BULLETIN **13** (3), 1970

THE COEFFICIENTS OF $\dfrac{\sinh x}{\cos x}$

BY
J. M. GANDHI[1]

1. **Introduction.** The purpose of the present paper is to investigate some of the properties of the coefficient K_{2n} defined by

(1.1) $$\frac{\sinh x}{\cos x} = \sum_{n=0}^{\infty} K_{2n} \frac{x^{2n+1}}{(2n+1)!}.$$

We prove

(1.2) $$K_{2n} \equiv 1 (\mathrm{mod}\ 2n+1) \quad \text{if } 2n+1 \text{ is prime.}$$

(1.3) $$K_{4n+2} \equiv 4 (\mathrm{mod}\ 10), \qquad K_{4n+4} \equiv 6 (\mathrm{mod}\ 10)$$

(1.4) $$K_{2n} = 2^{n+1} g(n) - \frac{2^n}{2n+2} \sum_{s=0}^{n} (-1)^{\frac{1}{2}(n+s)} \binom{2n+2}{2s} B_{2s}\, \alpha(n, s)$$

where $g(n) = (-1)^{(n+1)/2}$ if n is odd, $g(n) = (-1)^{n/2}$ if n is even, $\alpha(n, s) = 2$ if $n-s+1$ is odd, $\alpha(n, s) = 0$ if $n-s+1$ is even, and B_{2s} are the well-known Bernoulli's numbers.

As corollaries to (1.4) we prove

(1.5) $$K_{4n+2} \equiv 0 (\mathrm{mod}\ 2^{2n+2}), \qquad K_{4n+2} \not\equiv 0 (\mathrm{mod}\ 2^{2n+3})$$

(1.6) $$K_{4n} \equiv 0 (\mathrm{mod}\ 2^{2n}), \qquad K_{4n} \not\equiv 0 (\mathrm{mod}\ 2^{2n+1})$$

Also let

(1.7) $$K_{4n}/2^{2n} = K'_{4n} \quad \text{and} \quad K_{4n+2}/2^{2n+2} = K'_{4n+2}.$$

From (1.4) we prove the following interesting special cases.

(1.8) $$K'_{4m} \equiv 1 (\mathrm{mod}\ 4)$$

(1.9) $$K'_{4m+2} \equiv (-1)^m (\mathrm{mod}\ 4)$$

and

(1.10) $$K'_{4m} \equiv 2(-1)^m - \frac{(-1)^m}{2m+1} (\mathrm{mod}\ 32)$$

(1.11) $$K'_{4m+2} \equiv (-1)^m - \frac{(-1)^{m+1} 2(4m+3)}{3} (\mathrm{mod}\ 32)$$

For a large part of this paper we follow Carlitz's paper [3]. For other related coefficients, for example, sinh x/sin x, cosh x/cos x, etc., the papers listed may be referred to.

Received by the editors, in revised form, September 15, 1969.

[1] Supported by the Summer Research Institute of the Canadian Mathematical Congress, held at Montreal in 1969.

2. Now (1.1) can be written as

$$\sinh x = \cos x \sum_{n=0}^{\infty} \frac{K_{2n}x^{2n+1}}{(2n+1)!}.$$

Writing the expansions of $\sinh x$ and $\cos x$, and simplifying and equating the coefficients of x^{2n+1}, we have

$$(2.1)\qquad K_{2n} = \binom{2n+1}{2n+1}+\binom{2n+1}{2n-1}K_{2n-2} -\binom{2n+1}{2n-3}K_{2n-4}+\cdots+(-1)^{n+1}K_0\binom{2n+1}{1}.$$

Using (2.1) we calculate some values of K_{2n}.

TABLE 1

$K_0 = 1.$	$K_8 = 18256.$
$K_2 = 4.$	$K_{10} = 81, 41, 44.$
$K_4 = 36.$	$K_{12} = 51, 47, 57, 76.$
$K_6 = 624.$	

From (2.1) it is easy to prove that except K_0 all other coefficients are even positive integers.

