

COMPUTING STRENGTH OF STRUCTURES RELATED TO THE FIELD OF REAL NUMBERS

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Abstract. In [8], the third author defined a reducibility \leq_w^* that lets us compare the computing power of structures of any cardinality. In [6], the first two authors showed that the ordered field of reals \mathcal{R} lies strictly above certain related structures. In the present paper, we show that $(\mathcal{R}, \text{exp}) \equiv_w^* \mathcal{R}$. More generally, for the weak-looking structure $\mathcal{R}_{\mathbb{Q}}$ consisting of the real numbers with just the ordering and constants naming the rationals, all o -minimal expansions of $\mathcal{R}_{\mathbb{Q}}$ are equivalent to \mathcal{R} . Using this, we show that for any analytic function f , $(\mathcal{R}, f) \equiv_w^* \mathcal{R}$. (This is so even if (\mathcal{R}, f) is not o -minimal.)

§1. Introduction. The behavior of structures in generic extensions of the universe has been studied from a number of different angles; for example, Baldwin, Laskowski, and Shelah [2] studied the conditions under which nonisomorphic structures may become isomorphic, and Knight, Montalbán, and Schwebel [8] (and independently Kaplan and Shelah [7]) studied structures existing in every generic extension of the universe by some forcing. In the latter paper, general results about such “generically presentable” structures led to a new proof of a result of Harrington saying that if T is a counterexample to Vaught’s Conjecture, then T has models of cardinality \aleph_1 with arbitrarily large Scott ranks less than ω_2 . (There are now at least three new proofs of this result. In addition to the one in [8], there is one by Baldwin, S.-D. Friedman, Koerwien, and Laskowski [1] and one by Larson [10]; these other proofs do not use generically presentable structures directly, but do use related ideas.)

We can do more with generic extensions. In [8], the third author defined a notion that lets us compare the computing power of structures of any cardinality.

DEFINITION 1.1 (Schwebel). Let \mathcal{A} and \mathcal{B} be structures in V (of any cardinality). We say that $\mathcal{A} \leq_w^* \mathcal{B}$ if in a generic extension $V(G)$ in which both \mathcal{A} and \mathcal{B} are countable, every copy of \mathcal{B} computes a copy of \mathcal{A} .

In [8], there are a few examples comparing familiar structures. In particular, it is shown that $\mathcal{W} \leq_w^* \mathcal{R}$, where \mathcal{R} is the ordered field of real numbers, and \mathcal{W} represents the power set of ω . We have $\mathcal{W} = (P(\omega) \cup \omega, P(\omega), \omega, \in, S)$, where S is the successor relation on ω . In computability, the structures \mathcal{R} and \mathcal{W} are

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sometimes identified; both are referred to as “the reals.” Of course, they are not the same structure: \mathcal{R} is an archimedean ordered field, while \mathcal{W} is just a family of subsets of ω .

Let \mathcal{R}^* be an ω -saturated extension of \mathcal{R} . In [6], it is shown that $\mathcal{R}^* \equiv_w^* \mathcal{W}$ and that $\mathcal{R} \not\leq_w^* \mathcal{R}^*$, so $\mathcal{R} \not\leq_w^* \mathcal{W}$. We recall a little of the proof. First, \mathcal{R} is a residue field section of \mathcal{R}^* . After collapse, \mathcal{R}^* is no longer ω -saturated, but it is recursively saturated, and it realizes just the types in the Scott set that is the old $P(\omega)$. It is shown that for a countable recursively saturated real closed field K , with residue field k , some copy of K does not compute a copy of k . The proof of this involves a reduction. It is shown that if every copy of K computes a copy of k , then the set $FT(K)$ consisting of finite elements that are not infinitesimally close to any algebraic element must be defined in K by a computable Σ_2 formula. It is then shown that $FT(K)$ has no such definition.

In the present paper, we consider further structures related to the reals. Let $\mathcal{R}_{exp} = (\mathcal{R}, exp)$. We show that $\mathcal{R}_{exp} \leq_w^* \mathcal{R}$. This was a surprise to the authors. We had expected \mathcal{R}_{exp} to be strictly more powerful than \mathcal{R} . We generalize this result, replacing \mathcal{R} by an apparently very weak structure $\mathcal{R}_{\mathbb{Q}}$ consisting of the real numbers with the ordering and constants naming the rationals, and replacing \mathcal{R}_{exp} by certain expansions of $\mathcal{R}_{\mathbb{Q}}$.

GENERAL THEOREM. *Let \mathcal{M} be an o -minimal expansion of $\mathcal{R}_{\mathbb{Q}}$, in a countable language, with definable Skolem functions. Then $\mathcal{M} \leq_w^* \mathcal{R}_{\mathbb{Q}}$. In fact, after collapse, every copy K of $\mathcal{R}_{\mathbb{Q}}$ computes the complete diagram of a copy of \mathcal{M} .*

Using the General Theorem, we prove that if $\mathcal{R}_f = (\mathcal{R}, f)$, where f is total analytic on \mathcal{R} , then $\mathcal{R}_f \equiv_w^* \mathcal{R}$. As further applications, we have the fact that $(\mathcal{R}, (r_n)_{n \in \omega}) \equiv_w^* \mathcal{R}$, where $(r_n)_{n \in \omega}$ is an arbitrary countable sequence of reals, and $\mathcal{R}^+ \equiv_w^* \mathcal{R}$, where \mathcal{R}^+ is the reduct of \mathcal{R} in which multiplication is dropped. This last result was obtained independently by Downey, Greenberg, and Miller [3].

It is clear that $\mathcal{W} \equiv_w^* \mathcal{C} = (2^\omega, (R_n)_{n \in \omega})$, where $f \in R_n$ iff $f(n) = 1$. The structure \mathcal{C} represents Cantor space. To represent Baire space, we may take $\mathcal{B} = (\omega^\omega, (R_{n,k})_{n,k \in \omega})$, where $f \in R_{n,k}$ iff $f(n) = k$. Downey, Greenberg, and Miller [3] showed that $\mathcal{B} \equiv_w^* \mathcal{R}$. From Baire space, we derive a structure \mathcal{R}_{int} , consisting of the real numbers, with the ordering, and predicates for the half-open intervals $[q, q')$, where q, q' are dyadic rationals with $q < q'$. It is not difficult to show that $\mathcal{B} \equiv_w^* \mathcal{R}_{int}$.

