Proceedings of the Royal Society of Edinburgh, **152**, 879–911, *2022*
DOI:10.1017/prm.2021.40

On the fractional Lazer-McKenna conjecture with critical growth

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(Received 23 September 2020; accepted 14 July 2021)

This paper deals with the following fractional elliptic equation with critical exponent

$$\begin{cases}(-\Delta)^s u = u_+^{2_s^*-1} + \lambda u - \bar{\nu}\varphi_1, & \text{in } \Omega,\\ u = 0, & \text{in } \mathfrak{R}^N \backslash \Omega,\end{cases}$$

where λ, $\bar{\nu} \in \mathfrak{R}$, $s \in (0,1)$, $2_s^* = (2N/N-2s)\,(N > 2s)$, $(-\Delta)^s$ is the fractional Laplace operator, $\Omega \subset \mathfrak{R}^N$ is a bounded domain with smooth boundary and φ_1 is the first positive eigenfunction of the fractional Laplace under the condition $u = 0$ in $\mathfrak{R}^N \setminus \Omega$. Under suitable conditions on λ and $\bar{\nu}$ and using a Lyapunov-Schmidt reduction method, we prove the fractional version of the Lazer-McKenna conjecture which says that the equation above has infinitely many solutions as $|\bar{\nu}| \to \infty$.

Keywords: fractional Laplace; first eigenfunction; Lyapunov-Schmidt reduction

2020 *Mathematics subject classification* Primary: 35A01
Secondary: 35J60

1. Introduction

In this paper, we consider the following fractional problem

$$\begin{cases}(-\Delta)^s u = g(u) - \bar{\nu}\varphi_1, & \text{in } \Omega,\\ u = 0, & \text{in } \mathfrak{R}^N \backslash \Omega,\end{cases} \tag{1.1}$$

where $\bar{\nu} \in \mathfrak{R}$, $s \in (0,1)$, $\Omega \subset \mathfrak{R}^N\,(N > 2s)$ is a bounded domain with smooth boundary and φ_1 is the first positive eigenfunction of the fractional Laplace under the condition $u = 0$ in $\mathfrak{R}^N \setminus \Omega$. $g(t)$ has superlinear growth and satisfies

$$\lim_{t\to+\infty} \frac{g(t)}{t} = \alpha > \lim_{t\to-\infty} \frac{g(t)}{t} = \beta.$$

Here $\alpha = +\infty$ and $\beta = -\infty$ are allowed. For any $\Omega \subset \mathfrak{R}^N$ and $u \in C_0^\infty(\Omega)$, we have $u = 0$ in $\mathfrak{R}^N \backslash \Omega$. The fractional Laplace operator $(-\Delta)^s$ is defined as follows:

$$(-\Delta)^s u(y) = C_{N,s} P.V. \int_{\mathfrak{R}^N} \frac{u(y) - u(z)}{|y - z|^{N+2s}} \mathrm{d}z,$$

where $P.V.$ stands for the principle value and $C_{N,s}$ is a normalization constant (see for instance [**14**]).

In particular, if $s = 1$, equation (1.1) reduces to

$$\begin{cases} -\Delta u = g(u) - \bar{\nu}\varphi_1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Equation (1.2) has been studied first by Ambrosetti *et al.* in [**2**], and many results were obtained there, the readers can refer to [**5**–**7**, **10**–**12**, **22**]. It is well known (e.g. [**2**]) that the location of α, β with respect to the spectrum of $(-\Delta, H_0^1(\Omega))$ has great influence on the number of solutions to equation (1.2). Let $0 < \Lambda_1 < \Lambda_2 \leqslant \Lambda_3 \leqslant \cdots \leqslant \Lambda_i \leqslant \cdots$ be the eigenvalues of Laplace $-\Delta$ in $H_0^1(\Omega)$. In [**10**] Lazer and McKenna made a conjecture that equation (1.2) has an unbounded number of solutions as $\bar{\nu} \to +\infty$ if $\alpha = +\infty$, $\beta < \Lambda_1$, and $g(t)$ does not grow too rapidly.

There were several works related to the Lazer-McKenna conjecture. Breuer *et al.* [**3**] used numerical method to show that equation (1.2) has at least four solutions if $g(t) = t^2$ and Ω is a unit square in $\mathfrak{R}^2$. Dancer *et al.* [**5**] proved the Lazer-McKenna conjecture if $g(t) = |t|^p$, where $p \in (1, (N+2)/(N-2))$ and $N \geqslant 3$. Moreover, it is shown in [**6**] that the Lazer-McKenna conjecture is also true if $g(t) = t_+^p + \lambda t$, where $u_+ = \max(u, 0)$, $N \geqslant 3$, $p \in (1, (N+2)/(N-2))$, $\lambda < \Lambda_1$ or $\lambda \in (\Lambda_i, \Lambda_{i+1})$ for $i \geqslant 1$. Later on, Li *et al.* [**11**, **12**] and Wei *et al.* [**22**] proved the Lazer-McKenna conjecture if $g(t) = t_+^{2^*-1} + \lambda t$, where $N \geqslant 6$, $2^* = (2N)/(N-2)$ and $\lambda \in (0, \Lambda_1)$ or $\lambda \in (\Lambda_i, \Lambda_{i+1})$ for $i \geqslant 1$. Recently, in [**1**], Abdellaoui *et al.* extended the results in [**5**] to fractional Laplace and proved the fractional version of conjecture. This inspires us to consider problem (1.1). Our goal in this paper is to prove the fractional version of the Lazer-McKenna conjecture, extending the results in [**12**]. More precisely, we consider the following equation

$$\begin{cases} (-\Delta)^s u = u_+^{2_s^*-1} + \lambda u - \bar{\nu}\varphi_1, & \text{in } \Omega, \\ u = 0, & \text{in } \mathfrak{R}^N \backslash \Omega, \end{cases} \tag{1.3}$$

where $\lambda \in \mathfrak{R}$, $2_s^* = (2N/N-2s)$, $N > 2s$.

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Let $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots$ be the eigenvalues of fractional Laplace $(-\Delta)^s$ under the condition $u = 0$ in $\mathfrak{R}^N \setminus \Omega$. Indeed, it follows from [20] that

$$\lambda_1 = \min_{u\in X_0^s(\Omega)\setminus\{0\}} \frac{\int_{\mathfrak{R}^N}\int_{\mathfrak{R}^N}(|u(y)-u(z)|^2/|y-z|^{N+2s})\mathrm{d}y\mathrm{d}z}{\int_\Omega |u|^2},$$

where $X_0^s(\Omega)$ is given by

$$X_0^s(\Omega) = \Big\{u \in H^s(\mathfrak{R}^N) : u = 0 \text{ in } \mathfrak{R}^N\backslash\Omega\Big\},$$

with the norm

$$\|u\| := \left(\int_{\mathfrak{R}^N}\int_{\mathfrak{R}^N} \frac{|u(y)-u(z)|^2}{|y-z|^{N+2s}}\mathrm{d}y\mathrm{d}z\right)^{1/2}.$$

Furthermore, by [20], λ_1 is simple and $\varphi_1 > 0$ in Ω.

Throughout this paper, we always assume that λ and $\bar{\nu}$ satisfy one of the following conditions:

(C_1) $\lambda \in (0, \lambda_1)$ and $\bar{\nu} > 0$;

(C_2) $\lambda \in (\lambda_i, \lambda_{i+1})$ for some $i \geqslant 1$ and $\bar{\nu} < 0$.

The main result in this paper can be stated as follows:

THEOREM 1.1. *Suppose that* $0 < s < 1$*, (C_1) or (C_2) is satisfied. Then the number of the solutions for equation* (1.3) *tends to infinity as* $|\bar{\nu}| \to +\infty$ *if* $N > 6s$*.*

Obviously, $-(\bar{\nu}/\lambda_1 - \lambda)\varphi_1$ is a negative solution of equation (1.3). We will construct solutions of equation (1.3) redin the form

$$u = -\frac{\bar{\nu}}{\lambda_1 - \lambda}\varphi_1 + v.$$

Then v solves

$$\begin{cases} (-\Delta)^s v - \lambda v = (v - \nu\varphi_1)_+^{2_s^*-1}, & \text{in } \Omega, \\ v = 0, & \text{in } \mathfrak{R}^N\backslash\Omega, \end{cases} \tag{1.4}$$

where $\nu = (\bar{\nu}/\lambda_1 - \lambda)$. Thus $\nu \to +\infty$ as $|\bar{\nu}| \to +\infty$.

In the sequel, we mainly consider equation (1.4). We will use Lyapunov-Schmidt reduction method to construct peak solutions of equation (1.4). This method has been widely used to study elliptic problems, see for examples [15–18, 24] and the references therein. The advantage of this method is that we can not only prove the existence of many solutions but also obtain the profile of these solutions. Without loss of generality, we always assume that $\max\limits_{y\in\Omega}\varphi_1(y) = 1$ and we denote $S = \{z \in \Omega : \varphi_1(z) = 1\}$.

It is well known that

$$U_{x_j,\mu_j} = b_0\left(\frac{\mu_j}{1+\mu_j^2|y-x_j|^2}\right)^{N-2s/2},$$

with

$$b_0 = 2^{N-2s/2}\frac{\Gamma(N+2s/2)}{\Gamma(N-2s/2)},$$

solves equation

$$(-\Delta)^s u = u^{2_s^*-1}, \quad u>0, \quad \text{in } \mathfrak{R}^N, \tag{1.5}$$

where $x_j \in \mathfrak{R}^N$ and $\mu_j \in (0,\infty)$. In order to simplify notations, we denote $U = U_{0,1}$. Furthermore, in [8], it is shown that U is non-degenerate, in the sense that, if ϕ solves the linearized equation of equation (1.5)

$$(-\Delta)^s\phi = (2_s^*-1)U^{2_s^*-2}\phi, \quad \text{in } \mathfrak{R}^N,$$

then ϕ is a linear combination of

$$\frac{N-2s}{2}U + x\cdot\nabla U, \quad \frac{\partial U}{\partial x_i}, \quad i=1,\dots,N.$$

Let PU_{x_j,μ_j} be the solution of

$$\begin{cases}(-\Delta)^s PU_{x_j,\mu_j} = U_{x_j,\mu_j}^{2_s^*-1}, & \text{in } \Omega,\\ PU_{x_j,\mu_j} = 0, & \text{in } \mathfrak{R}^N\backslash\Omega.\end{cases} \tag{1.6}$$

We will choose PU_{x_j,μ_j} as a building block of approximate solution. Moreover, we have

THEOREM 1.2. *Let $k>0$ be an integer and $N>6s$. Then there exists $\nu_k>0$ such that for any $\nu \geqslant \nu_k$, equation (1.4) has a solution of the form*

$$v_\nu = \sum_{j=1}^k PU_{x_{\nu,j},\mu_{\nu,j}} + \phi_{\nu,k},$$

satisfying that as $\nu\to\infty$,

(*i*) $\phi_{\nu,k}\in X_0^s(\Omega)$ *and* $\|\phi_{\nu,k}\|\to 0$;

(*ii*) $\mu_{\nu,j}\nu^{-2/(N-6s)} \to t_0 > 0$;

(*iii*) $\nu^{2/(N-6s)}|x_{\nu,i}-x_{\nu,j}| \to +\infty$ *for* $i\neq j$;

(*iv*) $x_{\nu,j}\to x_j^*\in\Omega$ *with* $x_j^*\in S$;
where the constant t_0 *is defined in* (2.1).

https://doi.org/10.1017/prm.2021.40 Published online by Cambridge University Press

In order to expand energy (see appendix B), we have to estimate

$$\Psi_{x_j,\mu_j} = U_{x_j,\mu_j} - PU_{x_j,\mu_j}.$$

If $s = 1$, then Ψ_{x_j,μ_j} solves

$$\begin{cases} -\Delta\Psi_{x_j,\mu_j} = 0, & \text{in } \Omega, \\ \Psi_{x_j,\mu_j} = U_{x_j,\mu_j}, & \text{on } \partial\Omega. \end{cases}$$

Using comparison principle, we can easily obtain the leading term of Ψ_{x_j,μ_j}, for much details, the reader can see [19]. In order to overcome these difficulties due to the fractional Laplace, considering that many mathematics applied s-harmonic extension method (see [4]) and studied a new local problem

$$\begin{cases} div(y_{N+1}^{1-2s}\nabla\Phi) = 0, & \text{in } \mathfrak{R}_+^{N+1}, \\ \Phi = U_{x_j,\mu_j}, & \text{on } \mathfrak{R}^N\backslash\Omega \times \{0\}, \\ \lim\limits_{y_{N+1}\to 0^+} y_{N+1}^{1-2s}\dfrac{\partial\Phi}{\partial y_{N+1}} = 0, & \text{on } \Omega \times \{0\}, \end{cases}$$

we turn to obtain the leading term of Ψ_{x_j,μ_j} by convolution formula of Green function with $U_{x_j,\mu_j}^{2_s^*-1}$. This idea is mainly from [9, 13]. For much details, the readers can see appendix A. On the other hand, in order to solve critical points of $K(x,\mu)$ (see § 3), we will use a type of gradient flow method (see [22, 23]). Indeed, we cannot prove the existence of critical points of $K(x,\mu)$ by using maximization procedure as in [5, 6]. Furthermore, in [12], Li *et al.* proved that $K(x,\mu)$ has a saddle point such that $K(x,\mu)$ attained the minimum at μ_i direction and attained maximum at x_i direction. Compared with [12], the gradient flow method we used in this paper will simply be the procedure very much to obtain critical points.