Now we prove (1.2). Let $2n+1$ be a prime then $\binom{2n+1}{\gamma}$ will always have a factor $2n+1$ for all values of γ except when $\gamma=0$ and $\gamma=2n+1$. Hence when (2.1) is divided by $2n+1$, the remainder will be $\binom{2n+1}{2n+1}$ and hence we get $K_{2n}\equiv 1(\text{mod } 2n+1)$ if $2n+1$ is prime.

Proof of (1.3). Putting $2n+1$ for n in (2.1) we get

$$(2.2)\qquad K_{4n+2} = \binom{4n+3}{4n+3}+\binom{4n+3}{4n+1}K_{4n} -\binom{4n+3}{4n-1}K_{4n-2}+\cdots-K_2\binom{4n+3}{2}+K_0\binom{4n+3}{1}.$$

Assume that

$$(2.3)\qquad K_{4n} \equiv 6(\text{mod } 10) \quad\text{and}\quad K_{4n-2} \equiv 4(\text{mod } 10)$$

for $n=1, 2,\ldots, n$; then from (2.2) by elementary but lengthy discussions we can prove (2.4) $K_{4n+2} \equiv 4(\text{mod } 10)$.

Now substituting $2n$ for n in (2.1) and using (2.3) and (2.4) we can prove

$$K_{4n+4} \equiv 6(\text{mod } 10)$$

and the result follows by the usual method of induction.

https://doi.org/10.4153/CMB-1970-059-9 Published online by Cambridge University Press

3. We have

$$1/\cos x = \sum_{n=0}^{\infty} (-1)^n E_{2n} x^{2n}/(2n)! \tag{3.1}$$

where E_{2n} are the Euler numbers in the even suffix notation.

Using (3.1) and (1.1) it is easy to prove

$$\sum_{\gamma=0}^{n} (-1)^n \binom{2n+1}{2\gamma} E_{2\gamma} = K_{2n}. \tag{3.2}$$

Let $f(x)$ be an odd polynomial defined by

$$f(x) = \frac{1}{2n+2} \sum_{\gamma=0}^{n} (-1)^\gamma \binom{2n+2}{2\gamma+1} x^{2\gamma+1} \tag{3.3}$$

so that

$$f'(x) = \sum_{\gamma=0}^{n} \frac{(-1)^\gamma (2n+1)!}{(2\gamma)!\,(2n-2\gamma+1)!} x^{2\gamma}.$$

Therefore,

$$f'(E) = \sum_{\gamma=0}^{n} (-1)^\gamma \binom{2n+1}{2\gamma} E_{2\gamma} = K_{2n} \quad \text{by (3.3).}$$

Now it was proved by Carlitz [3] that

$$f'(E) = -f(4B+1) \tag{3.4}$$

where the B's are the well-known Bernoulli numbers. Thus

$$K_{2n} = -\frac{1}{2n+2} \sum_{\gamma=0}^{n} (-1)^\gamma \binom{2n+2}{2\gamma+1} (4B+1)^{2\gamma+1}. \tag{3.5}$$

Now

$$\sum_{\gamma=0}^{n} (-1)^\gamma \binom{2n+2}{2\gamma+1} (4B+1)^{2\gamma+1} = \sum_{\gamma=0}^{n} (-1)^\gamma \binom{2n+2}{2\gamma+1} \sum_{s=0}^{2\gamma+1} \binom{2\gamma+1}{s} 4^s B_s$$
$$= \sum_{s=0}^{2n+1} \binom{2n+2}{s} 4^s B_s \sum_{\gamma} (-1)^\gamma \binom{2n-s+2}{2n-2\gamma+1}.$$

Since

$$\sum_{\gamma=s-1}^{2n+1} \binom{2n-s+2}{2n-\gamma+1} x^\gamma = x^{s-1}(1+x)^{2n-s+2}$$

it is evident that

$$\sum_{\gamma} \binom{2n-s+1}{2n-2\gamma} (-1)^\gamma = \frac{i^{s-1}}{2} [(1+i)^{2n-s+2} + (-1)^{s-1}(1-i)^{2n-s+2}].$$

In particular we have

$$\sum_{\gamma=0}^{n} \binom{2n+1}{2n-2\gamma+1} (-1)^\gamma = \sum_{\gamma=0}^{n} (-1)^\gamma \binom{2n+1}{2\gamma}. \tag{3.6}$$