On its face, \mathcal{R}_{int} is even weaker than $\mathcal{R}_{\mathbb{Q}}$. Modifying the proof of the General Theorem, we could show that $\mathcal{R}_{\mathbb{Q}} \leq_w \mathcal{R}_{int}$. Hence, $\mathcal{R} \equiv_w^* \mathcal{R}_{int}$. Alternatively, we could modify our General Theorem, replacing $\mathcal{R}_{\mathbb{Q}}$ by \mathcal{R}_{int} . Applying this variant of the General Theorem, we would get the fact that $\mathcal{R} \leq_w^* \mathcal{R}_{int}$, so $\mathcal{R}_{\mathbb{Q}} \leq_w^* \mathcal{R}_{int}$. Before knowing the results in [3], the authors used this approach to show that $\mathcal{R} \equiv_w^* \mathcal{B}$.

In Section 2, we show that $\mathcal{R}_{exp} \equiv_w^* \mathcal{R}$. The proof combines ideas from computability (jumps and effective guessing strategies), computable structure theory (definability by computable infinitary formulas), and model theory (o -minimality). In Section 3, we generalize the result from Section 2 to prove the General Theorem. In Section 4, we apply the General Theorem from Section 3 to show that the expansions of \mathcal{R} by an analytic function or an arbitrary sequence of constants, and the reduct \mathcal{R}^+ are equivalent to \mathcal{R} . In Section 5, we show that $\mathcal{B} \equiv_w^* \mathcal{R}_{int}$.

We indicate briefly what would be involved in proving the modified version of the General Theorem, with $\mathcal{R}_{\mathbb{Q}}$ replaced by \mathcal{R}_{int} . In the remainder of the introduction, we give some background on o -minimality.

REMARK. If \mathcal{A} is an expansion of \mathcal{B} such that $\mathcal{A} \equiv_w^* \mathcal{B}$, it may not be the case that (in an appropriate generic extension) every copy of \mathcal{B} computes a copy of \mathcal{A} together with an isomorphism between the copy of \mathcal{B} and the reduct of the copy of \mathcal{A} . Indeed, that is the case with expansions of \mathcal{R} : for example, the functions that \mathcal{R} can compute in this sense are precisely the piecewise algebraic functions.

1.1. o -minimality.

DEFINITION 1.2. A structure \mathcal{M} with a dense linear ordering on the universe is o -minimal if each set definable by an elementary first order formula (with parameters) is a finite union of intervals (possibly trivial) with endpoints in \mathcal{M} .

The following is well-known [9].

PROPOSITION 1.3. *If T is the elementary first order theory of an o -minimal structure \mathcal{M} , then all models of T are o -minimal.*

We say that T is an o -minimal theory if some/all models of T are o -minimal.

EXAMPLES.

1. \mathcal{R} is o -minimal. Tarski [14] proved that $Th(\mathcal{R})$ is decidable. In the proof, Tarski gave an effective elimination of quantifiers. There is an algorithm (familiar to every school child) for deciding the truth of the quantifier-free sentences. As a side result, Tarski stated the fact that in \mathcal{R} , and the other models of the theory, the definable sets are finite unions of intervals.
2. \mathcal{R}^+ is o -minimal. It is clear from the definition that any reduct of an o -minimal structure that includes the ordering is o -minimal.
3. \mathcal{R}_{sin} is not o -minimal—think of the set of zeroes of $\sin(x)$.
4. \mathcal{R}_{exp} is o -minimal. Wilkie [17] showed that $T_{exp} = Th(\mathcal{R}_{exp})$ is model complete; i.e., if \mathcal{M}_1 and \mathcal{M}_2 are models of the theory, with $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $\mathcal{M}_1 \prec \mathcal{M}_2$. By results of Khovanskii [5], it follows that the theory is o -minimal.
5. \mathcal{R}_{an} is o -minimal, where this is the expansion of \mathcal{R} with the restrictions f_I of analytic functions f to compact intervals $I = [a, b]$. More precisely, f_I is the total function that agrees with f on I and has value 0 otherwise. By results of van den Dries [15], building on work of Gabrielov [4], \mathcal{R}_{an} is o -minimal.

We will use the following three facts. The first is due to van den Dries [17, p. 94].

PROPOSITION 1.4. *Any o -minimal expansion of \mathcal{R}^+ has definable Skolem functions.*

The second fact is due to Pillay [13].

PROPOSITION 1.5. *For an o -minimal theory with definable Skolem functions, definable closure is a good closure notion, satisfying the Exchange Property—if a is definable from \bar{b}, c and not from \bar{b} , then c is definable from \bar{b}, a .*

This means that independence, basis, and dimension are well-defined. The third fact is also due to Pillay [9].

PROPOSITION 1.6. *For an independent tuple \bar{b} in an o -minimal structure \mathcal{A} , if $\varphi(\bar{x})$ is a finitary formula true of \bar{b} , then there is an open box B around \bar{b} , with vertices having coordinates in \mathcal{A} , such that $\varphi(\bar{x})$ is valid on B .*

REMARK. If \mathcal{A} is an Archimedean model of $Th(\mathcal{R})$ (or $Th(\mathcal{R}^+)$), and \bar{a} is a tuple in an open box B with vertices having coordinates in \mathcal{A} , then there is another open box $B^* \subseteq B$ such that $\bar{a} \in B^*$ and B^* has vertices with rational coordinates. We refer to B^* as a *rational box*.

§2. $\mathcal{R}_{exp} \equiv_w^* \mathcal{R}$. In this section, our goal is to prove the following.

THEOREM 2.1. $\mathcal{R}_{exp} \equiv_w^* \mathcal{R}$.

Here is a brief overview of the proof. Clearly, $\mathcal{R} \leq_w^* \mathcal{R}_{exp}$. We show that $\mathcal{R}_{exp} \leq_w^* \mathcal{R}$. Let T_{exp} be the elementary first order theory of \mathcal{R}_{exp} . The theory T_{exp} may or may not be decidable, depending on Schanuel’s Conjecture, see [12]. In any case, the theory is coded in a real parameter. After collapse, let K be a copy of \mathcal{R} with universe a subset of ω . Since K is isomorphic to \mathcal{R} , there is an expansion K_{exp} satisfying T_{exp} . We show that this expansion is unique. Next, we show that independence is defined in a way that yields a basis for K_{exp} that is Δ_2^0 relative to K . Finally, we use a computable approximation to this basis in a finite injury priority construction in order to construct a copy of K_{exp} that is computable in K .

