacle problem with a lateral boundary $x = \underline{s}_k^L(t)$. By Theorem 4.1, with $T = T_k^L$, there exists a unique solution.

Step 4. Finally, we define

$$\hat{\underline{s}}_k^L(t) = \inf\{x \mid \underline{v}_k^L(x,t) = \gamma^L\} \quad \forall t > 0, \tag{5.7}$$

$$\underline{s}_{k+1}^H(t) = \inf\{x \mid \underline{v}_k^L(x,t) < \gamma^H\} \quad \forall t > 0, \tag{5.8}$$

$$T_{k+1} = \min\{T^H, T_k^L\} = \min\left\{T^H, T_k + \frac{1}{\delta^H} \ln \frac{\gamma^H}{\gamma^L}\right\}. \tag{5.9}$$

By Theorem 4.1, we know that

$$\bar{v}_k^L(-\infty, t) = e^{-\delta^L t} \quad \text{for } t \in [0, T_k^L), \quad \underline{s}_{k+1}^H \in \mathfrak{X}_{T_{k+1}}^H.$$

This completes the construction of the sequence $\{\underline{v}_k^H, \underline{s}_k^L, \underline{v}_k^L, \hat{\underline{s}}_k^L, \underline{s}_{k+1}^H, T_{k+1}\}_{k=0}^\infty$.

Note that, from (5.9), there exists an integer n such that $T_k = T^H$ and $T_k^L = T^L$ for every $k \geqslant n$.

5.2. Convergence of the subsolution sequence

Now we show that $\{\underline{v}_k^H, \underline{s}_k^L, \underline{v}_k^L, \hat{\underline{s}}_k^L, \underline{s}_k^H\}_{k=1}^{\infty}$ is increasing in k and the limit as $k \to \infty$ is a solution of (2.9).

Lemma 5.1. *Starting from* $\underline{s}_0^H(\cdot) \equiv -\infty$ *and* $T_0 = 0$, *let* $\{\underline{v}_k^H, \underline{s}_k^L, \underline{v}_k^L, \hat{\underline{s}}_k^L, \underline{s}_{k+1}^H, T_{k+1}\}_{k=0}^{\infty}$ *be the sequence defined successively by (5.1), (5.4), (5.6), (5.7), (5.8) and (5.9). Then the sequence is increasing in k in the sense that for each integer* $k \geq 1$,

$$\underline{s}_{k-1}^H(t) \leqslant \underline{s}_k^H(t) \leqslant \underline{s}_k^L(t) \leqslant \underline{s}_{k+1}^L(t) < s^L(0) \quad \forall\, t > 0,$$

$$0 < \underline{v}_k^H \leqslant \underline{v}_{k+1}^H \leqslant \gamma^H, \quad \gamma^L \leqslant \underline{v}_k^L \leqslant \underline{v}_{k+1}^L < 1 \quad \text{in } \mathbb{R} \times [0, \infty),$$

$$\underline{s}_k^H(t) \leqslant \hat{\underline{s}}_k^L(t) \leqslant \hat{\underline{s}}_{k+1}^L(t) \leqslant \underline{s}_{k+1}^L(t) \quad \forall t > 0.$$

Consequently, for each $t \geqslant 0$ *and* $x \in \mathbb{R}$, *the limit*

$$\begin{aligned} &\left[s^H(t), s^L(t), v^H(x,t), \hat{s}^L(t), v^L(x,t)\right] \\ &= \lim_{k\to\infty} \left[\underline{s}_k^H(t), \underline{s}_k^L(t), \underline{v}_k^H(x,t), \hat{s}_k^L(t), \underline{v}_k^L(x,t)\right] \end{aligned} \tag{5.10}$$

exists and forms a solution of the free boundary problem (2.9). In addition $s^H \in \mathfrak{X}_{T^H}^H$, $s^L \in \mathfrak{X}_{T^L}^L$ *and*

$$s^L(t) \geqslant s^H(t) + \ln \frac{\gamma^H}{\gamma^L} \quad \forall t \in [0, T^H).$$