To end this section, we introduce some notations. We define $H^s(\mathfrak{R}^N)$ the classical Sobolev space

$$H^s(\mathfrak{R}^N) = \left\{ u \in L^2(\mathfrak{R}^N) : \int_{\mathfrak{R}^N}\int_{\mathfrak{R}^N} \frac{|u(y)-u(z)|^2}{|y-z|^{N+2s}} \mathrm{d}y\mathrm{d}z < \infty \right\},$$

with the norm

$$\|u\|_{H^s(\mathfrak{R}^N)} = \left(\int_{\mathfrak{R}^N} u^2 + \int_{\mathfrak{R}^N}\int_{\mathfrak{R}^N} \frac{|u(y)-u(z)|^2}{|y-z|^{N+2s}} \mathrm{d}y\mathrm{d}z \right)^{1/2}.$$

We also define $D^{s,2}(\mathfrak{R}^N)$ as follows

$$D^{s,2}(\mathfrak{R}^N) = \left\{ u \in L^{2_s^*}(\mathfrak{R}^N) : \int_{\mathfrak{R}^N}\int_{\mathfrak{R}^N} \frac{|u(y)-u(z)|^2}{|y-z|^{N+2s}} \mathrm{d}y\mathrm{d}z < \infty \right\},$$

with the norm

$$\|u\| := \left(\int_{\mathfrak{R}^N}\int_{\mathfrak{R}^N} \frac{|u(y)-u(z)|^2}{|y-z|^{N+2s}} \mathrm{d}y\mathrm{d}z \right)^{1/2}.$$

We recall that $X_0^s(\Omega)$ is a Hilbert space with the product

$$\langle u, v\rangle = \int_{\mathfrak{R}^N}\int_{\mathfrak{R}^N} \frac{(u(y)-u(z))(v(y)-v(z))}{|y-z|^{N+2s}}\mathrm{d}y\mathrm{d}z.$$

By [14], we can see that

$$\int_{\mathfrak{R}^N}\int_{\mathfrak{R}^N} \frac{|u(y)-u(z)|^2}{|y-z|^{N+2s}}\mathrm{d}y\mathrm{d}z = 2C_{N,s}^{-1}\int_{\mathfrak{R}^N} |(-\Delta)^{(s/2)}u|^2,$$

where the constant $C_{N,s}$ is given by

$$C_{N,s} = \left(\int_{\mathfrak{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\right)^{-1}.$$

We can also refer to [20, 21] for more properties of $X_0^s(\Omega)$.

Our paper is organized as follows. In § 2, we will carry out the reduction procedure. Then, we will study the reduced finite dimensional problem and prove theorem 1.2 in § 3. Our notations are standard. We will use C to denote different positive constant from line to line.

2. Finite dimensional reduction

Define

$$D_{k,\nu} = \Big\{(x,\mu) : \mu_j \in \big[(t_0 - L\nu^{-s\tau})\nu^{2/(N-6s)}, (t_0 + L\nu^{-s\tau})\nu^{2/(N-6s)}\big],$$
$$|\varphi_1(x_j) - 1| \leqslant \nu^{-s\tau}, |x_i - x_j|^{N-2s} \geqslant \nu^{-(2N-8s)/(N-6s)+s\tau}, i \neq j\Big\},$$

where $x_j \in \Omega$, $j = 1, \ldots, k$, $x = (x_1, \ldots, x_k)$, $\mu = (\mu_1, \ldots, \mu_k)$, τ is a small constant, L is a fixed large positive constant and t_0 is given by

$$t_0 = \left(\frac{A_3(N-2s)}{4s\lambda A_2}\right)^{2/(N-6s)}. \tag{2.1}$$

Here the positive constants A_2 and A_3 are defined in lemma B.1. Let

$$\varepsilon_{ij} = \frac{1}{\mu_i^{(N-2s)/2}\mu_j^{(N-2s)/2}|x_i - x_j|^{N-2s}}, \quad i \neq j.$$

Then, for $(x,\mu) \in D_{k,\nu}$, we have

$$\varepsilon_{ij} \leqslant C\nu^{-4s/(N-6s)-s\tau}.$$

We set

$$E_{x,\mu,k} = \left\{\phi \in X_0^s(\Omega) : \left\langle \phi, \frac{\partial PU_{x_j,\mu_j}}{\partial x_{jl}}\right\rangle = \left\langle \phi, \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}\right\rangle = 0\right\},$$

where $x_j = (x_{j1}, \ldots, x_{jN}) \in \mathfrak{R}^N$, $j = 1, \ldots, k$, $l = 1, \ldots, N$.

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Define the energy functional corresponding to equation (1.4) as follows

$$I_\nu(v) = \frac{1}{2}\int_{\mathfrak{R}^N}\int_{\mathfrak{R}^N}\frac{|v(y)-v(z)|^2}{|y-z|^{N+2s}}\mathrm{d}y\mathrm{d}z - \frac{1}{2}\int_\Omega \lambda v^2 - \frac{1}{2_s^*}\int_\Omega (v-\nu\varphi_1)_+^{2_s^*}.$$

Let

$$J_\nu(x,\mu,\phi) = I_\nu\left(\sum_{j=1}^k PU_{x_j,\mu_j}+\phi\right),\ (x,\mu)\in D_{k,\nu},\ \phi\in E_{x,\mu,k}.$$

Then we expand $J_\nu(x,\mu,\phi)$ at $\phi = 0$ as follows

$$J_\nu(x,\mu,\phi) = J_\nu(x,0) + l_\nu(\phi) + \frac{1}{2}Q_\nu(\phi,\phi) - R_\nu(\phi),$$

where

$$l_\nu(\phi) = \sum_{j=1}^k \langle PU_{x_j,\mu_j},\phi\rangle - \sum_{j=1}^k\int_\Omega \lambda PU_{x_j,\mu_j}\phi - \int_\Omega\left(\sum_{j=1}^k PU_{x_j,\mu_j}-\nu\varphi_1\right)_+^{2_s^*-1}\phi, \tag{2.2}$$

$$Q_\nu(\phi,\psi) = \langle\phi,\psi\rangle - \int_\Omega\lambda\phi\psi - (2_s^*-1)\int_\Omega\left(\sum_{j=1}^k PU_{x_j,\mu_j}-\nu\varphi_1\right)_+^{2_s^*-2}\phi\psi, \tag{2.3}$$

and

$$\begin{aligned} R_\nu(\phi) = {}& \frac{1}{2_s^*}\int_\Omega\left(\sum_{j=1}^k PU_{x_j,\mu_j}+\phi-\nu\varphi_1\right)_+^{2_s^*} - \frac{1}{2_s^*}\int_\Omega\left(\sum_{j=1}^k PU_{x_j,\mu_j}-\nu\varphi_1\right)_+^{2_s^*} \\ & - \int_\Omega\left(\sum_{j=1}^k PU_{x_j,\mu_j}-\nu\varphi_1\right)_+^{2_s^*-1}\phi \\ & - \frac{2_s^*-1}{2}\int_\Omega\left(\sum_{j=1}^k PU_{x_j,\mu_j}-\nu\varphi_1\right)_+^{2_s^*-2}\phi^2. \end{aligned} \tag{2.4}$$

Now we estimate $l_\nu(\phi)$, $Q_\nu(\phi,\psi)$ and $R_\nu(\phi)$ respectively.

LEMMA 2.1. *For any $\phi\in X_0^s(\Omega)$, we have*

$$l_\nu(\phi) = O\left(\sum_{j=1}^k\frac{\lambda}{\mu_j^{2s}} + \sum_{j=1}^k\frac{\nu^{(1/2)+\sigma}}{\mu_j^{(N-2s/2)((1/2)+\sigma)}} + \sum_{i\neq j}\varepsilon_{ij}^{(1/2)+\sigma}\right)\|\phi\|.$$

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Proof. Rewrite $l_\nu(\phi)$ as follows:

$$l_\nu(\phi) = \left\{ \sum_{j=1}^{k} \int_\Omega U_{x_j,\mu_j}^{2_s^*-1}\phi - \int_\Omega \left(\sum_{j=1}^{k} PU_{x_j,\mu_j} - \nu\varphi_1 \right)_+^{2_s^*-1} \phi \right\} - \sum_{j=1}^{k} \int_\Omega \lambda PU_{x_j,\mu_j}\phi$$
$$=: l_1 - l_2.$$

Note that

$$(a-b)_+^{2_s^*-1} = a^{2_s^*-1} + O(a^{2_s^*-1-((1/2)+\sigma)} b^{(1/2)+\sigma}), \quad \forall a, b > 0,$$

where $\sigma > 0$ is a small constant. Then we have

$$\begin{aligned}
l_1 &= \sum_{j=1}^{k} \int_\Omega U_{x_j,\mu_j}^{2_s^*-1}\phi - \int_\Omega \left(\sum_{j=1}^{k} PU_{x_j,\mu_j} - \nu\varphi_1 \right)_+^{2_s^*-1} \phi \\
&= \sum_{j=1}^{k} \int_\Omega U_{x_j,\mu_j}^{2_s^*-1}\phi - \int_\Omega \left(\sum_{j=1}^{k} PU_{x_j,\mu_j} \right)^{2_s^*-1} \phi \\
&\quad + O\left(\int_\Omega \left(\sum_{j=1}^{k} PU_{x_j,\mu_j} \right)^{2_s^*-1-((1/2)+\sigma)} (\nu\varphi_1)^{(1/2)+\sigma}\phi \right) \\
&= \sum_{j=1}^{k} \int_\Omega U_{x_j,\mu_j}^{2_s^*-1}\phi - \int_\Omega \sum_{j=1}^{k} PU_{x_j,\mu_j}^{2_s^*-1}\phi + O\left(\sum_{i\neq j} \int_\Omega PU_{x_j,\mu_j}^{(2_s^*-1/2)} PU_{x_i,\mu_i}^{(2_s^*-1/2)}\phi \right) \\
&\quad + O\left(\int_\Omega \left(\sum_{j=1}^{k} U_{x_j,\mu_j} \right)^{2_s^*-1-((1/2)+\sigma)} \phi \right) \nu^{\frac{1}{2}+\sigma} \\
&= \sum_{j=1}^{k} \int_\Omega U_{x_j,\mu_j}^{2_s^*-1}\phi - \sum_{j=1}^{k} \int_\Omega PU_{x_j,\mu_j}^{2_s^*-1}\phi + O\left(\sum_{i\neq j} \int_\Omega U_{x_j,\mu_j}^{(2_s^*-1/2)} U_{x_i,\mu_i}^{(2_s^*-1/2)}\phi \right) \\
&\quad + O\left(\sum_{j=1}^{k} \int_\Omega U_{x_j,\mu_j}^{2_s^*-1-((1/2)+\sigma)}\phi \right) \nu^{(1/2)+\sigma} \\
&= \sum_{j=1}^{k} \int_\Omega U_{x_j,\mu_j}^{2_s^*-1}\phi - \sum_{j=1}^{k} \int_\Omega (U_{x_j,\mu_j} - \Psi_{x_j,\mu_j})^{2_s^*-1}\phi \\
&\quad + O\left(\sum_{i\neq j} \int_\Omega U_{x_j,\mu_j}^{(2_s^*-1/2)} U_{x_i,\mu_i}^{(2_s^*-1/2)}\phi \right) + O\left(\sum_{j=1}^{k} \int_\Omega U_{x_j,\mu_j}^{2_s^*-1-((1/2)+\sigma)}\phi \right) \nu^{(1/2)+\sigma}
\end{aligned}$$

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$$
\begin{aligned}
&= \sum_{j=1}^{k} \int_{\Omega} (2_s^* - 1) U_{x_j,\mu_j}^{2_s^*-2} \Psi_{x_j,\mu_j} \phi + O\left(\sum_{j=1}^{k} \int_{\Omega} \Psi_{x_j}^{2_s^*-1} \phi \right) \\
&\quad + O\left(\sum_{i \neq j} \int_{\Omega} U_{x_j,\mu_j}^{(2_s^*-1/2)} U_{x_i,\mu_i}^{(2_s^*-1/2)} \phi \right) + O\left(\sum_{j=1}^{k} \int_{\Omega} U_{x_j,\mu_j}^{2_s^*-1-((1/2)+\sigma)} \phi \right) \nu^{(1/2)+\sigma}.
\end{aligned}
\tag{2.5}
$$

Define $\Omega_j = \{z : \mu_j^{-1} z + x_j \in \Omega\}$, we choose $\sigma > 0$ small enough such that

$$
(N-2s)(2_s^* - 1 - \left(\frac{1}{2} + \sigma)\right) \frac{2N}{N+2s} = N\left(2 - \frac{N-2s}{N+2s}\right) - \sigma \frac{2N(N-2s)}{N+2s} > N.
$$

Then

$$
\begin{aligned}
&\int_{\Omega} |U_{x_j,\mu_j}^{2_s^*-1-(\frac{1}{2}+\sigma)} \phi| \\
&\quad \leqslant C \mu_j^{-\frac{N-2s}{2}(\frac{1}{2}+\sigma)} \left(\int_{\Omega_j} \left(\frac{1}{(1+|z|^2)^{(N-2s/2)(2_s^*-1-(\frac{1}{2}+\sigma))}} \right)^{\frac{2N}{N+2s}} \right)^{\frac{N+2s}{2N}} \|\phi\| \\
&\quad \leqslant C \mu_j^{-\frac{N-2s}{2}(\frac{1}{2}+\sigma)} \|\phi\|.
\end{aligned}
\tag{2.6}
$$

