CONCERNING UPPER SEMI-CONTINUOUS DECOMPOSITIONS OF E^n WHOSE NON-DEGENERATE ELEMENTS ARE POLYHEDRAL ARCS OR STAR-LIKE CONTINUA

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1. Introduction. In (**1**) Armentrout raised the question "Is there a monotone decomposition of E^3 into arcs?"[1] The analogous question for E^2 was answered negatively by Roberts in (**8**). Our aim in this paper is to give a partial answer to Armentrout's question by proving the following theorem.

THEOREM 1. *Suppose that G is an upper semi-continuous decomposition of Euclidean n-space E^n $(n \geqq 1)$ so that there is a positive integer m such that if g is a non-degenerate element of G, then g is a polygonal arc of the form $A_1A_2 \ldots A_m$. Then, if g is a non-degenerate element of G and ϵ is a positive number, there is a degenerate element of G which lies within ϵ of g.*

The next theorem is one whose proof is analogous to that of Theorem 1.

THEOREM 2. *We change the statement of Theorem 1 by requiring that each non-degenerate element of G be a compact continuum that is star-like relative to a unique point.*

Using an indirect argument, we prove Theorems 1 and 2 by showing that in either case, an application of the following theorem yields a contradiction.

THEOREM 3. *Suppose that G' is an upper semi-continuous decomposition of a closed geometric n-simplex T such that* (1) *each element of G' is compact and* (2) *there is an open subset U of B, the boundary of T, such that every element of G' which intersects U is a point or a straight line interval that intersects B in only one point. Then, some element of G' is a subset of* Int T.

In (**3**) Bing has shown that if G is an upper semi-continuous decomposition of E^3 having only countably many non-degenerate elements, then E^3/G is homeomorphic to E^3 if G satisfies one of the following conditions: (1) each element of G is point-like and the sum of the non-degenerate elements is a G_δ; (2) each non-degenerate element of G is star-like; (3) each non-degenerate element of G is a tame arc. In (**4**) Bing has given an example of an upper semi-continuous decomposition G of E^3 into points and a Cantor set of straight

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[1]Stephen Jones (**5**) has recently solved the problem for E^n.

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line intervals such that E^3/G is considered to be probably topologically different from E^3. In (**6**) McAuley has shown that if G is an upper semi-continuous decomposition of E^3 into points and straight line intervals and the intervals of G point in only countably many directions, then E^3/G is topologically E^3. For earlier work and basic theorems on upper semi-continuous decompositions, see Moore (**7**) and Whyburn (**9**). For an excellent expository paper with a fairly complete bibliography, see Armentrout (**1**).

2. Definitions. The statement that G is an *upper semi-continuous decomposition of the topological space* X means that G is a decomposition of X such that if $g \in G$ and U is an open set in X containing g, then the union of the elements of G contained in U is open in X. We consider here only decompositions into compact sets. A *monotone decomposition* of a space is an upper semi-continuous decomposition into compact continua. The *decomposition space* X/G associated with a space X and an upper semi-continuous decomposition G of X is the space whose points are elements of G and whose open sets are those subsets H of G such that $\bigcup H$ is open in X. The natural mapping $P: X \to X/G$ is a closed, continuous mapping. The decomposition G will be said to be *continuous* at the element g of G provided it is true that if C is a finite proper cover of g by open sets in X, then there is an open set V containing g such that every element of G which intersects V is a subset of $\bigcup C$ and intersects every member of C. A continuum M in E^n is said to be *point-like* provided $E^n - M$ is homeomorphic to the complement of a point in E^n. A *decomposition G of E^n is said to be point-like* provided every member of G is a point-like continuum.

An arc M in E^n will be said to be *polygonal* provided M is the union of a finite number of straight line intervals. We denote such an M by the symbol $A_1A_2 \ldots A_m$, where each A_iA_{i+1} is a straight line interval and where no straight line contains $A_{i-1}A_i$ and A_iA_{i+1} for any i, $2 \leqq i \leqq m - 1$. A compact continuum g is said to be *star-like* provided there is a point P of g such that if X is a point of $g - P$, then interval XP is a subset of g. Here, let us say g is *star-like relative to* P. If g_1 and g_2 are two point sets in the metric space (X, d) the Hausdorff distance $H(g_1, g_2)$ from g_1 to g_2 is $\text{lub}\{d(x, g_i) | i = 1 \text{ or } 2;\ x \in g_1 \cup g_2\}$. The word *compact* is used in the "finite cover" sense.