Proof. First, we show that the sequence is monotonic in k. We claim that $\underline{s}_k^H(t) \leqslant \underline{s}_{k+1}^H(t)$ for all integer $k \geqslant 0$. Since $\underline{s}_0^H(t) \equiv -\infty$, the claim holds when $k = 0$. Now let $n \geqslant 0$ be an integer and suppose that $\underline{s}_n^H(t) \leqslant \underline{s}_{n+1}^H(t)$, for all $t > 0$.

Comparing $\underline{v}_n^H$ and $\underline{v}_{n+1}^H$ we find that

$$\underline{v}_n^H(x,t) \leq \underline{v}_{n+1}^H(x,t) \qquad \text{for } t \geqslant 0,\ x \in \mathbb{R}.$$

Then by the definition of $\underline{s}_k^L$ we derive that

$$\underline{s}_n^L(t) \leq \underline{s}_{n+1}^L(t) \quad \forall\, t \geqslant 0.$$

Then comparing $\underline{v}_n^L$ and $\underline{v}_{n+1}^L$, we find that

$$\underline{v}_n^L(x,t) \leq \underline{v}_{n+1}^L(x,t) \quad \text{for } t \geqslant 0,\ x \in \mathbb{R}.$$

As $\underline{v}_{kx}^L \leqslant 0$, it follows that

$$\hat{s}_n^L(t) \leq \hat{s}_{n+1}^L(t) \text{ and } \underline{s}_n^H(t) \leq \underline{s}_{n+1}^H(t) \quad \forall t \geqslant 0.$$

This completes the induction argument for the monotonicity of the sequence.

Recall (5.5),

$$\underline{s}_k^H(t) + \ln \frac{\gamma^H}{\gamma^L} \leqslant \underline{s}_k^L(t) \leqslant \ln \frac{1}{\gamma^L} \quad \forall\, t \geqslant 0.$$

Also, $0 \leqslant \underline{v}_k^H \leqslant \gamma^H$ and $\gamma^L \leqslant \underline{v}_k^L \leqslant 1$. Thus, for each $t \geqslant 0$ and $x \in \mathbb{R}$, the limit in (5.10) exists. We now show that the limit is a solution of (2.9).

One can see that $v_t^L = \frac{1}{2}(\sigma^L)^2(v_{xx}^L + v_x^L) - \delta^L(v_x^L + v^L)$ in the set $\hat{Q}^L := \{(x,t) \mid t > 0, x < \hat{s}^L(t)\}$. In addition, from the strong maximum principle, $v_t^L < 0$ and $v_x^L < 0$ in $\hat{Q}^L$.

Similarly, we can show that $v_t^H = \frac{1}{2}(\sigma^H)^2(v_{xx}^H + v_x^H) - \delta^H(v_x^L + v^L)$ in $Q^H := \{(x,t) \mid t > 0, x > s^H(t)\}$, and $v_t^H < 0$ and $v_x^H < 0$ in Q^H.

Finally, for free boundary conditions, we have the following:

(1) $\underline{v}_k^H(\underline{s}_k^H(t), t) = \gamma^H$ for $t \in [0, T_k) \Rightarrow v^H(s^H(t), t) = \gamma^H$ for $t \in [0, T^H)$;

(2) $\underline{v}_k^H(\underline{s}_k^L(t), t) = \gamma^L$ for $t \in [0, T_k, + \frac{1}{\gamma^H} \ln \frac{\gamma^H}{\gamma^L}) \Rightarrow v^H(s^L(t), t) = \gamma^L$ for $t \in [0, T^L)$ and $s^L \in \mathfrak{X}_{T^L}^L$;

(3) $\underline{v}_k^L(\underline{s}_k^L(t), t) = \gamma^L$ for $t \in [0, T_k + \frac{1}{\gamma^H} \ln \frac{\gamma^H}{\gamma^L}) \Rightarrow v^L(s^L(t), t) = \gamma^L$ for $t \in [0, T^L)$;

(4) $\underline{v}_k^L(\underline{s}_{k+1}^H(t), t) = \gamma^H$ for $t \in [0, T_{k+1}) \Rightarrow v^L(s^H(t), t) = \gamma^H$ for $t \in [0, T^H)$ and $s^H \in \mathfrak{X}_{T^H}^H$.

