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PAPER

A unified approach to some non-Hausdorff topological properties

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Abstract
Sobriety, well-filteredness, and monotone convergence are three of the most important properties of topological spaces extensively studied in domain theory. Some other weak forms of sobriety and well-filteredness have also been investigated by some authors. In this paper, we introduce the notion of Θ-fine spaces, which provides a unified approach to such properties. In addition, this general approach leads to the definitions of some new topological properties.

Keywords: Sober space; d-space; Θ-fine space; PF-well-filtered space

1. Introduction

Sobriety, well-filteredness, and monotone convergence are three of the most important and extensively studied topological properties in domain theory. The original definitions of such properties look quite different. In this paper, we introduce the Θ-fine spaces, where Θ is a family of collections of subsets of the given space. We show that by choosing different Θ, we obtain the equivalent definitions of sobriety, well-filteredness, monotone convergence, and of several other related properties such as weak well-filteredness, weak sobriety, PF-well-filteredness, and PF-sobriety. This unified approach also leads to the studies of some new topological properties, which enrich the theory of non-Hausdorff topological spaces.

2. Preliminaries

In this paper, all topological spaces are assumed to be T_0.

Let $(P, \leq)$ be a poset. For $A \subseteq P$, we write $\downarrow A = \{y \in P \mid \exists x \in A, y \leq x\}$ and $\uparrow A = \{y \in P \mid \exists x \in A, x \leq y\}$. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). Let $(P, \leq)$ and $(Q, \sqsubseteq)$ be two posets with $P \cap Q = \emptyset$. The *linear sum* $P \oplus Q$ is defined by taking the following order relation $\preceq$ on $P \cup Q$: $x \preceq y$ iff $x \leq y$ in P, or $x \sqsubseteq y$ in Q, or $x \in P$ and $y \in Q$. If F is a finite set of P, then we denote by $F \subseteq_{fin} P$.

A nonempty subset D of P is *directed* if every finite subset of D has an upper bound in D. The poset P is a *directed complete partially ordered set* (dcpo, for short) if every directed subset of P has a supremum.

A subset A of a poset P is *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset $D \subseteq P$, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\bigvee D$ exists. The set $\sigma(P)$ of all Scott open sets of P forms a topology on P, called the *Scott topology* on P. The space $(P, \sigma(P))$ is denoted by ΣP, called the *Scott space* of P.

Let (X, τ) be a T_0 space. The *specialization order* $\leq_\tau$ on X is defined by $x \leq_\tau y$ iff $x \in \mathrm{cl}(\{y\})$, where cl is the closure operator. In the following, the specialization order $\leq_\tau$ on a T_0 space (X, τ) will be simply denoted by $\leq$, if no ambiguous occurs.

The *saturation* $\mathrm{sat}(A)$ of a subset $A \subseteq X$ is the intersection of all open sets containing A. A subset A of a space (X, τ) is called *saturated* if $A = \mathrm{sat}(A)$. It is a standard result that $A \subseteq X$ is saturated iff it is an upper set with respect to the specialization order, that is iff $A = \uparrow A = \{x \in X : \exists a \in A, a \leq_\tau x\}$ (Gierz et al. 2003; Goubault 2013).

A nonempty subset A of a space is *irreducible* if whenever $A \subseteq F_1 \cup F_2$ with F_1 and F_2 closed, then $A \subseteq F_1$ or $A \subseteq F_2$ holds. Each directed subset of $(X, \leq_\tau)$ is irreducible.

In the following, $\mathbb{Z}^+$ will denote the set of all positive integers and $\mathbb{Z}^-$ denote the set of all negative integers. The set of all natural numbers is denoted by $\mathbb{N}$. All these sets are posets under the usual order of numbers.

3. Θ-Fine Spaces

We now introduce the notion of Θ-fine spaces.

Definition 1. *Let Θ be a "function" which assigns a family $\Theta(X)$ of collections of subsets of X for each space X.*

A space X is called Θ-fine if for any open set U of X and $\mathcal{A} \in \Theta(X)$,

$$\bigcap\{\mathrm{sat}(A) : A \in \mathcal{A}\} \subseteq U \textit{ implies } A \subseteq U \textit{ for some } A \in \mathcal{A}.$$

Example 1.

(1) For each space X, let $\Theta_d(X)$ consist of $\mathcal{A} = \{\{x_i\} : i \in I\}$ such that $\{x_i : i \in I\}$ is a directed set with respect to the specialization order.
(2) For each space X, let $\Theta_s(X)$ consist of $\mathcal{A} = \{\{x_i\} : i \in I\}$ such that $\{x_i : i \in I\} \subseteq X$ is an irreducible set.
(3) For each space X, let $\Theta_w(X)$ consist of $\mathcal{A} = \{F_i : i \in I\}$, where every F_i is compact and $\{F_i : i \in I\}$ is directed (that is, for any F_i, F_j, there exists F_k such that $F_k \subseteq \uparrow F_i \cap \uparrow F_j$).
(4) For each space X, let $\Theta_{CK}(X)$ consist of $\mathcal{A} = \{F_i : i \in I\}$, where every F_i is countable compact and $\{F_i : i \in I\}$ is directed.

Remark 1.

(1) Since every directed subset D of a space (with respect to the specialization order) is irreducible, hence every Θ_s-fine space is Θ_d-fine.
(2) For any directed set $\{x_i : i \in I\}$, $\{\{x_i\} : i \in I\} \in \Theta_w(X)$. Hence every Θ_w-fine space is Θ_d-fine.

The following result should have been proved by other people. For readers convenience, we provide a brief proof.

Lemma 1. *Let A be a nonempty saturated compact subset of a T_0 space (X, τ) and $Min(A)$ be the set of all minimal elements of A. Then*

(1) $A = \uparrow Min(A)$.
(2) $Min(A)$ is compact.

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Proof. Note that every open set U is an upper set ($U = \uparrow U$), so if every element of A is above some minimal element(s), then every open cover of $\mathrm{Min}(A)$ is also an open cover of A. Thus, (2) follows from (1).

For (1), it is enough to show that for any $a \in A$, there is $m \in \mathrm{Min}(A)$ such that $m \leq a$. Chose a maximal chain C of $(A, \leq)$ that contains a. If C does not have the smallest element, then $\{X - \mathrm{cl}(\{y\}) : y \in C\}$ is an open cover of A that does not have a finite subcover, contradicting the compactness of A. Thus, C must have a smallest element, say m. Then clearly $m \in \mathrm{Min}(A)$ and $m \leq a$. □

Lemma 2. *Let $\mathcal{F}$ be a family of nonempty compact subsets of a space X such that for any open set U of X,*

$$\bigcap\{\uparrow F : F \in \mathcal{F}\} \subseteq U \text{ implies } F \subseteq U \text{ for some } F \in \mathcal{F}.$$

Let $A = \bigcap\{\uparrow F : F \in \mathcal{F}\}$. Then

(1) A is a nonempty compact saturated set and
(2) $A = \uparrow E$ for some compact T_1 subspace $E \subseteq X$.

Proof. (1) The set $A = \bigcap\{\uparrow F : F \in \mathcal{F}\}$ is nonempty. Otherwise $\bigcap\{\uparrow F : F \in \mathcal{F}\} \subseteq \emptyset$ would imply $F = \emptyset$ for some $F \in \mathcal{F}$. Now assume that $\mathcal{U}$ is an open cover of A. Then $\bigcap\{\uparrow F : F \in \mathcal{F}\} \subseteq \bigcup \mathcal{U}$. So there is $F \in \mathcal{F}$ such that $F \subseteq \bigcup \mathcal{U}$. But F is compact, there are $U_1, \ldots, U_n \in \mathcal{U}$ such that $F \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$. Then it follows that $A \subseteq \uparrow F \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$, showing the compactness of A.

As the intersection of any collection of saturated sets is saturated, A is clearly saturated (each $\uparrow F_i$ is a saturated set).

For (2), $A = \uparrow \mathrm{Min}(A)$ and $\mathrm{Min}(A)$ is compact by (1) and Lemma 1. The subspace $\mathrm{Min}(A)$ is clearly T_1. □

A T_0 space X is called *well-filtered* (Gierz et al. 2003) if for any open set U and filtered family $\mathcal{F}$ of saturated compact subsets of X, $\bigcap \mathcal{F} \subseteq U$ implies $F \subseteq U$ for some $F \in \mathcal{F}$.