3. Some lemmas. The first lemma is an immediate consequence of Theorem 1 of (**8**). An analogous theorem is also proved by Bing in Theorem 2 of (**2**).

LEMMA 1. *Suppose that G is an upper semi-continuous decomposition of the complete metric space (X, d) such that each element of G is compact. Then, the set K of all elements of G at which G is continuous is a dense G_δ in X/G.*

We now prove a lemma in slightly more general form than is needed here, but which should be useful in attacking the problem of showing that there is no monotone decomposition of E^n into polygonal arcs.

LEMMA 2. *Suppose that $A_1A_2 \ldots A_n$ is a polygonal arc in E^N and ϵ is a positive number. Then, there is a positive number h such that if $B_1, B_2, \ldots, B_m$ is a polygonal arc whose Hausdorff distance from $A_1A_2 \ldots A_n$ is less than h, then (1) $m \geqq n$, and (2) if $m = n$, then $d(A_i, B_i) < \epsilon$, $i = 1, \ldots, n$, or $d(A_i, B_{n+1-i}) < \epsilon$, $i = 1, \ldots, n$.*

Proof. Let k denote a positive number less than each of ϵ, $4^{-1}d(A_1, A_2), \ldots, 4^{-1}d(A_{n-1}, A_n)$. There exist points X_i, Y_i, $i = 1, \ldots, n-1$, such that $X_i, Y_i \in A_iA_{i+1}$ and $d(A_i, X_i) = d(Y_i, A_{i+1}) = k$. There is a positive number r such that (1) $r < k$, $r < \epsilon - k$, (2) $r < 4^{-1}d(A_iA_{i+1}, A_jA_{j+1})$, $1 \leqq i < i+1 < j < n$, and (3) no straight line intersects

$$\mathrm{Cl}\, N(A_iY_i, r), \mathrm{Cl}\, N(A_{i+1}, r), \text{ and } \mathrm{Cl}\, N(X_{i+1}A_{i+2}, r), \quad i = 1, \ldots, n-2.$$

(If $M \subset E^N$, then $N(M, r) = \{X | \, d(X, M) < r\}$.) Also, let $C = \{g_1, \ldots, g_k\}$ denote a simple chain of spherical open sets of the form $N(X, r)$ such that (1) each g_i is centred on a point of $A_1A_2 \ldots A_n$ and intersects g_j if and only if $|i - j| \leqq 1$, and (2) each A_i is the centre of some g_j, $g_1 = N(A_1, r)$ and $g_k = N(A_n, r)$. Note that C covers $A_1A_2 \ldots A_n$. Let h denote a positive number such that $\mathrm{Cl}\, N(A_1A_2 \ldots A_n, h) \subset \bigcup C$.

Now suppose that $B_1B_2 \ldots B_m$ is a polygonal arc such that $H(B_1B_2 \ldots B_m, A_1A_2 \ldots A_n) < h$. Let X_1' denote a point of $B_1B_2 \ldots B_m \cap g_1$ and Y_{n-1}' a point of $B_1B_2 \ldots B_m \cap g_k$ and suppose, for example, that X_1' precedes Y_{n-1}' in the order from B_1 to B_m on $B_1B_2 \ldots B_m$.

Let Y_1' denote the last point of subarc $X_1'\, Y_{n-1}'$ of $B_1B_2 \ldots B_m$ on $\mathrm{Bd}\, N(A_1Y_1, r)$. Let X_2' denote the first point of arc $Y_1'\, Y_{n-1}'$ which lies on $\mathrm{Bd}\, N(X_2A_3, r)$, and let Y_2' denote the last point of $X_2'\, Y_{n-1}'$ on $\mathrm{Bd}\, N(A_2Y_2, r)$. Consider a continuation of this process to obtain X_3', $Y_3', \ldots, X_{n-1}'$. Now, if we let $B_{n_i}B_{n_i+1}$ denote an interval of $B_1B_2 \ldots B_m$ containing X_i', then we see that $n_1 \leqq n_2 \leqq \ldots \leqq n_{m-1}$. But also, if some $n_i = n_{i+1}$, then $B_{n_i}B_{n_i+1}$ intersects $\mathrm{Cl}\, N(A_iY_i, r)$, $\mathrm{Cl}\, N(A_{i+1}, r)$, and $\mathrm{Cl}\, N(X_{i+1}A_{i+2}, r)$, a contradiction. Hence $n_1 < n_2 < \ldots < n_{m-1}$; thus $m \geqq n$.