In conclusion, (s^H, s^L, v^H, v^L) is a classical solution of (2.9). This completes the proof of Lemma 5.1. □

5.3. Supersolutions

For completion, here we construct a supersolution sequence. Starting from $\bar{s}_0^L(\cdot) \equiv s^L(0)$, we construct inductively $\{\bar{v}_k^L, \bar{s}_k^H, \bar{v}_k^H, \bar{s}_{k+1}^L\}_{k=0}^{\infty}$ as follows.

Let $k \geqslant 0$ be an integer and assume that $\bar{s}_k^L$ is known.

Step 1. We define $\bar{v}_k^L$ as the unique solution of

$$\begin{cases} \bar{v}_k^L = \gamma^L & \text{for } t > 0,\ x \geqslant \bar{s}_k^L(t), \\ \min\{\mathscr{L}^L \bar{v}_k^L, \bar{v}_k^L - \gamma^L\} = 0 & \text{for } t > 0,\ x < \bar{s}_k^L(t), \\ \bar{v}_k^L(\cdot, 0) = v_0^L(\cdot) & \text{on } \mathbb{R}. \end{cases} \tag{5.11}$$

From the solution, we define

$$\bar{s}_k^H(t) = \inf\{x | \bar{v}_k^L(x, t) < \gamma^H\}, \quad \forall\, t \geqslant 0. \tag{5.12}$$

We consider two cases (1) $k = 1$ (2) $k = 0$.

(1) Suppose $k \geqslant 1$. We make an induction assumption that when $k \geqslant 1$, $s_k^L \in \mathfrak{X}_{T^L}^L$ where $\mathfrak{X}_T^L$ is defined in (4.1).

By Theorem 4.1, there exists a unique solution $\bar{v}_k^L$ to (5.11). In addition $\bar{s}_k^H \in \mathfrak{X}_{T^H}^H$, where $\mathfrak{X}_T^H$ is defined in (4.2).

(2) Suppose $k = 0$. One can check that $v_k^L(-\infty, t) = e^{-\delta^L t} > \gamma^H$ for $t \in [0, T^H)$. Still $\bar{s}_0^H$ is well defined and $\bar{s}_0^H \in \mathfrak{X}_{T^H}^H$.

We remark that $\bar{s}_k^H(\cdot) \equiv -\infty$ in $[T^H, \infty)$.

Step 2. Define $\bar{v}_k^H$ as the solution of

$$\begin{cases} \mathscr{L}^H \bar{v}_k^H = 0 & \text{for } x > \bar{s}_k^H(t),\ t > 0, \\ \bar{v}_k^H(x, 0) = \gamma^H \wedge e^{-x} & \text{for } x \in \mathbb{R},\ t = 0, \\ \bar{v}_k^H(x, t) = \gamma^H & \text{for } x \leqslant \bar{s}_k^H(t),\ t > 0. \end{cases} \tag{5.13}$$

Using maximum principle, we find that $0 < \bar{v}_k^H \leqslant \gamma^H$. Thus, it is also the solution of the variational inequality

$$\max\{\mathscr{L}^H \bar{v}_k^H, \bar{v}_k^H - \gamma^H\} = 0 \quad \text{for } t > 0,\ x > \bar{s}_k^H(t).$$