By lemma A.2, we have

$$
\begin{aligned}
\int_{\Omega} |U_{x_j,\mu_j}^{2_s^*-2} \Psi_{x_j,\mu_j} \phi| &\leqslant C \mu_j^{-N+2s} \left(\int_{\Omega_j} \frac{1}{(1+|z|^2)^{(4sN/N+2s)}} \right)^{(N+2s/2N)} \|\phi\| \\
&\leqslant \frac{C}{\mu_j^{(N+2s/2)}} \|\phi\|,
\end{aligned}
\tag{2.7}
$$

the second inequality is because of

$$
\left(\int_{\Omega_j} \frac{1}{(1+|z|^2)^{(4sN/N+2s)}} \right)^{(N+2s/2N)} = O\left(\mu_j^{(N-6s/2)} \right).
$$

In fact, let $R > 0$ be such that $\Omega \subset B_R(x_j)$, we have $\Omega_j \subset B_{\mu_j R}(0)$. Thus,

$$
\begin{aligned}
\int_{\Omega_j} \frac{1}{(1+|z|^2)^{(4sN/N+2s)}} &\leqslant \int_{B_{\mu_j R}(0)} \frac{1}{(1+|z|^2)^{(4sN/N+2s)}} \\
&\leqslant \int_{B_1(0)} + \int_{B_{\mu_j R}(0) \setminus B_1(0)} \frac{1}{(1+|z|^2)^{(4sN/N+2s)}} \\
&\leqslant C + C \int_1^{\mu_j R} r^{N-1-(8sN/N+2s)} \mathrm{d}r \\
&\leqslant C \mu_j^{N-(8sN/N+2s)}.
\end{aligned}
$$

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By Hölder inequality, we have

$$\int_\Omega U_{x_j,\mu_j}^{(2_s^*-1/2)} U_{x_i,\mu_i}^{(2_s^*-1/2)} \phi \leqslant \left(\int_\Omega U_{x_j,\mu_j}^{(2_s^*/2)} U_{x_i,\mu_i}^{(2_s^*/2)} \right)^{(N+2s/2N)} \|\phi\|$$
$$\leqslant \frac{C[\log(\mu_i\mu_j)]^{(N+2s/2N)}}{\mu_i^{(N+2s/4)}\mu_j^{(N+2s/4)}|x_i-x_j|^{(N+2s/2)}}\|\phi\|, \tag{2.8}$$

the second inequality is because of

$$\int_\Omega U_{x_j,\mu_j}^{(2_s^*/2)} U_{x_i,\mu_i}^{(2_s^*/2)} \leqslant C|x_i-x_j|^{-N}\mu_j^{-(N/2)}\mu_i^{-(N/2)}\log(\mu_i\mu_j).$$

In fact, define

$$\tilde{\Omega}_1 = \{y\in\Omega \ : \ |y-x_i| \geqslant |y-x_j|\}, \quad \tilde{\Omega}_2 = \{y\in\Omega \ : \ |y-x_i| < |y-x_j|\}.$$

For all $y \in \tilde{\Omega}_1$, we have

$$|x_i-x_j| \leqslant |x_i-y| + |y-x_j| \leqslant 2|y-x_i|.$$

Hence, we have

$$\begin{aligned}
\int_{\tilde{\Omega}_1} U_{x_j,\mu_j}^{(2_s^*/2)} U_{x_i,\mu_i}^{(2_s^*/2)} &= b_0^{2_s^*}\int_{\tilde{\Omega}_1} \frac{\mu_j^{(N/2)}}{(1+\mu_j^2|y-x_j|^2)^{(N/2)}}\cdot\frac{\mu_i^{(N/2)}}{(1+\mu_i^2|y-x_i|^2)^{(N/2)}} \\
&\leqslant 2^N b_0^{2_s^*}\mu_i^{-(N/2)}|x_i-x_j|^{-N}\int_{\tilde{\Omega}_1}\frac{\mu_j^{(N/2)}}{(1+\mu_j^2|y-x_j|^2)^{(N/2)}} \\
&\leqslant 2^N b_0^{2_s^*}\mu_i^{-(N/2)}|x_i-x_j|^{-N}\int_{\Omega}\frac{\mu_j^{(N/2)}}{(1+\mu_j^2|y-x_j|^2)^{(N/2)}} \\
&\leqslant 2^N b_0^{2_s^*}\mu_i^{-(N/2)}|x_i-x_j|^{-N}\int_{B_R(x_j)}\frac{\mu_j^{(N/2)}}{(1+\mu_j^2|y-x_j|^2)^{(N/2)}} \\
&= 2^N b_0^{2_s^*}|x_i-x_j|^{-N}\mu_j^{-(N/2)}\mu_i^{(N/2)}\int_{B_{\mu_j R(0)}}\frac{1}{(1+|z|^2)^{(N/2)}} \\
&\leqslant C|x_i-x_j|^{-N}\mu_j^{-(N/2}\mu_i^{-(N/2)}\log\mu_j.
\end{aligned}$$

Similarly, we have

$$\int_{\tilde{\Omega}_2} U_{x_j,\mu_j}^{(2_s^*/2)} U_{x_i,\mu_i}^{(2_s^*/2)} \leqslant C|x_i-x_j|^{-N}\mu_j^{-(N/2)}\mu_i^{-(N/2)}\log\mu_i.$$

Combining (2.5)–(2.8) with lemma A.2, we can obtain that

$$l_1 = O\left(\sum_{j=1}^k \frac{1}{\mu_j^{(N+2s/2)}} + \sum_{j=1}^k \frac{\nu^{(1/2)+\sigma}}{\mu_j^{(N-2s/2((1/2)+\sigma)}} + \sum_{i\neq j}\varepsilon_{ij}^{(1/2)+\sigma}\right)\|\phi\|. \tag{2.9}$$

On the other hand,

$$\begin{aligned}|l_2| &= \left|\sum_{j=1}^{k}\int_\Omega \lambda PU_{x_j,\mu_j}\phi\right| \leqslant \sum_{j=1}^{k}\int_\Omega \lambda U_{x_j,\mu_j}|\phi| \\ &\leqslant \sum_{j=1}^{k}\lambda\left(\int_\Omega U_{x_j,\mu_j}^{(2N/N+2s)}\right)^{(N+2s/2N)}\|\phi\| \\ &\leqslant \sum_{j=1}^{k}\frac{\lambda}{\mu_j^{2s}}\left(\int_{\Omega_j}\frac{1}{(1+|z|^2)^{(N(N-2s)/N+2s)}}\right)^{(N+2s/2N)}\|\phi\| \\ &\leqslant \sum_{j=1}^{k}\frac{C\lambda}{\mu_j^{2s}}\|\phi\|, \end{aligned} \tag{2.10}$$

the last inequality follows from $(2N(N-2s)/N+2s) > N$ since $N > 6s$.
The result follows directly from (2.9) and (2.10). □

LEMMA 2.2. *For any $\phi, \psi \in X_0^s(\Omega)$, there exists constant $C > 0$ such that*

$$|Q_\nu(\phi,\psi)| \leqslant C\|\phi\|\|\psi\|,$$

where C is independent of ν.

Proof. Using Hölder inequality, we can easily check this conclusion. □

It follows from lemma 2.2 that $Q_\nu(\phi,\psi)$ is a bounded bi-linear functional in $X_0^s(\Omega)$. Then there exists a bounded linear operator Q_ν from $E_{x,\mu,k}$ to $E_{x,\mu,k}$ such that

$$Q_\nu(\phi,\psi) = \langle Q_\nu\phi,\psi\rangle, \quad \forall\phi,\psi \in E_{x,\mu,k}. \tag{2.11}$$

Now, we intend to prove that operator Q_ν is invertible in $E_{x,\mu,k}$.

PROPOSITION 2.3. *There exists constant $\rho > 0$, independent of ν and $(x,\mu) \in D_{k,\nu}$, such that*

$$\|Q_\nu\phi\| \geqslant \rho\|\phi\|, \quad \phi \in E_{x,\mu,k}.$$

Proof. We argue by contradiction. Assume that there exist $\nu_n \to \infty$, $(x_n,\mu_n) \in D_{\nu_n,k}$, $x_{j,n} \to x_j^* \in S$ and $\phi_n \in E_{x_n,\mu_n,k}$ such that

$$\|Q_\nu\phi_n\| = o(1)\|\phi_n\|.$$

Without loss of generality, we may assume that $\|\phi_n\| = 1$. Let $\tilde{\phi}_{i,n}(y) = \mu_{i,n}^{-(N-2s)/2}\phi_n(\mu_{i,n}^{-1}y + x_{i,n})$, $\Omega_{i,n} = \{y : \mu_{i,n}^{-1}y + x_{i,n} \in \Omega\}$. Then $\|\tilde{\phi}_{i,n}\| = \|\phi_n\| = 1$.

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Thus, we may assume that there exists $\tilde{\phi}_i \in D^{s,2}(\mathfrak{R}^N)$ such that

$$\begin{aligned} \tilde{\phi}_{i,n} &\rightharpoonup \tilde{\phi}_i \quad \text{in } D^{s,2}\left(\mathfrak{R}^N\right), \\ \tilde{\phi}_{i,n} &\to \tilde{\phi}_i \quad \text{in } L^2_{loc}\left(\mathfrak{R}^N\right), \\ \tilde{\phi}_{i,n} &\to \tilde{\phi}_i \quad a.e. \text{ on } \mathfrak{R}^N. \end{aligned}$$

We claim that $\tilde{\phi}_i$ solves

$$(-\Delta)^s \tilde{\phi}_i - (2^*_s - 1) U^{2^*_s - 2} \tilde{\phi}_i = 0. \tag{2.12}$$

In fact, it is sufficient to show that

$$\int_{\mathfrak{R}^N} (-\Delta)^s \tilde{\phi}_i \eta - \int_{\mathfrak{R}^N} (2^*_s - 1) U^{2^*_s - 2} \tilde{\phi}_i \eta = 0, \quad \text{for all } \eta \in C_0^\infty(\mathfrak{R}^N). \tag{2.13}$$

Since

$$\begin{aligned} &\langle \phi_n, \eta \rangle - \int_\Omega \lambda \phi_n \eta - (2^*_s - 1) \int_\Omega \left(\sum_{j=1}^k PU_{x_j,\mu_j} - \nu \varphi_1 \right)_+^{2^*_s - 2} \phi_n \eta \\ &= Q_\nu(\phi_n, \eta) = \langle Q_\nu \phi_n, \eta \rangle = o(1)\|\eta\|, \quad \forall \eta \in E_{x_n,\mu_n,k}, \end{aligned} \tag{2.14}$$

we have

$$\begin{aligned} &\int_{\mathfrak{R}^N} (-\Delta)^s \tilde{\phi}_{i,n} \tilde{\eta} - \frac{\lambda}{\mu_{i,n}^{2s}} \int_{\Omega_{i,n}} \tilde{\phi}_{i,n} \tilde{\eta} \\ &\quad - (2^*_s - 1) \int_{\Omega_{i,n}} \left(\sum_{j=1}^k \tilde{U}_{j,n} - \nu_n \mu_{i,n}^{-(N-2s/2)} \varphi_1(\mu_{i,n}^{-1} + x_{i,n}) \right)_+^{2^*_s - 2} \tilde{\phi}_{i,n} \tilde{\eta} \\ &= o(1)\|\tilde{\eta}\|, \quad \forall \tilde{\eta} \in \tilde{E}_{x_n,\mu_n,k}, \end{aligned} \tag{2.15}$$

where $\tilde{\eta}(y) = \eta(\mu_{i,n}^{-1} y + x_{i,n})$, $\tilde{U}_{j,n} = \mu_{i,n}^{-(N-2s)/2} PU_{x_{j,n},\mu_{j,n}}(\mu_{i,n}^{-1} y + x_{i,n})$,

$$\begin{aligned} \tilde{E}_{x_n,\mu_n,k} = \Bigg\{ \tilde{\eta} \in X_0^s(\Omega_{i,n}) : &\int_{\mathfrak{R}^N} (-\Delta)^{\frac{s}{2}} \tilde{V}_{j,l,n} (-\Delta)^{\frac{s}{2}} \tilde{\eta} \\ &= \int_{\mathfrak{R}^N} (-\Delta)^{\frac{s}{2}} \tilde{V}_{j,n} (-\Delta)^{\frac{s}{2}} \tilde{\eta} = 0 \Bigg\}, \end{aligned}$$

where $l = 1, \ldots, N$, $j = 1, \ldots, k$ and $\tilde{V}_{j,l,n}$, $\tilde{V}_{j,n}$ are given by

$$\tilde{V}_{j,l,n}(y) = \mu_{i,n}^{-(N-2s)/2} \mu_{i,n}^{-1} \frac{\partial PU_{x_{j,n},\mu_{j,n}}(z)}{\partial x_{jl}} \Big|_{z=\mu_{i,n}^{-1} y + x_{i,n}},$$

$$\tilde{V}_{j,n}(y) = \mu_{i,n}^{-(N-2s)/2} \mu_{i,n} \frac{\partial PU_{\varepsilon_n,x_{j,n}}(z)}{\partial \mu_j} \Big|_{z=\mu_{i,n}^{-1} y + x_{i,n}}.$$

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For any $\eta \in C_0^\infty(\mathfrak{R}^N)$, we can choose $a_{j,l,n}$ and $b_{j,n}$ such that

$$\tilde{\eta} = \eta - \sum_{j=1}^{k}\sum_{l=1}^{N} a_{j,l,n}\tilde{V}_{j,l,n} + \sum_{j=1}^{k} b_{j,n}\tilde{V}_{j,n} \in \tilde{E}_{x_n,\mu_n,k}.$$

