Math. Struct. in Comp. Science (2006), *vol.* 16, *pp.* 1085–1131.
doi:10.1017/S0960129506005706 Printed in the United Kingdom

Iterative algebras at work

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Received 21 October 2004; revised 21 April 2006

Iterative theories, which were introduced by Calvin Elgot, formalise potentially infinite computations as unique solutions of recursive equations. One of the main results of Elgot and his coauthors is a description of a free iterative theory as the theory of all rational trees. Their algebraic proof of this fact is extremely complicated. In our paper we show that by starting with 'iterative algebras', that is, algebras admitting a unique solution of all systems of flat recursive equations, a free iterative theory is obtained as the theory of free iterative algebras. The (coalgebraic) proof we present is dramatically simpler than the original algebraic one. Despite this, our result is much more general: we describe a free iterative theory on any finitary endofunctor of every locally presentable category $\mathcal{A}$.

Reportedly, a blow from the welterweight boxer Norman Selby, also known as *Kid McCoy*, left one victim proclaiming,
'*It's the real McCoy!*'.[¶]

1. Introduction

1.1. *Iterative* Σ*-algebras*

About a quarter of a century ago Evelyn Nelson and Jerzy Tiuryn obtained a very nice result by introducing the concept of an iterative Σ-algebra and proving that the theory of free iterative Σ-algebras is a free iterative theory on Σ in the sense of Calvin Elgot. This dramatically improved the original description of a free iterative theory devised by the group of researchers around Elgot: whereas the original proof, which was based on a technically highly involved approach working with algebraic theories, occupied more than 100 pages of the three articles Elgot (1975), Bloom and Elgot (1974) and Elgot *et al.* (1978), the new proof presented in Nelson (1983) and Tiuryn (1980), is short and intuitive.

§ The first and the third author acknowledge the support of the Grant Agency of the Czech Republic under the Grant No. 201/02/0148.
¶ The Phrase Thesaurus, http://phrases.shu.ac.uk.

Let us recall the concept of iterativity of a Σ-algebra A as introduced by Evelyn Nelson (Nelson 1983); see (Tiuryn 1980) for a related, but not quite equivalent, concept. We consider an arbitrary system of recursive equations

$$x_i \approx t_i, \qquad i = 1, \dots, n, \tag{1.1}$$

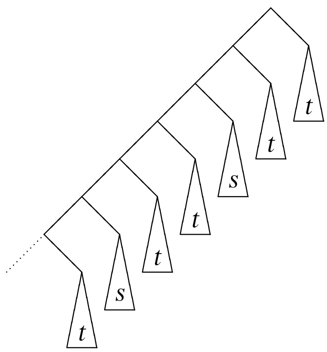
in an algebra A, where $X = \{x_1, x_2, \dots, x_n\}$ is a finite set of variables and $t_1, t_2, \dots, t_n$ are terms over $X + A$, none of which is a single variable x_i. This last condition is called *guardedness* of the system. The algebra A is called *iterative* provided that for every such system of equations there exists a unique *solution*. That is, there exists a unique n-tuple ${x_1}^\dagger, {x_2}^\dagger, \dots, {x_n}^\dagger$ of elements of A such that each of the formal equations in (1.1) becomes an equality after the substitution ${x_i}^\dagger/x_i$:

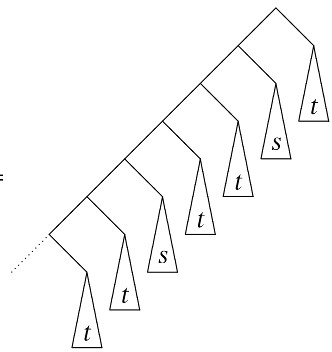
$${x_i}^\dagger = t_i({x_1}^\dagger/x_1, {x_2}^\dagger/x_2, \dots, {x_n}^\dagger/x_n), \qquad i = 1, \dots, n.$$

As an example, let Σ consist of a single binary operation symbol, $*$. Then the algebra A of all (finite and infinite) binary trees is iterative. For example, the system

$$\begin{aligned} x_1 &\approx x_2 * t \\ x_2 &\approx (x_1 * s) * t \end{aligned} \tag{1.2}$$

where s and t are binary trees in A has the unique solution

${x_1}^\dagger =$ 

${x_2}^\dagger =$ 

(1.3)

'Classical' Σ-algebras, such as groups and lattices, are seldom iterative. However, there are enough interesting iterative algebras to warrant their study. For example, the Σ-algebra

$$T_\Sigma$$

of all (finite and infinite) Σ-trees is iterative (we use the term Σ-*tree* to mean a tree labelled in Σ so that every node with a k-ary label has precisely k children; in particular, leaves are labelled by nullary symbols), and so is its subalgebra

$$R_\Sigma$$

of all *rational* Σ-trees, that is, trees that can be obtained by solving systems (1.1) of finitely many recursive equations. Such trees were characterised by Susanna Ginali (Ginali 1979) as precisely the Σ-trees that have (up to isomorphism) only finitely many subtrees. For example, the above tree ${x_1}^\dagger$ is rational whenever t and s are: the subtrees of ${x_1}^\dagger$ are ${x_1}^\dagger$, ${x_2}^\dagger$ and all subtrees of t and s—and nothing else.

https://doi.org/10.1017/S0960129506005706 Published online by Cambridge University Press

Evelyn Nelson proved in Nelson (1983) that R_Σ is an initial iterative Σ-algebra. More generally, given a set Y (of generators), a free iterative Σ-algebra on Y can be described as the algebra

$$R_\Sigma(Y)$$

of all rational Σ-*trees on* Y that differ from the above Σ-trees in allowing labels from Y (as well as nullary operation symbols) on the leaves. We thus obtain an algebraic theory, or, equivalently, a finitary monad on Set: the rational-tree theory

$$\mathbb{R}_\Sigma,$$

which is the theory of the adjunction of the forgetful functor of iterative Σ-algebras and the free-algebra functor $Y \longmapsto R_\Sigma(Y)$.

Theorem 1.1. $\mathbb{R}_\Sigma$ is a free iterative theory on Σ.

As mentioned above, this is the main result of Elgot *et al.* (1978), but a much clearer proof can be found in Nelson (1983) and Tiuryn (1980). The concept of Calvin Elgot's iterative theory is reviewed briefly in Section 5.

1.2. *Iterative H-algebras*

The main topic of the present paper is the iterativity of algebras for a given endofunctor H of a category $\mathcal{A}$ satisfying fairly mild assumptions: we require that $\mathcal{A}$ be locally finitely presentable (see Section 2) and that H be *finitary*, that is, preserve filtered colimits. (In particular, for $\mathcal{A} = \mathsf{Set}$, finitary means that every element of HA is contained in the union $\bigcup Hi[HB]$ where B ranges over finite subsets of A and $i : B \hookrightarrow A$ is the inclusion map.) An *algebra* is an object A of $\mathcal{A}$ together with a morphism $\alpha : HA \longrightarrow A$. The particular case of Σ-algebras is captured in $\mathcal{A} = \mathsf{Set}$ by the *polynomial functor* given on objects by

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \cdots .$$

We now need to ask how we can capture the concept of a system (1.1) of guarded equations and its solution. There are two 'natural' approaches, which we will prove to be equivalent.

One approach is to require that the systems (1.1) are *flat*, which is defined by the property that the terms $t_1, \ldots, t_n$ on the right-hand sides are flat. A *flat term* is either of the form

$$t_i = \sigma(y_1, \ldots, y_k),$$

where σ is a k-ary symbol of Σ and $y_1, \ldots, y_k$ are variables from $X = \{x_1, \ldots, x_n\}$, or of the form

$$t_i = a, \qquad \text{an element of } A.$$

It is easy to see that every system (1.1) can be flattened by using additional variables to give a flat system with the same solution. For example, we can add variables to the

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system (1.2) to give the flat system

$$\begin{array}{lll} x_1 \approx x_2 * x_3 & x_3 \approx t & \\ x_2 \approx x_4 * x_3 & x_4 \approx x_1 * x_5 & x_5 \approx s. \end{array}$$

Therefore, an algebra is iterative iff every flat equation system has a unique solution.

Now, flat systems (1.1) have their right-hand sides in the set $H_\Sigma X + A$. Therefore, we can view (1.1) as a function $e : X \longrightarrow H_\Sigma X + A$, $e(x_i) = t_i$. A *solution* of e is then a function $e^\dagger : X \longrightarrow A$, $e^\dagger(x_i) = x_i{}^\dagger$, with the property that the square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [\alpha, A]} \\ H_\Sigma X + A & \xrightarrow[H_\Sigma e^\dagger + A]{} & H_\Sigma A + A \end{array} \tag{1.4}$$

commutes. This makes it possible to introduce flat systems and their solutions for H-algebras over an arbitrary endofunctor H, see Definition 2.5 below. And an H-algebra A is called *iterative* if for every finitary flat equation morphism there exists a unique solution.

The other approach to generalising iterativity of algebras is to work with all guarded systems (1.1). Here we recall that the terms t_i (on the right-hand sides) are simply elements of a free Σ-algebra on $X + A$. Thus, if $Z \longmapsto FZ$ denotes the free-algebra functor for H, a finitary equation morphism can be viewed as a morphism

$$e : X \longrightarrow F(X + A),$$

where X is finitely presentable[§]. In Section 4 we explain how guardedness of e is 'naturally' formulated in abstract categories, and prove that every iterative algebra has a unique solution of all guarded finitary equation morphisms. Thus, the two definitions of iterativity (using flat or guarded finitary equation morphisms) are again equivalent.

The above concept of iterative algebra is analogous to the concept of a *completely iterative algebra*, cia for short, defined in Milius (2005). The difference is that infinite systems of equations are allowed in a cia. Thus, a cia is an algebra A such that every flat equation morphism $e : X \longrightarrow HX + A$ has a unique solution $e^\dagger : X \longrightarrow A$.

1.3. *Free iterative H-algebras*

In Section 2 we show that iterative algebras are abundant. For example, all limits and all filtered colimits of iterative algebras are iterative. As a consequence, we see that iterative algebras form a full reflective subcategory of the category of all H-algebras, that is, every algebra has a universal iterative modification. For example, a free H-algebra $F(Z)$ yields by the modification a free iterative algebra

$$R(Z)$$

on Z for every object Z. In particular, an initial iterative algebra, R, exists.

[§] Finite presentability means that the hom-functor $\mathcal{A}(X, \_)$ preserves filtered colimits. In Set this says that X is finite.

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The main technical result of our paper, which is presented in Section 3, is a coalgebraic construction of R (and, more generally $R(Z)$): we prove that R is a colimit of the diagram of all coalgebras $e : X \longrightarrow HX$ on finitely presentable objects X. For every object Z we obtain an analogous description of $R(Z)$: here we consider all coalgebras $e : X \longrightarrow HX + Z$ for the endofunctor $H(-) + Z$ carried by finitely presentable objects X. The result that $R(Z)$ is a colimit of these coalgebras is presented in Section 3 also.