That $\bar{v}_k^H \leqslant \gamma^H$ implies that $\frac{\partial}{\partial x} \bar{v}_k^H(\bar{s}_k^H(t), t) \leqslant 0$ for $t \in [0, T^H)$. Thus, by the maximum principle

$$(\bar{v}_k^H)_x < 0 \quad \text{in } \bar{Q}_k^H = \{(x, t) \mid t > 0, x > \bar{s}_k^H(t)\}.$$

Now differentiating $\bar{v}_k^H(\bar{s}_k^H(t), t) = 0$ and using $(\bar{s}_k^H)'(t) \leqslant 0$ for $t \in [0, T^H)$, we find that

$$(\bar{v}_k^H)_t < 0 \quad \text{in } \bar{Q}_k^H.$$

Finally, by construction sub-supersolutions (c.f. the proof of Theorem 4.1), we can show that

$$\lim_{x \to -\infty} \bar{v}_k^H(x, t) = \begin{cases} \gamma^H & \text{if } t \in [0, T^H), \\ \gamma^H e^{\delta^H (t - T^H)} & \text{if } t > T^H. \end{cases}$$

Now define

$$\bar{s}_{k+1}^L(t) := \inf\{x | \bar{v}_k^H(x, t) \leqslant \gamma^L\} \quad \forall t \geqslant 0. \tag{5.14}$$

One can verify that $\bar{s}_{k+1}^L \in \mathfrak{X}_{T^L}^L$. This completes the inductive construction of $\{\bar{v}_k^L, \bar{s}_k^H, \bar{v}_k^H, \bar{s}_{k+1}^L\}$.

5.4. The limit of the supersolution sequence

Furthermore, we can show the following, whose proof is omitted.

Lemma 5.2. *Start from $\bar{s}_0^L(\cdot) \equiv s^L(0)$. Define successively a sequence $\{\bar{v}_k^L, \bar{s}_k^H, \bar{v}_k^H, \bar{s}_{k+1}^L\}_{k=0}^\infty$ by (5.11), (5.12), (5.13), (5.14). Then, the sequence is decreasing in k in the sense that for each integer $k \geq 0$,*

$$\underline{s}_k^H(t) \leqslant \bar{s}_{k+1}^H(t) \leq \bar{s}_k^H(t) \leq \bar{s}_{k+1}^L(t) \leq \bar{s}_k^L(t) \leqslant s^L(0) \quad \forall\, t > 0,$$

$$\underline{v}_k^H \leqslant \bar{v}_{k+1}^H \leqslant \bar{v}_k^H \leqslant \gamma^H, \quad \underline{v}_k^L \leqslant \bar{v}_{k+1}^L \leqslant \bar{v}_k^L \leqslant 1 \quad \text{ in } \mathbb{R} \times [0, \infty).$$

Consequently, for each $t \geqslant 0$ and $x \in \mathbb{R}$, the limit

$$\left[s^H(t), s^L(t), v^H(x,t), v^L(x,t)\right] = \lim_{k\to\infty} \left[\bar{s}_k^H(t), \bar{s}_k^L(t), \bar{v}_k^H(x,t), \bar{v}_k^L(x,t)\right]$$

exists and forms a solution of the free boundary problem (2.9) .

6. Main theorem and uniqueness

Now we present our main mathematical result of our paper.

Theorem 6.1 (Main theorem). *The free boundary problem (2.9) admits a unique solution $(s^H, s^L, \hat{s}^L, v^H, v^L)$. The solution satisfies*

$$s^L \in \mathfrak{X}_{T^L}^L, \quad s^H \in \mathfrak{X}_{T^H}^H,$$

$$v^L \in C^1\left(\overline{Q^L} \setminus (0,0)\right) \cup C^\infty\left(\hat{Q}^L \cup ((-\infty, 0) \cap (0, s^L(0)) \times \{0\})\right),$$