2.1. Expanding K to a model of T_{exp} . We first show that for a countable Archimedean real closed ordered field K with an added function f satisfying T_{exp} , the expansion is unique. Moreover, if the function exp^K is defined by a computable Π_1 formula. The same is true if we substitute for exp an arbitrary continuous function.

LEMMA 2.2 (Uniqueness). *Let f be a continuous function on the reals, and let $T_f = Th(\mathcal{R}, f)$. If K is a copy of \mathcal{R} (after collapse), then there is a unique expansion (K, f^K) satisfying T_f . Moreover, the function f^K is defined by a computable Π_1 formula with a real parameter r . Hence, it is Δ_2^0 relative to K .*

PROOF. Let $a \in K$. For each open interval I containing $f(a)$, and having rational endpoints, there is an open interval J containing a , also with rational endpoints, such that f (as a function on \mathcal{R}) maps J to I . For each such pair of intervals I and J , with rational endpoints, there is a sentence in T_f saying that f maps J into I . Then the function f^K must map J to I in K . This implies that $f^K(a)$ in K must match $f(a)$ in \mathcal{R} . This proves uniqueness.

Suppose r is a parameter coding T_f . This means that $r \in [0, 1]$, and in its “preferred” binary expansion, r has 1 in the k^{th} place iff k is the Gödel number of a sentence of T_{exp} . The preferred binary expansion has infinitely many 0’s. Most reals in the interval $[0, 1]$ have a unique binary expansion, with infinitely many 0’s and infinitely many 1’s. However, the dyadic rationals in the interval $[0, 1]$ have two binary expansions, one ending in an infinite string of 1’s, with only finitely many 0’s, and the other ending in an infinite string of 0’s, with only finitely many 1’s.

We show that there is a computable Π_1 formula, with parameter r , defining f^K in K . We have $y = f(x)$ iff for all pairs of rational intervals J and I such that T_f contains the sentence saying that $f : J \rightarrow I$, if $x \in J$, then $y \in I$. This is naturally expressed as the conjunction of finitary quantifier-free formulas over a set that is c.e. relative to T_f . We can replace this by a c.e. conjunction involving the parameter $r \in [0, 1]$. Let $c_k(u)$ be a finitary quantifier-free formula saying of $u \in [0, 1]$ that its preferred binary expansion has 1 in the k^{th} place. For all k , we define a finitary

quantifier-free formula $\rho_k(u, x, y)$. If k is the Gödel number of a sentence saying that $f : J \rightarrow I$, then $\rho_k(u, x, y)$ says $c_k(u) \rightarrow (x \in I \rightarrow y \in J)$, and if k is not the Gödel number of such a sentence, then $\rho_k(u, x, y) = \top$. Then the computable Π_1 formula $\bigwedge_k \rho_k(r, x, y)$ holds just in case $y = f(x)$. \dashv

REMARK. Not all Archimedean real closed ordered fields can be expanded to models of T_{exp} . In particular, since $e = exp(1)$ is transcendental, the ordered field of real algebraic numbers cannot be expanded in this way.

By Proposition 1.6, for a tuple \bar{a} that is independent in \mathcal{R}_{exp} , each formula $\varphi(\bar{x})$ true of \bar{a} is valid on a rational box B around \bar{a} . We need the converse of this.

LEMMA 2.3. *Let \bar{a} be a tuple of reals. Suppose that for every formula $\varphi(\bar{x})$ true of \bar{a} in \mathcal{R}_{exp} , there is a rational box B around \bar{a} such that T_{exp} contains the sentence saying that $\varphi(\bar{x})$ is valid on B . Then \bar{a} is independent in \mathcal{R}_{exp} .*

PROOF. Suppose not. Say a_k is defined from a_1, \dots, a_{k-1} in \mathcal{R}_{exp} . Let $\varphi(\bar{x})$ be a formula saying that x_k is defined in this way from x_1, \dots, x_{k-1} . This cannot be valid on an open box. \dashv

2.2. Independence relations on \mathcal{R}_{exp} .

DEFINITION 2.4. Suppose K is an Archimedean real closed ordered field with an expansion (K, exp) satisfying T_{exp} . Let $IND_n(K)$ be the set of n -tuples in K that are independent in (K, exp) .

We show that the relations $IND_n(\mathcal{R})$ are defined in \mathcal{R} by computable sequences of computable Π_2 and computable Σ_2 formulas. The computable Π_2 definitions are easy.

LEMMA 2.5 (Computable Π_2 definition of IND_n). *For each n , we can effectively find a computable Π_2 definition of IND_n , with a parameter r coding T_{exp} .*

PROOF. We have $\bar{a} \in IND_n$ iff for each formula $\varphi(\bar{x})$, there is a rational box B around \bar{a} such that T_{exp} contains one of the sentences $(\forall \bar{x} \in B)\varphi(\bar{x})$ or $(\forall \bar{x} \in B)\neg\varphi(\bar{x})$. We can express this as a computable Π_2 formula. Let $c_k(u)$ be the formula saying that the k^{th} place in the preferred binary expansion of u is 1. For each formula φ in the appropriate variables, and each rational box B , let $k(\varphi, B)$ be the Gödel number of the sentence saying $(\forall \bar{x} \in B)\varphi(\bar{x})$. We have $\bar{a} \in IND_n$ iff

$$\bigwedge_{\varphi} \bigvee_B (\bar{a} \in B \ \& \ (c_{k(\varphi, B)}(r) \vee c_{k(\neg\varphi, B)}(r))),$$

where the conjunction is over all φ with appropriate variables, and the disjunction is over all rational boxes B . This is computable Π_2 , with the parameter r , as required. \dashv

The computable Σ_2 definition for the relation IND_n is less obvious.