Example 1.2 (Regular languages as an iterative algebra). Recall that a sequential automaton

$$Q \times I \xrightarrow{\delta} Q \xrightarrow{\mathsf{accept?}} \mathsf{Bool}$$

with the input alphabet I can be viewed as a coalgebra for the functor $HX = X^I \times \mathsf{Bool}$. In fact, if $\overline{\delta} : Q \longrightarrow Q^I$ is the curryfication of δ, the automaton is represented by the coalgebra structure

$$\langle \overline{\delta}, \mathsf{accept?} \rangle : Q \longrightarrow Q^I \times \mathsf{Bool}.$$

Also, an initial iterative algebra can be described as the algebra Reg of all regular languages over I, see Example 3.7. The next-state function of Reg is given by the Brzozowski derivatives

$$\delta(L, s) = \{w \in I^* \mid sw \in L\},$$

and the accepting states are precisely the regular languages containing the empty word.

1.4. *Free iterative theories and monads*

In sections 4 and 5 we deal with the monad $\mathbb{R}$ of free iterative H-algebras, which is called the *rational monad* of H prove that $\mathbb{R}$ is a free iterative monad on H.

The concept of iterative algebraic theory, introduced by Calvin Elgot (Elgot 1975), works in the category of sets. It is well known that algebraic theories correspond precisely to finitary monads on Set, and the translation of Elgot's concept to the language of monads was provided in Aczel *et al.* (2003).

A monad $\mathbb{S}$ is *ideal* if the unit $\eta : Id \longrightarrow S$ ('injection of generators') is part of a coproduct, that is, we have a subfunctor $S' \hookrightarrow S$ with $S = S' + Id$ correlating well with the monad multiplication. Iterativity of $\mathbb{S}$ means that for guarded equation morphisms of the form $e : X \longrightarrow S(X + Z)$, with X finite, a unique solution $e^\dagger : X \longrightarrow SZ$ exists. The guardedness states that e factors through the summand $S'(X + Z) + Z$ of $S(X + Z)$. For more details, see Section 5, where the concept of an iterative monad is recalled.

The coalgebraic construction of free iterative H-algebras $R(Z)$ mentioned above makes it easy to prove that the monad $\mathbb{R}$ of free iterative H-algebras is iterative and can be characterised as the free iterative monad on H. Again, all our results described in Subsections 1.3 and 1.4 hold for every finitary endofunctor of every locally finitely presentable category.

In a subsequent work, Adámek *et al.* (2006), we describe the Eilenberg–Moore category of all algebras of the rational monad.

1.5. *Related work*

In the classical setting, that is, for polynomial endofunctors of Set, iterative algebras were introduced by Evelyn Nelson (Nelson 1983) to obtain a short proof of the existence of Elgot's free iterative theories. Our paper can be seen as a categorical generalisation of that paper with a distinctive coalgebraic 'flavour'. Also, Jerzy Tiuryn introduced a concept of iterative algebra in Tiuryn (1980) with the same aim as ours: to relate Elgot's iterative theories to properties of algebras. But the approach of Tiuryn (1980) is different from ours: for example, the trivial, one-element algebra is not iterative in the sense of Tiuryn, so his iterative algebras are not closed under limits.

The coalgebraic construction of a free iterative monad via a colimit of 'finitary coalgebras' appears first in Adámek *et al.* (2003a). This construction was later generalised in Ghani *et al.* (2002).

The present paper is a dramatic improvement on our previous description of the rational monad in Adámek *et al.* (2003a; 2003b) where we assumed that the endofunctor preserves monomorphisms and the underlying category satisfies three rather technical conditions; also, the proof was much more involved. The current approach includes all equationally defined algebraic categories as base categories (whereas in Adámek *et al.* (2003b) we still needed strong side conditions, which only hold in a very small number of algebraic categories). We believe that in the current paper we have the 'real McCoy'.

We have already mentioned the related concept of a completely iterative algebra. The present paper and the paper Milius (2005) were written simultaneously, and some parts overlap: the proof of Theorem 5.12 below is identical with the proof of Theorem 5.14 in *loc. cit.*; we present it here for the convenience of the reader. The present paper and *loc. cit.* follow a closely related pattern of ideas, but the technical details are rather different.

2. Iterative algebras

In the present section we introduce the concept of an iterative H-algebra for an arbitrary finitary endofunctor H filtered colimits of a locally finitely presentable category, and illustrate it with some examples. These can be skipped; the technical results we prove, beginning with Proposition 2.18, show what morphisms we need to choose and that there exist 'enough' iterative algebras. We then prove that free iterative algebras always exist, and introduce the rational monad of H as the monad of free iterative algebras for H.

In order to define the concept of a flat equation morphism as in the Introduction (a morphism $e : X \longrightarrow HX + A$ in Set where X is finite) in a general category, we need the appropriate generalisation of finiteness. A set is finite if and only if its hom-functor is finitary. This has inspired Peter Gabriel and Friedrich Ulmer (Gabriel and Ulmer 1971) to the following definition.

Definition 2.1. An object A of a category $\mathcal{A}$ is *finitely presentable* if its hom-functor $\mathcal{A}(A, -) : \mathcal{A} \longrightarrow \mathsf{Set}$ is finitary.

A category $\mathcal{A}$ is said to be *locally finitely presentable* if it has colimits and a (small) set of finitely presentable objects whose closure under filtered colimits is all of $\mathcal{A}$.

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Assumption 2.2. Throughout this paper we use $\mathcal{A}$ to denote a locally finitely presentable category (see Definition 2.1) and H to denote a finitary endofunctor. We use inl and inr to denote the coproduct injections of binary coproducts $A + B$, and $\mathsf{can} : HA + HB \longrightarrow H(A + B)$ for the canonical morphism $\mathsf{can} = [H\mathsf{inl}, H\mathsf{inr}]$.

Notation 2.3. $\mathsf{Alg}\, H$ denotes the category of *H-algebras*, that is, pairs (A, α) where A is an object and $\alpha : HA \longrightarrow A$. Morphisms of $\mathsf{Alg}\, H$ from (A, α) to (B, β) are called *homomorphisms*; they are morphisms $f : A \longrightarrow B$ with $f \cdot \alpha = \beta \cdot Hf$. We write $\mathsf{Coalg}\, H$ for the category of *H-coalgebras* and their homomorphisms, where objects are pairs (A, α) with $\alpha : A \longrightarrow HA$, and homomorphisms $f : (A, \alpha) \longrightarrow (B, \beta)$ are morphisms $f : A \longrightarrow B$ with $\beta \cdot f = Hf \cdot \alpha$.

Examples 2.4.

(1) In Set, finitely presentable means finite, and Set is locally finitely presentable.
(2) A poset is finitely presentable in Pos, the category of posets and order-preserving functions, if and only if it is finite. Pos is a locally finitely presentable category.
(3) The category CPO of complete partial orders and continuous functions is not locally finitely presentable: it has no non-trivial finitely presentable objects.
(4) Every variety of finitary algebras is locally finitely presentable. The categorical concept of finitely presentable object coincides with the algebraic one (of having finitely many generators and finitely many presenting equations), see Adámek and Rosický (1994).
(5) Let H be a finitary endofunctor of a locally finitely presentable category $\mathcal{A}$. Then the category $\mathsf{Alg}\, H$ of H-algebras and homomorphisms is also locally finitely presentable, see Adámek and Rosický (1994).

Definition 2.5. We use the term *finitary flat equation morphism* (later just: *equation morphism*) in an object A to mean a morphism $e : X \longrightarrow HX + A$ of $\mathcal{A}$, where X is a finitely presentable object of $\mathcal{A}$.

Suppose that A is an underlying object of an H-algebra $\alpha : HA \longrightarrow A$. Then by a *solution* of e in the algebra A we mean a morphism $e^\dagger : X \longrightarrow A$ in $\mathcal{A}$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle [\alpha, A]} \\ HX + A & \xrightarrow[\quad He^\dagger + A \quad]{} & HA + A \end{array} \tag{2.1}$$

commutes.

An H-algebra is said to be *iterative* if every finitary flat equation morphism has a unique solution.

Example 2.6 (Milius 2005). Terminal coalgebras are iterative algebras. That is, if H has a final coalgebra $\tau : T \longrightarrow HT$, then τ is invertible (due to Lambek's Lemma (Lambek 1968)) and $\tau^{-1} : HT \longrightarrow T$ is an iterative algebra. In fact, it is even a cia. By applying this to H_Σ, we conclude that the coalgebra T_Σ of all Σ-trees (see Introduction) is iterative. More generally, given a set Y, the algebra $T_\Sigma Y$ of all *Σ-trees on Y* (that is, trees with

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leaves labelled by constant symbols in Σ_0 or by elements of Y, and inner nodes with n children labelled in Σ_n) is iterative. This is the final coalgebra for $H_\Sigma(-) + Y$.

Example 2.7. The subalgebra $R_\Sigma Y$ of $T_\Sigma Y$ formed by all rational trees (see Introduction) is iterative. In fact, we present a proof that $R_\Sigma Y$ is a free iterative algebra on Y in Section 3. The original proof is in Nelson (1983) and Tiuryn (1980).

Example 2.8. Groups, lattices, and so on, considered as Σ-algebras, are seldom iterative. For example, if a group is iterative, its unique element is the unit element 1, since the system of recursive equations $x \approx x \cdot y$, $y \approx 1$ has a unique solution. Analogously, if a lattice is iterative, it has a unique element: consider $x \approx x \vee x$.

Example 2.9. The algebra of addition on the set

$$\widetilde{\mathbb{N}} = \{1, 2, 3, \dots\} \cup \{\infty\}$$

is iterative. (Observe that 0 is not included. This is forced by the uniqueness of solutions of $x \approx x + x$.)

To prove the iterativity of $\widetilde{\mathbb{N}}$, we use $h : T_\Sigma \widetilde{\mathbb{N}} \longrightarrow \widetilde{\mathbb{N}}$ to denote the homomorphism that to every finite tree assigns the result of computing the corresponding term in $\widetilde{\mathbb{N}}$ and to every infinite tree assigns ∞. Observe that the canonical embedding $\eta : \widetilde{\mathbb{N}} \longrightarrow T_\Sigma \widetilde{\mathbb{N}}$ satisfies $h \cdot \eta = id$. Let

$$e : X \longrightarrow X \times X + \widetilde{\mathbb{N}}$$

be an equation morphism. The derived equation morphism

$$\overline{e} \equiv X \xrightarrow{\;e\;} X \times X + \widetilde{\mathbb{N}} \xrightarrow{\;X \times X + \eta\;} X \times X + T_\Sigma \widetilde{\mathbb{N}}$$

has a unique solution $\overline{e}^\dagger : X \longrightarrow T_\Sigma \widetilde{\mathbb{N}}$ in the tree algebra. This yields a solution $e^\dagger$ in $\widetilde{\mathbb{N}}$ as follows:

$$e^\dagger \equiv X \xrightarrow{\;\overline{e}^\dagger\;} T_\Sigma \widetilde{\mathbb{N}} \xrightarrow{\;h\;} \widetilde{\mathbb{N}}.$$

To prove that solutions in $\widetilde{\mathbb{N}}$ are unique, let $e^\ddagger : X \longrightarrow \widetilde{\mathbb{N}}$ be a solution of e. For every $x \in X$ with $\overline{e}^\dagger(x)$ finite, we have $e^\ddagger(x)$ as the computation of $\overline{e}^\dagger(x)$, that is, $e^\ddagger(x) = e^\dagger(x)$ (this follows from an easy proof by induction on the cardinality of the set of nodes of $\overline{e}^\dagger(x)$). And for every x with $\overline{e}^\dagger(x)$ infinite, we prove $e^\ddagger(x) = \infty$ $(= e^\dagger(x))$. This follows from the next lemma since $\overline{e}^\dagger(x)$ has either infinitely many leaves or a complete binary subtree.