$$v^H \in C^\infty\left(\overline{Q^H} \setminus (s^H(0), 0)\right),$$

where $Q^H := \{(x,t) \mid t > 0, x > s^H(t)\}$, $Q^L := \{(x,t) \mid t > 0, \ x < s^L(t)\}$ and $\hat{Q}^L := \{(x,t) \mid t > 0, \ x < \hat{s}^L(t)\}$. In addition,

$$v_x^H < 0,\ v_t^H < 0 \ \text{ in } \ Q^H; \quad v_x^L < 0,\ v_t^L < 0 \ \text{ in } \ \hat{Q}^L;$$

$$s^H \equiv -\infty \ \text{ in } [T^H, \infty); \quad s^L \equiv -\infty \ \text{ in } [T^L, \infty);$$

$$\frac{d\hat{s}^L}{dt} \leqslant 0 \ \text{ in } (0, T^L); \quad \frac{ds^H}{dt} < 0 \ \text{ in } (0, T^H); \quad \frac{ds^L}{dt} < 0 \ \text{ in } (0, T^L);$$

$$s^L(t) - s^H(t) > \ln \frac{\gamma^H}{\gamma^L}, \quad s^H(t) < \hat{s}^L(t) \leqslant s^L(t), \quad \text{for } \ t \in [0, T^H).$$

Proof. The existence comes from the limit of the sequence and lemmas in Section 5.

For the uniqueness, let $(v_i^H, v_i^L, s_i^H, s_i^L)$, $i = 1, 2$, be two solutions of (2.9). Fix an arbitrary positive ϵ. Define $v_\epsilon^l(x,t) = v_2^l(x+\epsilon, t)$ and $s_\epsilon^l(t) = s_2^l(t) - \epsilon$ for $l = H, L$. Define

$$T_\epsilon = \sup\{t \in [0, T^H] \mid s_1^H(\tau) > s_2^H(\tau) - \epsilon \quad \forall \tau \in [0, t)\}.$$

Clearly, from $s_1^H(0) = s_2^H(0)$, we have $T_\epsilon > 0$. By strong comparison,

$$v_1^H(x,t) > v_\epsilon^H(x,t) \quad \forall t \in [0, T_\epsilon],\ x > s_\epsilon^H := s_2^H(t) - \epsilon.$$

This implies $s_1^L(t) > s_\epsilon^L(t)\ \forall t \in [0, T_\epsilon]$. Now we claim that $T_\epsilon = T^H$. In fact, supposed not, then

$$s_1^H(T_\epsilon) = s_2^H(T_\epsilon) - \epsilon = s_\epsilon^H(T_\epsilon).$$

By comparison, $v_1^L \geqslant v_\epsilon^L$ in $\mathbb{R} \times [0, T_\epsilon]$. It follows from strong comparison that $v_1^L > v_\epsilon^L$ in $\{(x, t) \mid x < \hat{s}^L(t), 0 < t \leqslant T_\epsilon\}$. But this implies

$$\gamma^H = v_1^L(s_1^H(T_\epsilon), T_\epsilon) > v_\epsilon^L(s_\epsilon^H(T_\epsilon), T_\epsilon) = v_2^L(s_2^H(T_\epsilon), T_\epsilon)) = \gamma^H,$$

a contradiction. Thus, $T_\epsilon = T^H$. It follows that

$$v_1^L(x, t) > v_2^L(x + \epsilon, t), \quad s_1^L(t) > s_2^L(t) - \epsilon \quad \forall t \in [0, T^H), \ x > s_1^H(t),$$
$$v_1^H(x, t) > v_2^H(x + \epsilon, t), \quad s_1^H(t) > s_2^H(t) - \epsilon \quad \forall t \in [0, T^H), \ x < s_1^L(t).$$

Sending $\epsilon \to 0$, we have, for $l = H, L$,

$$v_1^l(x, t) \geqslant v_2^l(x, t), \quad s_1^l(t) \geqslant s_2^l(t) \qquad \forall t \in [0, T^H], \ x \in \mathbb{R}.$$