Noting that η has compact support and the support of $\tilde{V}_{j,l,n}$ and $\tilde{V}_{j,n}$ moves to infinity as $n \to \infty$ if $i \neq j$. Thus, we can see that $a_{j,l,n} \to 0$ and $b_{j,n} \to 0$ if $i \neq j$. Furthermore, we can check that $a_{i,l,n}$ and $b_{i,n}$ are bounded. Since $(x_n, \mu_n) \in D_{\nu_n,k}$, then $\nu_n \mu_{i,n}^{-(N-2s)/2} \to 0$. Substituting $\tilde{\eta}$ in (2.15) and letting $n \to \infty$, we derive that

$$\begin{aligned}
&\int_{\mathfrak{R}^N} (-\Delta)^s \tilde{\phi}_i \eta - (2_s^* - 1)\int_{\mathfrak{R}^N} U^{2_s^*-2}\tilde{\phi}_i \eta \\
&= \sum_{l=1}^{k} a_{il}\left(\int_{\mathfrak{R}^N} (-\Delta)^{\frac{s}{2}}\tilde{\phi}_i(-\Delta)^{\frac{s}{2}}\frac{\partial U}{\partial x_l} - (2_s^*-1)U^{2_s^*-2}\tilde{\phi}_i\frac{\partial U}{\partial x_l}\right) \\
&\quad + b_i\left(\int_{\mathfrak{R}^N} (-\Delta)^{\frac{s}{2}}\tilde{\phi}_i(-\Delta)^{\frac{s}{2}}\frac{\partial U}{\partial \mu} - (2_s^*-1)U^{2_s^*-2}\tilde{\phi}_i\frac{\partial U}{\partial \mu}\right), \qquad (2.16)
\end{aligned}$$

where $a_{i,l} = \lim_{n\to\infty} a_{i,l,n}$ and $b_i = \lim_{n\to\infty} b_{i,n}$. On the other hand,

$$\int_{\mathfrak{R}^N} (-\Delta)^{\frac{s}{2}}\tilde{\phi}_i(-\Delta)^{\frac{s}{2}}\frac{\partial U}{\partial x_l} - (2_s^*-1)U^{2_s^*-2}\tilde{\phi}_i\frac{\partial U}{\partial x_l} = 0. \qquad (2.17)$$

and

$$\int_{\mathfrak{R}^N} (-\Delta)^{\frac{s}{2}}\tilde{\phi}_i(-\Delta)^{\frac{s}{2}}\frac{\partial U}{\partial \mu} - (2_s^*-1)U^{2_s^*-2}\tilde{\phi}_i\frac{\partial U}{\partial \mu} = 0. \qquad (2.18)$$

Thus, (2.13) follows from (2.16)–(2.18). Therefore, we have proved this claim.

We recall that U is non-degenerate, that is, if $\tilde{\phi}_i$ solves (2.12), then there exists some constants $\bar{c}_l$ and $\bar{c}$ such that

$$\tilde{\phi}_i = \sum_{l=1}^{N} \bar{c}_l \frac{\partial U}{\partial x_l} + \bar{c}\frac{\partial U}{\partial \mu}.$$

Note that $\phi_n \in E_{x_n,\mu_n,k}$, then $\tilde{\phi}_{i,n} \in \tilde{E}_{x_n,\mu_n,k}$. So we can obtain that

$$\int_{\mathfrak{R}^N} (-\Delta)^{\frac{s}{2}}\tilde{\phi}_i(-\Delta)^{\frac{s}{2}}\frac{\partial U}{\partial \mu} = 0,$$

and

$$\int_{\mathfrak{R}^N} (-\Delta)^{\frac{s}{2}}\tilde{\phi}_i(-\Delta)^{\frac{s}{2}}\frac{\partial U}{\partial x_l} = 0, \quad l = 1, \ldots, N,$$

which imply that $\tilde{\phi}_i = 0$. Then for any $R > 0$,

$$\int_{B_{\mu_{i,n}^{-1}R}(x_{i,n})} \phi_n^2 = \mu_{i,n}^{2s}\int_{B_R(0)} \tilde{\phi}_{i,n}^2 = o(\mu_{i,n}^{2s}).$$

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Moreover, we have

$$\int_{B_{\mu_{i,n}^{-1}R}(x_{i,n})}\left(\sum_{j=1}^{k}PU_{x_j,\mu_j}-\nu\varphi_1\right)_+^{2_s^*-2}\phi_n^2\leqslant C\int_{B_{\mu_{i,n}^{-1}R}(x_{i,n})}U_{x_{i,n},\mu_{i,n}}^{2_s^*-2}\phi_n^2=o(1). \tag{2.19}$$

Thus,

$$\begin{aligned}
&\int_{\Omega}\left(\sum_{j=1}^{k}PU_{x_j,\mu_j}-\nu\varphi_1\right)_+^{2_s^*-2}\phi_n^2\\
&=\sum_{i=1}^{k}\int_{\Omega\setminus B_{\mu_{i,n}^{-1}R}(x_{i,n})}\left(\sum_{j=1}^{k}PU_{x_j,\mu_j}-\nu\varphi_1\right)_+^{2_s^*-2}\phi_n^2+o(1)\\
&\leqslant C\sum_{i=1}^{k}\int_{\Omega\setminus B_{\mu_{i,n}^{-1}R}(x_{i,n})}(PU_{x_i,\mu_i}-\nu\varphi_1)_+^{2_s^*-2}\phi_n^2+o(1)\\
&\leqslant C\sum_{i=1}^{k}\int_{\Omega\setminus B_{\mu_{i,n}^{-1}R}(x_{i,n})}U_{x_i,\mu_i}^{2_s^*-2}\phi_n^2+o(1)\\
&\leqslant C\sum_{i=1}^{k}\left(\int_{\Omega_{i,n}\setminus B_{R(0)}}U^{2_s^*}\right)^{(s/N)}\|\phi_n\|^2+o(1)=o_R(1)+o(1),
\end{aligned} \tag{2.20}$$

where $o_R(1)\to 0$ as $R\to\infty$. Combining (2.14) with (2.20), we deduce that

$$\int_{\mathfrak{R}^N}(-\Delta)^{\frac{s}{2}}\phi_n(-\Delta)^{\frac{s}{2}}\eta-\int_{\Omega}\lambda\phi_n\eta=o(1)\|\eta\|,\quad \forall\eta\in E_{x_n,\mu_n,k}. \tag{2.21}$$

Note that $\phi_n\in X_0^s(\Omega)$ and $\|\phi_n\|=1$. We may assume that there exits $\phi\in X_0^s(\Omega)$ such that

$$\begin{aligned}
&\phi_n\rightharpoonup\phi\quad\text{weakly in } X_0^s(\Omega),\\
&\phi_n\to\phi\quad\text{strongly in } L^2(\Omega).
\end{aligned}$$

We claim that $\phi=0$. Indeed, for any $\eta\in C_0^\infty(\Omega)$, we choose $c_{j,l,n}$ such that

$$\bar\eta=\eta-\sum_{j=1}^{k}\sum_{l=1}^{N}c_{j,l,n}\frac{\partial PU_{x_{j,n},\mu_{j,n}}}{\partial x_{jl}}-d_{j,n}\frac{\partial PU_{x_{j,n},\mu_{j,n}}}{\partial\mu_j}\in E_{x_n,\mu_n,k}.$$

In order to estimate $c_{j,l,n}$ and $d_{j,n}$, multiplying (2.21) by $\frac{\partial PU_{x_{i,n},\mu_{i,n}}}{\partial x_{ih}}$ and $\frac{\partial PU_{x_{i,n},\mu_{i,n}}}{\partial\mu_i}$ respectively, we have

$$\left\langle\eta,\frac{\partial PU_{x_{i,n},\mu_{i,n}}}{\partial x_{ih}}\right\rangle-\sum_{j=1}^{k}\sum_{l=1}^{N}c_{j,l,n}\left\langle\frac{\partial PU_{x_{j,n},\mu_{j,n}}}{\partial x_{jl}},\frac{\partial PU_{x_{i,n},\mu_{i,n}}}{\partial x_{ih}}\right\rangle=0, \tag{2.22}$$

and

$$\left\langle \eta, \frac{\partial PU_{x_{i,n},\mu_{i,n}}}{\partial \mu_i} \right\rangle - \sum_{j=1}^{k} d_{j,n} \left\langle \frac{\partial PU_{x_{j,n},\mu_{j,n}}}{\partial \mu_j}, \frac{\partial PU_{x_{i,n},\mu_{i,n}}}{\partial \mu_i} \right\rangle = 0. \tag{2.23}$$

Note that

$$\left\langle \eta, \frac{\partial PU_{x_{j,n},\mu_{j,n}}}{\partial x_{jh}} \right\rangle = (2_s^* - 1) \int_\Omega U_{x_{j,n},\mu_{j,n}}^{2_s^*-2} \frac{\partial U_{x_{j,n},\mu_{j,n}}}{\partial x_{jh}} \eta = O\left(\frac{\mu_j}{\mu_j^{(N-2s/2)}} \right), \tag{2.24}$$

and

$$\left\langle \eta, \frac{\partial PU_{x_{j,n},\mu_{j,n}}}{\partial \mu_j} \right\rangle = (2_s^* - 1) \int_\Omega U_{x_{j,n},\mu_{j,n}}^{2_s^*-2} \frac{\partial U_{x_{j,n},\mu_{j,n}}}{\partial \mu_j} \eta = O\left(\frac{\mu_j^{-1}}{\mu_j^{(N-2s/2)}} \right). \tag{2.25}$$

Combining (2.22) with (2.24), we see that $c_{j,l,n} = O(\mu_j^{-1}\mu_j^{-(N-2s)/2})$. By (2.23) and (2.25), we obtain that $d_{j,n} = O(\mu_j \mu_j^{-(N-2s)/2})$. Thus,

$$\begin{aligned}
&\left| c_{j,l,n} \left(\left\langle \frac{\partial PU_{x_{j,n},\mu_{j,n}}}{\partial x_{jl}}, \phi_n \right\rangle - \lambda \int_\Omega \frac{\partial PU_{x_{j,n},\mu_{j,n}}}{\partial x_{jl}} \phi_n \right) \right| \\
&\quad \leqslant C \frac{1}{\mu_j^{(N-2s/2)}} \left(\int_\Omega U_{x_{j,n},\mu_{j,n}}^{2_s^*-1} |\phi_n| + \int_\Omega U_{x_{j,n},\mu_{j,n}} |\phi_n| \right) \\
&\quad \leqslant C \frac{1}{\mu_j^{(N-2s/2)}} \left[\left(\int_\Omega U_{x_{j,n},\mu_{j,n}}^{2_s^*} \right)^{(N+2s/2N)} + \left(\int_\Omega U_{x_{j,n},\mu_{j,n}}^{(2N/N+2s)} \right)^{(N+2s/2N)} \right] \|\phi_n\| \\
&\quad \leqslant C \left[\frac{1}{\mu_j^{(N-2s/2)}} + \frac{1}{\mu_j^{(N+2s/2)}} \left(\int_{\Omega_{j,n}} \frac{1}{(1+|y|^2)^{(N(N-2s)/N+2s)}} \right)^{(N+2s/2N)} \right] \|\phi_n\| \\
&\quad \leqslant C \frac{1}{\mu_j^{(N-2s/2)}} + \frac{1}{\mu_j^{(N+2s/2)}},
\end{aligned} \tag{2.26}$$

the last inequality follows from $(2N(N-2s)/N+2s) > N$ since $N > 6s$. Using the similar computation, we also have

$$\left| d_{j,n} \left(\left\langle \frac{\partial PU_{x_{j,n},\mu_{j,n}}}{\partial \mu_j}, \phi_n \right\rangle - \lambda \int_\Omega \frac{\partial PU_{x_{j,n},\mu_{j,n}}}{\partial \mu_j} \phi_n \right) \right| = O\left(\frac{1}{\mu_j^{(N-2s/2)}} \right). \tag{2.27}$$

Inserting $\bar{\eta}$ into (2.21), combining (2.26) with (2.27) and letting $n \to \infty$, we can deduce that

$$\int_{\mathfrak{R}^N} (-\Delta)^{\frac{s}{2}} \phi (-\Delta)^{\frac{s}{2}} \eta - \int_\Omega \lambda \phi \eta = 0. \tag{2.28}$$

Since $\lambda \neq \lambda_i$, $\phi = 0$. Hence, the claim is completed.

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Taking $\eta = \phi_n$ in (2.21), we derive that

$$\int_{\mathfrak{R}^N} |(-\Delta)^{\frac{s}{2}}\phi_n|^2 = \int_{\Omega} \lambda\phi_n^2 + o(1)\|\phi_n\| = o(1) + o(1)\|\phi_n\|$$

which contradicts with $\|\phi_n\| = 1$. □

LEMMA 2.4. *For any $\phi \in X_0^s(\Omega)$, it holds*

$$D^i R_\nu(\phi) = O(\|\phi\|^{2_s^*-i}), \quad i = 0, 1, 2.$$

Proof. Using the fact that for $a, b > 0$,

$$(a+b)^p = a^p + pa^{p-1}b + \frac{p(p-1)}{2}a^{p-2}b^2 + O(b^p), \quad p > 2,$$

we can easily check this conclusion. Here, we omit it. □

PROPOSITION 2.5. *There exists $\nu_k > 0$ such that for $\nu \geqslant \nu_k$, there exists a C^1-map $\phi_{\nu,x,\mu} : D_{k,\nu} \to X_0^s(\Omega)$, such that $\phi_{\nu,x,\mu} \in E_{x,\mu,k}$ satisfies*

$$\left\langle I_\nu'\left(\sum_{j=1}^k PU_{x_j,\mu_j} + \phi_{\nu,x,\mu}\right), \eta\right\rangle = 0, \quad \forall\eta \in E_{x,\mu,k}. \tag{2.29}$$

Furthermore,

$$\|\phi_{\nu,x,\mu}\| = O\left(\sum_{j=1}^k \frac{\lambda}{\mu_j^{2s}} + \sum_{j=1}^k \frac{\nu^{(1/2)+\sigma}}{\mu_j^{(N-2s/2)((1/2)+\sigma)}} + \sum_{i\neq j} \varepsilon_{ij}^{(1/2)+\sigma}\right),$$

where σ is a small positive constant.