A dcpo P is called well-filtered if its Scott space $\Sigma P = (P, \sigma(P))$ is a well-filtered space.

The following result is trivial now.

Proposition 1. *A T_0 space is well-filtered iff it is Θ_w-fine.*

By Lemma 2, we deduce the following known result.

Corollary 1. *If $\mathcal{F}$ is a filtered family of nonempty saturated compact subsets of a well-filtered space, then $\bigcap \mathcal{F}$ is a nonempty compact set.*

A T_0 space X is called a *d-space* (or *monotone convergence space* in Gierz et al. 2003) if

(i) for any directed set $D \subseteq X$, $\bigvee D = \sup D$ exists and
(ii) for any open set U and directed set D, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$.

Note that for any x in a T_0 space X, $\mathrm{cl}(\{x\}) = \downarrow x$. If x is an upper bound of $D \subseteq X$, then $D \subseteq \downarrow x$, hence $\mathrm{cl}(D) \subseteq \downarrow x$.

Theorem 1. *A T_0 space X is a d-space iff it is Θ_d-fine.*

Proof. Assume that X is a d-space. Let $D \subseteq X$ be a directed subset and $U \subseteq X$ an open set such that

$$\bigcap\{\uparrow x : x \in D\} \subseteq U.$$

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Then $\bigvee D \in \bigcap\{\uparrow x : x \in D\}$, implying $\bigvee D \in U$. Since X is a d-space, there is $d \in D \cap U$. Then $\uparrow d \subseteq U$. Therefore X is Θ_d-fine.

Conversely, assume that X is Θ_d-fine. Let D be a directed subset of X. Then $\{\{x\} : x \in D\} \in \Theta_d$. By Lemma 2, we have that $\bigcap\{\uparrow x : x \in D\} = \uparrow E$, where $E = \mathrm{Min}(\bigcap\{\uparrow x : x \in D\})$ is nonempty. We claim that E has exactly one element. In fact, assume that E has at least two different elements, say a, b. Then $\mathrm{cl}(D) \subseteq \bigcap\{\mathrm{cl}(\{m\}) : m \in E\}$. Let $A = \bigcap\{\mathrm{cl}(\{m\}) : m \in E\}$. Because the intersection of closed sets of topological space is again closed, we have that A is a closed set and for any $m \in E$, $m \notin A$ (otherwise $m \leq a$ and $m \leq b$ would imply $m = a$, $m = b$, thus $a = b$). Therefore $E \subseteq X - A$. But $X - A$ is an upper set so $\uparrow E \subseteq X - A$. Then $X - \mathrm{cl}(D) \supseteq X - A \supseteq \uparrow E = \bigcap\{\uparrow x : x \in D\}$. Since X is Θ_d-fine, there is $d_0 \in D$ and $\uparrow d_0 \subseteq X - \mathrm{cl}(D)$, implying $d_0 \in X - \mathrm{cl}(D)$. But this is impossible because $d_0 \in D \subseteq \mathrm{cl}(D)$. This contradiction shows that $E = \{a\}$ for some $a \in X$. This then implies that $a = \bigvee D$. Hence $(X, \leq_\tau)$ is a directed complete poset. Now for any open set U of X and directed set $D \subseteq X$, if $\bigvee D \in U$, then

$$\bigcap\{\uparrow x : x \in D\} = \uparrow \bigvee D \subseteq U.$$

Since X is Θ_d-fine, there is $d_0 \in D$, $\uparrow d_0 \subseteq U$, implying $d_0 \in U$.

All these together show that (X, τ) is a d-space. □

A T_0 space X is *sober* if for any irreducible closed subset F of X, there is a (unique) point $a \in X$ such that $F = \mathrm{cl}(\{a\})$.

Theorem 2. *A space X is sober iff it is Θ_s-fine.*

Proof. Assume that the space X is sober and F is an irreducible set. Since the closure of every irreducible set is irreducible, $\mathrm{cl}(F)$ is a closed irreducible set. Thus, there is a unique x_0 such that $\mathrm{cl}(\{x_0\}) = \mathrm{cl}(F)$. Let U be an open set and

$$\bigcap\{\uparrow x : x \in F\} \subseteq U.$$

Then as $x \leq x_0$ holds for all $x \in F$, $x_0 \in U$. So $\mathrm{cl}(F) \cap U \neq \emptyset$, implying $F \cap U \neq \emptyset$. Pick $x \in F \cap U$. Thus $\uparrow x \subseteq U$. Hence X is Θ_s-fine.

Now assume that X is Θ_s-fine. For any closed irreducible set F, by the assumption on X and Lemma 2, through the same process as the proof of Theorem 1, we have that $\bigcap\{\uparrow x : x \in F\} = \uparrow a$ for some $a \in X$. Thus $F \subseteq \downarrow a$. We claim that $F = \mathrm{cl}(\{a\})$, otherwise $a \in X - F$, there is some $x \in F$ such that $x \in X - F$ for X is Θ-fine and F is closed, which is a contradiction. Hence X is sober. □

Corollary 2.

(1) Every sober space is a d-space.
(2) Every well-filtered space is a d-space.

For any space X, let $\mathcal{Q}(X)$ be the set of all nonempty compact saturated subsets of X. The *upper Vietoris topology* on $\mathcal{Q}(X)$ is the topology that has $\{\Box U : U \in \mathcal{O}(X)\}$ as a base, where $\Box U = \{K \in \mathcal{Q}(X) : K \subseteq U\}$. The sets of the form $\Diamond C = \{K \in \mathcal{Q}(X) : K \cap C \neq \emptyset\}$ for closed set C of X form a base for the closed sets of $\mathcal{Q}(X)$. The set $\mathcal{Q}(X)$ equipped with the upper Vietoris topology is called the *Smyth power space* or *upper space* of X (cf. Heckmann and Keimel 2013; Schalk 1993).

The specialization order on the upper space $\mathcal{Q}(X)$ is the reverse inclusion order $\supseteq$. In what follows, the partial order on $\mathcal{Q}(X)$ we will concern is just the reverse inclusion order.

For each space X, let $\mathcal{S}(X)$ be the collection of $\mathcal{F}$ which is an irreducible set of the upper space $\mathcal{Q}(X)$. Then by Theorem 3.13 of Heckmann and Keimel (2013) we have the following.

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Proposition 2. *A space X is sober if and only if it is $\mathcal{S}$-fine.*

For each space X, let Θ_f be the collection of all directed families $\mathcal{A}$ of nonempty finite subsets of X, that is, $\mathcal{A} = \{A_i : i \in I\} \in \Theta_f(X)$ if each A_i is a nonempty finite subset of X and for any A_i, A_j there is a A_k such that $A_k \subseteq \uparrow A_i \cap \uparrow A_j$.

Theorem 3. *A T_0 space is a d-space iff it is Θ_f-fine.*

Proof. Clearly every Θ_f-fine space is a d-space. We now show that every d-space is Θ_f-fine. Let (X, τ) be a d-space. Then $(X, \leq_\tau)$ is a dcpo and every open set $U \in \tau$ is a Scott open set of $(X, \leq_\tau)$. Assume that $\mathcal{A} = \{A_i : i \in I\}$ is a directed family of nonempty finite subsets of X and $U \in \tau$, such that

$$\bigcap\{\uparrow A_i : i \in I\} \subseteq U.$$

Assume that $A_i \not\subseteq U$ for every $i \in I$. Then $\{A_i - U : i \in I\}$ is a family of nonempty finite subsets.

Also for any A_i, A_j, there exists A_k such that $A_k \subseteq (\uparrow A_i) \cap (\uparrow A_j)$. It follows that $A_k - U \subseteq (\uparrow A_i - U) \cap (\uparrow A_j - U)$. As a matter of fact, for any $b \in A_k - U$, $b \in \uparrow A_i$ and $b \notin U$. Let $b \geq a_i$ for some $a_i \in A_i$, then $a_i \in A_i - U$ (otherwise, $a_i \in U$ would imply $b \in U$). Thus $b \in \uparrow(A_i - U)$. Therefore $A_k - U \subseteq \uparrow(A_i - U)$. Similarly we have that $A_k - U \subseteq \uparrow(A_j - U)$, consequently, $A_k - U \subseteq \uparrow(A_i - U) \cap \uparrow(A_j - U)$.