LEMMA 2.6 (Computable Σ_2 definition of IND_n). *For each n , we can effectively find a computable Σ_2 definition of IND_n , with a parameter r coding T_{exp} .*

PROOF. Fix n . Let $(\varphi_m(\bar{x}))_{m \in \omega}$ be a computable list of formulas in the variables $\bar{x} = (x_1, \dots, x_n)$, in the language of \mathcal{R}_{exp} . We build a tree \mathcal{T} , computable in T_{exp} , consisting of finite sequences of rational boxes B_1, B_2, \dots, B_s such that $B_1 \supseteq B_2 \supseteq \dots \supseteq B_s$ and for each $k < s$, one of the sentences $(\forall \bar{x} \in B_{k+1})\varphi_k(\bar{x})$ or $(\forall \bar{x} \in B_{k+1})\neg\varphi_k$ is in T_{exp} . By Proposition 1.6 and Lemma 2.3, $\bar{a} \in IND_n$ iff there

is a path $\pi = B_1, B_2, \dots$ through \mathcal{T} such that for each s , \bar{a} is in the box B_s . We show that this definition can be expressed by a computable Σ_2 formula in the language of real closed ordered fields, with the parameter r .

CLAIM: There is a computable Π_1 formula, with parameter r , saying that x codes a path through \mathcal{T} .

PROOF OF LEMMA. The preferred binary expansion of $x \in [0,1]$ is the characteristic function f_x of a set $S_x \subseteq \omega$. The set S_x may be finite, but it cannot be cofinite. We consider a path through \mathcal{T} to be a set S with the following properties:

1. each element of S is a code for a finite sequence (B_1, \dots, B_s) in \mathcal{T} ,
2. if $(B_1, \dots, B_s, B_{s+1})$ is in S , then so is (B_1, \dots, B_s) ,
3. if two sequences in S have length s , then they are equal,
4. S is infinite.

We show that each of the four properties above can be expressed by a computable Π_1 formula. For Property 1, we say that S_x has no elements not in \mathcal{T} . Since \mathcal{T} is computable in T_{exp} and T_{exp} is coded by r , there is a c.e. set C of pairs (σ, k) , with $\sigma \in 2^{<\omega}$, such that $k \notin \mathcal{T}$ iff for some $(\sigma, k) \in C$, r agrees with σ , where this means that the preferred binary expansion of r extends σ ; i.e., $c_k(r)$ holds for $\sigma(k) = 1$ and $\neg c_k(r)$ holds for $\sigma(k) = 0$. To say that S_x has no elements not in \mathcal{T} , we take the c.e. conjunction over $(\sigma, k) \in C$ of formulas saying that if r agrees with σ , then $\neg c_k(x)$. This is computable Π_1 with parameter r .

For Property 2, we say that S_x (a set of codes for finite sequences), is closed under initial segments. We take the conjunction of formulas $c_k(x) \rightarrow c_{k'}(x)$, for the pairs (k, k') such that for some s , k is the code for a sequence of length $s + 1$ and k' is the code for the initial segment of length k . This is computable Π_1 , with no parameter.

For Property 3, we say that if two sequences in S_x have length s , then they are equal. We take the conjunction of formulas saying $\neg(c_k(x) \ \& \ c_{k'}(x))$, for the pairs (k, k') coding distinct sequences of the same length. This is computable Π_1 , with no parameter.

For Property 4, we must say that S_x is infinite. We recall that the elements x of $[0, 1)$ that code finite sets are just the dyadic rationals. We have a computable Π_1 formula saying that r is not equal to any of these rationals.

Putting the four statements together, we have a computable Π_1 formula, with parameter r , saying that x codes a path through \mathcal{T} . This proves the Claim. \dashv

Knowing that x codes a path through \mathcal{T} , we want a computable Π_1 formula saying that an n -tuple \bar{u} lies in the boxes on this path. We take the conjunction over k coding a finite sequence of rational boxes (B_1, \dots, B_s) of the formulas saying $c_k(x) \rightarrow \bar{u} \in B_s$. To say that \bar{u} is independent, we have a computable Σ_2 formula saying that there exists x such that S_x is a path through \mathcal{T} and \bar{u} lies in the boxes corresponding to this path. \dashv

Thanks to the computable Π_2 and computable Σ_2 definitions, we know that for any copy K of \mathcal{R} , the relations $IND_n(K)$ are Δ_2^0 relative to K , uniformly in n .

LEMMA 2.7 (Basis). *Suppose K is a copy of \mathcal{R} . Then we have a sequence b_1, b_2, \dots , Δ_2^0 relative to K , and forming a basis for K_{exp} .*

PROOF. Applying a procedure that is Δ_2^0 relative to K , we run through the elements, and we use the relations IND_n to choose a basis. We let b_1 be first satisfying $IND_1(u_1)$, we let b_2 be first such that (b_1, b_2) satisfies $IND_2(u_1, u_2)$, etc. \dashv

To complete the proof that $\mathcal{R}_{exp} \leq_w^* \mathcal{R}$, we show the following:

PROPOSITION 2.8 (Enumerating the complete diagram of the expansion). *After collapse, let $K \cong \mathcal{R}$. Then there is a copy \mathcal{C} of \mathcal{K}_{exp} with complete diagram computable in K .*

PROOF. Let b_1, b_2, \dots be a basis for \mathcal{K}_{exp} that is Δ_2^0 relative to K , determined as in the previous lemma. Guessing at this basis, and using T_{exp} , we enumerate the complete diagram of a copy \mathcal{C} of \mathcal{K}_{exp} . The universe of \mathcal{C} will be ω , which we think of as a set of constants. We fix a computable enumeration of the sentences $\varphi(\bar{c})$, where $\varphi(\bar{x})$ is a formula in the language of \mathcal{R}_{exp} and \bar{c} is a tuple of constants. We suppose that the language includes symbols for the definable Skolem functions. We fix a computable enumeration of terms $\tau(\bar{c})$, where $\tau(\bar{x})$ is a term in variables \bar{x} and \bar{c} is a corresponding tuple of constants.

At each stage s , we have enumerated into the complete diagram of \mathcal{C} a finite set δ_s of sentences. The set δ_s includes sentences saying that the constants mentioned are all distinct. We start with $\delta_0 = \emptyset$, and $\delta_s \subseteq \delta_{s+1}$. We will arrange that for each sentence $\varphi(\bar{c})$, one of $\pm\varphi(\bar{c})$ is in δ_s for some s . We will also arrange that for each term $\tau(\bar{c})$, a sentence of the form $\tau(\bar{c}) = c'$ appears in δ_s for some s . To determine an isomorphism f from \mathcal{C} onto \mathcal{K}_{exp} , it is enough to determine $f^{-1}(b_n)$ for all n , since the rest of the elements are definable from the basis. We have the following requirements.