Proof. Set

$$\mathcal{N}_{x,\mu,k} = \left\{\phi : \phi \in E_{x,\mu,k}, \|\phi\| \leqslant \sum_{j=1}^k \frac{\lambda}{\mu_j^{2s-\sigma}} + \sum_{j=1}^k \frac{\nu^{(1+\sigma/2)}}{\mu_j^{(N-2s/2)(1+\sigma/2)}} + \sum_{i\neq j} \varepsilon_{ij}^{(1+\sigma/2)}\right\}.$$

First, by lemma 2.1, we see that $l_\nu(\phi)$ is a bounded linear functional in $E_{x,\mu,k}$. Then, there exists l_ν such that

$$l_\nu(\phi) = \langle l_\nu, \phi\rangle, \quad \forall\phi \in E_{x,\mu,k}.$$

Combining this with (2.11), we can obtain that (2.29) is equivalent to

$$l_\nu + Q_\nu\phi + R_\nu'(\phi) = 0. \tag{2.30}$$

It follows from proposition 2.3 that Q_ν is invertible in $E_{x,\mu,k}$ and

$$\|Q_\nu^{-1}\| \leqslant \rho^{-1}.$$

Thus, (2.30) can be written as

$$\phi = \mathcal{A}\phi := -Q_\nu^{-1}l_\nu - Q_\nu^{-1}R_\nu'(\phi). \tag{2.31}$$

Now, we prove that $\mathcal{A}$ is a contraction map from $\mathcal{N}_{x,\mu,k}$ to $\mathcal{N}_{x,\mu,k}$.

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On one hand, for any ϕ_1, $\phi_2 \in \mathcal{N}_{x,\mu,k}$, using lemma 2.4, we have

$$\begin{aligned}\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\| &= \|Q_\nu^{-1}R'_\nu(\phi_1) - Q_\nu^{-1}R'_\nu(\phi_2)\| \\ &\leqslant \rho^{-1}\|R'_\nu(\phi_1) - R'_\nu(\phi_2)\| \\ &\leqslant C(\|\phi_1\|^{2_s^*-2} + \|\phi_2\|^{2_s^*-2})\|\phi_1 - \phi_2\| \\ &\leqslant \frac{1}{2}\|\phi_1 - \phi_2\|,\end{aligned}$$

if ν is large enough. Hence, $\mathcal{A}$ is a contraction map.

On the other hand, for any $\phi \in \mathcal{N}_{x,\mu,k}$, applying lemma 2.1 and lemma 2.4 again, we have

$$\begin{aligned}\|\mathcal{A}\phi\| &\leqslant \rho^{-1}\|l_\nu\| + \rho^{-1}\|R'_\nu(\phi)\| \leqslant C(\|l_\nu\| + \|\phi\|^{2_s^*-1}) \\ &\leqslant C\left(\sum_{j=1}^{k}\frac{\lambda}{\mu_j^{2s}} + \sum_{j=1}^{k}\frac{\nu^{(1/2)+\sigma}}{\mu_j^{(N-2s/2)(1/2+\sigma)}} + \sum_{i\neq j}\varepsilon_{ij}^{(1/2)+\sigma}\right) \\ &\leqslant \sum_{j=1}^{k}\frac{\lambda}{\mu_j^{2s-\sigma}} + \sum_{j=1}^{k}\frac{\nu^{(1+\sigma/2)}}{\mu_j^{(N-2s/2)(1+\sigma/2)}} + \sum_{i\neq j}\varepsilon_{ij}^{(1+\sigma/2)},\end{aligned}$$

if ν is large enough. Therefore, $\mathcal{A}$ is a contraction map from $\mathcal{N}_{x,\mu,k}$ to $\mathcal{N}_{x,\mu,k}$. By contraction mapping theorem, there exists a unique $\phi_{\nu,x,\mu} \in \mathcal{N}_{x,\mu,k}$ such that (2.31) holds. Moreover,

$$\|\phi_{\nu,x,\mu}\| \leqslant C\left(\sum_{j=1}^{k}\frac{\lambda}{\mu_j^{2s}} + \sum_{j=1}^{k}\frac{\nu^{(1/2)+\sigma}}{\mu_j^{(N-2s/2)(1/2+\sigma)}} + \sum_{i\neq j}\varepsilon_{ij}^{(1/2)+\sigma}\right).$$

□

3. Proof of main result

In this section, we will choose suitable $(x, \mu) \in D_{k,\nu}$ such that

$$v_\nu = \sum_{j=1}^{k} PU_{x_j,\mu_j} + \phi_{\nu,x,\mu}$$

is a solution of equation (1.4). We define

$$K(x, \mu) = J_\nu(x, \mu, \phi_{\nu,x,\mu}), \quad (x, \mu) \in D_{k,\nu},$$

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where $\phi_{\nu,x,\mu}$ is obtained in proposition 2.5. Using proposition 2.5 and lemma B.2, we deduce that

$$\begin{aligned} K(x,\mu) &= J_\nu(x,\mu,0) + O(\|\phi_{\nu,x,\mu}\|^2) \\ &= kA_1 - \sum_{j=1}^{k}\left(\frac{\lambda A_2}{\mu_j^{2s}} - \frac{\varphi_1(x_j)A_3\nu}{\mu_j^{(N-2s/2)}}\right) - \sum_{i\neq j}^{k}\frac{1}{2}b_0(A_3 + b_0\lambda A(x_i,x_j))\varepsilon_{ij} \\ &\quad + O\left(\sum_{j=1}^{k}\left(\frac{1}{\mu_j^{2s+2\sigma}} + \frac{\nu}{\mu_j^{\frac{N}{2}}} + \frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}}\right) + \sum_{i\neq j}\varepsilon_{ij}^{1+\sigma}\right) \\ &= kA_1 - \sum_{j=1}^{k}\left(\frac{\lambda A_2}{\mu_j^{2s}} - \frac{\varphi_1(x_j)A_3\nu}{\mu_j^{(N-2s/2)}}\right) - \sum_{i\neq j}^{k}\frac{1}{2}b_0(A_3 + b_0\lambda A(x_i,x_j))\varepsilon_{ij} \\ &\quad + O\left(\frac{1}{\nu^{(4s(1+\sigma)/N-6s)}}\right), \end{aligned} \tag{3.1}$$

where the positive constants A_1, A_2, A_3 $A(x_i,x_j)$ are defined in lemma B.2 and $\sigma > 0$ is a small constant.

Now, we intend to estimate the derivative of $K(x,\mu)$. It follows from proposition 2.5 that there exist constants c_{ih}, d_i, $i = 1,\dots,k$, $h = 1,\dots,N$ such that

$$\frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial \phi_{\nu,x,\mu}} = \sum_{i=1}^{k}\sum_{h=1}^{N} c_{ih}\frac{\partial PU_{x_i,\mu_i}}{\partial x_{ih}} + \sum_{i=1}^{k} d_i \frac{\partial PU_{x_i,\mu_i}}{\partial \mu_i}. \tag{3.2}$$

Thus

$$\begin{aligned} \frac{\partial K(x,\mu)}{\partial \mu_j} &= \frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial \mu_j} + \left\langle \frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\phi_{\nu,x,\mu}}, \frac{\partial \phi_{\nu,x,\mu}}{\partial \mu_j}\right\rangle \\ &= \frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial \mu_j} + \sum_{i=1}^{k}\sum_{h=1}^{N} c_{ih}\left\langle \frac{\partial PU_{x_i,\mu_i}}{\partial x_{ih}}, \frac{\partial \phi_{\nu,x,\mu}}{\partial \mu_j}\right\rangle \\ &\quad + \sum_{i=1}^{k} d_i \left\langle \frac{\partial PU_{x_i,\mu_i}}{\partial \mu_i}, \frac{\partial \phi_{\nu,x,\mu}}{\partial \mu_j}\right\rangle. \end{aligned}$$

Hence, we have to estimate $(\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})/\partial \mu_j)$, c_{ih} and d_i.

Lemma 3.1. *Let $\phi_{\nu,x,\mu}$ be obtained in proposition 2.5. Then*

$$\frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial \mu_j} = \frac{2s\lambda A_2}{\mu_j^{2s+1}} - \frac{N-2s}{2}\frac{\varphi_1(x_j)A_3\nu}{\mu_j^{(N-2s/2)+1}} + \frac{1}{\mu_j}O\left(\frac{1}{\nu^{(4s(1+\sigma)/N-6s)}}\right), \tag{3.3}$$

and

$$\frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial x_{ji}} = \mu_j O\left(\frac{1}{\mu_j^{N-2s}} + \frac{\nu}{\mu_j^{(N/2)}} + \frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}} + \sum_{i\neq j}\varepsilon_{ij}\right), \tag{3.4}$$

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where $\sigma > 0$ is a small constant, constants A_2, A_3 are defined in lemma B.1.

Proof. By a direct computation, we have

$$\begin{aligned}\frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial\mu_j} &= \left\langle I'_\nu\left(\sum_{i=1}^k PU_{x_i,\mu_i}+\phi_{\nu,x,\mu}\right), \frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j}\right\rangle \\ &= \left\langle I'_\nu\left(\sum_{i=1}^k PU_{x_i,\mu_i}\right), \frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j}\right\rangle - \lambda\int_\Omega \phi_{\nu,x,\mu}\frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j} \\ &\quad - \int_\Omega\left(\sum_{i=1}^k PU_{x_i,\mu_i}+\phi_{\nu,x,\mu}-\nu\varphi_1\right)_+^{2_s^*-1}\frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j} \\ &\quad + \int_\Omega\left(\sum_{i=1}^k PU_{x_i,\mu_i}-\nu\varphi_1\right)_+^{2_s^*-1}\frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j} \end{aligned} \tag{3.5}$$

It follows from proposition 2.5 and (2.10) that

$$\begin{aligned}\left|\int_\Omega \phi_{\nu,x,\mu}\frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j}\right| &\leqslant \frac{C}{\mu_j}\left(\int_\Omega |\phi_{\nu,x,\mu}|U_{x_j,\mu_j}\right) \\ &\leqslant \frac{C}{\mu_j}\frac{\|\phi_{\nu,x,\mu}\|}{\mu_j^{2s}} \leqslant \frac{C}{\mu_j}\left(\frac{1}{\nu^{(4s(1+\sigma)/N-6s)}}\right). \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned}&\int_\Omega\left[\left(\sum_{i=1}^k PU_{x_i,\mu_i}+\phi_{\nu,x,\mu}-\nu\varphi_1\right)_+^{2_s^*-1} - \left(\sum_{i=1}^k PU_{x_i,\mu_i}-\nu\varphi_1\right)_+^{2_s^*-1}\right]\frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j} \\ &= (2_s^*-1)\int_\Omega\left(\sum_{i=1}^k PU_{x_i,\mu_i}-\nu\varphi_1\right)_+^{2_s^*-2}\phi_{\nu,x,\mu}\frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j} \\ &\quad + O\left(\int_\Omega \phi_{\nu,x,\mu}^{2_s^*-1}\frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j}\right). \end{aligned} \tag{3.7}$$

Then, using lemma A.2, proposition 2.5 and Hölder inequality, we obtain that

$$\begin{aligned}\left|\int_\Omega \phi_{\nu,x,\mu}^{2_s^*-1}\frac{\partial PU_{x_j,\mu_j}}{\partial\mu_j}\right| &\leqslant \frac{C}{\mu_j}\int_\Omega |\phi_{\nu,x,\mu}^{2_s^*-1}U_{x_j,\mu_j}| \leqslant \frac{C}{\mu_j}\|\phi_{\nu,x,\mu}\|^{2_s^*-1} \\ &\leqslant \frac{C}{\mu_j}\left(\frac{1}{\nu^{(4s(1+\sigma)/N-6s)}}\right). \end{aligned} \tag{3.8}$$

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On the other hand, by a similar computation in (B.7), we can obtain

$$
\begin{aligned}
&\int_\Omega \left(\sum_{i=1}^k PU_{x_i,\mu_i} - \nu\varphi_1\right)_+^{2_s^*-2} \phi_{\nu,x,\mu} \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j} \\
&= \int_\Omega \left(\sum_{i=1}^k U_{x_i,\mu_i} - \nu\varphi_1\right)_+^{2_s^*-2} \phi_{\nu,x,\mu} \frac{\partial U_{x_j,\mu_j}}{\partial \mu_j} + \frac{1}{\mu_j} O\left(\frac{\|\phi_{\nu,x,\mu}\|}{\mu_j^{(N-2s/2)}}\right) \\
&= \int_\Omega \left(\sum_{i=1}^k U_{x_i,\mu_i}\right)^{2_s^*-2} \phi_{\nu,x,\mu} \frac{\partial U_{x_j,\mu_j}}{\partial \mu_j} + \frac{1}{\mu_j} O\left(\frac{\|\phi_{\nu,x,\mu}\|}{\mu_j^{(N-2s/2)}}\right) \\
&\quad + O\left(\frac{1}{\mu_j}\int_\Omega \left(\sum_{i=1}^k U_{x_i,\mu_i}\right)^{2_s^*-2-((1/2)+\sigma)} |\nu\varphi_1|^{(1/2)+\sigma} |\phi_{\nu,x,\mu}| U_{x_j,\mu_j}\right) \\
&= \int_\Omega \left[\left(\sum_{i=1}^k U_{x_i,\mu_i}\right)^{2_s^*-2} - U_{x_j,\mu_j}^{2_s^*-2}\right] \phi_{\nu,x,\mu} \frac{\partial U_{x_j,\mu_j}}{\partial \mu_j} + \frac{1}{\mu_j} O\left(\frac{\|\phi_{\nu,x,\mu}\|}{\mu_j^{(N-2s/2)}}\right) \\
&\quad + \frac{1}{\mu_j} O\left(\sum_{i=1}^k \frac{\nu^{(1/2)+\sigma}\|\phi_{\nu,x,\mu}\|}{\mu_i^{((N-2s)/2)(1/2+\sigma)}}\right) \\
&= \frac{1}{\mu_j}\int_\Omega \sum_{i\neq j} U_{x_i,\mu_i}^{(2_s^*-1/2)} U_{x_j,\mu_j}^{(2_s^*-1/2)} |\phi_{\nu,x,\mu}| \\
&\quad + \frac{1}{\mu_j} O\left(\frac{\|\phi_{\nu,x,\mu}\|}{\mu_j^{(N-2s/2)}}\right) + \frac{1}{\mu_j} O\left(\sum_{i=1}^k \frac{\nu^{(1/2)+\sigma}\|\phi_{\nu,x,\mu}\|}{\mu_i^{((N-2s)/2)(1/2)+\sigma)}}\right) \\
&= \frac{1}{\mu_j}\sum_{i\neq j} \varepsilon_{ij}^{(1/2)+\sigma} \|\phi_{\nu,x,\mu}\| + \frac{1}{\mu_j} O\left(\frac{\|\phi_{\nu,x,\mu}\|}{\mu_j^{(N-2s/2)}}\right) \\
&\quad + \frac{1}{\mu_j} O\left(\sum_{i=1}^k \frac{\nu^{(1/2)+\sigma}\|\phi_{\nu,x,\mu}\|}{\mu_i^{((N-2s)/2)(1/2+\sigma)}}\right). \qquad (3.9)
\end{aligned}
$$

Thus, (3.3) follows from lemma B.3 and (3.5)–(3.9).