All these show that $\{A_i - U : i \in I\}$ is a directed family of nonempty finite subsets of poset $(X, \leq_\tau)$. By Rudin's Lemma (Lemma III-3.3 of Gierz et al. 2003), there is a directed subset $D \subseteq \bigcup\{A_i - U : i \in I\}$ such that $D \cap (A_i - U) \neq \emptyset$ for each $i \in I$. Let $e = \bigvee D$. Then, as D is a subset of the Scott closed set $X - U$, we have $e = \bigvee D \in X - U$. On the other hand, $e \in \bigcap\{\uparrow(A_i - U) : i \in I\} \subseteq \bigcap\{\uparrow A_i : i \in I\} \subseteq U$, implying $e \in U$.

This contradiction proves that there must be a A_i such that $A_i \subseteq U$. Therefore X is Θ_f-fine. $\square$

The above result gives another equivalent definition of d-spaces.

Since the Scott space ΣP of every dcpo P is a d-space, we deduce the follow:

Corollary 3. *Let P be a dcpo and $\mathcal{F}$ a filtered family of nonempty finite sets of P. If $\bigcap_{F \in \mathcal{F}} \uparrow F$ is a subset of a Scott open set U, then already some member of $\mathcal{F}$ is a subset of U.*

From the above results, we see that every well-filtered space is Θ_{CK}-fine, and every Θ_{CK}-fine space is a d-space.

4. Weak Well-Filtered Spaces and Weak Sober Spaces

For each space X, let $\Theta_{w_0}(X)$ consist of $\mathcal{A} = \{F_i : i \in I\}$, where each F_i is compact, $\{F_i : i \in I\}$ is directed and $\bigcap_{i \in I} \uparrow F_i \neq \emptyset$. In Lu and Li (2017), the authors introduced the weak well-filtered spaces and proved many nice properties of such spaces. From their definition, we immediately deduce the following.

Proposition 3. *A space is weak well-filtered if and only if it is Θ_{w_0}-fine.*

By Lemma 2, if $\{K_i : i \in I\} \subseteq \mathcal{Q}(X)$ is a filtered family, $\bigcap_{i \in I} K_i \neq \emptyset$ and X is weak well-filtered, then $\bigcap_{i \in I} K_i \in \mathcal{Q}(X)$.

Lemma 3 (Topological Rudin Lemma Heckmann and Keimel 2013). *Let X be a topological space and $\mathcal{A}$ an irreducible subset of $\mathcal{Q}(X)$. Any closed set $C \subseteq X$ that meets all members of $\mathcal{A}$ contains an irreducible closed subset A that still meets all members of $\mathcal{A}$.*

Proposition 4. *A topological space X is weak well-filtered iff its upper space $\mathcal{Q}(X)$ is weak well-filtered.*

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Proof. Let $\mathcal{Q}(X)$ be weak well-filtered. Then for any intersecting nonempty filtered subset $\mathcal{F} = \{A_\alpha : \alpha \in I\}$ of $\mathcal{Q}(X)$, $\{\uparrow_{\mathcal{Q}(X)} A_\alpha : \alpha \in I\}$ is a filtered family of compact saturated sets of $\mathcal{Q}(X)$ which has nonempty intersection. Thus for any open set U of X with $\bigcap \mathcal{F} \subseteq U$,

$$\bigcap \{\uparrow_{\mathcal{Q}(X)} A_\alpha : \alpha \in I\} \subseteq \Box U.$$

By the weak well-filteredness, there exists A_α such that $\uparrow_{\mathcal{Q}(X)} A_\alpha \subseteq \Box U$. Therefore, $A_\alpha \subseteq U$ for some $\alpha \in I$ in X.

Conversely, let X be a weak well-filtered space. Suppose that $\{\mathcal{F}_j : j \in J\}$ is a filtered family of saturated compact subsets of $\mathcal{Q}(X)$ and $\bigcap_{j\in J} \mathcal{F}_j \neq \emptyset$. Then for any open set $\mathcal{U} = \bigcup \{\Box U_i : i \in I\}$ of $\mathcal{Q}(X)$ with $\bigcap_{j \in J} \mathcal{F}_j \subseteq \mathcal{U}$, there is $U_{i_0} \neq \emptyset$ for some $i_0 \in I$. Thus we can pick some $x \in U_{i_0}$. Assume that $\mathcal{F}_j \nsubseteq \mathcal{U}$ for all $j \in J$. Then by Lemma 3, there exists a minimal closed irreducible subset $\mathcal{C} \subseteq \mathcal{Q}(X) - \mathcal{U}$ that meets every $\mathcal{F}_j$ for $j \in J$. For each $j \in J$, let $K_j = \bigcup (\mathcal{F}_j \bigcap \mathcal{C})$. Note that $\mathcal{F}_j \bigcap \mathcal{C} \neq \emptyset$ for all $j \in J$. As $\mathcal{F}_j$ is compact and $\mathcal{C}$ is closed in $\mathcal{Q}(X)$, we have $\mathcal{F}_j \bigcap \mathcal{C}$ is compact in $\mathcal{Q}(X)$. It is easy to show that $K_j = \bigcup (\mathcal{F}_j \bigcap \mathcal{C}) \in \mathcal{Q}(X)$. Also, $\{K_j : j \in J\}$ is a filtered family of saturated compact subsets of X. It follows that $\{K_j \cup \uparrow x : j \in J\}$ is filtered in $\mathcal{Q}(X)$.

Take $K = \bigcap_{j \in J} K_j$. We just need to show that $K \neq \emptyset$ because the rest of the proof is similar to that of Theorem 4 in Xu et al. (2019). Assume, on the contrary, that $K = \emptyset$. Then $K = \emptyset \subseteq U_{i_0}$ for $i_0 \in I$. As a result,

$$\bigcap_{j \in J} (K_j \cup \uparrow x) = \uparrow x \cup \bigcap_{j \in J} K_j = \uparrow x \subseteq U_{i_0}.$$

Since X is weak well-filtered, we obtain that $K_{j_0} \cup \uparrow x \subseteq U_{i_0}$ for some $j_0 \in J$. Thus $K_{j_0} \subseteq U_{i_0}$. However, $K_{j_0} = \bigcup (\mathcal{F}_{j_0} \bigcap \mathcal{C}) \subseteq U_{i_0}$ implies that there exists some $G \in \mathcal{F}_{j_0} \bigcap \mathcal{C}$ such that $G \subseteq K_{j_0} \subseteq U_{i_0}$. Therefore $G \in \Box U_{i_0} \subseteq \mathcal{U}$, which contradicts the fact that $\mathcal{C} \subseteq \mathcal{Q}(X) - \mathcal{U}$. Hence, $K = \bigcap_{j \in J} K_j$ is not empty. $\Box$

By Zhao and Fan (2010), a space X is said to be *bounded sober* if for any upper bounded closed irreducible set C of X, there is a unique element x such that $C = \mathrm{cl}(\{x\})$. A space X is called *k-bounded sober* (Zhao and Ho 2015) if for any irreducible closed set F whose supremum exists, there is a unique point $x \in X$ such that $F = \mathrm{cl}(\{x\})$.

For each space X, let $\Theta_b(X)$ consist of $\mathcal{A} = \{\{x_i\} : i \in I\}$ such that $\{x_i : i \in I\} \subseteq X$ is an irreducible set with an upper bound. Let $\Theta_k(X)$ consist of $\mathcal{A} = \{\{x_i\} : i \in I\}$ such that $\{x_i : i \in I\} \subseteq X$ is an irreducible set whose supremum exists. Let $\mathcal{S}_0(X)$ consist of all irreducible subsets $\mathcal{F}$ of $\mathcal{Q}(X)$ such that $\bigcap \mathcal{F} \neq \emptyset$. Obviously, every $\mathcal{S}_0$-fine space is weak well-filtered.