By elementary methods it can be shown that the right-hand side of (3.6) equals $2^n(-1)^{(n+1)/2}$ when n is odd and $2^n(-1)^{n/2}$ when n is even.

https://doi.org/10.4153/CMB-1970-059-9 Published online by Cambridge University Press

Also

$$\sum_{\gamma=s}^{n}(-1)^{\gamma}\binom{2n-2s+2}{2n-2\gamma+1} = i^{(2s-1)/2}[(1+i)^{2n-2s+2}+(-1)^{2s-1}(1-i)^{2n-2s+2}]$$
$$= 2^{n-s}(-1)^{1/2(n+s)}\alpha(n, s).$$

where $\alpha(n)=2$ if $n-s+1$ is odd and $\alpha(n)=0$ if $n-s+1$ is even.

It follows that

$$\sum_{\gamma=0}^{n}(-1)^{\gamma}\binom{2n+2}{2\gamma+1}(4B+1)^{2\gamma+1} = \sum_{s=0}^{2n+1}\binom{2n+2}{s}4^{s}B_{s}\sum_{\gamma}(-1)^{\gamma}\binom{2n-s+2}{2n-2\gamma+1}.$$

Since $B_{2n+1}=0$ we have

$$= -2^{n+1}(2n+2)g(n)+\sum_{s=0}^{n}\binom{2n+2}{2s}2^{4s}B_{2s}2^{n-s}(-1)^{(n+s)/2}\alpha(n, s)$$

$$= -2^{n+1}(2n+2)g(n)+2^{n}\sum_{s=0}^{n}(-1)^{(n+s)/2}\binom{2n+2}{2s}2^{3s}B_{2s}\alpha(n, s),$$

where $g(n)=(-1)^{(n+1)/2}$ if n is odd and $g(n)=(-1)^{n/2}$ if n is even.

Then (3.5) becomes

$$\text{(3.7)}\qquad K_{2n} = 2^{n+1}g(n)-\frac{2^{n}}{2n+2}\sum_{s=0}^{n}(-1)^{(n+s)/2}\binom{2n+2}{2s}2^{3s}B_{2s}\alpha(n, s).$$

Substituting $n=2m$ and $n=2m+1$ we respectively get

$$\text{(3.8)}\qquad K_{4m} = 2^{2m+1}g(2m)-\frac{2^{2m}}{4m+2}\sum_{s=0}^{2m}(-1)^{(2m+s)/2}\binom{4m+2}{2s}2^{3s}B_{2s}\alpha(2m, s)$$

and

$$\text{(3.9)}\qquad K_{4m+2} = \binom{2m+2}{2}g(2m+1)$$
$$-\frac{2^{2m+1}}{4m+4}\sum_{s=0}^{2m+1}(-1)^{(2m+1+s)/2}\binom{4m+4}{2s}2^{3s}B_{2s}\alpha(2m+1, s).$$

From (3.8) we have

$$\frac{K_{4m}}{2^{2m}} = 2(-1)^{m}-\frac{1}{4m+2}\sum_{s=0}^{2m}(-1)^{(2m+s)/2}\binom{4m+2}{2s}2^{3s}B_{2s}\alpha(2m, s)$$

since all terms on right are even except the term with $s=0$, which is $-1/(2m+1)$, and hence

$$K_{4m} \equiv 0 \pmod{2^{2m}} \quad\text{and}\quad K_{4m} \not\equiv 0 \pmod{2^{2m+1}}.$$

Similarly from (3.9) we get

$$K_{4m+2} \equiv 0 \pmod{2^{2m+2}} \quad\text{and}\quad K_{4m+2} \not\equiv 0 \pmod{2^{2m+3}}.$$

Now letting

$$\text{(3.10)}\qquad \frac{K_{4n}}{2^{2n}} = K'_{4n} \quad\text{and}\quad \frac{K_{4n+2}}{2^{2n+2}} = K'_{4n+2}$$

https://doi.org/10.4153/CMB-1970-059-9 Published online by Cambridge University Press