Lemma 2.10. Let $e^\ddagger$ be a solution of e.

(1) Suppose that the tree $\overline{e}^\dagger(x)$ has (at least) k leaves labelled by $r_1, \dots, r_k \in \widetilde{\mathbb{N}}$. Then

$$e^\ddagger(x) \geqslant r_1 + \cdots + r_k.$$

(2) Suppose that the tree $\overline{e}^\dagger(x)$ has a node whose subtree is a complete binary tree (no leaves), then $e^\ddagger(x) = \infty$.

Proof.

(1) This part is proved by induction on the maximum depth d of the k leaves. The case $d = 0$ means that $\overline{e}^\dagger(x)$ is a single root labelled by r_1. Then we must have $e(x) = r_1$, and

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it follows that $e^{\ddagger}(x) = r_1$. In the induction step let $d > 0$. Then, certainly, $e(x) \in X \times X$, say $e(x) = (y_1, y_2)$, and each of the k leaves is a leaf of $\overline{e}^{\dagger}(y_i)$, $i = 1$ or 2. Since the maximum depth in $\overline{e}^{\dagger}(y_i)$ is one less than that in $\overline{e}^{\dagger}(x)$, we can use the induction hypothesis to conclude

$$e^{\ddagger}(y_1) + e^{\ddagger}(y_2) \geqslant r_1 + \cdots + r_k.$$

And from $e(x) = (y_1, y_2)$, since $e^{\ddagger} = [\alpha, id] \cdot (H_\Sigma e^{\ddagger} + id) \cdot e$, we obtain

$$e^{\ddagger}(x) = e^{\ddagger}(y_1) + e^{\ddagger}(y_2) \geqslant r_1 + \cdots + r_k.$$

(2) This part is proved by induction on the depth of the given node j. The case $d = 0$ means that $\overline{e}^{\dagger}(x)$ is a complete binary tree. Consider the subset $X_\infty \subseteq X$ of all those variables x from X for which $\overline{e}^{\dagger}(x)$ is a complete binary tree. Then, for every $x \in X_\infty$, we have $e(x) = (x_0, x_1)$ with $x_0, x_1 \in X_\infty$. Therefore, $e(x_i) = (x_{i0}, x_{i1})$ with $x_{i0}, x_{i1} \in X_\infty$, $i = 0, 1$, and so on. Continuing, we obtain variables x_w for every binary word w, from X_∞ with $e(x_w) = (x_{w0}, x_{w1})$ with $x_{w0}, x_{w1} \in X_\infty$. As $e^{\ddagger}$ is a solution of e, we must have $e^{\ddagger}(x_w) = e^{\ddagger}(x_{w0}) + e^{\ddagger}(x_{w1})$. Then one easily shows by induction that for every prefix v of a binary word w we have $e^{\ddagger}(x_v) = e^{\ddagger}(x_w) + k$ for some $k \in \widetilde{\mathbb{N}}$. But since X_∞ is a finite set, there exist binary words v and w, with v a prefix of w, with $x_v = x_w$. It follows that $r = e^{\ddagger}(x_v) = e^{\ddagger}(x_w)$ satisfies $r = r + k$ for some $k \in \widetilde{\mathbb{N}}$. This implies that $r = \infty$, and another easy argument by induction then shows that $e^{\ddagger}(x_w) = \infty$ for all binary words. In particular, the empty word yields $e^{\ddagger}(x) = \infty$.

Finally, for $d > 0$ we have $e(x) = (y, z)$ and the node j lies in $\overline{e}^{\dagger}(y)$ or $\overline{e}^{\dagger}(z)$ where it has smaller depth than in $\overline{e}^{\dagger}(x)$, thus $e^{\ddagger}(y) = \infty$ or $e^{\ddagger}(z) = \infty$. Consequently,

$$e^{\ddagger}(x) = e^{\ddagger}(y) + e^{\ddagger}(z) = \infty. \qquad \square$$

Example 2.11. The algebra of addition of extended real numbers of the interval

$$I = (0, \infty]$$

is iterative.

The proof that equation morphisms have solutions is completely analogous to Example 2.9 above. The uniqueness is proved as follows. We first establish the above lemma. Next we use (unlike in Example 2.9!) the finiteness of the set X: since X is finite, the tree $\overline{e}^{\dagger}(x)$ is rational. If it has a subtree that is a complete binary tree, then $e^{\dagger}(x) = \infty$. Otherwise, every subtree of $\overline{e}^{\dagger}(x)$ contains a leaf, and the rationality of $\overline{e}^{\dagger}(x)$ then implies that infinitely many leaves of $\overline{e}^{\dagger}(x)$ carry the same label, say, $r \in I$. The lemma, applied to k of these leaves, implies $e^{\dagger}(x) \geqslant k \cdot r$, for any $k = 1, 2, 3, \ldots$, and thus $e^{\dagger}(x) = \infty$.

Remark 2.12. The uniqueness of solutions is sometimes subtle. In Example 2.9 above we need not assume that X is a finite set, but Example 2.11 would be false without this assumption: consider the system

$$x_0 \approx x_1 + \frac{1}{2} \qquad x_1 \approx x_2 + \frac{1}{4} \qquad x_2 \approx x_3 + \frac{1}{8} \qquad \cdots$$

One solution is $x_n^{\dagger} = \infty$ $(n \in \mathbb{N})$, another is $x_n^{\dagger} = 2^{-n}$ $(n \in \mathbb{N})$.

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Example 2.13 (Unary algebras in Set). Consider the endofunctor

$$HA = \Sigma \times A$$

corresponding to unary algebras: every algebra $\alpha : \Sigma \times A \longrightarrow A$ is given by unary operations

$$s^A = \alpha(s, -) : A \longrightarrow A \quad \text{for } s \in \Sigma.$$

Such an algebra is iterative if and only if the operation

$$s_1^A \cdot s_2^A \cdot \dots \cdot s_n^A : A \longrightarrow A$$

has a unique fixed point for every non-empty word $s_1 s_2 \cdots s_n$ over Σ.

In fact, the above condition is necessary because the solution of the system

$$e : \{x_0, \dots, x_{n-1}\} \longrightarrow \Sigma \times \{x_0, \dots, x_{n-1}\} + A$$

where $e(x_i) = (s_i, x_{i+1})$ for $i < n-1$, and $e(x_{n-1}) = (s_n, x_0)$ is nothing else than a fixed point, a, of $s_1^A \cdot \dots \cdot s_n^A$. More precisely, the corresponding map $e^\dagger : \{x_0, \dots, x_{n-1}\} \longrightarrow A$ with

$$e^\dagger(x_i) = s_{i+1}^A \cdot \dots \cdot s_n^A(a) \quad (i = 0, \dots, n-1)$$

solves e.

To prove that the above condition is sufficient, consider a finitary equation morphism

$$e : X \longrightarrow \Sigma \times X + A.$$

We will say a variable $x_0 \in X$ is *cyclic* if the values of e always stay in the first summand, that is, we have

$$e(x_i) = (s_{i+1}, x_{i+1}) \qquad i = 0, 1, 2, \dots$$

for an infinite sequence $(s_n, x_n) \in \Sigma \times X$. Since X is finite, there exists $p < q$ with $x_p = x_q$.

Every solution $e^\dagger : X \longrightarrow A$ assigns to x_i elements $a_i = e^\dagger(x_i)$ such that

$$a_i = \alpha(s_{i+1}, a_{i+1});$$

in other words

$$a_i = s_{i+1}^A(a_{i+1}).$$

Therefore, $a_p = a_q$ implies that a_p is a fixed point of $s_{p+1}^A \cdot \dots \cdot s_q^A$, and this fixed point determines the value

$$a_0 = s_1^A \cdot \dots \cdot s_p^A(a_p).$$

Consequently, if the fixed point is unique, $e^\dagger(x_0)$ is uniquely determined.

The non-cyclic variables x_0 present no problem: here we have, for some $k \geqslant 0$,

$$\begin{aligned} e(x_i) &= (s_{i+1}, x_{i+1}) \qquad i = 0, \dots, k-1 \\ e(x_k) &= a \in A, \end{aligned}$$

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which implies

$$e^\dagger(x_0) = s_1^A \cdot \dots \cdot s_k^A(a).$$

Remark 2.14. In particular, for $Id : \mathsf{Set} \longrightarrow \mathsf{Set}$, an algebra $\alpha : A \longrightarrow A$ is iterative if and only if α has a unique fixed point and none of α^n, $n \geqslant 2$, has a different fixed point.

Example 2.15 (Ordered unary algebras). Here we consider, for a set Σ with discrete ordering, the endofunctor

$$HA = \Sigma \times A$$

on the category Pos of partially ordered sets and order-preserving functions. An ordered unary Σ-algebra is iterative if and only if the operation $s_1^A \cdot \dots \cdot s_n^A$ has a unique fixed point for every non-empty word $s_1 \cdots s_n$ over Σ.

The argument is as before, we just have to verify that the function

$$e^\dagger(x_0) = \begin{cases} s_1^A \cdot \dots \cdot s_p^A(a_p), & x_0 \text{ cyclic} \\ s_1^A \cdot \dots \cdot s_p^A(a), & \text{otherwise} \end{cases}$$

is order-preserving (whenever $e : X \longrightarrow \Sigma \times X + A$ is), which is easy.