Theorem 4. *A space X is bounded sober iff it is Θ_b-fine.*

Proof. Similar to the proof of Theorem 2. $\Box$

For k-bounded sober, we give a simple proof.

Theorem 5. *A space X is k-bounded sober iff it is Θ_k-fine.*

Proof. Let $\mathcal{A} = \{\{x_i\} : i \in I\}$ be a family of singletons such that $F = \{x_i : i \in I\}$ is an irreducible set whose supremum exists in the k-bounded sober space X and $U \subseteq X$ open such that $\bigcap \{\mathrm{sat}(\{x_i\}) : i \in I\} = \bigcap \{\uparrow x_i : i \in I\} \subseteq U$. Then the closure $\mathrm{cl}(F)$ of F is irreducible and $\bigvee \mathrm{cl}(F) = \bigvee F$. Thus, there is a unique $a \in X$ such that $\mathrm{cl}(F) = \mathrm{cl}(\{a\})$. Since $a \in \bigcap \{\uparrow x : x \in F\} \subseteq U$ for open set U, we obtain $\mathrm{cl}(F) \cap U \neq \emptyset$, therefore $F \cap U \neq \emptyset$. This implies $\{x_i\} \subseteq U$ for some $\{x_i\} \in \mathcal{A}$. Hence X is Θ_k-fine.

Conversely, assume that X is Θ_k-fine. Let $F \subseteq X$ be an irreducible closed set with $\bigvee F$ existing. Then for any open neighborhood V of $\bigvee F$, $\uparrow \bigvee F = \bigcap \{\uparrow x : x \in F\} \subseteq V$. As X is Θ_k-fine, $\uparrow x_0 \subseteq V$ for some $x_0 \in F$. Thus $\bigvee F \in \mathrm{cl}(F) = F$. Therefore, $F = \mathrm{cl}(\{\bigvee F\})$. $\Box$

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The following results reveal more links between sobriety and well-filterdness. By Theorem 4, every $\mathcal{S}_0$-fine space is bounded sober. For clarity, we will also give a direct proof.

Proposition 5. *Every $\mathcal{S}_0$-fine space is bounded sober.*

Proof. Let X be a $\mathcal{S}_0$-fine space and $C \subseteq X$ be a closed irreducible set with an upper bound x_0. Then $\{\uparrow x : x \in C\} \subseteq \mathcal{Q}(X)$ is an irreducible set of $\mathcal{Q}(X)$ with $\uparrow x_0$ as an upper bound. Hence

$$\mathcal{A} = \mathrm{cl}_{\mathcal{Q}(X)}(\{\uparrow x : x \in C\})$$

is a closed irreducible set of $\mathcal{Q}(X)$. The set $K = \bigcap \mathcal{A}$ contains $\uparrow x_0$, thus $K \neq \emptyset$. Since X is $\mathcal{S}_0$-fine, $K \in \mathcal{Q}(X)$ by Lemma 2. Hence K is an upper bound of $\mathcal{A}$. In addition, for any open set $U \subseteq X$, $K \in \Box U$ (i.e. $K \subseteq U$) iff $\mathcal{A} \bigcap \Box U \neq \emptyset$ (because X is $\mathcal{S}_0$-fine), we have $K \in \mathrm{cl}_{\mathcal{Q}(X)}(\mathcal{A})$, hence $K \in \mathcal{A} = \mathrm{cl}_{\mathcal{Q}(X)}(\mathcal{A})$. As K is an upper bound of $\mathcal{A}$, it follows that

$$\mathrm{cl}_{\mathcal{Q}(X)}(\{K\}) = \downarrow_{\mathcal{Q}(X)} K = \mathcal{A}.$$

We claim that $K \cap \bigcap\{\downarrow y : y \in K\} \neq \emptyset$. If not, then $K \subseteq \bigcup_{y\in K} (X - \downarrow y)$, so $K \in \Box \bigcup_{y\in K} (X - \downarrow y)$, therefore $\{\uparrow x : x \in C\} \bigcap \Box \bigcup_{y\in K} (X - \downarrow y) \neq \emptyset$ because $K \in \mathcal{A} = \mathrm{cl}_{\mathcal{Q}(X)}(\{\uparrow x : x \in C\}$. Then there exist $x \in C$ and $y \in K$ such that $\uparrow x \subseteq X - \downarrow y$, which implies $y \in K \subseteq \uparrow x \subseteq X - \downarrow y$, a contradiction.

Take one $t \in K \cap \bigcap_{y\in K} \downarrow y$. Then, as K is saturated, $\uparrow t \subseteq K$. Also $t \in \downarrow y$ for all $y \in K$, we have $K \subseteq \uparrow t$, hence $K = \uparrow t$. For any open set $U \subseteq X$, $C \cap U \neq \emptyset$ iff $\uparrow x \subseteq U$ for some $x \in C$ iff $\mathcal{A} \bigcap \Box U \neq \emptyset$ iff $\mathrm{cl}_{\mathcal{Q}(X)}(\{K\}) \bigcap \Box U \neq \emptyset$ iff $K \in \Box U$ iff $\uparrow t \subseteq U$ iff $t \in U$. It follows that $t \in \mathrm{cl}(C) = C$. Clearly, $x \leq t$ for all $x \in C$, so $C \subseteq \mathrm{cl}(\{t\})$. Hence $C = \mathrm{cl}(\{t\})$. Therefore X is bounded sober. $\Box$

Remark 2. (1) It is well-known that well-filtered is strictly weaker than sober, which was first proved by Kou (2001).

(2) For any complete lattice L, let $X = \Sigma L$. It is easily seen that $\mathcal{S}(X) = \mathcal{S}_0(X)$. Hence ΣL is sober if and only if it is $\mathcal{S}_0$-fine.

Remark 3. The converse of Proposition 5 does not hold. Let $P = ((\mathbb{N} \oplus \{\top\}) \times \{0\}) \sqcup (\mathbb{N} \times \{1\})$ with the Scott topology, where $\sqcup$ denotes the disjoint union. Consider the filtered family of compact saturated sets $\mathcal{K} = \{\uparrow\{(\top, 0), (i, 1)\} : i \in \mathbb{N}\}$. Then the intersection of $\mathcal{K}$ is the singleton set $\{(\top, 0)\}$. However, no element of $\mathcal{K}$ is contained in $(\mathbb{N} \oplus \{\top\}) \times \{0\}$. Thus ΣP is not weak well-filtered. It is easy to show that ΣP is bounded sober since every upper bounded irreducible closed set is of the form $\downarrow x$ for some $x \in P$.

Example 2. Isbell's complete lattice L with its Scott topology is well-filtered (Isbell 1982; Xi and Lawson 2017). But ΣL is not $\mathcal{S}_0$-fine. Let $M = [0, 1)$ with the usual order of real numbers. Then $(M, \sigma(M))$ is $\mathcal{S}_0$-fine but not well-filtered.

Therefore, we have the following diagram for implications:

$$\begin{tikzcd}
\text{sober} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \mathcal{S}_0\text{-fine} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{bounded sober} \arrow[r, Rightarrow] & \text{k-bounded sober} \\
& \text{well-filtered} \arrow[r, Rightarrow] & \text{weak well-filtered} &
\end{tikzcd}$$

A space is called *weak sober* if each proper irreducible closed set is the closure of a unique point. This concept is introduced by Lu et al. (2019). We find that this sobriety can also be characterized as a Θ-fine property.

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Theorem 6. *A space is weak sober if and only if it is $\mathcal{S}_0$-fine.*

Proof. Let X be a weak sober space and $\mathcal{F} = \{K_i : i \in I\}$ be an irreducible set of $\mathcal{Q}(X)$ with $\bigcap \mathcal{F} \neq \emptyset$. For every open set U such that $\bigcap \mathcal{F} \subseteq U$, if $K_i \nsubseteq U$ for all $i \in I$, then there is an irreducible closed set $A \subseteq X - U \subseteq X$ such that A meets all members of $\mathcal{F}$ by Lemma 3. Since A is a proper subset of X and X is weak sober, we have $A = \mathrm{cl}(\{a\})$ for some $a \in X$. This implies $a \in \bigcap \mathcal{F} \subseteq U$ ($A \cap K_i \neq \emptyset$ and K_i is saturated), deriving a contradiction.