The similar computation and lemma B.4 yield (3.4). □

LEMMA 3.2. *Let c_{jl} and d_j be defined in (3.2). Then*

$$
c_{jl} = \mu_j^{-1} O\left(\frac{1}{\mu_j^{N-2s}} + \frac{\nu}{\mu_j^{N/2}} + \frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}} + \sum_{i\neq j} \varepsilon_{ij}\right),
$$

and

$$
d_j = \mu_j O\left(\frac{1}{\mu_j^{2s}} + \frac{\nu}{\mu_j^{(N-2s/2)}} + \frac{1}{\nu^{(4s(1+\sigma)/N-6s)}}\right).
$$

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Proof. Multiplying $(\partial PU_{x_j,\mu_j}/\partial\mu_j)$ and $(\partial PU_{x_j,\mu_j}/\partial x_{jl})$ by (3.2), respectively, we obtain

$$\sum_{i=1}^{k}\sum_{h=1}^{N} c_{ih}\left\langle \frac{\partial PU_{x_i,\mu_i}}{\partial x_{ih}}, \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}\right\rangle + \sum_{i=1}^{k} d_i\left\langle \frac{\partial PU_{x_i,\mu_i}}{\partial \mu_i}, \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}\right\rangle = \left\langle \frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial \phi_{\nu,x,\mu}}, \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}\right\rangle, \tag{3.10}$$

and

$$\sum_{i=1}^{k}\sum_{h=1}^{N} c_{ih}\left\langle \frac{\partial PU_{x_i,\mu_i}}{\partial x_{ih}}, \frac{\partial PU_{x_j,\mu_j}}{\partial x_{jl}}\right\rangle + \sum_{i=1}^{k} d_i\left\langle \frac{\partial PU_{x_i,\mu_i}}{\partial \mu_i}, \frac{\partial PU_{x_j,\mu_j}}{\partial x_{jl}}\right\rangle = \left\langle \frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial \phi_{\nu,x,\mu}}, \frac{\partial PU_{x_j,\mu_j}}{\partial x_{jl}}\right\rangle. \tag{3.11}$$

On the other hand, by a direct computation, we have

$$\left\langle \frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial \phi_{\nu,x,\mu}}, \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}\right\rangle = \frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial \mu_j} \tag{3.12}$$

and

$$\left\langle \frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial \phi_{\nu,x,\mu}}, \frac{\partial PU_{x_j,\mu_j}}{\partial x_{jl}}\right\rangle = \frac{\partial J_\nu(x,\mu,\phi_{\nu,x,\mu})}{\partial x_{jl}} \tag{3.13}$$

Combining (3.10)–(3.13) with lemma 3.1, we can complete the proof. □

Based on lemma 3.1 and lemma 3.2, we can conclude the following conclusion.

PROPOSITION 3.3. *Let $(x,\mu)\in D_{k,\nu}$. Then*

$$\frac{\partial K(x,\mu)}{\partial \mu_j} = \frac{2s\lambda A_2}{\mu_j^{2s+1}} - \frac{N-2s}{2}\frac{\varphi_1(x_j)A_3\nu}{\mu_j^{(N-2s/2)+1}} + \frac{1}{\mu_j}O\left(\frac{1}{\nu^{(4s(1+\sigma)/N-6s)}}\right), \tag{3.14}$$

where $\sigma>0$ is a small constant.

Define

$$f(t) = -\frac{\lambda A_2}{t^{2s}} + \frac{A_3}{t^{(N-2s/2)}}.$$

Then it is easy to check that $f(t)$ has unique minimum point

$$t_0 = \left(\frac{A_3(N-2s)}{4s\lambda A_2}\right)^{2/(N-6s)}$$

on $(0,\infty)$. Let

$$\alpha_2 = kA_1+\eta, \quad \alpha_1 = kA_1 + kf(t_0)\nu^{-(4s/N-6s)} - \nu^{-(4s/N-6s)-(3s/2)\tau},$$

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where $\eta > 0$ is a small fixed constant. Denote

$$K^{\alpha} = \left\{ (x, \mu) \in D_{k,\nu}, K(x, \mu) \leqslant \alpha \right\}.$$

Consider the following flow

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = -D_x K(x(t), \mu(t)), & t > 0, \\ \dfrac{\mathrm{d}\mu(t)}{\mathrm{d}t} = -D_\mu K(x(t), \mu(t)), & t > 0, \\ (x(0), \mu(0)) = (x_0, \mu_0) \in K^{\alpha_2}. \end{cases}$$

We have

Proposition 3.4. *Let $N > 6s$. Then the flow does not leave $D_{k,\nu}$ before it reaches K^{α_1}.*

Proof. Denote

$$\mu_j = t_j \nu^{2/(N-6s)}, \quad t_j \in [t_0 - L\nu^{-s\tau}, t_0 + L\nu^{-s\tau}], \quad j = 1, \ldots, k.$$

Suppose that $\mu_j = (t_0 + L\nu^{-s\tau})\nu^{2/(N-6s)}$ for some j. Then by (3.14), we have

$$\begin{aligned} \frac{\partial K(x, \mu)}{\partial \mu_j} &= f'(t_j)\nu^{-(4s+2/N-6s)} + O\left((1 - \varphi_1(x_j))\nu^{-(4s+2/N-6s)} \right) \\ &\quad + O\left(\nu^{-(2+4s+4s\sigma/N-6s)} \right) \\ &= f'(t_j)\nu^{-(4s+2/N-6s)} + O\left(\nu^{-(4s+2/N-6s)-s\tau} \right) \\ &= f''(t_0)\nu^{-(4s+2/N-6s)} L\nu^{-s\tau} + O\left(L^2 \nu^{-2s\tau} \nu^{-(4s+2/N-6s)} \right) \\ &\quad + O\left(\nu^{-(4s+2/N-6s)-s\tau} \right) > 0, \end{aligned}$$

if we choose $L > 0$ is large.

Similarly, if $\mu_j = (t_0 - L\nu^{-s\tau})\nu^{2/(N-6s)}$ for some j. Then by (3.14), we have

$$\begin{aligned} \frac{\partial K(x, \mu)}{\partial \mu_j} &= -f''(t_0)\nu^{-(4s+2/N-6s)} L\nu^{-s\tau} + O\left(L^2 \nu^{-2s\tau} \nu^{-(4s+2/N-6s)} \right) \\ &\quad + O\left(\nu^{-(4s+2/N-6s)-s\tau} \right) < 0. \end{aligned}$$

So the flow does not leave $D_{k,\nu}$.

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Now, we suppose that $|x_i - x_j|^{N-2s} = \nu^{-(2N-8s)/(N-6s)+s\tau}$. Then we have

$$\varepsilon_{ij} \geqslant C_1 \nu^{-(4s/N-6s)-s\tau}.$$

Thus, applying (3.1), we can derive

$$\begin{aligned} K(x,\mu) &\leqslant kA_1 - \sum_{j=1}^{k}\left(\frac{\lambda A_2}{\mu_j^{2s}} - \frac{A_3\nu}{\mu_j^{(N-2s/2)}}\right) \\ &\quad - \sum_{i\neq j}^{k}\frac{1}{2}b_0(A_3 + b_0\lambda A(x_i,x_j))\varepsilon_{ij} + O\left(\nu^{-(4s+4s\sigma/N-6s)}\right) \\ &\leqslant kA_1 + \sum_{j=1}^{k} f(t_j)\nu^{(4s/N-6s)} - C_1\nu^{-(4s/N-6s)-s\tau} + O\left(\nu^{-(4s/N-6s)-2s\tau}\right) \\ &= kA_1 + kf(t_0)\nu^{-(4s/N-6s)} + O\left(\nu^{-(4s/N-6s)}L^2\nu^{-2s\tau}\right) \\ &\quad - C_1\nu^{-(4s/N-6s)-s\tau} + O\left(\nu^{-(4s/N-6s)-2s\tau}\right) < \alpha_1, \end{aligned} \tag{3.15}$$

where the last equality follows from the fact that $f(t_j) = f(t_0) + O(|t - t_j|^2)$.

On the other hand, if $|\varphi_1(x_j) - 1| = \nu^{-s\tau}$, then using (3.1), we can obtain

$$\begin{aligned} K(x,\mu) &= kA_1 - \sum_{j=1}^{k}\left(\frac{\lambda A_2}{\mu_j^{2s}} - \frac{A_3\nu}{\mu_j^{(N-2s/2)}}\right) - \sum_{j=1}^{k}\frac{1-\varphi_1(x_j)A_3\nu}{\mu_j^{(N-2s/2)}} \\ &\quad - \sum_{i\neq j}^{k}\frac{1}{2}b_0(A_3 + b_0\lambda A(x_i,x_j))\varepsilon_{ij} + O\left(\nu^{-(4s+4s\sigma/N-6s)}\right) \\ &= kA_1 + \sum_{j=1}^{k} f(t_j)\nu^{-(4s/N-6s)} - \sum_{j=1}^{k}\frac{A_3}{t_j^{(N-2s/2)}}\nu^{-(4s/N-6s)-s\tau} \\ &\quad - \sum_{i\neq j}^{k}\frac{1}{2}b_0(A_3 + b_0\lambda A(x_i,x_j))\varepsilon_{ij} + O\left(\nu^{-(4s+4s\sigma/N-6s)}\right) \\ &\leqslant kA_1 + \sum_{j=1}^{k} f(t_j)\nu^{-(4s/N-6s)} - \sum_{j=1}^{k}\frac{A_3}{t_j^{(N-2s/2)}}\nu^{-(4s/N-6s)-s\tau} \\ &\quad + O\left(\nu^{-(4s/N-6s)-2s\tau}\right) \end{aligned}$$

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$$= kA_1 + kf(t_0)\nu^{-(4s/N-6s)} + O\left(\nu^{-(4s/N-6s)}L^2\nu^{-2s\tau}\right)$$

$$- \sum_{j=1}^{k} \frac{A_3}{t_j^{(N-2s/2)}}\nu^{-(4s/N-6s)-s\tau} + O\left(\nu^{-(4s/N-6s)-2s\tau}\right) < \alpha_1. \quad (3.16)$$

Thus, we complete the proof. □

Proof of theorem 1.2 We will prove that $K(x, \mu)$ has a critical point in $D_{k,\nu}$. Define

$$\Gamma = \left\{h : h(x, \mu) = (h_1(x, \mu), h_2(x, \mu)) \in D_{k,\nu}, (x, \mu) \in D_{k,\nu}\right\},$$

where $h_1(x, \mu) = x$ if $x \in \partial D^1_{k,\nu}$. Here we denote $D^1_{k,\nu} = \{x : (x, \mu) \in D_{k,\nu}\}$ and define the boundary of $D^1_{k,\nu}$ by $\partial D^1_{\nu,k}$. Furthermore, we denote $D^2_{k,\nu} = \{\mu : (x, \mu) \in D_{k,\nu}\}$.

Let

$$c_\nu = \inf_{h\in\Gamma} \max_{(x,\mu)\in D_{k,\nu}} K(h(x, \mu)).$$

We claim that c_ν is a critical value of K. In order to prove this claim, it is sufficient to prove that

(i) $\alpha_1 < c_\nu < \alpha_2$,

(ii) $\sup_{(x,\mu)\in\partial D^1_{\nu,k}\times D^2_{\nu,k}} K(h(x, \mu)) < \alpha_1, \forall\, h \in \Gamma$.

Obviously, (ii) directly follows from (3.15) and (3.16).

Now, we prove (i). Using (3.1), we can easily check that $c_\nu < \alpha_2$.