Now, suppose that $m = n$ and, as above, that X_1' precedes Y_{m-1}' in the order from B_1 to B_m on $B_1B_2 \ldots B_m$. Some B_{j_i} lies between Y_{i-1}' and X_i' on $B_1B_2 \ldots B_m$ for $i = 2, \ldots, m-1$. Clearly, the open ball $N(A_i, k + r)$ contains the segment $Y_{i-1}'X_i'$ of $B_1B_2 \ldots B_m$, $i = 2, \ldots, m-1$, and has radius less than ϵ; which implies that $d(B_{j_i}, A_i) < \epsilon$. Now let B_pB_{p+1} contain Y_1', where p is as small as possible, and let B_qB_{q+1} contain X_{n-1}', where q is as large as possible. If $p \geqq 2$, we see that $j_{m-1} \geqq m = n$ and that $q + 1 > m$, a contradiction. Thus $p = 1$, and analogously, $q = m - 1$. Therefore, we see that $d(A_i, B_i) < k + r < \epsilon$, $2 \leqq i \leqq m - 1$. If B_1 is not an element of g_1, then we find that there must be two points of B_1B_2 on

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Bd g_1, where only one of them can belong to $\text{Int}(\bigcup C)$. Analogously, $B_m \in g_k$. This completes the proof of Lemma 2.

The next lemma is not really needed in this paper, but would be of use to anyone attacking the problem (stated above) of filling up E^n with polygonal arcs.

LEMMA 3. *Suppose that G is an upper semi-continuous decomposition of E^n such that each non-degenerate element of G is a polygonal arc and that G_1 is a collection of non-degenerate elements of G which is open in X/G. Then, there is an open subset V of G_1 and a positive integer N such that if $g = A_1A_2 \ldots A_m$ is an element of V and G is continuous at g, then $m \leqq N$.*

Proof. Suppose the contrary. Let $g_1 = A_{11}A_{12} \ldots A_{1n_1}$ denote an element of G_1 at which G is continuous. By Lemma 2 there is a positive number $h_1 < 1$ such that (1) $\text{Cl } N(g_1, h_1) \subset \bigcup G_1$ and (2) if g is an element of G_1 lying in $N(g_1, h_1)$, then $g = B_1B_2 \ldots B_p$, where $p \geqq n_1$. Let $g_2 = A_{21}A_{22} \ldots A_{2n_2}$ denote an element of G_1 such that (1) G is continuous at g_2, (2) $g_2 \subset N(g_1, 2^{-1}h_1)$, and (3) $n_2 > n_1$. There is a positive number h_2 less than one half the distance from g_1 to g_2 such that $\text{Cl } N(g_2, h_2) \subset N(g_1, 2^{-1}h_1)$ and such that if g is any element of G lying in $N(g_2, h_2)$, then $g = B_1B_2 \ldots B_n$, where $n \geqq n_2$. Now, let $g_3 = A_{31}A_{32} \ldots A_{3n_3}$ denote an element of G_1 such that (1) G is continuous at g_3, (2) $g_3 \subset N(g_2, 2^{-1}h_2)$, and (3) $n_3 > n_2$. Consider a continuation of this process to obtain $h_3, g_4, n_4, h_4, g_5, n_5, \ldots$. The continua $g_1, g_2, \ldots$ converge to a subcontinuum of an element $g = B_1B_2 \ldots B_m$ of G. Since $g \subset N(g_p, h_p)$ for each positive integer p, then $m \geqq p$ for $p = 1, 2, \ldots$. This is a contradiction.