On the other hand, let X be a $\mathcal{S}_0$-fine space. If C is a proper irreducible closed set, then there exists $y \in X$ such that $y \in X - C$. Thus $\bigcap_{x \in C} (\uparrow x \cup \uparrow y) \neq \emptyset$. Suppose that $\{\uparrow x \cup \uparrow y : x \in C\} \bigcap \Box U_j \neq \emptyset$ for $j = 1, 2$. Then there exist $x_j \in C$ such that $\uparrow x_j \cup \uparrow y \subseteq U_j$ for $j = 1, 2$. It follows that $C \cap U_j \neq \emptyset$ for $j = 1, 2$. Since C is irreducible, we have $C \cap U_1 \cap U_2 \neq \emptyset$. Pick $x_3 \in C \cap U_1 \cap U_2$. Note that $y \in U_j$ for $j = 1, 2$. We obtain $\uparrow x_3 \cup \uparrow y \in \Box U_j$ for $j = 1, 2$, i.e. $\{\uparrow x \cup \uparrow y : x \in C\} \bigcap \Box U_1 \bigcap \Box U_2 \neq \emptyset$. Hence $\{\uparrow x \cup \uparrow y : x \in C\}$ is irreducible in $\mathcal{Q}(X)$.

We claim that $\bigcap \{\uparrow x : x \in C\} \nsubseteq X - C$. If not, $\bigcap \{\uparrow x : x \in C\} \subseteq X - C$, then

$$\bigcap \{\uparrow x \cup \uparrow y : x \in C\} = \uparrow y \cup \bigcap_{x \in C} \uparrow x \subseteq X - C$$

for some $y \in X - C$. So there is some $x \in C$ such that $\uparrow x \cup \uparrow y \subseteq X - C$ because X is $\mathcal{S}_0$-fine, which contradicts $x \in C$. Thus $\bigcap \{\uparrow x : x \in C\} \cap C \neq \emptyset$. Take one element $a \in \bigcap \{\uparrow x : x \in C\} \cap C$, then $C = \mathrm{cl}(\{a\})$ due to the definition of specialization order. The uniqueness of a is guaranteed by the T_0 property of X. □

Recall that a space X is called *locally compact* if every neighborhood of a point contains a compact neighborhood of the point.

Theorem 7. *Let X be a locally compact space. Then the following statements are equivalent:*

(1) X is $\mathcal{S}_0$-fine.
(2) X is weak sober.
(3) X is weak well-filtered.

Proof. (1) ⇔ (2): By Theorem 6.

(1) ⇒ (3): It is straightforward.

(3) ⇒ (1): Suppose that $\mathcal{F} = \{K_i : i \in I\}$ is an irreducible subset of $\mathcal{Q}(X)$ with $\bigcap \mathcal{F} \neq \emptyset$ and U is open such that $\bigcap \mathcal{F} \subseteq U$. For any $i \in I$, assume that $K_i \nsubseteq U$. Then by Lemma 3, we have an irreducible closed subset $A \subseteq X - U$ such that $A \cap K_i \neq \emptyset$ for all $i \in I$. Let $\mathcal{H} = \{K \in \mathcal{Q}(X) : A \cap \mathrm{int}(K) \neq \emptyset\}$. $\mathcal{H}$ is not empty since X is locally compact. Moreover, $\mathcal{H}$ is a filtered family of compact saturated sets due to irreducibility of A. For weak well-filtered space, according to Proposition 3.2 in Lu and Li (2017), we obtain $\bigcap \mathcal{H} \cap A \neq \emptyset$. Take $a \in \bigcap \mathcal{H} \cap A \neq \emptyset$. It is easy to show that $A = \mathrm{cl}(\{a\})$. Then $a \in K_i$ for all $i \in I$ since K_i is an upper set. Thus, $a \in \bigcap \mathcal{F} \subseteq U$, contradicting $a \in A \subseteq X - U$. □

There is a noteworthy connection between nonempty open sets and nonempty compact saturated sets in $\mathcal{S}_0$-fine space X.

Lemma 4. *Let X be a $\mathcal{S}_0$-fine space and $\mathcal{F}$ a Scott open filter of open sets of X with $\bigcap \mathcal{F} \neq \emptyset$. Then*

(1) each open set U containing $\bigcap \mathcal{F}$ belongs to $\mathcal{F}$;
(2) the intersection $\bigcap \mathcal{F}$ is a compact saturated set.

Proof. (1) Assume that $U \notin \mathcal{F}$. Let $\mathcal{C} = \{V \in \mathcal{O}(X) - \mathcal{F} : \bigcap \mathcal{F} \subseteq V\}$. By Zorn's Lemma (Theorem 2.4.2 in Goubault 2013) and that $\mathcal{F}$ is a Scott open filter, there is a maximal open neighborhood V containing $\bigcap \mathcal{F}$ that is not in $\mathcal{F}$, i.e. V is maximal in $\mathcal{O}(X) - \mathcal{F}$.

We claim that $X - V$ is an irreducible closed set. Let M, N be two closed sets and $X - V \subseteq M \cup N$. Then $(X - M) \cap (X - N) \subseteq V$. Assume that $X - V \nsubseteq M$ and $X - V \nsubseteq N$. Then there exist x, y such that $x \in X - V$ but $x \notin M$ and $y \in X - V$ but $y \notin N$, i.e. $x \notin V$ but $x \in X - M$ and $y \notin V$ but $y \in X - N$. Thus V is not only the proper subset of $V \cup (X - M)$ but also the proper subset of $V \cup (X - N)$. Since V is maximal in $\mathcal{O}(X) - \mathcal{F}$, we obtain $V \cup (X - M) \in \mathcal{F}$ and $V \cup (X - N) \in \mathcal{F}$. Thus $(V \cup (X - M)) \cap (V \cup (X - N)) = V \cup ((X - M) \cap (X - N)) = V \in \mathcal{F}$ because $\mathcal{F}$ is filtered, which is a contradiction. Therefore $X - V \subseteq M$ or $X - V \subseteq N$, so $X - V$ is irreducible closed.

Note that $V \neq X$ since $\mathcal{F}$ is a filter. Then by Theorem 6, $X - V = \mathrm{cl}(\{a\})$ for some $a \in X$. Moreover, $a \notin \bigcap \mathcal{F}$ due to $\bigcap \mathcal{F} \subseteq V$. Thus, there exists $W \in \mathcal{F}$ such that $a \notin W$. It follows that $(X - V) \cap W = \emptyset$, that is, $W \subseteq V$. Hence $V \in \mathcal{F}$, a contradiction.

(2) It is straightforward by the Scott openness of $\mathcal{F}$. □

So we obtained an analogous Hofmann–Mislove theorem for $\mathcal{S}_0$-fine spaces, which explains that intersecting nonempty Scott open filters correspond to nonempty compact saturated subsets.

Theorem 8. *Let X be a $\mathcal{S}_0$-fine space. For each Scott open filter $\mathcal{F}$ with $\bigcap \mathcal{F} \neq \emptyset$, $\bigcap \mathcal{F}$ is compact saturated and the elements of $\mathcal{F}$ are precisely the open neighborhoods of $\bigcap \mathcal{F}$.*

From Theorem 8, we deduce the following theorem immediately.

Theorem 9. *Every core-compact $\mathcal{S}_0$-fine space is locally compact.*

5. PF-Well-Filtered Space

In this section, we continue the investigation of Θ-fine spaces initiated in the previous sections. Here we make use of filtered (irreducible) $\mathcal{F} \subseteq \mathcal{Q}(X)$ with $\bigcap \mathcal{F} = \uparrow x$ for some $x \in X$, we find that such families give rise to new topological properties that enrich the theory of non-Hausdorff topology.

For each space X, let $\Theta_{w_p}(X)$ consist of all filtered sets $\mathcal{A}$ of $\mathcal{Q}(X)$ such that $\bigcap \mathcal{A} = \uparrow x$ for some $x \in X$, and let $\mathcal{S}_p(X)$ consist of all irreducible subsets $\mathcal{F}$ of $\mathcal{Q}(X)$ such that $\bigcap \mathcal{F} = \uparrow x$ for some $x \in X$.