Example 2.16 (Unary algebras in Un). Here the base category Un is that of unary algebras on one operation $\sigma_A : A \longrightarrow A$ and homomorphisms. We consider H-algebras for the identity endofunctor Id_{Un}. That is, we work with algebras

$$\alpha : (A, \sigma_A) \longrightarrow (A, \sigma_A),$$

where α is another unary operation on A, and since α is a homomorphism, it commutes with σ_A:

$$\alpha \cdot \sigma_A = \sigma_A \cdot \alpha.$$

Finitely presentable objects of Un are precisely the unary algebras given by finitely many generators and finitely many equations. For example, free algebras on n generators for $n \in \mathbb{N}$. We prove that an algebra is iterative if and only if

$$\sigma_A^k \alpha^n : A \longrightarrow A \text{ has a unique fixed point for all } n \geqslant 1 \text{ and } k \geqslant 0. \qquad (*)$$

The necessity of $(*)$ follows from solutions of the equation morphisms

$$e : X \longrightarrow X + A$$

where X is a free unary algebra on n generators, $x_1, \dots, x_n$, and e is determined by

$$e(x_i) = x_{i+1} \quad \text{for } i < n, \quad e(x_n) = \sigma_X^k(x_1).$$

In fact, a solution $e^\dagger : X \longrightarrow A$ is given by elements $a_i = e^\dagger(x_i)$, $i = 1, \dots, n$ satisfying

$$a_i = \alpha(a_{i+1}) \quad \text{for } i < n,\ a_n = \alpha\sigma_A^k(a_1).$$

Thus, a_1 is a fixed point of $\sigma_A^k \alpha^n$, and, conversely, every fixed point corresponds to a solution of e.

To show the sufficiency of (∗), given an equation morphism

$$e : X \longrightarrow X + A \quad \text{with } X \text{ generated by } y_1, \dots, y_r,$$

we can describe a solution analogously to Example 2.13 above. Given a 'non-cyclic' variable $x_0 \in X$, that is, one with

$$\begin{aligned} e(x_i) &= x_{i+1} \qquad i = 0, \dots, k-1 \\ e(x_k) &= a \in A, \end{aligned}$$

we necessarily have $e^\dagger(x_k) = a$, $e^\dagger(x_{k-1}) = \alpha(a)$ and so on, thus here

$$e^\dagger(x_0) = \alpha^k(a).$$

For a 'cyclic' variable $x_0 \in X$ we have an infinite sequence x_0, x_1, $x_2, \dots$ in X with $e(x_i) = x_{i+1}$. A solution $e^\dagger$ assigns to x_i an element $a_i \in A$ with

$$a_i = \alpha(a_{i+1}) = \alpha^2(a_{i+2}) = \dots$$

On the other hand, we can express each x_i via the generators $y_1, \dots, y_r$ in the form

$$x_i = \sigma_X^{c(i)}(y_{d(i)}) \quad c(i) \geqslant 0,\ d(i) \in \{1, \dots, r\}.$$

This implies $a_i = \sigma_A^{c(i)}(b_{d(i)})$, where $b_1, \dots, b_r$ are the elements $e^\dagger(y_1), \dots, e^\dagger(y_r)$. We can certainly choose $p < q$ such that

$$d(p) = d(q) \quad \text{and} \quad c(p) \leqslant c(q).$$

Then the equality $a_p = \alpha^{q-p}(a_q)$ yields

$$\sigma_A^{c(p)}(b_{d(p)}) = \alpha^{q-p}\sigma_A^{c(q)}(b_{d(p)}).$$

We now put $n = q - p$ and $k = c(q) - c(p)$ to conclude that $a_p = \sigma_A^{c(p)}(b_{d(p)})$ is a fixed point of $\alpha^n\sigma_A^k$. Consequently, if a^* denotes the unique fixed point of $\alpha^n\sigma_A^k$, we conclude $a_1 = \alpha^p(a_p) = \alpha^p(a^*)$. Thus, we have to define

$$e^\dagger(x_0) = \alpha^p(a^*).$$

To summarise, the unique solution of e is defined as follows:

$$e^\dagger(x_0) = \begin{cases} \alpha^k(a), & \text{if } x_0 \text{ is not cyclic} \\[2ex] \alpha^p(a^*), & \text{if } x_0 \text{ is cyclic.} \end{cases}$$

Remark 2.17. We use

$$\mathsf{Alg}_{it}\ H$$

to denote the category of all iterative algebras and all homomorphisms. The following proposition shows that this choice of morphisms is the 'right' one.

Proposition 2.18 (Homomorphisms = solutions-preserving morphisms). Let A and B be iterative algebras and $h : A \longrightarrow B$ be a morphism of $\mathcal{A}$. Then h is a homomorphism if and only if it preserves solutions in the following sense. For every equation morphism

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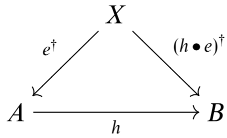
$e : X \longrightarrow HX + A$ the solution of e in A yields a solution of the equation morphism

$$h \bullet e \equiv X \xrightarrow{e} HX + A \xrightarrow{HX+h} HX + B$$

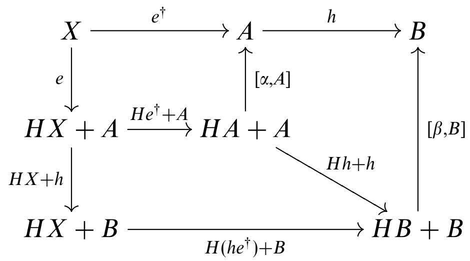
in B via the commutative triangle



(2.2)

Proof.

(1) Let h be a homomorphism. The following commutative diagram shows that $h \cdot e^\dagger$ solves $h \bullet e$:

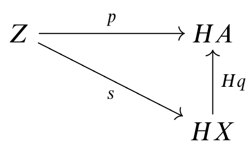


The upper left-hand part commutes since $e^\dagger$ is a solution of e, the right-hand part commutes since h is a homomorphism, and the lower part is obvious. Thus, by the uniqueness of solutions, we know that the triangle (2.2) commutes.

(2) Let h preserve solutions, let $\mathcal{A}_{fp}$ be a set of representative finitely presentable objects of $\mathcal{A}$, and let $\mathcal{A}_{fp}/A$ be the comma-category of all arrows $q : X \longrightarrow A$ with X in $\mathcal{A}_{fp}$. Since $\mathcal{A}$ is locally finitely presentable, A is a filtered colimit of the canonical diagram $D_A : \mathcal{A}_{fp}/A \longrightarrow \mathcal{A}$ given by $(q : X \longrightarrow A) \longmapsto X$.
Now $\mathcal{A}_{fp}$ is a generator of $\mathcal{A}$, thus, in order to complete the proof it is sufficient to show that for every morphism $p : Z \longrightarrow HA$ with Z in $\mathcal{A}_{fp}$ we have

$$h \cdot \alpha \cdot p = \beta \cdot Hh \cdot p. \tag{2.3}$$

Since H is finitary, it preserves the above colimit of D_A. This implies, since $\mathcal{A}(Z, -)$ preserves filtered colimits, that p has a factorisation



for some $q : X \longrightarrow A$ in $\mathcal{A}_{fp}/A$ and some s. For the equation morphism

$$e \equiv Z + X \xrightarrow{s+X} HX + X \xrightarrow{H\mathsf{inr}+q} H(Z + X) + A$$

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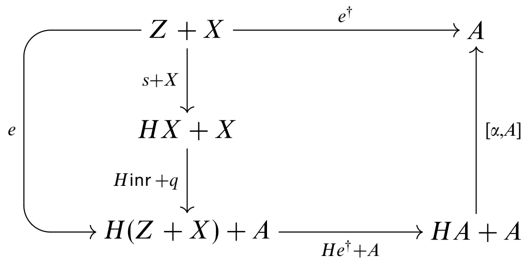
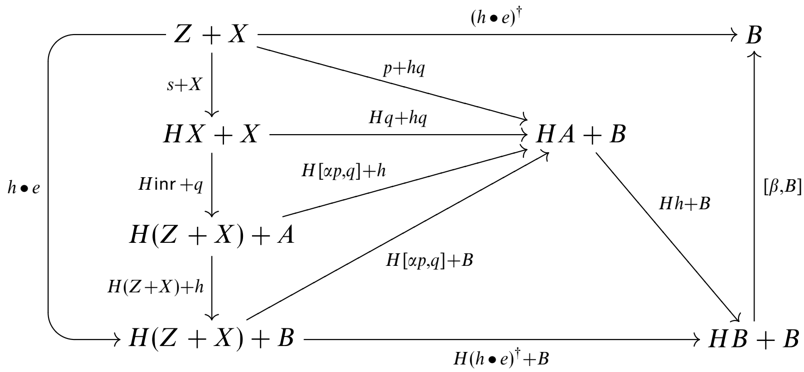
we have a commutative square



Consequently, $e^{\dagger} \cdot \mathsf{inr} = q$, and this implies $e^{\dagger} \cdot \mathsf{inl} = \alpha \cdot H(e^{\dagger} \cdot \mathsf{inr}) \cdot s = \alpha \cdot p$. By (2.2), we have $h \cdot e^{\dagger} = (h \bullet e)^{\dagger}$, and, therefore,

$$(h \bullet e)^{\dagger} = [h \cdot \alpha \cdot p, h \cdot q]. \tag{2.4}$$

On the other hand, consider the diagram



This commutes because the outer square commutes since $(h \bullet e)^{\dagger}$ is a solution, for the lower triangle use equation (2.4), and the remaining triangles are trivial. Thus, the upper right-hand part commutes:

$$(h \bullet e)^{\dagger} = [\beta \cdot Hh \cdot p, h \cdot q]. \tag{2.5}$$

The left-hand components of (2.4) and (2.5) establish the desired equality (2.3). □

Remark 2.19. Note that it follows from part (1) of the proof of Proposition 2.18 that homomorphisms always preserve solutions in the following sense. Let A and B be H-algebras (not necessarily iterative), let $h : A \longrightarrow B$ be a homomorphism, and let $e : X \longrightarrow HX + A$ by an equation morphism. Then we have that if $e^{\dagger}$ is a solution of e, then $h \cdot e^{\dagger}$ is a solution of $h \bullet e = (HX + h) \cdot e$.

Proposition 2.20. Iterative algebras are closed under limits and filtered colimits in $\mathsf{Alg}\, H$.

Proof.

(1) Let (A, α) be a limit, in $\mathsf{Alg}\, H$, of iterative algebras with a limit cone $h_i : (A, \alpha) \longrightarrow (A_i, \alpha_i)$, $i \in I$. It then easily follows that $A = \lim A_i$ in $\mathcal{A}$ with the limit cone $(h_i)_{i \in I}$.