R_n: Determine $f^{-1}(b_n)$.

At stage s , we have tentatively mapped some constants \bar{d}_s to a tuple \bar{v}_s in K which we believe to be an initial segment of the basis b_1, b_2, \dots . In δ_s , we have mentioned the constants \bar{d}_s , plus some further constants \bar{c}_s . Each $c_i \in \bar{c}_s$ has been given a definition $\tau_i(\bar{d}_s)$, and the sentence $c_i = \tau_i(\bar{d}_s)$ is in δ_s . We will maintain the condition that what we have said in δ_s about \bar{d}_s is valid on a rational box B_s around \bar{v}_s . We must make this precise.

Let $\chi_s(\bar{d}_s, \bar{c}_s)$ be the conjunction of the sentences in δ_s . Let \bar{u} be a tuple of variables corresponding to \bar{d}_s . We suppose that these variables do not appear in the sentences of δ_s . Let $\chi_s^*(\bar{u})$ be the formula obtained from $\chi_s(\bar{d}_s, \bar{c}_s)$ by replacing each $d_i \in \bar{d}_s$ by the corresponding variable u_i , and replacing each $c_i \in \bar{c}_s$ by $\tau_i(\bar{u})$, where c_i is defined to be $\tau_i(\bar{d}_s)$. Note that $\chi_s^*(\bar{u})$ has conjuncts saying that the terms u_i and $\tau_i(\bar{u})$ are all distinct. Now, $\chi_s^*(\bar{u})$ expresses what we have said about \bar{d}_s in δ_s . We say how to check that this is true on a rational box B_s around \bar{v}_s . We write $\chi_s^*(B_s)$ for the sentence saying $(\forall \bar{u} \in B_s) \chi_s^*(\bar{u})$. We check that $\bar{v}_s \in B_s$ and that $\chi_s^*(B_s) \in T_{exp}$. We can check that $\bar{v} \in B_s$ using K . We can check, using the real coding T_{exp} , that the sentence $\chi_s^*(B_s) \in T_{exp}$.

At stage $s + 1$, if our stage s guess \bar{v}_s at the initial segment \bar{b}_s of the basis seems correct, then $\bar{v}_{s+1} = \bar{v}_s, v'$, where v' appears to be the next element of the basis. If at stage s our guess \bar{v}_s at \bar{b}_s changes, then \bar{v}_{s+1} is the restriction of \bar{v}_s to the part that still seems to be an initial segment of the basis. In this case, any constants in \bar{d}_s which were tentatively mapped to elements of \bar{v}_s that are not in \bar{v}_{s+1} will now be given definitions in terms of \bar{d}_{s+1} . At all future stages t , these elements will be

part of the definable tuple \bar{c}_t . In the event that at a later stage some elements of \bar{v}_s appear to return to the basis, we will map new constants to those elements.

At stage $s + 1$, assuming that our \bar{v}_{s+1} has a new element v' , we map a new constant d' to it. We put into δ_{s+1} sentences saying that d is not equal to any element of \bar{d}_s or \bar{c}_s . We decide the next sentence φ that mentions only the constants from \bar{d}_s, \bar{c}_s . Also, for the next term $\tau(\bar{d}_s)$ not already given a name, we add a sentence $c = \tau(\bar{d}_s)$, where c is either in \bar{c}_s or the first constant not yet mentioned. The lemmas below guarantee that we can do all of this, while maintaining the condition that what we have said in δ_{s+1} about \bar{d}_{s+1} is valid on a rational box around \bar{v}_{s+1} . We need some terminology.

DEFINITION 2.9. We say that $(\delta; \bar{d}; \bar{c})$ is a *good triple* if

1. δ is a finite set of sentences with constants split into disjoint sets \bar{d} , and \bar{c} ,
2. δ includes sentences saying that the constants are all distinct,
3. for each $c \in \bar{c}$, δ includes a sentence $\tau(\bar{d}) = c$.

For a good triple $(\delta; \bar{d}; \bar{c})$, a *test formula* $\chi^*(\bar{u})$ is obtained in the way we obtained $\chi_s^*(\bar{u})$ from δ_s above.

DEFINITION 2.10. For a good triple $(\delta; \bar{d}; \bar{c})$, we say that $\chi^*(\bar{u})$ is a *test formula* if it is obtained by the following steps:

1. Let χ be the conjunction of δ , and let \bar{u} be a sequence of new variables corresponding to \bar{d} .
2. Let $\chi^*(\bar{u})$ be the formula obtained from χ by replacing d_i by u_i and replacing c_i by $\tau_i(\bar{u})$, where $c_i = \tau_i(\bar{d})$ is a sentence of δ defining c_i in terms of \bar{d} .

For our construction, at stage s , we will have a good triple $(\delta_s; \bar{d}_s; \bar{c}_s)$ with a test formula χ_s^* that is valid on a rational box B_s , so that the sentence $\chi_s^*(B_s)$ is in T_{exp} . Moreover, we will have f tentatively mapping \bar{d}_s to $\bar{v}_s \in K$, where $\bar{v}_s \in B_s$. We believe that \bar{v}_s is an initial segment of the basis.

LEMMA 2.11. Let $(\delta; \bar{d}; \bar{c})$ be a good triple with test formula $\chi^*(\bar{u})$ valid on a rational box B containing an independent tuple \bar{b} . Let b' be a further element independent over \bar{b} . Let δ' be the result of adding to δ sentences saying of a new constant d that it is not equal to any mentioned in δ . Then $(\delta'; \bar{d}; d; \bar{c})$ is a good triple, with test formula $\chi'^*(\bar{u}, u')$ that is valid on a rational box B' around \bar{b}, b' . (We may suppose that the projection of B' on the initial coordinates, omitting the last one, is contained in B .)

LEMMA 2.12. Let $(\delta; \bar{d}; \bar{c})$ be a good triple with test formula $\chi^*(\bar{u})$ valid on a rational box B containing an independent tuple \bar{b} . Let φ be a sentence with constants among \bar{d}, \bar{c} . There is a good triple $(\delta'; \bar{d}; \bar{c})$, where δ' is the result of adding $\pm\varphi$ to δ , with test formula $\chi'^*(\bar{u})$ valid on a rational box $B' \subseteq B$ containing \bar{b} .