Definition 2. *A space X is called PF-well-filtered if it is Θ_{w_p}-fine, and PF-sober if it is $\mathcal{S}_p$-fine.*

Remark 4. Clearly every $\mathcal{S}_0$-fine space is PF-sober and every PF-sober space is PF-well-filtered. For any complete lattice L, ΣL is sober iff it is $\mathcal{S}_0$-fine iff it is PF-sober, and ΣL is PF-well-filtered iff it is weak well-filtered iff it is well-filtered. The non-sober complete lattice constructed by Isbell (1982) illustrates the difference between PF-well-filtered space and PF-sober space.

Using Theorem 5, we can prove the following.

Proposition 6. *Every PF-sober space is k-bounded sober.*

By the example given in Remark 3, we see that a k-bounded sober spaces need not be PF-sober.

It is easy to check that each weak well-filtered space is PF-well-filtered. The following two examples show that the converse implication is not true.

Example 3. Consider the poset $E = \{\pm \frac{1}{n} : n \in \mathbb{Z}^+\}$ equipped with the Alexandroff topology τ. It is clear that (E, τ) is a T_0 space. Let $\mathcal{K} = \{\uparrow(-\frac{1}{n}) : n \in \mathbb{Z}^+\}$. Then $\mathcal{K}$ is a filtered family of compact saturated subsets and $\bigcap \mathcal{K} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ is an open set. Since there is no element of $\mathcal{K}$ contained in $\{\frac{1}{n} : n \in \mathbb{Z}^+\}$, (E, τ) is not weak well-filtered (not bounded sober), but it is clearly a PF-sober and locally compact space. The space (E, τ) is not a d-space because E is not a dcpo.

A poset P is said to be PF-well-filtered (PF-sober, weak well-filtered) if the Scott space of P is a PF-well-filtered (PF-sober, weak well-filtered) space. Using Johnstone's example of non-sober dcpo, we can construct a PF-well-filtered dcpo that is not weak well-filtered.

Example 4. Let $L = \mathbb{J} \oplus \mathbb{Z}^-$, where $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ is the non-sober dcpo by Johnstone with the partial order defined by $(j, k) \leq (m, n)$ iff $j = m$ and $k \leq n$, or $n = \omega$ and $k \leq m$ (see Johnstone 1981). Then L is a dcpo and can be depicted in Figure 1.

We claim that L with the Scott topology $\sigma(L)$ is PF-well-filtered, but not weak well-filtered. Indeed, since $\mathcal{K}_0 = \{\uparrow(\mathbb{N} \times \{\omega\}) - F \mid F \subseteq_{fin} \mathbb{N} \times \{\omega\}\}$ is a filtered family of compact saturated sets and $\bigcap \mathcal{K}_0 = \mathbb{Z}^-$, there is no element $K \in \mathcal{K}_0$ such that $K \subseteq \mathbb{Z}^-$.

Note that if K is a nonempty compact saturated subset of L with $K \not\subseteq \uparrow(\mathbb{N} \times \{\omega\})$ and some Scott open set U with $U \cap \mathbb{J} \neq \emptyset$, then we have that $K = \uparrow \mathrm{Min}(K)$ by Lemma 1, $\mathrm{Min}(K)$ and $\uparrow(\mathbb{N} \times \{\omega\}) - U$ are finite by the proof of Example 3.1 in Lu and Li (2017). Suppose that $\mathcal{K}$ is a filtered family of compact saturated subsets and $U \in \sigma(L)$ with $\bigcap \mathcal{K} = \uparrow x \subseteq U$. Each member of $\mathcal{K}$ is nonempty. If there is a $K \in \mathcal{K}$ such that $K \cap \mathbb{J} = \emptyset$, then there must be $x \in \mathbb{Z}^-$ and $\uparrow x \in \mathcal{K}$. Assume that $K \not\subseteq \mathbb{Z}^-$ for all $K \in \mathcal{K}$, this can be divided into two cases. One is $K \not\subseteq \uparrow(\mathbb{N} \times \{\omega\})$ for each member of $\mathcal{K}$, then $\bigcap \mathcal{K} = \bigcap_{K \in \mathcal{K}} \uparrow \mathrm{Min}(K) = \uparrow x \subseteq U$. Hence, there exists $K \in \mathcal{K}$ such that $K \subseteq U$ by Corollary 3. The other is that there is a $K_0 \in \mathcal{K}$ such that $\mathrm{Min}(K_0) \subseteq \mathbb{N} \times \{\omega\}$. By assumption, we have $U \cap \mathbb{J} \neq \emptyset$. For any $(m, \omega) \in \uparrow(\mathbb{N} \times \{\omega\}) - U$, there exists $K_m \in \mathcal{K}$ such that $(m, \omega) \notin K_m$. Because $\mathcal{K}$ is filtered and $\uparrow(\mathbb{N} \times \{\omega\}) - U$ is finite, we obtain $(\uparrow(\mathbb{N} \times \{\omega\}) - U) \cap K_1 = \emptyset$ for some $K_1 \in \mathcal{K}$. Then there is a $K \in \mathcal{K}$ with $K \subseteq K_0 \cap K_1$ such that $(\uparrow(\mathbb{N} \times \{\omega\}) - U) \cap K = \emptyset$. It follows that $K \subseteq U$. Therefore, L is a PF-well-filtered poset.

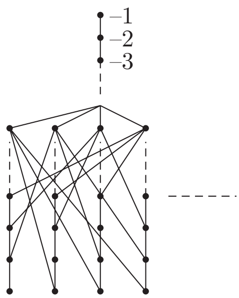

Figure 1. dcpo L.

Example 5. We may consider the dcpo P of $\mathbb{J} \oplus \{\top_1, \top_2\}$, where $\mathbb{J}$ is the Johnstone dcpo and $\top_1, \top_2$ are two incomparable maximal elements. By a similar proof as in Example 4, we can verify that P is PF-well-filtered.

There are many PF-well-filtered posets, however, not every dcpo is PF-well-filtered. The following example shows that the linear sum of two PF-well-filtered posets need not PF-well-filtered.

Example 6. (Lu and Li 2017) Let $M = \mathbb{J} \oplus \{T\}$, where $\mathbb{J}$ is the Johnstone dcpo. M can be depicted as in Figure 2. Consider a filtered family $\mathcal{K} = \{(\mathbb{N} \times \{\omega\} - F) \cup \{\top\} \mid F \subseteq_{fin} \mathbb{N} \times \{\omega\}\}$ of compact saturated subsets and $\{\top\} \in \sigma(M)$. Since the intersection of $\mathcal{K}$ is the upper set $\{\top\}$, we have that M is a dcpo which is neither weak well-filtered nor PF-well-filtered.

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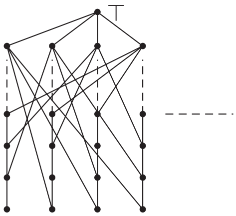

Figure 2. dcpo M.

The following is the refined diagram showing the relationship between the relevant properties:

$$
\begin{tikzcd}
 & & \text{bounded sober} \arrow[dr, Rightarrow] & \\
\text{sober} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \mathcal{S}_0\text{-fine} \arrow[r, Rightarrow] \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \text{PF-sober} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{k-bounded sober} \\
 & \text{well-filtered} \arrow[r, Rightarrow] & \text{weak well-filtered} \arrow[r, Rightarrow] & \text{PF-well-filtered}
\end{tikzcd}
$$

In Section 3, we proved that for locally compact spaces, $\mathcal{S}_0$-fine property and weak well-filteredness are equivalent. But we do not know the answer to the following problem.

Problem 1. *Is the PF-sobriety equivalent to the PF-well-filteredness for locally compact space?*

Note that any two upper sets of a chain have inclusion relation. Let P be a chain. Suppose that A_1 and A_2 are two upper sets of P with $A_1 \not\subseteq A_2$. Then there exists $x \in A_1$ such that $y \not\leq x$ for all $y \in A_2$. Thus $x < y$ for all $y \in A_2$ since P is a chain, that is, $A_2 \subseteq \uparrow x$. Hence $A_2 \subseteq A_1$. Similarly, we obtain $A_1 \subseteq A_2$ if $A_2 \not\subseteq A_1$.