For any $h = (h_1, h_2) \in \Gamma$, by the definition of h, we have $h_1(x, \mu) = x$ if $x \in \partial D^1_{\nu,k}$. Define

$$\tilde{h}_1(x) = h_1(x, t_0\nu^{-2/(N-6s)}).$$

Then $\tilde{h}_1(x) = x$ for any $x \in \partial D^1_{\nu,k}$. Thus, by degree argument, we can obtain

$$\deg(\tilde{h}_1, D^1_{\nu,k}, \xi) = 1, \quad \forall\, \xi \in D^1_{\nu,k}.$$

Hence, for any $\xi \in D^1_{\nu,k}$, there exists $\tilde{x} \in D^1_{\nu,k}$ such that

$$\tilde{h}_1(\tilde{x}) = \xi.$$

Let $\tilde{\mu} = h_2(\tilde{x}, t_0\nu^{-2/(N-6s)})$. Then we have

$$\max_{(x,\mu)\in D_{\nu,k}} K(h(x, \mu)) \geqslant K(h(\tilde{x}, t_0\nu^{-2/(N-6s)})) = K(\xi, \tilde{\mu}).$$

Choose $\xi_j \in \Omega$ such that $dist(\xi_j, S) \leqslant \nu^{-s\tau}$ and $|\xi_i - \xi_j| \geqslant c_1\nu^{-s\tau}$, where c_1 is a small positive constant. By a direct computation, we have

$$\tilde{\varepsilon}_{ij} := \frac{1}{\mu_i^{(N-2s/2)}\mu_j^{(N-2s/2)}|\xi_i - \xi_j|^{N-2s}} = O\left(\nu^{(N-2s)\tau-(2(N-2s)/N-6s)}\right). \quad (3.17)$$

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Combining (3.1) with (3.17), we obtain

$$\begin{aligned}
K(\xi,\mu) &= kA_1 - \sum_{j=1}^{k}\left(\frac{\lambda A_2}{\mu_j^{2s}} - \frac{A_3\nu}{\mu_j^{(N-2s/2)}}\right) - \sum_{j=1}^{k}\frac{1-\varphi_1(x_j)A_3\nu}{\mu_j^{(N-2s/2)}}\\
&\quad - \sum_{i\neq j}^{k}\frac{1}{2}b_0(A_3+b_0\lambda A(x_i,x_j))\tilde{\varepsilon}_{ij} + O\left(\nu^{-(4s+4s\sigma/N-6s)}\right)\\
&= kA_1 + \sum_{j=1}^{k} f(t_j)\nu^{-(4s/N-6s)} + O(\nu^{-(4s/N-6s)-2s\tau})\\
&\quad + O\left(\nu^{(N-2s)\tau-(2(N-2s)/N-6s)}\right)\\
&= kA_1 + kf(t_0)\nu^{-(4s/N-6s)} + O\left(\nu^{-(4s/N-6s)}L^2\nu^{-2s\tau}\right)\\
&\quad + O(\nu^{-(4s/N-6s)-2s\tau})\\
&\geqslant \alpha_1 + \frac{1}{2}\nu^{-(4s/N-6s)-(3s/2)\tau} > \alpha_1.
\end{aligned}$$

Consequently, we complete the proof.

Appendix A. Basic estimate

In this section, we always suppose that $dist(x_j,\partial\Omega)\geqslant\delta>0$, where δ is a small constant. Let $G(y,z)$ be the Green's function of $(-\Delta)^s$ in Ω. That is, G satisfies

$$\begin{cases}(-\Delta)^s G(y,\cdot)=\delta_y, & \text{in } \Omega,\\ G(y,\cdot)=0, & \text{in } \mathfrak{R}^N\backslash\Omega,\end{cases}$$

where δ_y denotes the Dirac mass at the point y. The regular part of G is given by

$$H(y,z)=\Gamma(y,z)-G(y,z),$$

where $\Gamma(y,z)$ is given by

$$\Gamma(y,z)=\frac{C_{N,s}}{|y-z|^{N-2s}}.$$

Let

$$\Psi_{x_j,\mu_j}=U_{x_j,\mu_j}-PU_{x_j,\mu_j}.$$

Then we have

Lemma A.1. *It holds*

$$0\leqslant PU_{x_j,\mu_j}\leqslant U_{x_j,\mu_j},\quad 0\leqslant\Psi_{x_j,\mu_j}\leqslant U_{x_j,\mu_j}.$$

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Proof. By proposition 2.4 in [13], we can see that $G(y,z)\geqslant 0$ and $H(y,z)\geqslant 0$. Then

$$PU_{x_j,\mu_j}=\int_\Omega G(y,z)U_{x_j,\mu_j}^{2_s^*-1}dz\geqslant 0$$

and

$$\begin{aligned}\Psi_{x_j,\mu_j}&=\int_{\mathfrak{R}^N}\Gamma(y,z)U_{x_j,\mu_j}^{2_s^*-1}\mathrm{d}z-\int_\Omega G(y,z)U_{x_j,\mu_j}^{2_s^*-1}\mathrm{d}z\\&=\int_{\mathfrak{R}^N\setminus\Omega}\Gamma(y,z)U_{x_j,\mu_j}^{2_s^*-1}\mathrm{d}z+\int_\Omega H(y,z)U_{x_j,\mu_j}^{2_s^*-1}\mathrm{d}z\geqslant 0.\end{aligned}$$

□

Lemma A.2. *We have*

$$\Psi_{x_j,\mu_j}=\frac{c_0}{\mu_j^{(N-2s/2)}}(H(y,x_j)+o(1)),\tag{A.1}$$

$$\frac{\partial\Psi_{x_j,\mu_j}}{\partial\mu_j}=-\frac{N-2s}{2}\frac{c_0}{\mu_j\mu_j^{(N-2s/2)}}(H(y,x_j)+o(1)),\tag{A.2}$$

$$\frac{\partial\Psi_{x_j,\mu_j}}{\partial x_{ji}}=\frac{c_0}{\mu_j^{(N-2s/2)}}\left(\frac{\partial H(y,x_j)}{\partial x_{ji}}+o(1)\right),\tag{A.3}$$

where $j=1,\dots,k$, $i=1,\dots,N$ *and* $c_0=\int_{\mathfrak{R}^N}U^{2_s^*-1}$.

Proof. Note that

$$U_{x_j,\mu_j}(y)=\int_{\mathfrak{R}^N}\Gamma(y,z)U_{x_j,\mu_j}^{2_s^*-1}(z)\mathrm{d}z,$$

and

$$PU_{x_j,\mu_j}(y)=\int_\Omega G(y,z)U_{x_j,\mu_j}^{2_s^*-1}(z)\mathrm{d}z.$$

Then, we have

$$\begin{aligned}\Psi_{x_j,\mu_j}&=\int_{\mathfrak{R}^N}\Gamma(y,z)U_{x_j,\mu_j}^{2_s^*-1}(z)\mathrm{d}z-\int_\Omega G(y,z)U_{x_j,\mu_j}^{2_s^*-1}(z)\mathrm{d}z\\&=\int_{\mathfrak{R}^N\setminus\Omega}\Gamma(y,z)U_{x_j,\mu_j}^{2_s^*-1}(z)\mathrm{d}z+\int_\Omega H(y,z)U_{x_j,\mu_j}^{2_s^*-1}(z)\mathrm{d}z\\&=O\left(\frac{1}{\mu_j^{(N+2s/2)}}\int_{\mathfrak{R}^N\setminus\Omega}\frac{1}{|z-y|^{N-2s}}\frac{1}{|z-x_j|^{N+2s}}\mathrm{d}z\right)\\&\quad+\frac{1}{\mu_j^{(N-2s/2)}}\int_{\Omega_j}H(y,\mu_j^{-1}z+x_j)U^{2_s^*-1}(z)\mathrm{d}z\end{aligned}$$

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$$= O\left(\frac{1}{\mu_j^{(N+2s/2)}}\right) + \frac{H(y,x_j)}{\mu_j^{(N-2s/2)}}\int_{\mathfrak{R}^N} U^{2_s^*-1}$$
$$+ O\left(\frac{1}{\mu_j^{(N-2s/2)}}\int_{\Omega_j}\frac{\mu_j^{-1}|z|}{(1+|z|)^{N+2s}\mathrm{d}z}\right)$$
$$= O\left(\frac{1}{\mu_j^{(N+2s/2)}}\right) + \frac{H(y,x_j)}{\mu_j^{(N-2s/2)}}\int_{\mathfrak{R}^N} U^{2_s^*-1} + O\left(\frac{1}{\mu_j^{(N/2)}}\right),$$

the last equality follows from (B.3), where $\Omega_j = \{z : \mu_j^{-1}z + x_j \in \Omega\}$. By a similar argument above, we can prove (A.2) and (A.3). □

Appendix B. Energy expand

In this section, we will expand $J_\nu(x,\mu,0)$ and its derivatives. First, we recall that the energy function corresponding to equation (1.4) is given by

$$I_\nu(v) = \frac{1}{2}\int_{\mathfrak{R}^N}\int_{\mathfrak{R}^N}\frac{|v(y)-v(z)|^2}{|y-z|^{N+2s}}\mathrm{d}y\mathrm{d}z - \frac{1}{2}\int_\Omega \lambda v^2 - \frac{1}{2_s^*}\int_\Omega (v-\nu\varphi_1)_+^{2_s^*}.$$

LEMMA B.1. *We have*

$$I_\nu(PU_{x_j,\mu_j}) = A_1 - \frac{\lambda A_2}{\mu_j^{2s}} + \frac{\varphi_1(x_j)A_3\nu}{\mu_j^{(N-2s/2)}} + O\left(\frac{1}{\mu_j^{N-2s}}\right) + O\left(\frac{\nu}{\mu_j^{(N/2)}}\right)$$
$$+ O\left(\frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}}\right),$$

where σ is a small positive constant and A_1, A_2, A_3 are defined by

$$A_1 = \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\int_{\mathfrak{R}^N} U^{2_s^*},\quad A_2 = \frac{1}{2}\int_{\mathfrak{R}^N} U^2,\quad A_3 = \int_{\mathfrak{R}^N} U^{2_s^*-1}.$$

Proof. By a direct computation, we have

$$I_\nu(PU_{x_j,\mu_j}) = \frac{1}{2}\int_\Omega U_{x_j,\mu_j}^{2_s^*-1}PU_{x_j,\mu_j} - \frac{\lambda}{2}\int_\Omega (PU_{x_j,\mu_j})^2 - \frac{1}{2_s^*}\int_\Omega (PU_{x_j,\mu_j}-\nu\varphi_1)_+^{2_s^*}$$
$$= \frac{1}{2}\int_\Omega U_{x_j,\mu_j}^{2_s^*} - \frac{1}{2}\int_\Omega U_{x_j,\mu_j}^{2_s^*-1}\Psi_{x_j,\mu_j} - \frac{\lambda}{2}\int_\Omega U_{x_j,\mu_j}^2$$
$$+ O\left(\int_\Omega U_{x_j,\mu_j}\Psi_{x_j,\mu_j}\right)$$
$$- \frac{1}{2_s^*}\int_\Omega (U_{x_j,\mu_j}-\nu\varphi_1)_+^{2_s^*} + O\left(\int_\Omega (U_{x_j,\mu_j}-\nu\varphi_1)_+^{2_s^*-1}\Psi_{x_j,\mu_j}\right)$$

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$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathfrak{R}^N} U^{2_s^*} - \frac{\lambda}{2\mu_j^{2s}}\int_{\mathfrak{R}^N} U^2 + O\bigg(\frac{1}{\mu_j^{N-2s}}\bigg) \\
&\quad + O\bigg(\frac{1}{\mu_j^{N-2s}}\int_\Omega \frac{1}{|y-x_j|^{N-2s}}\bigg) \\
&\quad - \frac{1}{2_s^*}\int_\Omega (U_{x_j,\mu_j} - \nu\varphi_1)_+^{2_s^*} + O\bigg(\int_\Omega U_{x_j,\mu_j}^{2_s^*-1}\Psi_{x_j,\mu_j}\bigg) \\
&= \frac{1}{2}\int_{\mathfrak{R}^N} U^{2_s^*} - \frac{\lambda A_2}{\mu_j^{2s}} - \frac{1}{2_s^*}\int_\Omega (U_{x_j,\mu_j} - \nu\varphi_1)_+^{2_s^*} + O\bigg(\frac{1}{\mu_j^{N-2s}}\bigg). \qquad \text{(B.1)}
\end{aligned}
$$

Let $R > 0$ such that $\Omega \subset B_R(x_j)$. Choose σ small such that $N + 2s - \sigma(N-2s) > N$. Using the following estimate

$$
(a-b)_+^{2_s^*} = a^{2_s^*} - 2_s^* a^{2_s^*-1} b + O(a^{2_s^*-1-\sigma} b^{1+\sigma}), \quad a, b > 0,
$$

we find

$$
\begin{aligned}
\int_\Omega (U_{x_j,\mu_j} - \nu\varphi_1)_+^{2_s^*} &= \int_\Omega U_{x_j,\mu_j}^{2_s^*} - 2_s^* \int_\Omega U_{x_j,\mu_j}^{2_s^*-1}\nu\varphi_1 + O\bigg(\nu^{1+\sigma}\int_\Omega U_{x_j,\mu_j}^{2_s^*-1-\sigma}\bigg) \\
&= \int_{\mathfrak{R}^N} U^{2_s^*} - \frac{2_s^*\varphi_1(x_j)\nu}{\mu_j^{(N-2s/2)}}\int_{\mathfrak{R}^N} U^{2_s^*-1} + O\bigg(\frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}}\bigg) \\
&\quad + \frac{\nu}{\mu_j^{(N-2s/2)}} O\bigg(\int_{B_{\mu_j R(0)}} \frac{\mu_j^{-1}|y|}{1+|y|^{N+2s}}\bigg) \\
&= \int_{\mathfrak{R}^N} U^{2_s^*} - \frac{2_s^*\varphi_1(x_j)A_3\nu}{\mu_j^{(N-2s/2)}} \\
&\quad + O\bigg(\frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}}\bigg) + O\bigg(\frac{\nu}{\mu_j^{(N/2)}}\bigg), \qquad \text{(B.2)}
\end{aligned}
$$

the last equality is due to the following estimate

$$
\int_{B_{\mu_j R(0)}} \frac{\mu_j^{-1}|y|}{1+|y|^{N+2s}} = \begin{cases} \mu_j^{-1} & \text{if } \frac{1}{2} < s < 1, \\ \mu_j^{-1}\log\mu_j & \text{if } s = \frac{1}{2}, \\ \mu_j^{-2s} & \text{if } 0 < s < \frac{1}{2}. \end{cases} \qquad \text{(B.3)}
$$