LEMMA 2.13. Let $(\delta; \bar{d}; \bar{c})$ be a good triple with test formula $\chi^*(\bar{u})$ valid on a rational box B containing an independent tuple \bar{b} . For a term $\tau(\bar{d})$, there is a good triple $(\delta'; \bar{d}'; \bar{c}')$, with a test formula $\chi'^*(\bar{u})$ valid on a rational box $B' \subseteq B$ containing \bar{b} , where δ' and \bar{c}' satisfy one of the following:

1. δ' is the result of adding to δ a sentence $c = \tau(\bar{d})$, for some $c \in \bar{c}$, and $\bar{c}' = \bar{c}$,
2. δ' is the result of adding to δ a sentence $c' = \tau(\bar{d})$, where c' is new, along with sentences saying that c' is not equal to any of the constants in \bar{d}, \bar{c} , and \bar{c}' is \bar{c}, c' .

In our construction, it may be that at stage $s + 1$, our guess at the initial segment of the basis changes. Then \bar{v}_{s+1} is the restriction of \bar{v}_s to the part that seems correct. We must give the extra elements of \bar{d}_s definitions in terms of \bar{d}_{s+1} . The following lemma says that we can do this.

LEMMA 2.14. *Let $(\delta, \bar{d}; d'; \bar{c})$ be a good triple with test formula $\chi^*(\bar{u}, u')$ valid on a rational box B containing a tuple \bar{b}, b' , where \bar{b} is independent. There is a good triple $(\delta'; \bar{d}'; \bar{c}, d')$ with test formula $\chi(\bar{u}, u')$ valid on a rational box containing \bar{b} , where δ' is the result of adding to δ a sentence $d' = \tau(\bar{d})$. We may take B' to be the projection of B on the initial coordinates, omitting the one that corresponds to u' .*

PROOF OF LEMMA. The box B is a cross product of rational intervals. Say that I is the interval corresponding to the coordinate u' , and take $q \in I$. There is a term τ in our language naming q . Let δ' be the result of adding to δ the defining sentence $d' = \tau$, and modifying the definitions $c_i = \tau(\bar{u}, u')$, by replacing u' by τ . We have in δ sentences saying that d' is distinct from all constants in \bar{d}, \bar{c} . The formulas of δ are valid on B , and they guarantee that for $\bar{u} \in B'$, nothing in I can be equal to any u_i . Also, for c_i with a definition $c_i = \tau(\bar{d})$ in δ , for $\bar{u} \in B'$, nothing in I can be equal to $\tau(\bar{u})$. ⊖

We begin at stage 0 with the good triple $(\emptyset, \emptyset, \emptyset)$. Our guess at an initial segment of the basis is \emptyset , and f is not defined on any elements. Suppose at stage s , our guess at an initial segment of the basis is \bar{v}_s , we have the good triple $(\delta_s, \bar{d}_s; \bar{c}_s)$, with test formula $\chi_s^*(\bar{u})$ valid on a box B_s around \bar{v}_s , and we have f mapping \bar{d}_s to \bar{v}_s .

We must say what happens at stage $s + 1$. Supposing \bar{v}_s still appears to be an initial segment of the basis, and that v' is the next element of the basis, we consider letting $\bar{v}_{s+1} = \bar{v}_s, v'$ and extending f to map a new constant d' to v' , and letting $\bar{d}_{s+1} = \bar{d}_s, d'$. Assuming that we can find an appropriate rational box on which the test formula is valid, we let δ_{s+1} be an extension of δ_s , with some sentences added as follows:

- STEP 1. We add sentences saying that d' is not equal to anything in \bar{d}_s, \bar{c}_s .
- STEP 2. We add one of the sentences $\pm\varphi$, where φ is the first sentence on our list that involves only constants from \bar{d}_s, \bar{c}_s .
- STEP 3. For the first term $\tau(\bar{d}_s)$ such that δ_s does not include a defining sentence $c = \tau(\bar{d})$, we add such a sentence. Here c may be an element of \bar{d}_s , or \bar{c}_s or a new constant.

Lemma 2.11 says that we can carry out Step 1, finding a rational box on which the appropriate test formula is valid, provided that our guess the initial segment of the basis is correct. Lemma 2.12 says that we can carry out Step 2, provided that our guess at the initial segment of the basis is correct. Lemma 2.13 says that we can carry out Step 3, provided that our guess at the initial segment of the basis is correct.

Running our approximations ahead, either \bar{v}_s will no longer seem to be an initial segment of the basis, or else we will arrive at \bar{v}_{s+1} the result of adding a single element to \bar{v}_s and a good triple $(\delta_{s+1}, \bar{d}_{s+1}, \bar{c}_{s+1})$, carrying out all three steps, with a test formula that is valid on an appropriate rational box B_{s+1} containing \bar{v}_{s+1} . We do not add to the diagram unless this happens.

If it appears that \bar{v}_s is not an initial segment of the basis, then we apply Lemma 2.14 finitely many times, to give definitions to the elements of \bar{d}_s that are mapped to the elements of \bar{v}_s that are not in \bar{v}_{s+1} . This lemma tells us how to arrive at an appropriate next good triple and a rational box B_{s+1} . If those elements of \bar{v}_s

later return to our approximation for \bar{b} , the construction will create new elements that will be mapped to those elements.

Eventually, our guess at the initial segment of the basis of length n is correct. Say this happens at stage s . The initial segment of the basis of length n is \bar{v}_s , and for all stages $t \geq s$, the stage t version of f will map \bar{d}_s to \bar{v}_s . What we say about \bar{d}_s is true about \bar{v}_s . Taking the limit, f gives preimages to all elements of the basis. Each element of our \mathcal{C} that is not the preimage of a basis element under f has a definition in terms of some elements that preimages of the basis elements. We have arranged that if $\varphi(\bar{d}, \bar{c})$ is in δ_s , where $f(\bar{d}) = \bar{v}$ is part of the basis, and c_i has been given a definition $\tau_i(\bar{d})$ in δ_s , then $\varphi(\bar{v}, \bar{\tau}(\bar{v}))$ is true in K_{exp} . Thus, f is an isomorphism. This completes the proof of Proposition 2.8. \dashv

§3. Generalizing. In Section 2, we proved that $\mathcal{R}_{exp} \leq_w^* \mathcal{R}$. In this section, we use essentially the same proof to obtain the General Theorem stated in the introduction. Recall that $\mathcal{R}_{\mathbb{Q}}$ consists of the set of real numbers, with the ordering and constants for the rational numbers. This is a very weak structure in terms of elementary first order definability. The theorem says that it has as much computing power as various expansions of the reals, including \mathcal{R}_{exp} .