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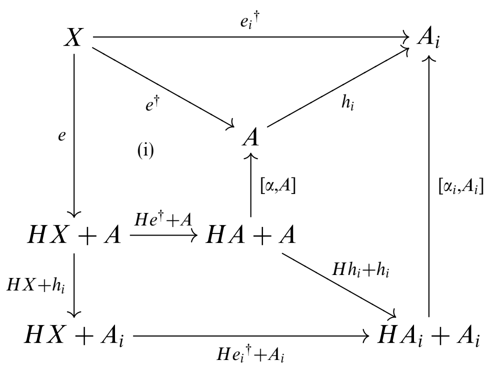
For every equation morphism $e : X \longrightarrow HX + A$, the uniqueness of its solution in A follows from Remark 2.19: given a solution $e^\dagger : X \longrightarrow A$, each $h_i e^\dagger$ is the unique solution of $e_i = (HX + h_i) \cdot e$ in A_i, thus, $h_i e^\dagger$ is unique, and since $(h_i)_{i \in I}$ is a limit cone in $\mathcal{A}$, we conclude that $e^\dagger$ is unique. To prove the existence, let ${e_i}^\dagger : X \longrightarrow A_i$ denote the solution of e_i in A_i. This is a cone of the given diagram, that is, for every connecting homomorphism $f : (A_i, \alpha_i) \longrightarrow (A_j, \alpha_j)$ we have

$$f{e_i}^\dagger = {e_j}^\dagger.$$

This follows from Proposition 2.18 and $fh_i = h_j$ (which implies $(HX + f) \cdot e_i = e_j$). Thus, there exists a unique morphism $e^\dagger : X \longrightarrow A$ with

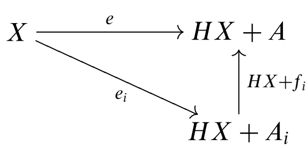
$${e_i}^\dagger = h_i e^\dagger \quad (i \in I).$$

To prove that $e^\dagger$ solves e, it is sufficient to verify that $h_i e^\dagger = h_i \cdot [\alpha, A] \cdot (He^\dagger + A) \cdot e$ for all $i \in I$. In fact, the outer square of the following diagram



commutes, and so do the upper triangle, the right-hand and lower parts. Thus, part (i) commutes when extended by h_i, as desired.

(2) Let (A, α) be a filtered colimit, in $\mathsf{Alg}\ H$, of iterative algebras with a colimit cocone $f_i : (A_i, \alpha_i) \longrightarrow (A, \alpha)$, $i \in I$. Since H is finitary, filtered colimits of H-algebras are formed on the level of $\mathcal{A}$. Given an equation morphism $e : X \longrightarrow HX + A = \mathrm{colim}(HX + A_i)$, since X is finitely presentable, e factors through one of the colimit morphisms $HX + f_i$:



If ${e_i}^\dagger : X \longrightarrow A_i$ is the solution of e_i in A_i, then $f_i{e_i}^\dagger : X \longrightarrow A$ is a solution of e in A by Remark 2.19.

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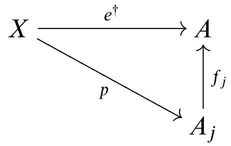
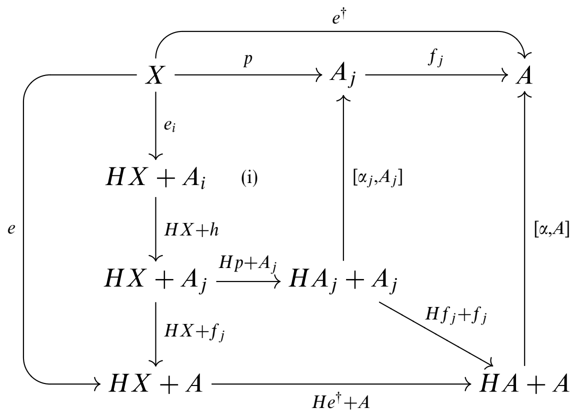
Conversely, for every solution $e^\dagger : X \longrightarrow A$ of e in A we prove $e^\dagger = f_i e_i{}^\dagger$, so we factorise $e^\dagger$ through one of the colimit morphisms:



Since the given diagram is filtered, we can suppose that the choice of $j \in I$ is such that a connecting homomorphism $h : (A_i, \alpha_i) \longrightarrow (A_j, \alpha_j)$ of our diagram exists. Then the morphism $e_j = (HX + h) \cdot e_i : X \longrightarrow HX + A_j$ has the solution $e_j{}^\dagger = p$. To see this, notice that all parts of the following diagram



except (i) commute. Therefore (i) commutes when extended by f_j. By filteredness, we can therefore suppose that (i) commutes (otherwise choose a connecting morphism $g : (A_j, \alpha_j) \longrightarrow (A_k, \alpha_k)$ equating the sides of (i) and work with k instead of j). But it follows from Proposition 2.18 that $e_j{}^\dagger = he_i{}^\dagger$, therefore $p = he_i{}^\dagger$. This proves

$$e^\dagger = f_j p = f_j h e_i{}^\dagger = f_i e_i{}^\dagger,$$

as desired. □

Corollary 2.21. The category $\mathsf{Alg}_{it}\, H$ is a reflective subcategory of $\mathsf{Alg}\, H$.

Proof. In fact, $\mathsf{Alg}\, H$ is locally finitely presentable, see Example 2.4(4). Thus, we can apply the Reflection Theorem of Adámek and Rosický (1994), which states that every full subcategory of a locally finitely presentable category closed under limits and filtered colimits is reflective. □

Corollary 2.22. Every object of $\mathcal{A}$ generates a free iterative H-algebra.

In other words, the natural forgetful functor $U : \mathsf{Alg}_{it}\, H \longrightarrow \mathcal{A}$ has a left adjoint.

Definition 2.23. The finitary monad on $\mathcal{A}$ formed by free iterative H-algebras is called the *rational monad* of H and is denoted by $\mathbb{R} = (R, \eta, \mu)$.

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Thus, $\mathbb{R}$ is the monad of the above adjunction

$$\mathsf{Alg}_{it}\, H \underset{U}{\overset{R}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{A}.$$

In more detail, for every object Z of $\mathcal{A}$ we use RZ to denote a free iterative H-algebra on Z with the universal arrow

$$\eta_Z : Z \longrightarrow RZ,$$

and the algebra structure

$$\rho_Z : HRZ \longrightarrow RZ.$$

Then $\mu_Z : RRZ \longrightarrow RZ$ is the unique homomorphism of H-algebras with $\mu_Z \cdot \eta_{RZ} = id$.

Before turning to concrete examples of free iterative algebras, we note in the following proposition that it is sufficient to describe the initial one.

Proposition 2.24. For any object Z of $\mathcal{A}$ the following are equivalent:

(1) RZ is an initial iterative algebra of $H(-) + Z$.
(2) RZ is a free iterative H-algebra on Z.

In fact, this was proved for completely iterative algebras in Milius (2005); the proof for iterative algebras is the same.

Remark 2.25. A special case of a recursive equation morphism arises when no parameters appear, that is, we simply have coalgebras $e : X \longrightarrow HX$ with X finitely presentable. We should explain here why solutions of these special equation morphisms are not sufficient for our purposes. Let us (for the duration of this remark only) say an algebra $\alpha : HA \longrightarrow A$ is *weakly iterative* if every equation morphism $e : X \longrightarrow HX$, with X finitely presentable, has a unique solution $e^\dagger : X \longrightarrow A$ (that is, $e^\dagger = \alpha \cdot He^\dagger \cdot e$). For example, when $H_\Sigma : \mathsf{Set} \longrightarrow \mathsf{Set}$ represents a binary operation, $H_\Sigma X = X \times X$, the free iterative algebra $R_\Sigma\{a\}$ on one generator has the property that every equation $e : X \longrightarrow X \times X$ has the solution $e^\dagger : x \longmapsto t_0$, the constant function to the complete binary tree t_0. Consequently, every subalgebra of $R_\Sigma\{a\}$ containing t_0 and all finite trees is weakly iterative, although $R_\Sigma\{a\}$ has no proper iterative subalgebra containing finite trees.

3. A coalgebraic construction

The aim of this section is to describe an initial iterative H-algebra as a colimit of all finitary coalgebras; and to describe a free iterative algebra on Z analogously as a colimit of all finitary equation morphisms $e : X \longrightarrow HX + Z$. The idea of using such colimits originates in Adámek *et al.* (2003a), see also Ghani *et al.* (2002) for a generalisation.

We will continue to assume throughout this section that $\mathcal{A}$ is a locally finitely presentable category, see Definition 2.1, and H is a finitary endofunctor of $\mathcal{A}$.

We choose a set $\mathcal{A}_{fp}$ of representatives of finitely presentable objects of $\mathcal{A}$ with respect to isomorphism.

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Recall that our setting allows a simple description of the *initial H-algebra* as a colimit of the ω-chain

$$0 \xrightarrow{t} H0 \xrightarrow{Ht} HH0 \xrightarrow{HHt} \cdots$$

where t is the unique morphism from 0, which is an initial object of $\mathcal{A}$. More precisely, if $I = \operatorname{colim}_{n<\omega} H^n 0$ is this colimit, then the chain above defines a canonical morphism $i : I \longrightarrow HI$. One then proves that i is invertible, yielding an initial H-algebra structure on I, see Adámek (1974).

Analogously, the initial iterative algebra will be proved to be a colimit of the diagram

$$\mathrm{Eq} : \mathsf{EQ} \longrightarrow \mathcal{A}, \qquad (X \xrightarrow{e} HX) \longmapsto X.$$

The objects of EQ are all H-coalgebras carried by finitely presentable objects of $\mathcal{A}$,

$$e : X \longrightarrow HX \quad \text{with } X \text{ in } \mathcal{A}_{fp},$$

and the morphisms are the usual coalgebra homomorphisms. That is, EQ is a full subcategory of $\mathsf{Coalg}\, H$. The functor Eq is the obvious forgetful functor.

A colimit

$$R_0 = \operatorname{colim} \mathrm{Eq}$$

of this diagram (with colimit morphisms $e^\sharp : X \longrightarrow R_0$ for all $e : X \longrightarrow HX$ in EQ) yields, again, a canonical morphism

$$i : R_0 \longrightarrow HR_0.$$

Namely, i is the unique morphism such that every $e^\sharp$ becomes a coalgebra homomorphism, that is, the squares

$$\begin{array}{ccc} X & \xrightarrow{\quad e \quad} & HX \\ {\scriptstyle e^\sharp}\downarrow & & \downarrow{\scriptstyle He^\sharp} \\ R_0 & \xrightarrow[\quad i \quad]{} & HR_0 \end{array} \tag{3.1}$$

commute. This determines i uniquely since the forgetful functor $\mathsf{Coalg}\, H \longrightarrow \mathcal{A}$ creates colimits.

Remark 3.1. The diagram Eq is filtered. In fact, the category $\mathsf{Coalg}\, H$ of all coalgebras is cocomplete, with colimits formed at the level of $\mathcal{A}$. Since $\mathcal{A}_{fp}$ is well-known to be closed under finite colimits, it follows that the category EQ is closed under finite colimits in $\mathsf{Coalg}\, H$ – so EQ is finitely cocomplete, and thus filtered.

Consequently, H preserves the colimit of Eq,

$$HR_0 = \operatorname{colim} H \cdot \mathrm{Eq},$$

with the colimit cocone $He^\sharp$.

We prove first that the coalgebra R_0 is 'almost final' among coalgebras on finitely presentable objects. As R_0 is not finitely presentable itself, it cannot be final, but we have the following proposition.