LEMMA 4. *Suppose that G is an upper semi-continuous decomposition of E^n such that each non-degenerate element g of G is a compact continuum which is star-like relative to exactly one of its points, say P_g. Then, if G is continuous at the non-degenerate element g and ϵ is a positive number, there is a positive number δ such that if h is an element of G and $H(g, h) < \delta$, then $d(P_h, P_g) < \epsilon$.*

Proof. Suppose the contrary. There is a sequence $h_1, h_2, \ldots$ of elements of G such that (1) $H(g, h_p) < p^{-1}$ for $p = 1, 2, \ldots$, and (2) there is a point Q of g distinct from P_g such that $P_{h_1}, P_{h_2}, \ldots$ converges to Q. This means, however, that if X is a point of $g - Q$ and k is a positive number, there is an integer N such that if p is an integer larger than N, then (1) there is a point X_p of h_p such that $d(X_p, X) < k$, (2) $d(Q, P_{h_p}) < k$, and (3) interval $X_pP_{h_p} \subset h_p$. This implies that interval $XQ \subset g$, and hence that g is star-like relative to Q, a contradiction.

4. Proofs of the theorems.

Proof of Theorem 3. Suppose that every element of G' intersects B. We give

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the proof only for $n > 1$. Suppose also that W is a convex open subset of U which is a subset of a face F of T and such that (1) the boundary of W relative to B is an $(n-2)$-sphere S of radius r, (2) $\bar{W} \subset U$, and (3) AC is an interval in G', where C is the centre of S. (Note that if there is an open subset W of U such that every element of G' which intersects W is degenerate, then the proof given here can be simplified considerably to cover that case.) Let $m = \text{lub}\{d(P, Q) | PQ \in G' \text{ and } Q \in \bar{W}\}$, and let b denote a positive number less than the length of AC.

We now define a mapping $f: T \to T$ as follows: If X belongs to an element g of G' such that g intersects $B - W$ or if $X \in B$, then let $f(X) = X$. Define $F: W \to E^1$ such that if $Q \in W$, then $F(Q) = b + r^{-1}(m - b)(r - d(Q, S))$. If X belongs to an interval PQ in G', where $Q \in W$, then (a) if $d(P, Q) \leqq F(Q)$, then let $f(X) = X$, and (b) if $d(P, Q) > F(Q)$, let R denote the point of PQ such that $d(R, Q) = F(Q)$ and (i) let $f(X) = R$ for $X \in PR$, and (ii) let $f(X) = X$ for $X \in RQ$.

We observe that f maps no point onto A and that $f(X) = X$ for $X \in B$. We now proceed toward showing that f is continuous. Let $X_1, X_2, \ldots$ denote a sequence of elements of T converging to a point X_0 of T and suppose that $X_i \in g_i \in G'$, $i = 0, 1, 2, \ldots$.

Case 1. $f(X_0) = X_0$. Since $f(X) \neq X$, only for some elements of T belonging to intervals of G' which intersect W, we may as well assume that each g_i $(i = 1, 2, \ldots)$ is of the form P_iQ_i, where $Q_i \in W$ and that $f(X_i) \neq X_i$. In fact, let us assume that there is a positive number ϵ such that $d(X_i, f(X_i)) \geqq \epsilon$, $i = 1, 2, \ldots$. We first show that g_0 intersects W. For, if g_0 intersects $B - W$, then some subsequence of $Q_1, Q_2, \ldots$ converges to $Q \in g_0 \cap (B - W)$. Thus, for some large n we have that $m - \epsilon/2 \leqq F(Q_n)$, and since we also know that $d(X_n, R_n) \leqq d(P_n, Q_n) - d(R_n, Q_n)$ $(R_n = f(X_n))$ and

$$d(P_n, Q_n) - d(R_n, Q_n) = d(P_n, Q_n) - F(Q_n),$$

we then know that $d(X_n, R_n) \leqq d(P_n, Q_n) - (m - \epsilon/2) \leqq \epsilon/2$. This is a contradiction, so we know that g_0 intersects W and is of the form P_0Q_0, where $Q_0 \in W$.

Since G' is upper semi-continuous, then $Q_1, Q_2, \ldots$ must converge to Q_0, and since $d(X_n, Q_n) > \epsilon/2 + F(Q_n)$, $n = 1, 2, \ldots$, then

$$d(X_0, Q_0) \geqq \epsilon/2 + F(Q_0),$$

which implies that $f(X_0) \neq X_0$, a contradiction. Since $d(X_i, f(X_i)) \to 0$ and $d(X_i, X_0) \to 0$, it follows easily that $f(X_i) \to X_0 = f(X_0)$.