THEOREM 3.1 (General Theorem). *Let \mathcal{M} be an o -minimal expansion of $\mathcal{R}_{\mathbb{Q}}$, in a countable language, and with definable Skolem functions. Then $\mathcal{M} \leq_w^* \mathcal{R}_{\mathbb{Q}}$. In fact, after collapse, every copy K of $\mathcal{R}_{\mathbb{Q}}$ computes the complete diagram of a copy of \mathcal{M} .*

PROOF. The proof of Theorem 3.1 consists of a sequence of lemmas, following the outline from Section 2. Most of these are direct analogues of lemmas from Section 2, and are proved in the same way; we omit their proofs here. The greatest difference is in the first lemma below, Lemma 3.2. In Section 2, the proof of Lemma 2.2, on uniqueness of the expansion, did not use o -minimality, just the fact that exp is a continuous function, together with the fact that the theory T_{exp} is coded by a real. Here o -minimality plays an important role.

Let $T_{\mathcal{M}} = Th(\mathcal{M})$.

LEMMA 3.2 (Uniqueness). *For $K \cong \mathcal{R}_{\mathbb{Q}}$, there is a unique expansion $K_{\mathcal{M}}$ to a model of $T_{\mathcal{M}}$.*

PROOF. Since \mathcal{M} is o -minimal, with definable Skolem functions, definable closure is a good closure notion. Since $K \cong \mathcal{R}_{\mathbb{Q}}$, there is at least one expansion of K to a model of $T_{\mathcal{M}}$, say K_1 . Let b_1, b_2, \dots be a basis for K_1 . Suppose K_2 is another expansion of K to a model of $T_{\mathcal{M}}$. To show that $K_1 = K_2$, we first show that for all tuples \bar{b} in the K_1 -basis, \bar{b} satisfies the same formulas in K_1 and K_2 . Suppose $\varphi(\bar{x})$ is true in K_1 of a basis tuple \bar{b} . Since \bar{b} is independent in K_1 , by Proposition 1.6 there is a rational box B around \bar{b} such that the sentence $(\forall \bar{x} \in B)\varphi(\bar{x})$ is in $T_{\mathcal{M}}$. Then $\varphi(\bar{x})$ must be true of \bar{b} in K_2 .

To complete the proof that $K_1 = K_2$, we show that every element c has a definition from a tuple in the K_1 -basis that is good in both K_1 and K_2 . For an element c , we know that there is a definition in K_1 from a tuple \bar{b} of basis elements. Say $c = \tau(\bar{b})$. Let c' be the element satisfying the definition $\tau(\bar{b})$ in K_2 . We can show that $c = c'$. If c is in a rational interval I , then the formula saying $\tau(\bar{x}) \in I$ is true of \bar{b} in K_1 . This formula is also true of \bar{b} in K_2 , so $c' \in I$. This shows that $c = c'$, completing the proof that $K_1 = K_2$. \dashv

We write $K_{\mathcal{M}}$ for the unique expansion of K to a model of $T_{\mathcal{M}}$. Let IND_n be the set of n -tuples in K that are independent in the expansion $K_{\mathcal{M}}$. The following lemmas are then proved similarly as their counterparts (Lemmas 2.5, 2.6, 2.7, and Proposition 2.8, respectively) in Section 2:

LEMMA 3.3 (Computable Π_2 definition of IND_n). *For each n , we can effectively find a computable Π_2 definition of IND_n , with a real parameter r coding $T_{\mathcal{M}}$.*

LEMMA 3.4 (Computable Σ_2 definition of IND_n). *For each n , we can effectively find a computable Σ_2 definition of IND_n , with a real parameter r coding $T_{\mathcal{M}}$.*

LEMMA 3.5 (Basis). *There is a basis b_1, b_2, \dots for $K_{\mathcal{M}}$ that is Δ_2^0 relative to K .*

LEMMA 3.6 (Enumerating the complete diagram of the expansion). *Any copy of K computes the complete diagram of a copy of the expansion $K_{\mathcal{M}}$.*

The proof of Theorem 3.1 is thus complete. ⊣

COROLLARY 3.7. $\mathcal{R}_{\mathbb{Q}} \equiv_w^* \mathcal{R}$.

PROOF. It is easy to see that $\mathcal{R}_{\mathbb{Q}} \leq_w^* \mathcal{R} \leq_w^* (\mathcal{R}, (q)_{q \in \mathbb{Q}})$, where the structure on the right is the expansion of the ordered field of reals with constants for the rationals. By Theorem 3.1, $(\mathcal{R}, (q)_{q \in \mathbb{Q}}) \leq_w^* \mathcal{R}_{\mathbb{Q}}$. ⊣

§4. Applying the general result. In this section, we apply Theorem 3.1 to show that various structures are equivalent to \mathcal{R} in computing power. We begin with $\mathcal{R}_f = (\mathcal{R}, f)$, where f is analytic. In Section 2, we considered the case where f is the exponential function. In this case, \mathcal{R}_f is o -minimal, but in general, in particular, if f is the sine function, \mathcal{R}_f will not be o -minimal.

PROPOSITION 4.1. *Let f be analytic on \mathcal{R} . Then $\mathcal{R}_f \equiv_w^* \mathcal{R}$.*

PROOF. Clearly, $\mathcal{R} \leq_w^* \mathcal{R}_f$. Let $\mathcal{R}_{\text{bounded } f}$ be the expansion of \mathcal{R} by the family of functions f_z , for $z \in \mathbb{Z}$, where

$$f_z(x) = \begin{cases} f(x) & \text{if } x \in [z, z + 1], \\ 0 & \text{otherwise.} \end{cases}$$

Since \mathcal{R}_{an} is o -minimal, so is $\mathcal{R}_{\text{bounded } f}$.

LEMMA 4.2. $\mathcal{R}_f \leq_w^* \mathcal{R}_{\text{bounded } f}$.