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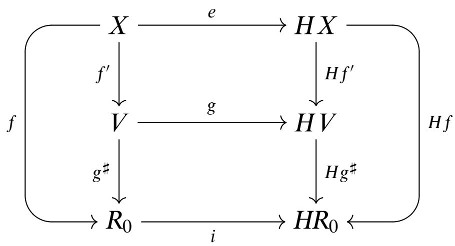
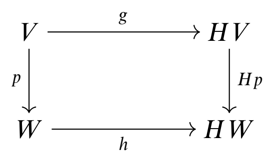
Proposition 3.2. Every coalgebra $e : X \longrightarrow HX$ with X finitely presentable has a unique homomorphism $e^\sharp$ into the coalgebra $i : R_0 \longrightarrow HR_0$.

Proof. We are to prove that the coalgebra homomorphisms of (3.1) are unique: given an object $e : X \longrightarrow HX$ of EQ and a coalgebra homomorphism f from (X, e) into (R_0, i), we have $f = e^\sharp$. In fact, since X is finitely presentable, the morphism $f : X \longrightarrow \operatorname{colim} \mathrm{Eq}$ factors through the colimit morphism $g^\sharp$ for some $g : V \longrightarrow HV$: $f = g^\sharp f'$. In the diagram



the outer square commutes, and so do all inner parts, except possibly for the upper square. This implies that $Hg^\sharp$ merges the two sides of that square. Now $Hg^\sharp$ is a colimit morphism of $HR_0 = H \operatorname{colim} \mathrm{Eq} = \operatorname{colim} H\mathrm{Eq}$ (recall that Eq is a filtered diagram, so H preserves its colimit). Since X is finitely presentable, $\mathcal{A}(X, -)$ preserves the colimit of $H\mathrm{Eq}$. Thus, if $Hg^\sharp$ merges two morphisms, then so does one of the connecting maps Hp, where p is a morphism in EQ, that is, the square



commutes. That is, we have

$$Hp \cdot (Hf' \cdot e) = Hp \cdot (g \cdot f'),$$

from which we conclude that pf' is a morphism of EQ from e to h since

$$H(p \cdot f') \cdot e = Hp \cdot g \cdot f' = h \cdot (p \cdot f').$$

Thus, $e^\sharp = h^\sharp \cdot (pf')$. Now p being a morphism of EQ implies $g^\sharp = h^\sharp \cdot p$, and, consequently,

$$f = g^\sharp f' = h^\sharp pf' = e^\sharp. \qquad \square$$

Theorem 3.3. R_0 is the initial iterative H-algebra. More precisely, the morphism i is an isomorphism and $i^{-1} : HR_0 \longrightarrow R_0$ is an initial iterative H-algebra.

Before proving Theorem 3.3, we need to establish some auxiliary facts.

Lemma 3.4. $i : R_0 \longrightarrow HR_0$ is an isomorphism.

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Proof.

(a) *Define a morphism* $j : HR_0 \longrightarrow R_0$.
We use the fact that in a locally finitely presentable category the given object HR_0 is a colimit of the diagram of all arrows $p : P \longrightarrow HR_0$ where P is in $\mathcal{A}_{fp}$. More precisely, let $\mathcal{A}_{fp}/HR_0$ denote the comma-category (of all these arrows p). Then the forgetful functor $D_{HR_0} : \mathcal{A}_{fp}/HR_0 \longrightarrow \mathcal{A}$ has, in $\mathcal{A}$, the colimit cocone formed by all $p : P \longrightarrow HR_0$. Thus, in order to define j, we need to define morphisms $jp : P \longrightarrow R_0$ forming a cocone of the diagram D_{HR_0}. We know that HR_0 is a filtered colimit of $H \cdot \mathrm{Eq}$ and that $\mathcal{A}(P, -)$ preserves this colimit, since P is in $\mathcal{A}_{fp}$. Therefore, p factors through one of the colimit morphisms

$$\begin{array}{ccc} P & \xrightarrow{\;p\;} & HR_0 \\ & \searrow^{p'} & \uparrow{\scriptstyle Hg^\sharp} \\ & & HW \end{array} \tag{3.2}$$

for some $g : W \longrightarrow HW$ in EQ. We form a new object

$$e_{p'} \equiv P + W \xrightarrow{[p',g]} HW \xrightarrow{H\mathsf{inr}} H(P + W)$$

of EQ and define j to be the unique morphism such that the square

$$\begin{array}{ccc} P & \xrightarrow{\;\mathsf{inl}\;} & P + W \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle e_{p'}{}^\sharp} \\ HR_0 & \xrightarrow[\;j\;]{} & R_0 \end{array} \tag{3.3}$$

commutes for every p in $\mathcal{A}_{fp}/HR_0$. To prove that j is well-defined we need to show that:

(i) $e_{p'}{}^\sharp \cdot \mathsf{inl}$ is independent of the choice of the factorisation (3.2).

(ii) The morphisms $e_{p'}{}^\sharp \cdot \mathsf{inl}$ form a cocone of $\mathcal{A}_{fp}/HR_0$.

These are proved as follows:

(i) Consider another factorisation

$$\begin{array}{ccc} P & \xrightarrow{\;p\;} & HR_0 \\ & \searrow^{q'} & \uparrow{\scriptstyle Hf^\sharp} \\ & & HV \end{array}$$

with $f : V \longrightarrow HV$ in EQ. Using the fact that the diagram $H\mathrm{Eq}$ is filtered, we conclude that, without loss of generality, this new factorisation can be assumed to

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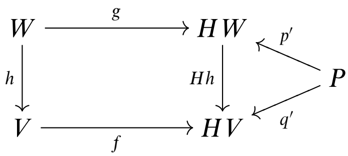
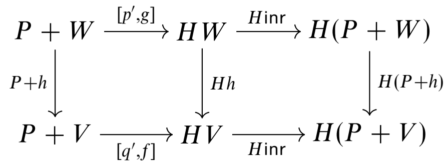
possess a morphism $h : W \longrightarrow V$ of EQ from the first one (3.2) with $q' = Hh \cdot p'$:



Then $P + h$ is a morphism of EQ from $e_{p'}$ to $e_{q'}$

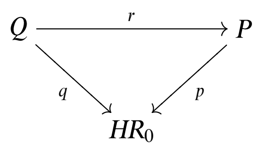


which proves ${e_{p'}}^{\sharp} = {e_{q'}}^{\sharp} \cdot (P + h)$. Consequently,

$$ {e_{p'}}^{\sharp} \cdot \mathsf{inl} = {e_{q'}}^{\sharp} \cdot (P + h) \cdot \mathsf{inl} = {e_{q'}}^{\sharp} \cdot \mathsf{inl}\,, $$

as required.

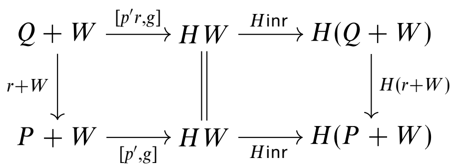
(ii) Consider a morphism r in $\mathcal{A}_{fp}/HR_0$:



We have defined $jp = {e_{p'}}^{\sharp} \cdot \mathsf{inl}$ for the factorisation (3.2) and, from (i) above, we can use the factorisation $q = Hg^{\sharp} \cdot (p' \cdot r)$ for the definition of $jq = {e_{p'r}}^{\sharp} \cdot \mathsf{inl}$. We now need to prove the equation

$$ {e_{p'}}^{\sharp} \cdot \mathsf{inl} \cdot r = {e_{p'r}}^{\sharp} \cdot \mathsf{inl}\,. \tag{3.4} $$

Observe that $r + W$ is a morphism of EQ from $e_{p'r}$ to $e_{p'}$:



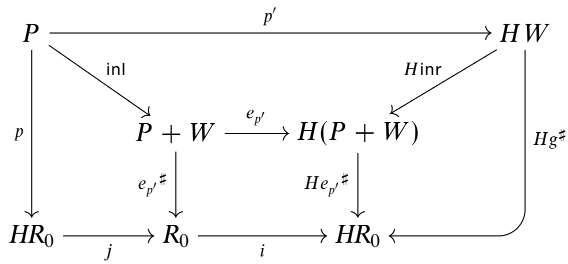
Thus ${e_{p'r}}^{\sharp} = {e_{p'}}^{\sharp} \cdot (r + W)$, which proves (3.4).

(b) *The proof of $ij = id$.*

We need to prove that $ijp = p$ for every $p : P \longrightarrow HR_0$ in $\mathcal{A}_{fp}/HR_0$. Observe that $\mathsf{inr} : W \longrightarrow P + W$ is a morphism of EQ from $g : W \longrightarrow HW$ to $e_{p'} : P + W \longrightarrow H(P + W)$, hence

$$ g^{\sharp} = {e_{p'}}^{\sharp} \cdot \mathsf{inr}\,. $$

The desired equality $ijp = p$ follows from (3.2) and the fact that the diagram



commutes.

(c) *The proof of $ji = id$.*
We need to prove that $jie^\sharp = e^\sharp$ for every $e : X \longrightarrow HX$ in EQ. In order to do this, we apply (3.3) to $p = He^\sharp \cdot e : X \longrightarrow HR_0$ with $p' = e$ and $g = e$ to obtain

$$j \cdot He^\sharp \cdot e = {e_{p'}}^\sharp \cdot \mathsf{inl} \tag{3.5}$$

for $e_{p'} \equiv X + X \xrightarrow{[e,e]} HX \xrightarrow{H\mathsf{inr}} H(X + X)$. It is easy to check that the codiagonal $\nabla = [id, id] : X + X \longrightarrow X$ is a morphism of EQ from $e_{p'}$ to e, so

$$e^\sharp \cdot \nabla = {e_{p'}}^\sharp.$$

We now use $i \cdot e^\sharp = He^\sharp \cdot e$, see (3.1), and (3.5) to conclude

$$j \cdot (i \cdot e^\sharp) = j \cdot He^\sharp \cdot e = {e_{p'}}^\sharp \cdot \mathsf{inl} = e^\sharp \cdot \nabla \cdot \mathsf{inl} = e^\sharp.$$ □

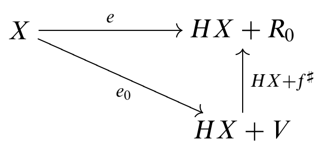
Lemma 3.5. The H-algebra $i^{-1} : HR_0 \longrightarrow R_0$ is iterative.

Proof.