The result directly follows from (B.1) and (B.2). □

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LEMMA B.2. *We have*

$$I_\nu\left(\sum_{j=1}^k PU_{x_j,\mu_j}\right) = kA_1 - \sum_{j=1}^k\left(\frac{\lambda A_2}{\mu_j^{2s}} - \frac{\varphi_1(x_j)A_3\nu}{\mu_j^{(N-2s/2)}}\right)$$
$$- \sum_{i\neq j}^k \frac{1}{2}b_0(A_3 + b_0\lambda A(x_i,x_j))\varepsilon_{ij}$$
$$+ O\left(\sum_{j=1}^k\left(\frac{1}{\mu_j^{N-2s}} + \frac{\nu}{\mu_j^{\frac{N}{2}}} + \frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}}\right) + \sum_{i\neq j}\varepsilon_{ij}^{1+\sigma}\right),$$

where A_2, A_3 *are defined in lemma* B.1, σ *is a small positive constant and* $A(x_i,x_j)$ *is defined by*

$$A(x_i,x_j) = \int_\Omega\left(\frac{1}{|y-x_i|^{N-2s}} + \frac{1}{|y-x_j|^{N-2s}}\right) < \infty.$$

Proof. By a direct computation, we have

$$I_\nu\left(\sum_{j=1}^k PU_{x_j,\mu_j}\right) = \sum_{j=1}^k I_\nu(PU_{x_j,\mu_j}) + \frac{1}{2}\sum_{i\neq j}\int_\Omega U_{x_j,\mu_j}^{2_s^*-1}PU_{x_i,\mu_i}$$
$$- \frac{\lambda}{2}\sum_{i\neq j}\int_\Omega PU_{x_j,\mu_j}PU_{x_i,\mu_i}$$
$$- \frac{1}{2_s^*}\int_\Omega\left(\left(\sum_{j=1}^k PU_{x_j,\mu_j} - \nu\varphi_1\right)_+^{2_s^*} - \sum_{j=1}^k(PU_{x_j,\mu_j} - \nu\varphi_1)_+^{2_s^*}\right). \tag{B.4}$$

Noting that for $i \neq j$, we have

$$\int_\Omega U_{x_j,\mu_j}^{2_s^*-1}PU_{x_i,\mu_i} = \int_\Omega U_{x_j,\mu_j}^{2_s^*-1}U_{x_i,\mu_i} - \int_\Omega U_{x_j,\mu_j}^{2_s^*-1}\Psi_{x_i,\mu_i}$$
$$= \int_\Omega U_{x_j,\mu_j}^{2_s^*-1}U_{x_i,\mu_i} + O\left(\frac{1}{\mu_j^{(N-2s/2)}\mu_i^{(N-2s/2)}}\right)$$
$$= \frac{b_0}{\mu_j^{(N-2s/2)}\mu_i^{(N-2s/2)}|x_i-x_j|^{N-2s}}\int_{\mathfrak{R}^N} U^{2_s^*-1}$$
$$+ O\left(\frac{1}{\mu_j^{(N-2s/2)}\mu_i^{(N-2s/2)}}\right)$$
$$= b_0A_3\varepsilon_{ij} + O\left(\frac{1}{\mu_j^{(N-2s/2)}\mu_i^{(N-2s/2)}}\right), \tag{B.5}$$

and

$$\begin{aligned}\int_\Omega PU_{x_j,\mu_j}PU_{x_i,\mu_i} &= \int_\Omega U_{x_j,\mu_j}U_{x_i,\mu_i} + O\left(\frac{1}{\mu_j^{(N-2s/2)}\mu_i^{(N-2s/2)}}\right)\\ &= b_0^2\varepsilon_{ij}\int_\Omega\left(\frac{1}{|y-x_i|^{N-2s}}+\frac{1}{|y-x_j|^{N-2s}}\right)\\ &\quad + O\left(\frac{1}{\mu_j^{(N-2s/2)}\mu_i^{(N-2s/2)}}\right)\\ &= b_0^2\varepsilon_{ij}A(x_i,x_j) + O\left(\frac{1}{\mu_j^{(N-2s/2)}\mu_i^{(N-2s/2)}}\right). \end{aligned} \tag{B.6}$$

Now we estimate the last term in (B.4). By a similar computation as (B.2), we obtain that

$$\begin{aligned}&\int_\Omega\left(\left(\sum_{j=1}^k PU_{x_j,\mu_j}-\nu\varphi_1\right)_+^{2_s^*} - \sum_{j=1}^k (PU_{x_j,\mu_j}-\nu\varphi_1)_+^{2_s^*}\right)\\ &= \int_\Omega\left(\left(\sum_{j=1}^k U_{x_j,\mu_j}-\nu\varphi_1\right)_+^{2_s^*} - \sum_{j=1}^k (U_{x_j,\mu_j}-\nu\varphi_1)_+^{2_s^*}\right) + O\left(\sum_{j=1}^k\frac{1}{\mu_j^{N-2s}}\right)\\ &= \int_\Omega\left(\sum_{j=1}^k U_{x_j,\mu_j}\right)^{2_s^*} - 2_s^*\int_\Omega\left(\sum_{j=1}^k U_{x_j,\mu_j}\right)^{2_s^*-1}\nu\varphi_1 + O\left(\sum_{j=1}^k\frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}}\right)\\ &\quad - \sum_{j=1}^k\int_\Omega\left(U_{x_j,\mu_j}^{2_s^*} - 2_s^* U_{x_j,\mu_j}^{2_s^*-1}\nu\varphi_1\right) + O\left(\sum_{j=1}^k\frac{1}{\mu_j^{N-2s}}\right)\\ &= \int_\Omega\left(\left(\sum_{j=1}^k U_{x_j,\mu_j}\right)^{2_s^*} - \sum_{j=1}^k\int_\Omega\left(\sum_{j=1}^k U_{x_j,\mu_j}\right)^{2_s^*}\right) + O\left(\sum_{j=1}^k\frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}}\right)\\ &\quad - 2_s^*\int_\Omega\left(\left(\sum_{j=1}^k U_{x_j,\mu_j}\right)^{2_s^*-1} - \sum_{j=1}^k U_{x_j,\mu_j}^{2_s^*-1}\right)\nu\varphi_1 + O\left(\sum_{j=1}^k\frac{1}{\mu_j^{N-2s}}\right)\\ &= 2_s^*\sum_{i\neq j}^k\int_\Omega U_{x_j,\mu_j}^{2_s^*-1}U_{x_i,\mu_i} + O\left(\sum_{i\neq j}^k\int_\Omega U_{x_j,\mu_j}^{\frac{2_s^*}{2}}U_{x_i,\mu_i}^{\frac{2_s^*}{2}}\right)\\ &\quad + O\left(\sum_{j=1}^k\frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}}\right) + O\left(\int_\Omega\sum_{i\neq j}^k U_{x_j,\mu_j}^{\frac{2_s^*-1}{2}}U_{x_i,\mu_i}^{\frac{2_s^*-1}{2}}\nu\varphi_1\right)\\ &\quad + O\left(\sum_{j=1}^k\frac{1}{\mu_j^{N-2s}}\right)\end{aligned}$$

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$$= \sum_{i\neq j}^{k} b_0 2_s^* A_3 \varepsilon_{ij} + O\left(\sum_{i\neq j}^{k} \frac{\log(\mu_i\mu_j)}{\mu_j^{(N/2)}\mu_i^{(N/2)}|x_i - x_j|^N}\right)$$
$$+ O\left(\sum_{j=1}^{k} \frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}}\right)$$
$$+ O\left(\sum_{i\neq j}^{k} \frac{\nu}{\mu_j^{(N+2/4)}\mu_i^{(N+2/4)}|x_i - x_j|^{(N+2/2)}}\right) + O\left(\sum_{j=1}^{k} \frac{1}{\mu_j^{N-2s}}\right). \tag{B.7}$$

The result follows from lemma B.1 and (B.4)–(B.7). □

LEMMA B.3. *We have*

$$\frac{\partial J_\nu(x,\mu,0)}{\partial \mu_j} = \frac{2s\lambda A_2}{\mu_j^{2s+1}} - \frac{N-2s}{2}\frac{\varphi_1(x_j)A_3\nu}{\mu_j^{(N-2s/2)+1}}$$
$$+ \frac{1}{\mu_j} O\left(\frac{1}{\mu_j^{N-2s}} + \frac{\nu}{\mu_j^{(N/2)}} + \frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}} + \sum_{i\neq j}\varepsilon_{ij}\right),$$

where A_2, A_3 are defined in lemma B.1 and $\sigma > 0$ is a small constant.

Proof. By a direct computation, we have

$$\frac{\partial J_\nu(x,\mu,0)}{\partial \mu_j} = \left\langle I'_\nu\left(\sum_{i=1}^{k} PU_{x_i,\mu_i}\right), \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}\right\rangle$$
$$= \int_\Omega \sum_{i=1}^{k} U_{x_i,\mu_i}^{2_s^*-1}\frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j} - \lambda \sum_{i=1}^{k}\int_\Omega PU_{x_i,\mu_i}\frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}$$
$$- \int_\Omega \left(\sum_{i=1}^{k} PU_{x_i,\mu_i} - \nu\varphi_1\right)_+^{2_s^*-1}\frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}$$
$$= \int_\Omega U_{x_j,\mu_j}^{2_s^*-1}\frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j} - \lambda\int_\Omega PU_{x_j,\mu_j}\frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}$$
$$+ O\left(\sum_{i\neq j}\int_\Omega PU_{x_i,\mu_i}\frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}\right)$$
$$- \int_\Omega (PU_{x_j,\mu_j} - \nu\varphi_1)_+^{2_s^*-1}\frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j} + O\left(\sum_{i\neq j}\int_\Omega U_{x_i,\mu_i}^{2_s^*-1}\frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j}\right)$$

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$$
\begin{aligned}
&= \int_\Omega U_{x_j,\mu_j}^{2_s^*-1} \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j} - \lambda \int_\Omega PU_{x_j,\mu_j} \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j} + O\left(\sum_{i\neq j} \frac{\varepsilon_{ij}}{\mu_j} \right) \\
&\quad - \int_\Omega (PU_{x_j,\mu_j} - \nu\varphi_1)_+^{2_s^*-1} \frac{\partial PU_{x_j,\mu_j}}{\partial \mu_j} \\
&= \int_\Omega U_{x_j,\mu_j}^{2_s^*-1} \frac{\partial U_{x_j,\mu_j}}{\partial \mu_j} - \lambda \int_\Omega U_{x_j,\mu_j} \frac{\partial U_{x_j,\mu_j}}{\partial \mu_j} \\
&\quad - \int_\Omega (U_{x_j,\mu_j} - \nu\varphi_1)_+^{2_s^*-1} \frac{\partial U_{x_j,\mu_j}}{\partial \mu_j} \\
&\quad + \frac{1}{\mu_j} O\left(\frac{1}{\mu_j^{N-2s}} \right) + O\left(\sum_{i\neq j} \frac{\varepsilon_{ij}}{\mu_j} \right) \\
&= \frac{2s\lambda A_2}{\mu_j^{2s+1}} + (2_s^*-1) \int_\Omega U_{x_j,\mu_j}^{2_s^*-2} \frac{\partial U_{x_j,\mu_j}}{\partial \mu_j} \nu\varphi_1 + \frac{1}{\mu_j} O\left(\frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}} \right) \\
&\quad + \frac{1}{\mu_j} O\left(\frac{1}{\mu_j^{N-2s}} \right) + O\left(\sum_{i\neq j} \frac{\varepsilon_{ij}}{\mu_j} \right) \\
&= \frac{2s\lambda A_2}{\mu_j^{2s+1}} - \frac{N-2s}{2} \frac{A_3 \varphi_1(x_j)\nu}{\mu_j^{(N-2s/2)+1}} + O\left(\frac{\nu}{\mu_j \mu_j^{(N/2)}} \right) \\
&\quad + \frac{1}{\mu_j} O\left(\frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}} \right) + O\left(\sum_{i\neq j} \frac{\varepsilon_{ij}}{\mu_j} \right).
\end{aligned}
$$

□

Using the similar computation, we can conclude the following result.

LEMMA B.4. *We have*

$$
\frac{\partial J_\nu(x,\mu,0)}{\partial x_{ji}} = \mu_j O\left(\frac{1}{\mu_j^{N-2s}} + \frac{\nu}{\mu_j^{(N/2)}} + \frac{\nu^{1+\sigma}}{\mu_j^{((N-2s)(1+\sigma)/2)}} + \sum_{i\neq j} \varepsilon_{ij} \right).
$$

Acknowledgments

The authors would like to express their gratitude to the reviewers for careful reading and helpful suggestions which led to an improvement of the original manuscript. This research was supported by the National Natural Science Foundation of China (No.11831009; No.12071169).

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