PROOF. Let $(K, (f_z^K)_{z \in \omega})$ be a copy of $\mathcal{R}_{\text{bounded } f}$. We define f^K such that (K, f^K) is isomorphic to \mathcal{R}_f . Given $a \in K$, we can find, effectively in the field K , the integer z that is the “floor” of a ; i.e., $z \leq a < z + 1$. Then $f^K(a) = f_z^K(a)$. ⊣

We can now complete the proof of Proposition 4.1. It is clear that $\mathcal{R}_{\text{bounded } f} \leq_w^* (\mathcal{R}_{\text{bounded } f}, (q)_{q \in \mathbb{Q}})$. Using Theorem 3.1, we get $(\mathcal{R}_{\text{bounded } f}, (q)_{q \in \mathbb{Q}}) \leq_w^* \mathcal{R}_{\mathbb{Q}} \leq_w^* \mathcal{R}$. This shows that $\mathcal{R}_f \leq_w^* \mathcal{R}$. ⊣

Recall that \mathcal{R}^+ is the reduct of \mathcal{R} without multiplication, but including addition, the ordering, and the constants 0 and 1. The result below is also proved in [3].

PROPOSITION 4.3. $\mathcal{R} \equiv_w^* \mathcal{R}^+$.

PROOF. By Theorem 3.1, we have $\mathcal{R} \leq_w^* \mathcal{R}_{\mathbb{Q}}$. It is easy to see that $\mathcal{R}_{\mathbb{Q}} \leq_w^* (\mathcal{R}^+, (q)_{q \in \mathbb{Q}}) \leq_w^* \mathcal{R}^+ \leq_w^* \mathcal{R}$. ⊣

The final example of this section is simple. Let $(r_n)_{n \in \omega}$ be any sequence of elements of \mathcal{R} , and consider $(\mathcal{R}, (r_n)_{n \in \omega})$, the expansion of the ordered field \mathcal{R} with constants for those elements.

PROPOSITION 4.4. $\mathcal{R} \equiv_w^* (\mathcal{R}, (r_n)_{n \in \omega})$.

PROOF. Let \mathcal{M} be the expansion of $\mathcal{R}_{\mathbb{Q}}$ with all of the structure of \mathcal{R} and the constants r_n . We have $\mathcal{R} \leq_w^* \mathcal{M} \leq_w^* \mathcal{R}_{\mathbb{Q}} \leq_w^* \mathcal{R}$. We use Theorem 3.1 for the second reduction. The other reductions are clear. \dashv

§5. The structure \mathcal{R}_{int} . Recall that the structure \mathcal{R}_{int} has just the set of real numbers, with the ordering, and unary predicates for the half-open intervals $[q, q')$ with dyadic rational endpoints. This can be thought of as the minimal structure that is able to recover the (preferred) binary expansions of the real numbers, as each initial segment of the binary expansion of a number corresponds exactly to the number being in a half-open interval of this sort. For instance, knowing that the binary expansion of x begins 0.10 corresponds exactly to knowing that $x \in [\frac{1}{2}, \frac{3}{4})$.

In this section, we will show that $\mathcal{R}_{int} \equiv_w^* \mathcal{B}$, where \mathcal{B} is the structure representing Baire space. In [3], Downey, Greenberg, and Miller showed that $\mathcal{B} \equiv_w^* \mathcal{R}$, by a proof which resembles our proofs from Sections 2 and 3, although the work was done independently. This, together with the fact that $\mathcal{R}_{int} \equiv_w^* \mathcal{B}$, implies that $\mathcal{R}_{int} \equiv_w^* \mathcal{R}$. Independently of Downey, Greenberg, and Miller, the authors had arrived at the fact that $\mathcal{B} \equiv_w \mathcal{R}$. We end the section by saying just a little about our reasoning. In particular, we state a variant of Theorem 3.1 in which the $\mathcal{R}_{\mathbb{Q}}$ is replaced by \mathcal{R}_{int} .

PROPOSITION 5.1. $\mathcal{B} \equiv_w^* \mathcal{R}_{int}$.

We split the proof of Proposition 5.1 into two lemmas.

LEMMA 5.2. $\mathcal{B} \leq_w^* \mathcal{R}_{int}$.

PROOF. Given a copy K of \mathcal{R}_{int} , we can enumerate the preferred binary expansions of the reals in the interval $[0, 1)$. For each such real, we get a function $f \in 2^\omega$ such that f has infinitely many 0's. Given such an f , we pass to a function $g \in \omega^\omega$, where $g(0)$ is the number of 1's before the first 0, and for $k > 0$, $g(k)$ is the number of 1's between the k^{th} 0 and the $(k + 1)^{st}$. This gives a copy of \mathcal{B} . \dashv

LEMMA 5.3. $\mathcal{R}_{int} \leq_w^* \mathcal{B}$.

PROOF. Given a copy of \mathcal{B} , we can enumerate the functions $g \in \omega^\omega$. From each g , we pass effectively to a function $f \in 2^\omega$ such that $g(0)$ is the number of 1's before the first 0 in f , and $g(k + 1)$ is the number of 1's between the $(k + 1)^{st}$ and $(k + 2)^{nd}$ 0's in f . The functions $f \in 2^\omega$ are just the preferred binary expansions of reals in the interval $[0, 1)$. The ordering on these reals corresponds to the lexicographic ordering on the functions f . For each dyadic rational q , we give a name in which we mark the first in the infinite sequence of 0's.

For a function f that is the preferred binary expansion of a real r in the interval $[0, 1)$, we cannot effectively determine whether $r = q$. However, for a pair $q < q' \in D$, we can effectively determine whether $r \in [q, q')$. We have a copy of the restriction of \mathcal{R}_{int} to the interval $[0, 1)$. For the full structure, we take pairs (z, f) , where $z \in \mathbb{Z}$ and $f \in 2^\omega$ has infinitely many 0's. We take the lexicographic ordering on these pairs. The full set of dyadic rationals consists of the elements $z + q$, for q with a special name. We can determine membership in intervals with these endpoints. This gives a copy of \mathcal{R}_{int} . \dashv

COROLLARY 5.4. $\mathcal{R}_{\mathbb{Q}} \equiv_w^* \mathcal{R}_{int}$.

PROOF. By Corollary 3, $\mathcal{R}_{\mathbb{Q}} \equiv_w^* \mathcal{R}$. As we said above, Downey, Greenberg, and Miller [3] showed that $\mathcal{R} \equiv_w^* \mathcal{B}$. By Proposition 5.1, $\mathcal{B} \equiv_w^* \mathcal{R}_{int}$. \dashv

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