From Example 3, not every chain is weak well-filtered. But for PF-well-filteredness, we have the following result.

Proposition 7. *Every chain P is PF-sober.*

Proof. Let $\mathcal{K} = \{K_i \mid i \in I\}$ be an irreducible set of compact saturated sets in ΣP and U be Scott open with $\bigcap \mathcal{K} = \uparrow x \subseteq U$. Then for any $K_{i_0}, K_{j_0} \in \mathcal{K}$, we have $K_{i_0} \subseteq K_{j_0}$ or $K_{j_0} \subseteq K_{i_0}$. That is, $\mathcal{K}$ is a chain of upper sets of P. Assume that $K_i \not\subseteq U$ for all $K_i \in \mathcal{K}$. If $K_{i_0} \subseteq K_{j_0}$, then there exist $x_{j_0} \in K_{j_0} - K_{i_0}$ and $x_{i_0} \in K_{i_0} - U$ such that $x_{j_0} < x_{i_0}$. Thus $x_i \in K_i$ for all $i \in I$, i.e. $\uparrow x_i \subseteq K_i$. By the hypothesis, we have $\bigcap \uparrow x_i \subseteq \bigcap \mathcal{K} = \uparrow x$. It follows that x is an upper bound of $\{x_i\}_{i \in I}$. If y is an upper bound of $\{x_i\}_{i \in I}$, then $y \in \bigcap \uparrow x_i \subseteq \bigcap \mathcal{K} = \uparrow x$. We obtain that x is the least upper bound of $\{x_i\}_{i \in I}$. So $x = \bigvee_{i \in I} x_i$ with $\{x_i\}_{i \in I} \subseteq P$ a chain. Hence, there exists some $x_i \in U$ because U is Scott open and $x \in U$, a contradiction. $\square$

The *upper topology* $\nu(P)$ on poset P is generated by sets of the form $P - \downarrow x$ for $x \in P$.

Proposition 8. *For any poset P, $(P, \nu(P))$ is a PF-sober space.*

Proof. Let $\mathcal{F}$ be an irreducible set of $\mathcal{Q}(P)$ with $\bigcap \mathcal{F} = \uparrow x \subseteq \bigcap_{y \in A} (P - \downarrow y)$ for some finite set A of P. Assume that $K \not\subseteq \bigcap_{y \in A} (P - \downarrow y)$ for all $K \in \mathcal{F}$. Then there is $y_K \in A$ such that $K \cap \downarrow y_K \neq \emptyset$, implying $K \in \Diamond \downarrow y_K$. Hence $\mathcal{F} \subseteq \bigcup_{y \in A} \Diamond \downarrow y$. Thus $\mathcal{F} \subseteq \Diamond \downarrow y_0$ for some $y_0 \in A$ since $\mathcal{F}$ is irreducible. It follows that $y_0 \in K$ for all $K \in \mathcal{F}$ because K is saturated. Clearly, $y_0 \in \uparrow x$, this contradicts the fact that $\uparrow x \subseteq \bigcap_{y \in A} (P - \downarrow y) \subseteq P - \downarrow y_0$. $\square$

A T_0 space (X, τ) is called a *weak monotone convergence space* if $\tau \subseteq \sigma(X)$.

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Proposition 9. *Every PF-well-filtered space is a weak monotone convergence space.*

Theorem 10. *For any first countable and PF-well-filtered space (X, τ), U is an open set of X if and only if U is an upper set such that for any filtered family $\{K_i\}_{i\in I}$ of compact saturated subsets with $\bigcap_{i\in I} K_i = \uparrow x \subseteq U$, there exists $K_i \subseteq U$ for some $i \in I$.*

Proof. We prove the nontrivial direction. Let U be a set that satisfies the condition. For any $x \in U$, since X is first countable, we may take $\{V_i \mid i \in \mathbb{N}\}$ as a neighborhood base of x with $V_{i+1} \subseteq V_i$. If $V_i \nsubseteq U$ for all $i \in \mathbb{N}$, then we can pick $x_i \in V_i - U$ for each $i \in \mathbb{N}$. Obviously, the sequence $(x_i)_{i\in\mathbb{N}}$ converges to x. Let $A_i = \uparrow\{x_j \mid j \in \mathbb{N} \text{ and } j \geq i\} \cup \uparrow x$. Then $\{A_i \mid i \in \mathbb{N}\}$ is a filtered family of compact saturated sets and $\bigcap_{i\in\mathbb{N}} A_i = \uparrow x$. Thus we have $A_i \subseteq U$ for some $i \in \mathbb{N}$ by the assumption on U, a contradiction. Hence, there exists some $V_i \in \tau$ such that $x \in V_i \subseteq U$. Therefore U is open. □

Example 7. The uncountable set $\mathbb{R}$ of all real numbers equipped with the co-countable topology τ_{coc} is not first countable. Since $\tau_{coc} = \{U \subseteq \mathbb{R} : \mathbb{R} - U \textit{ is countable}\} \cup \{\emptyset\}$, we have that $\{x\}$ is a closed set for all $x \in \mathbb{R}$. Thus $(\mathbb{R}, \tau_{coc})$ is T_1. It follows that its specialization order $\leq$ is equality, i.e. $(\mathbb{R}, \leq)$ is an antichain and $\uparrow x = \{x\}$ for all $x \in \mathbb{R}$.

We claim that only finite subsets of $\mathbb{R}$ are compact. If K is compact and not finite, then there exists countably infinite set A such that $A \subseteq K$. Thus $\mathbb{R} - A \in \tau_{coc}$. Note that $\{(\mathbb{R} - A) \cup F : F \subseteq_{fin} K\}$ is a directed set of τ_{coc} as $\mathbb{R} - ((\mathbb{R} - A) \cup F) \subseteq A$ is countable. Since $K \subseteq \bigcup_{F\subseteq_{fin} K} ((\mathbb{R} - A) \cup F)$ and K is compact, we obtain that $A \subseteq K \subseteq (\mathbb{R} - A) \cup F_0$ for some $F_0 \subseteq_{fin} K$, a contradiction. It is easy to prove that $(\mathbb{R}, \tau_{coc})$ is PF-well-filtered. For any $y \in \mathbb{R}$, $\{y\}$ is an upper set. Suppose that $\mathcal{K}$ is a filtered set of $\mathcal{Q}(\mathbb{R})$ and $\bigcap \mathcal{K} = \{y\}$. Then $\{y\} \in \mathcal{K}$ because all compact sets of $(\mathbb{R}, \tau_{coc})$ are finite. But $\{y\}$ is not open. Therefore, the first countability in Theorem 10 is necessary.

We may also use closed subsets to characterize PF-well-filtered spaces.

Proposition 10. *A space (X, τ) is PF-well-filtered iff for every nonempty proper closed subset C of X and filtered family $\{K_i\}_{i\in I}$ of compact saturated sets with $\bigcap_{i\in I} K_i = \uparrow x$, $K_i \cap C \neq \emptyset$ for all $i \in I$ implies $x \in C$.*

Proof. Suppose that $\{K_i\}_{i\in I}$ is a filtered subset of compact saturated sets with $\bigcap_{i\in I} K_i = \uparrow x$, and C is a nonempty proper closed subset with $K_i \cap C \neq \emptyset$ for all $i \in I$. Assume that $\uparrow x \cap C = \emptyset$. Then $\uparrow x \subseteq X - C$. Thus there exists $i \in I$ such that $K_i \subseteq X - C$ because X is PF-well-filtered, a contradiction. Conversely, if $U \in \tau$ and $K_i \nsubseteq U$ for all $i \in I$, then $K_i \cap (X - U)$ is nonempty. It follows that $x \in X - U$, which contradicts the assumption $\bigcap_{i\in I} K_i = \uparrow x \subseteq U$. □

In Heckmann and Keimel (2013), it is proved that a space X is sober iff the upper space $\mathcal{Q}(X)$ is sober. This characterization also applied to well-filtered space in Xu et al. (2019). Naturally, a question arises: Can PF-sober space be characterized by its upper space? The answer is No.