Case 2. $f(X_0) \neq X_0$. Since g_0 must be of the form P_0Q_0 for $Q_0 \in W$, then there is an integer N such that if n is an integer $> N$, then g_n does not intersect $B - W$. Otherwise, there would exist a sequence of points of $B - W$ which converges to a point of g_0, and g_0 contains only Q_0 in common with B.

So we may as well assume that each g_i ($i = 1, 2, \ldots$) is of the form P_iQ_i. Furthermore, since $X_i \to X_0$, $Q_i \to Q_0$, and $d(X_0, Q_0) > F(Q_0)$, we may as well assume that $d(X_n, Q_n) > F(Q_n)$ for $n = 1, 2, \ldots$. As above, for $n = 0, 1, \ldots$, let R_n denote the point of P_nQ_n such that $d(R_n, Q_n) = F(Q_n)$; $R_n = f(X_n)$.

Suppose that $R_{n1}, R_{n2}, \ldots$ is a subsequence of $R_1, R_2, \ldots$ which converges to a point R' of P_0Q_0. But $d(R', Q_0) = \lim_{p\to\infty} d(R_{np}, Q_{np}) = \lim_{p\to\infty} F(Q_{np})$, and this limit is $F(Q_0)$, which implies that $R_0 = R'$. Therefore $f(X_i) \to f(X_0)$; this completes the proof of the continuity of f.

Let g: $T - A \to B$ be the radial projection from A and let $r = gf$: $T \to B$. Since g is also continuous, it follows that r is a retraction, which is impossible. We conclude then that some element of G' is a subset of Int T.

Proof of Theorem 1. Suppose that every element of G which intersects $N(g, \epsilon)$ is non-degenerate. By Lemma 1, there is an element $h = A_1A_2 \ldots A_m$ of G such that $h \subset N(g, \epsilon)$ and G is continuous at h.

There is a positive number δ such that (1) $\delta < 4^{-1}d(A_i, A_{i+1})$, $i = 1, \ldots, m - 1$, and (2) $\delta < 4^{-1}d(A_1W, A_2A_3 \ldots A_m)$, where $W = (A_1 + A_2)/2$. There is a positive number $k < \delta$ such that if $B_1B_2 \ldots B_m$ is an element of G which intersects $N(A_1A_2 \ldots A_m, k)$, then

$$H(A_1A_2 \ldots A_m, B_1B_2 \ldots B_m) < \delta \quad \text{and} \quad d(A_i, B_i) < \delta, \quad i = 1, \ldots, m.$$

We let T denote a closed solid geometric n-simplex such that (1) the diameter of $T < k$, (2) $A_1 \in$ Int T, and (3) A_1A_2 intersects B, the boundary of T, in a point C which is in the open simplex determined by an $(n - 1)$-face F of T. We now let G' denote the collection of all sets of the form $g' \cap T$ for $g' \in G$. It is an easy matter to verify that the hypotheses of Theorem 3 are satisfied, but that no element of G' is a subset of Int T.

Proof of Theorem 2. As in Theorem 1, we suppose the contrary and let h denote an element of G such that $h \subset N(g, \epsilon)$ and G is continuous at h. For each non-degenerate element k of G, let $M(k)$ be the collection of all intervals in k having one endpoint at P_k, but which are contained in no larger such interval. It is easy to verify that $\bigcup M(k) = k$ and that two different intervals of $M(k)$ meet only in P_k.

Let AP_h denote an element of $M(h)$. There is a positive number δ such that if g' is an element of G which is a subset of $N(h, \delta)$, then (1) $g' \subset N(g, \epsilon)$, (2) $H(g', h) < 4^{-1}d(A, P_h)$, and (3) $d(P_{g'}, P_h) < 4^{-1}d(A, P_h)$. Let U denote the union of all elements of G which are subsets of $N(h, \delta)$ and let T denote a closed solid geometric n-simplex such that (1) the diameter of $T < 4^{-1}d(A, P_h)$, (2) $T \subset U$, (3) $A \in$ Int T, and (4) AP_h intersects Bd T in a point C which is in the open simplex determined by an $(n - 1)$-face F of T. We let G' denote the set of all intersections of the form $T \cap XP_k$, where $XP_k \in M(k)$ and $k \in G$. The hypotheses of Theorem 3 are satisfied but each element of G' intersects Bd T.

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