(1) *Existence of solutions.*
For every equation morphism

$$e : X \longrightarrow HX + R_0 = \mathrm{colim}(HX + \mathrm{Eq}),$$

there exists, since X is finitely presentable, a factorisation through the colimit morphism $HX + f^\sharp$ (for some $f : V \longrightarrow HV$ in EQ):



(3.6)

Recall from Assumption 2.2 that $\mathsf{can} : HX + HV \longrightarrow H(X + V)$ denotes the canonical morphism. Define a new object, $\overline{e}$, of EQ as follows:

$$\overline{e} \equiv X + V \xrightarrow{[e_0, \mathsf{inr}]} HX + V \xrightarrow{HX + f} HX + HV \xrightarrow{\mathsf{can}} H(X + V). \tag{3.7}$$

Observe that

$$f^\sharp = \overline{e}^\sharp \cdot \mathsf{inr} \tag{3.8}$$

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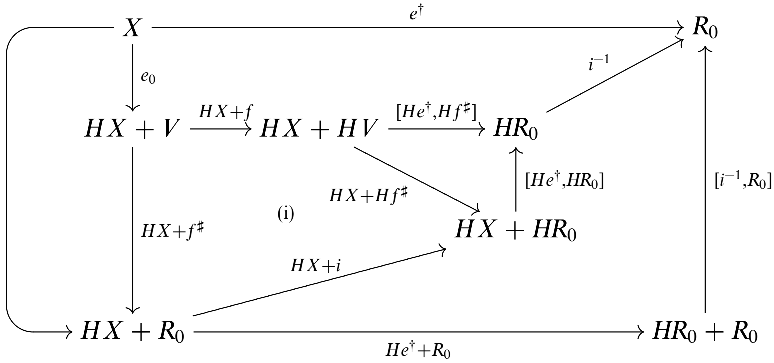
because $\mathsf{inr} : V \longrightarrow X + V$ is a coalgebra morphism (in EQ) from f to $\overline{e}$. We define a solution of e by

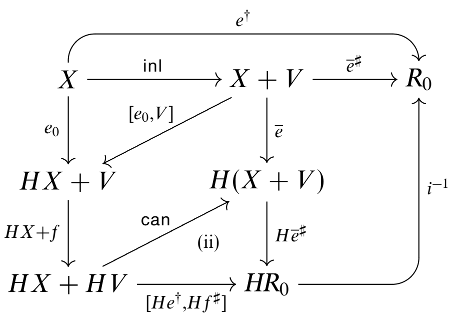
$$e^\dagger \equiv X \xrightarrow{\mathsf{inl}} X + V \xrightarrow{\overline{e}^\sharp} R_0. \tag{3.9}$$

In fact, in the diagram



(3.10)

all inner parts commute: see (3.6) for the left-hand part; (3.1) for part (i); the right-hand part commutes trivially (analyse the two components separately); and so does the middle triangle. It remains to verify the upper part: here we use (3.1) and (3.7) to conclude that the diagram



commutes. In fact, the left-hand component of (ii) commutes by definition of $e^\dagger$ and the right-hand one does by (3.8). Thus, (3.10) commutes, proving that $e^\dagger$ is a solution of e.

(2) *Uniqueness.*
Suppose that $e^\dagger : X \longrightarrow R_0$ is a solution of e. Then in (3.10) the outer square commutes. Since all the inner parts except the upper one commute, this proves that the upper part commutes, too. Consequently,

$$i \cdot e^\dagger = [He^\dagger, Hf^\sharp] \cdot (HX + f) \cdot e_0 = H[e^\dagger, f^\sharp] \cdot \overline{e} \cdot \mathsf{inl}\,.$$

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This equality implies that in the square

$$\begin{array}{ccc} X+V & \xrightarrow{\ \overline{e}\ } & H(X+V) \\ {\scriptstyle [e^\dagger, f^\sharp]}\downarrow & & \downarrow{\scriptstyle H[e^\dagger, f^\sharp]} \\ R_0 & \xrightarrow[\ i\ ]{} & HR_0 \end{array}$$

the left-hand components commute. Since $\overline{e}\cdot \mathsf{inr} = H\mathsf{inr}\cdot f$, the right-hand components commute by (3.1). Therefore, the square commutes, which, by Proposition 3.2, proves

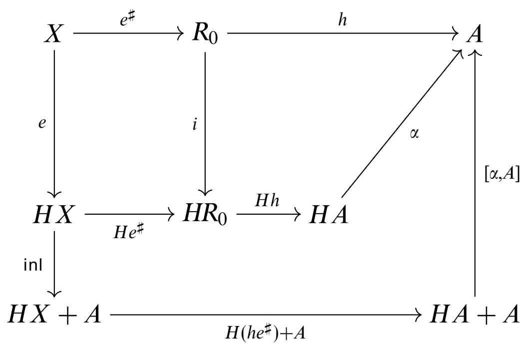
$$\overline{e}^\sharp = [e^\dagger, f^\sharp].$$

Thus, the given solution is the previous one: $e^\dagger = \overline{e}^\sharp \cdot \mathsf{inl}$. □

Proof of Theorem 3.3. Let $\alpha : HA \longrightarrow A$ be an iterative H-algebra. We first prove that there is at most one H-algebra homomorphism from R_0. Let

$$\begin{array}{ccc} HR_0 & \xrightarrow{\ i^{-1}\ } & R_0 \\ {\scriptstyle Hh}\downarrow & & \downarrow{\scriptstyle h} \\ HA & \xrightarrow[\ \alpha\ ]{} & A \end{array}$$

be a homomorphism. For every object $e : X \longrightarrow HX$ of EQ the diagram



(3.11)

commutes, see (3.1), which proves that $he^\sharp$ is a solution of $\mathsf{inl}\, e$ in A.

This determines h uniquely, since the $e^\sharp$'s form a colimit cocone of $R_0 = \operatorname{colim} \mathrm{Eq}$.

Conversely, let us define a morphism $h : R_0 \longrightarrow A$ by the above rule:

$$he^\sharp = (\mathsf{inl}\, e)^\dagger \quad \text{for all } e : X \longrightarrow HX \text{ in } \mathsf{EQ},$$

where $(-)^\dagger$ is the unique solution in A. This is well defined since the morphisms $(\mathsf{inl}\, e)^\dagger$ form a cocone of the diagram Eq: in fact, let $p : (X, e) \longrightarrow (Y, f)$ be a morphism of EQ.

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We prove that $(\mathsf{inl}\, f)^\dagger p$ is a solution of $\mathsf{inl}\, e$ by considering the corresponding diagram:

$$\begin{array}{ccccc}
X & \xrightarrow{\ p\ } & Y & \xrightarrow{(\mathsf{inl}\, f)^\dagger} & A \\
\big\downarrow{\scriptstyle e} & & \big\downarrow{\scriptstyle f} & & \big\uparrow \\
HX & \xrightarrow{\ Hp\ } & HY & & {\scriptstyle [\alpha, A]} \\
\big\downarrow{\scriptstyle \mathsf{inl}} & & \big\downarrow{\scriptstyle \mathsf{inl}} & & \big| \\
HX + A & \xrightarrow[Hp+A]{} & HY + A & \xrightarrow[H(\mathsf{inl}\, f)^\dagger + A]{} & HA + A
\end{array}$$

This proves $(\mathsf{inl}\, e)^\dagger = (\mathsf{inl}\, f)^\dagger p$.

The morphism h above is a homomorphism of algebras because the diagram (3.11) commutes: the outer square commutes by definition of h, the upper left-hand square by (3.1), and the lower part is obvious. This shows that the upper right-hand part commutes when precomposed with $e^\sharp$, e in EQ. Since the $e^\sharp$'s form a colimit cocone, it follows that h is a homomorphism. □

Example 3.6. The algebra R_Σ of rational Σ-trees from the Introduction is an initial iterative Σ-algebra. This follows from the above construction – the original proof in Nelson (1983) and Tiuryn (1980) is entirely different.

Our aim is to describe the filtered colimit of all finite coalgebras $e : X \longrightarrow H_\Sigma X$. Let $e^\dagger : X \longrightarrow T_\Sigma$ be the unique homomorphism into the terminal coalgebra T_Σ of all Σ-trees, that is, $e^\dagger$ is the solution of e in T_Σ. Then, by the definition of R_Σ (as all those Σ-trees obtained by solving systems (1.1) of recursive equations), we know that $e^\dagger[X] \subseteq R_\Sigma$. The codomain restrictions

$$e^\sharp : X \longrightarrow R_\Sigma, \qquad x \longmapsto e^\dagger(x) \qquad \text{for } x \in X,$$

form a cocone of the diagram Eq. In fact, given a morphism $h : (X, e) \longrightarrow (Y, f)$ of EQ, that is, a coalgebra homomorphism between finite coalgebras, we have $e^\dagger = f^\dagger \cdot h$ because T_Σ is terminal. Thus, $e^\sharp = f^\sharp \cdot h$. To prove that this cocone is a colimit cocone we only need to verify that:

(a) Every element of R_Σ has the form $e^\sharp(x)$ for some $e : X \longrightarrow HX$ from EQ and $x \in X$.
(b) Given $e^\sharp(x) = e^\sharp(y)$ for elements $x, y \in X$ there exists a morphism $h : (X, e) \longrightarrow (Y, f)$ of EQ with $h(x) = h(y)$.

The definition of R_Σ implies (a). For (b) we use the fact that since H_Σ preserves weak pullbacks, the kernel of $e^\dagger$ is a bisimulation equivalence $\sim$ on X; see Rutten (2000). Let $h : (X, e) \longrightarrow (X/{\sim}, f)$ be the quotient homomorphism of this equivalence. The corresponding coalgebra on $Y = X/{\sim}$ then lies in EQ, and $h(x) = h(y)$. This proves that $R_\Sigma = \operatorname{colim} \mathrm{Eq}$ is an initial iterative H_Σ-algebra.

Example 3.7 (The algebra Reg of regular languages). Here we prove the claim of Example 1.2 that Reg is an initial iterative algebra for the functor $HX = X^I \times \mathsf{Bool}$ representing automata as coalgebras. This is a special case of Example 3.6: if I has n elements, then $HX \cong X^I + X^I$ is the polynomial functor of the signature of two n-ary

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operations. A terminal coalgebra is the coalgebra $T = \exp I^*$ of all formal languages (with coalgebra structure given, as in Example 1.2, by the Brzozowski derivatives and the languages containing the empty word), as proved by Michael Arbib and Ernest Manes (Arbib and Manes 1986). From Example 3.6 we know that an initial iterative algebra is the subalgebra $R \subseteq T$ on all $e^\dagger(x)$ for all finite coalgebras $X \longrightarrow HX$ (that is, finite automata) and all states $x \in X$. Now $e^\dagger(x)$ is the language the automaton accepts provided x is its initial state, so R is precisely the subalgebra of all regular languages.

Example 3.8. The initial iterative algebra of $\mathcal{P}_{\text{fin}}$, the finite-power-set functor, can be described analogously to the description of a final coalgebra due to James Worrell (Worrell 2005).

Recall that a finitely branching non-ordered tree, considered as a coalgebra for $\mathcal{P}_{\text{fin}}$, is called *strongly extensional* if the subtrees corresponding to two distinct siblings (that is, nodes with the same mother) are never bisimilar. The set T of all strongly extensional finitely branching trees forms a final coalgebra for $\mathcal{P}_{\text{fin}}$ (whose coalgebra map is the inverse of tree tupling).

Now form the subalgebra R of all rational trees in T, that is, those with finitely many subtrees up to isomorphism. This is an initial iterative algebra. One can prove this using our construction analogously to Example 3.6. A different proof is presented in Adámek and Milius (2006).