Example 8. Consider the poset $P = \mathbb{N} \oplus \{a, b\}$ where a and b are incomparable. Then ΣP is a PF-sober space. Note that $\{\uparrow_{\mathcal{Q}(P)}(\uparrow i) : i \in \mathbb{N}\}$ is a filtered family of the upper space $\mathcal{Q}(P)$ and $\bigcap_{i\in\mathbb{N}} \uparrow_{\mathcal{Q}(P)}(\uparrow i) = \uparrow_{\mathcal{Q}(P)}(\uparrow\{a, b\})$. Also $\bigcap_{i\in\mathbb{N}} \uparrow_{\mathcal{Q}(P)}(\uparrow i) \subseteq \Box(\uparrow\{a, b\})$ and no member $\uparrow_{\mathcal{Q}(P)}(\uparrow i)$ is contained in $\Box(\uparrow\{a, b\})$.

In Jia et al. (2016), the coherence of well-filtered dcpos is equivalent to the compactness of the intersections of any two principal filters. The result of Jia et al. is also true for weak well-filtered posets in Lu and Li (2017). Unfortunately, this result does not hold for PF-well-filtered posets.

Example 9. Let $B = \{b_i \mid i \in \mathbb{N}\}$, where $b_i's$ are distinct elements, and $\mathcal{W}$ be the disjoint union of P in Example 5 and B. We define a partial order on $\mathcal{W}$ as follows: $x \leq y$ in $\mathcal{W}$ iff $x \leq y$ in P, or $x = (m, n) \in \mathbb{J}$ and $y = b_i$ with $n \leq i$.

Then $\mathcal{W}$ is a noncoherent dcpo by Example 4.1.2 in Jia (2018). It is easy to show that the intersection of any two principal filters in $\mathcal{W}$ is compact, since either the intersection is empty, or a principal filter, or the set $\{\top_1, \top_2\}$, or contains some elements contained in the set $\mathbb{N} \times \{\omega\}$. It follows from Lemma 3.1 in Lu and Li (2017), $\mathcal{W}$ cannot be weak well-filtered.

We show that $\mathcal{W}$ is a PF-well-filtered poset. Let $\mathcal{K}$ be a filtered family of compact saturated sets with $\bigcap\mathcal{K} = \uparrow x$ and U a Scott open set with $\bigcap\mathcal{K} = \uparrow x \subseteq U$. Then $\mathrm{Min}(K) \cap B$ is a finite set since K is compact for all $K \in \mathcal{K}$. If $x \in P$, then $\{K \cap P : K \in \mathcal{K}\}$ is a cofinal subfamily of $\mathcal{K}$. Since P is PF-well-filtered, we have that $\mathcal{W}$ is a PF-well-filtered poset. If $x \in B$, i.e. $x = b_t$ for some $b_t \in B$, then either $\{b_t\} \in \mathcal{K}$ or $K \nsubseteq B$ for all $K \in \mathcal{K}$. For the second case, there exists $y \in P \cap K$ such that $y \leq \top_1, \top_2$ for all $K \in \mathcal{K}$. This implies $\{\top_1, \top_2\} \subseteq \bigcap\mathcal{K} = \uparrow b_t$, a contradiction. Hence, $\mathcal{W}$ is PF-well-filtered.

A retract of a sober (respectively, well-filtered, weak well-filtered) space is still sober (respectively, well-filtered, weak well-filtered). However, as the following example shows, this is not true for PF-well-filtered spaces.

Example 10. We first define functions between L and M, corresponding to Figures 1 and 2, respectively. Let $s : M \to L$ send $\top$ to -1 and the remainder to itself, and define $r : L \to M$ by the following:

$$r(y) = \begin{cases} \top, & y \in \mathbb{Z}^-, \\ y, & y \in \mathbb{J}. \end{cases}$$

The trivially $r \circ s = \mathrm{id}_M$. For every $U \in \sigma(L)$, $s^{-1}(U) = (U - \mathbb{Z}^-) \cup \{\top\}$ if $U \cap \mathbb{J} \neq \emptyset$ and $s^{-1}(U) = \{\top\}$ whenever $U \cap \mathbb{J} = \emptyset$. Then s is continuous with respect to the Scott topology. So is r. Hence $(M, \sigma(M))$ is a retract of the PF-well-filtered space $(L, \sigma(L))$. But M is not PF-well-filtered as shown in Example 6.

Next, we discuss the mappings that preserve the PF-well-filteredness.

Let P and Q be two posets. We say that a pair (g, d) of functions $g : P \to Q$ and $d : Q \to P$ is a *Galois connection* or an *adjunction* (Gierz et al. 2003) between P and Q provided that

(i) both g and d are monotone and
(ii) the relations $g(s) \geq t$ and $s \geq d(t)$ are equivalent for all pairs of elements $(s, t) \in P \times Q$.

Theorem 11. *Let (g, d) be a pair of continuous maps and an adjunction such that g is surjective between poset Q and PF-well-filtered poset P. Then Q is PF-well-filtered.*

Proof. Suppose that $\mathcal{K} = \{K_i\}_{i \in I}$ is a filtered family of compact saturated subsets of Q and $U \in \sigma(Q)$ with $\bigcap\mathcal{K} = \bigcap_{i \in I} K_i = \uparrow x \subseteq U$. Then $d(K_i)$ is compact in P by d is continuous. Moreover, the family $\{\uparrow d(K_i) : K_i \in \mathcal{K}\}$ is a filtered set of compact saturated subsets of P by $\uparrow d(K_i) = \mathrm{sat}(d(K_i))$. Since $\uparrow d(x) = g^{-1}(\uparrow x)$ and $g^{-1}(U)$ is a Scott open set of P, we obtain that $\uparrow d(x) \subseteq g^{-1}(U)$. Obviously, $\uparrow d(x) \subseteq \bigcap_{i \in I} \uparrow d(K_i)$. For any $t \in \bigcap_{i \in I} \uparrow d(K_i)$, there exists $y_i \in K_i$ such that $d(y_i) \leq t$, i.e. $y_i \leq g(t)$ for each $i \in I$. Then we have $g(t) \in \bigcap\mathcal{K}$. It follows that $d(x) \leq t$. Thus, $\uparrow d(x) = \bigcap_{i \in I} \uparrow d(K_i)$. As P is PF-well-filtered, there is some $i_0 \in I$ such that $\uparrow d(K_{i_0}) \subseteq g^{-1}(U)$. Hence, $K_{i_0} = g(d(K_{i_0})) \subseteq U$. Therefore, Q is a PF-well-filtered poset. □

Proposition 11. *Given an adjunction (g, d) between poset Q and PF-sober poset P, if g and d are continuous mappings and d is injective, then Q is a PF-sober space.*

Proof. Let $\mathcal{K} = \{K_i : i \in I\}$ be an irreducible subset of compact saturated sets of Q. Here we only conform that $\{\uparrow d(K_i) : K_i \in \mathcal{K}\}$ is an irreducible set of $\mathcal{Q}(P)$, the rest of the proof is the same as that of Theorem 11. Indeed, if $\{\uparrow d(K_i) : i \in I\} \bigcap \Box U_j \neq \emptyset$ with $U_j \in \sigma(P)$ for

$j = 1, 2$, then there exists $K_j \in \mathcal{K}$ such that $\uparrow d(K_j) \subseteq U_j$ for $j = 1, 2$. Thus $\mathcal{K} \bigcap \Box d^{-1}(U_j) \neq \emptyset$. Therefore $\mathcal{K} \bigcap \Box d^{-1}(U_1) \bigcap \Box d^{-1}(U_2) \neq \emptyset$ by the irreducibility of $\mathcal{K}$. Hence, $\{\uparrow d(K_i) : K_i \in \mathcal{K}\} \bigcap \Box U_1 \bigcap \Box U_2 \neq \emptyset$. $\Box$

Corollary 4. *Let Y be a PF-well-filtered (PF-sober) space. If there exist continuous maps $f : X \to Y$ and surjection $g : Y \to X$ such that (g, f) is an adjunction between $(Y, \leq)$ and $(X, \leq)$, then X is PF-well-filtered (PF-sober).*

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