(3.8) and (3.9) respectively become

$$(3.11)\qquad K'_{4m} = 2(-1)^m - \frac{1}{4m+2}\sum_{s=0}^{2m}(-1)^{(2m+s)/2}\binom{4m+2}{2s}2^{3s}B_{2s}\alpha(2m,s)$$

$$(3.12)\qquad K'_{4m+2} = (-1)^{m+1} - \frac{1}{2(4m+4)}\sum_{s=0}^{2m+1}(-1)^{(2m+1+s)/2} \times\binom{4m+4}{2s}2^{3s}B_{2s}\alpha(2m+1,s).$$

From (3.12) it follows that

$$K'_{4m+2} \equiv (-1)^{m+1} - \frac{(-1)^{m+1}}{2(4m+4)}\binom{4m+4}{2}2^3\times\frac{1}{6}\times 2 \pmod 4$$
$$\equiv -(-1)^{m+1} \equiv (-1)^m \pmod 4.$$

whereby (1.9) is being proved.

From (3.10) we have

$$K'_{4m} \equiv 2(-1)^m - \frac{1}{4m+2}(-1)^m\times 2 \pmod 4$$

or

$$(2m+1)K'_{4m} \equiv (-1)^m \pmod 4.$$

from which it follows that if m is even then $K'_{4m} \equiv 1 \pmod 4$, while if m is odd then $3K'_{4m} \equiv -1 \pmod 4$ or $K'_{4m} \equiv 1 \pmod 4$, i.e. $K'_{4m} \equiv 1 \pmod 4$ for all values of m, whereby (1.8) is being proved. Since in (3.11) and (3.12), the terms in summations are divisible by 32 except the first, and hence

$$K'_{4m} \equiv 2(-1)^m - \frac{(-1)^m}{2m+1} \pmod{32}$$

and

$$K'_{4m+2} \equiv (-1)^{m+1} - \frac{(-1)^{m+1}2(4m+3)}{3} \pmod{32}.$$

Before concluding we remark that using (1.7) and the fact that the last digit of K_{4n+2} is 4 and the last digit of K_{4n+4} is 6 it can be easily proved that $K'_{8n+2} \equiv 1 \pmod{10}$, $K'_{8n} \equiv 1 \pmod{10}$, $K'_{8n+4} \equiv 9 \pmod{10}$ and $K'_{8n+6} \equiv 9 \pmod{10}$. We list some values of K'.

TABLE 2

$K'_0 = 1.$	$K'_8 = 11,41.$
$K'_2 = 1.$	$K'_{10} = 12,721.$
$K'_4 = 9.$	$K'_{12} = 80,43,09.$
$K'_6 = 39.$	

ACKNOWLEDGEMENT. I am grateful to Professor L. Carlitz for many helpful discussions.

https://doi.org/10.4153/CMB-1970-059-9 Published online by Cambridge University Press

References

1. L. Carlitz, *The coefficients of* $\frac{\sinh x}{\sin x}$, Math. Mag. **29** (1956), 193–197.

2. ———, *Note on coefficients of* $\frac{\cosh x}{\cos x}$, Math. Mag. **32** (1955), 132 and 136.

3. ———, *The coefficients of* $\frac{\cosh x}{\cos x}$, Monatsh. Math. **69** (1965), 123–135.

4. J. M. Gandhi, *The coefficients of* $\frac{\cosh x}{\cos x}$ *and a note on Carlitz's coefficients of* $\frac{\sinh x}{\sin x}$, Math. Mag. **31** (1958), 185–191.

5. J. M. Gandhi and A. Singh, *Fourth interval formula for the coefficients of* $\frac{\cosh x}{\cos x}$, Monatsh. Math. (4) **70** (1966), 327–330.

6. N. E. Norlund, *Vorlesungen uber Differenzenrechung*, Berlin, 1924.

7. H. Salie, *Arithmetische Eigenschaffen der koeffizienten speziellen Hurwitzchen Potenzreihen*, Wissenchaftliche Zeitschrift der Karl Marx Universitat, Leipzig **12** Jahrgung (1963) Math. Naturwiss Reihe, Heft **3**, 617–618.

York University,
Toronto, Ontario
Western Illinois University,
Macomb, Illinois