Reversing the roles of the two solutions, we see that $v_1^H = v_2^H$ and $v_1^L = v_2^L$ in $\mathbb{R} \times [0, T^H]$. Since $\mathscr{L}^H v_i^H = 0$ in $\mathbb{R} \times [T^H, \infty)$, we have $v_1^H = v_2^H$ in $\mathbb{R} \times [T^H, \infty)$, which implies $s_1^L \equiv s_2^L$ on $[T^H, T^L)$, and $v_1^L \equiv v_2^L$ on $\mathbb{R} \times [T^H, \infty)$. We then have proved the uniqueness. □

Remark 6.1. Note that e^{-x} is a steady state. This implies that $\frac{\partial^k}{\partial t^k} v^H(x, 0) = 0$ when $x \in (s^H(0), \infty)$ for any integer $k \geqslant 1$. This further implies that $\frac{d^k}{dt^k} s^L(0) = 0$ for any $k \geqslant 1$. Similarly, we have $\frac{d^k}{dt^k} s^H(0) = 0$ for any $k \geqslant 1$.

Remark 6.2. Note that if $\hat{s}^L(t) < s^L(t)$, then $v_x^L(\hat{s}^L(t), t) = 0$. Since $v_x^L(\hat{s}^L(0)-, 0) < 0$, we see that there exist $T > 0$ such that $\hat{s}^L(t) = s^L(t)$ for $t \in [0, T]$.

Remark 6.3. In the case $\delta^L > \delta^H$, we can see clear the need of the variational inequality. First of all, we have

$$v^H(-\infty, t) = \begin{cases} \gamma^H & \text{if } 0 \leqslant t < T^H, \\ \gamma^H e^{-\delta^H(t - T^H)} & \text{if } t \geqslant T^H. \end{cases}$$

This implies, using the notation (2.8), that $s^L \in C^\infty([0, T^L))$. However, we know that $v^L < e^{-\delta^L t}$ for $t \in [0, \hat{T}^L)$. Thus, we have $v^L(\cdot, t) \equiv \gamma^L$, $\hat{s}^L(t) = -\infty < s^L(t)$ for $t \in [\hat{T}^L, T^L)$. Therefore, we must replace the PDEs in [3] by variational inequalities.

7. Summary

Mathematical analyses of credit rating migration using structure frameworks start from [14], where the migration boundary is assumed to be known. The work is extended in [10, 12] by assuming that the migration boundary is a free boundary determined by the equation $R = \gamma$, where R is the ratio of debt and asset value and γ is a threshold. Since a Brownian motion $W_t - W_{t_0}$ can change sign infinite many time in $[t_0, t_0 + \epsilon)$ for any $\epsilon > 0$, the free boundary models in [10, 12] may result in the change of credit rating infinite many times in short time. Thus, for realistic application, an introduction of a buffer zone (dead band in engineering) is necessary. In our previous paper [3], a credit rating migration model with a buffer zone is introduced. However, we discover that the model works only for the case of nondividend. In this paper, we revise the credit rating migration model in [3], so that it covers the non-zero dividend case. It is not a trivial extension as the PDE problem will bring up the possibility of the solution violates the rating migration condition inside of the region; i.e. if we only use PDE, then the solution $\mathscr{L}^L v^L = 0$ in $\{(x, t) | t \in [0, T^L), x < s^L(t)\}$ may not satisfy the fundamental requirement that $v^L \geqslant \gamma^L$, see Remark 6.3. For this reason, we modify the model by replacing the parabolic equation by a variational inequality. This renders to a new free boundary problem where the PDE free boundary problem in [3] is a special case of the new model. By constructing two monotone sequences, existence, uniqueness and some properties are obtained.

The model established in this paper can be extended without essential difficulties to the cases of multiple ratings. It can be expected to be used by credit rating companies, bond issuing agencies and corresponding financial institutions.

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