Corollary 3.9. A free iterative H-algebra on an object Z is a colimit

$$RZ = \operatorname{colim} \mathrm{Eq}_Z$$

of the diagram

$$\mathrm{Eq}_Z : \mathsf{EQ}_Z \longrightarrow \mathcal{A},$$

where EQ_Z consists of all equation morphisms $e : X \longrightarrow HX + Z$, $X \in \mathcal{A}_{fp}$, and all coalgebra homomorphisms with respect to $H(-) + Z$, and Eq_Z sends e to X.

In fact, this is a consequence of Proposition 2.24 and Theorem 3.3.

Remark 3.10. We again denote the colimit morphisms of Eq_Z by

$$e^\sharp : X \longrightarrow RZ$$

for all $e : X \longrightarrow HX + Z$ in EQ_Z. The appropriate isomorphism is denoted by

$$i_Z : RZ \longrightarrow HRZ + Z.$$

It is characterised by the fact that the two coproduct injections of $HRZ + Z$ are (in the notation of Definition 2.23)

$$\mathsf{inl} = i_Z \rho_Z \quad \text{and} \quad \mathsf{inr} = i_Z \eta_Z.$$

In other words, $i_Z = [\rho_Z, \eta_Z]^{-1}$.

Example 3.11. The rational monad of a polynomial functor H_Σ is the monad $\mathbb{R}_\Sigma$ of rational trees from the Introduction. This follows from Example 3.6: in order to describe

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a free iterative algebra $R_\Sigma Z$, we know from Proposition 2.24 that we only need an initial iterative algebra for $H_\Sigma(-) + Z$, which is the polynomial endofunctor of the signature extending Σ by constant symbols from Z. The original proof in Nelson (1983) and Tiuryn (1980) is entirely different.

Example 3.12. The rational monad of $\mathcal{P}_{\mathrm{fin}}$ is the monad $\mathbb{R}$ that assigns to a set Y the coalgebra of all rational, strongly extensional finitely branching trees on Y. That is, RY consists of all rational, strongly extensional finitely branching trees where some leaves are labelled in Y.

Example 3.13 (The rational monad of unary algebras). Here H is the identity functor of the given category $\mathcal{A}$.

(1) For $\mathcal{A} = \mathsf{Set}$ the rational monad is given by

$$RY = \mathbb{N} \times Y + 1.$$

This follows from Corollary 3.9: the only rational trees on Y are $\sigma^n y$, $n \in \mathbb{N}$ and $y \in Y$, for the unary operation σ, and the infinite tree $\sigma\sigma\sigma\ldots$.

(2) For $\mathcal{A} = \mathsf{Pos}$ we have

$$R(Z, \leqslant) = \mathbb{N} \times (Z, \leqslant) + 1 \quad \text{with } \mathbb{N} \text{ discretely ordered.}$$

This follows from Example 2.15.

(3) For $\mathcal{A} = \mathsf{Un}$ the rational monad can, using Example 2.16, be obtained as follows: given an object (Z, σ_Z) of Un, we first freely 'add' a unary operation α that commutes with σ_Z by forming the algebra $Z \times \mathbb{N}$ with the operations σ given by $(z, n) \longmapsto (\sigma_Z(z), n)$ and α given by $(z, n) \longmapsto (z, n+1)$. Then we add a single element, a_0 say, which is the joint fixed point of both operations. Thus,

$$R(Z, \sigma_Z) = (Z \times \mathbb{N} + 1, \sigma_{R(Z,\sigma_Z)})$$

where

$$\sigma_{R(Z,\sigma_Z)} : \begin{cases} (z, n) \longmapsto (\sigma_Z(z), n) \\ a_0 \longmapsto a_0 \quad \text{where } 1 = \{a_0\}, \end{cases}$$

and with $\eta_{(Z,\sigma_Z)} : z \longmapsto (z, 0)$ and $\rho_{(Z,\sigma_Z)} : (z, n) \longmapsto (z, n+1)$, $a_0 \longmapsto a_0$.

4. Finitary and rational equations

The aim of this section is to prove that every iterative algebra has unique solutions of finitary (or even rational) guarded equation morphisms. In the Introduction we considered non-flat systems (1.1) of recursive equations for Σ-algebras, and argued that, due to the possibility of flattening such a system, we only need to consider the flat equation morphism $e : X \longrightarrow H_\Sigma X + A$. Here we are going to make that statement precise by showing that in iterative algebras (in general, not just in Set) much more general systems of recursive equations than the flat ones are uniquely solvable. This implies that, for polynomial endofunctors of Set, our definition of iterative algebras coincides with that presented by

Evelyn Nelson (Nelson 1983). And as we explain in the next section, this also implies that the rational monad is iterative in the sense of Calvin Elgot (Elgot 1975).

We first note that the guardedness condition stated for (1.1) in the Introduction (that no right-hand side be a single variable) is substantial: the equation $x \approx x$ has a unique solution only in the trivial terminal algebras.

We first consider guarded systems where the right-hand sides live in the free H-algebra (that is, they are finite trees when $H = H_\Sigma$). Such systems are called *finitary*.

Remark 4.1. Since H is finitary, free H-algebras exist (Adámek 1974). We denote for every object X in $\mathcal{A}$ a free algebra by $\varphi^0_X : HFX \longrightarrow FX$ with universal arrow $\eta^0_X : X \longrightarrow FX$. This defines a monad $\mathbb{F} = (F, \eta^0, \mu^0)$ where the component μ^0_X is the unique homomorphism $\mu^0_X : FFX \longrightarrow FX$ with $\mu^0_X \cdot \eta^0_{FX} = id$. It is easy to see that analogously to Proposition 2.24, FX is an initial algebra of $H(-) + X$, and thus by Lambek's Lemma (Lambek 1968),

$$FX = HFX + X. \tag{4.1}$$

More precisely, the morphism

$$j_X = [\varphi^0_X, \eta^0_X] : HFX + X \longrightarrow FX$$

is an isomorphism. For every H-algebra $\alpha : HA \longrightarrow A$ we have the unique homomorphism

$$\widehat{\alpha} : FA \longrightarrow A \qquad \text{with} \quad \widehat{\alpha} \cdot \eta_A = id$$

(which, in the case of H_Σ is the computation of (finite) terms over A in the Σ-algebra A). This allows us to define solutions of finitary equations morphisms in A as follows.

Definition 4.2.

(1) We define a *finitary equation morphism* in an object A to be a morphism

$$e : X \longrightarrow F(X + A), \quad X \text{ finitely presentable.}$$

(2) We say e is *guarded* if it factors through the summand $HF(X + A) + A$ of $F(X + A) = HF(X + A) + X + A$ (see (4.1) above):

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & F(X+A) \\ & \searrow & \uparrow{\scriptstyle [\varphi^0, \eta^0 \cdot \mathsf{inr}]} \\ & & HF(X+A) + A \end{array}$$

(3) Suppose that A is an underlying object of an H-algebra $\alpha : HA \longrightarrow A$. Then we define a *solution* of e in the algebra A to be a morphism $e^\dagger : X \longrightarrow A$ in $\mathcal{A}$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{\ e^\dagger\ } & A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \widehat{\alpha}} \\ F(X+A) & \xrightarrow[F[e^\dagger, A]]{} & FA \end{array} \tag{4.2}$$

commutes.

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Remark 4.3. The square (4.2) in Definition 4.2 means, for polynomial functors, that the assignment $e^\dagger$ of variables $x \in X$ to elements of A has the following property. We first form the 'substitution' mapping $[e^\dagger, A] : X + A \longrightarrow A$ (which interprets the variables as $e^\dagger$ does, and leaves elements of A unchanged), and then extend it to the unique homomorphism

$$\widehat{\alpha} \cdot F[e^\dagger, A] : F(X + A) \longrightarrow A$$

of the free algebra. Then the (formal) equations $x \approx e(x)$ become actual identities in A after the substitution $x \longmapsto e^\dagger(x)$ is performed for all $x \in X$, and the right-hand sides are computed in A. This is precisely the definition of a solution of (1.1) in the Introduction.

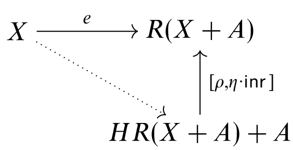
Theorem 4.4. An H-algebra A is iterative if and only if every guarded finitary equation morphism in A has a unique solution.

The proof of Theorem 4.4 will be derived from the next result, which generalises 'finitary' to 'rational'.

Definition 4.5. We define a *rational equation morphism* in an object A to be a morphism

$$e : X \longrightarrow R(X + A), \qquad X \text{ finitely presentable},$$

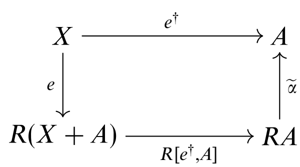
where R is the rational monad of H, see Definition 2.23. And e is called *guarded* if it factors through the summand $HR(X + A) + A$ of $R(X + A) = HR(X + A) + X + A$ (see Remark 3.10):



Suppose that A is the underlying object of an iterative H-algebra $\alpha : HA \longrightarrow A$. We use (by analogy with $\widehat{\alpha}$ above)

$$\widetilde{\alpha} : RA \longrightarrow A$$

to denote the unique homomorphism of H-algebras with $\widetilde{\alpha} \cdot \eta_A = id$. Then we define a *solution* of e in the iterative algebra A to be a morphism $e^\dagger : X \longrightarrow A$ in $\mathcal{A}$ such that the square below commutes:



Theorem 4.6. In an iterative algebra every guarded rational equation morphism has a unique solution.

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Proof. Let $\alpha : HA \longrightarrow A$ be an iterative algebra. Given a guarded rational equation morphism

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & R(X+A) \\ & {\scriptstyle e_0} \searrow & \uparrow {\scriptstyle [\rho_{X+A},\eta_{X+A}\cdot \mathsf{inr}]} \\ & & HR(X+A)+A \end{array}$$

we will prove that e has a unique solution $e^\dagger$.

(1) *Existence.*
Recall from Corollary 3.9 that $R(X+A) = \operatorname{colim} \mathrm{Eq}_{X+A}$ with colimit cocone $g^\sharp : W \longrightarrow R(X+A)$ for all $g : W \longrightarrow HW + X + A$ in EQ_{X+A}. Since this colimit is filtered and H is finitary, we have a filtered colimit

$$HR(X+A) + A = \operatorname{colim} H\mathrm{Eq}_{X+A} + A$$

with the colimit cocone formed by all $Hg^\sharp + A$. Since X is a finitely presentable object, the morphism

$$e_0 : X \longrightarrow \operatorname{colim} H\mathrm{Eq}_{X+A} + A$$

factors through the colimit cocone

$$\begin{array}{ccc} X & \xrightarrow{\;e_0\;} & HR(X+A)+A \\ & {\scriptstyle w} \searrow & \uparrow {\scriptstyle Hg^\sharp + A} \\ & & HW+A \end{array}$$

for some object $g : W \longrightarrow HW + X + A$ of EQ_{X+A} and some morphism w.
We define a finitary flat equation morphism $\langle e\rangle : W + X \longrightarrow H(W+X) + A$ as follows:

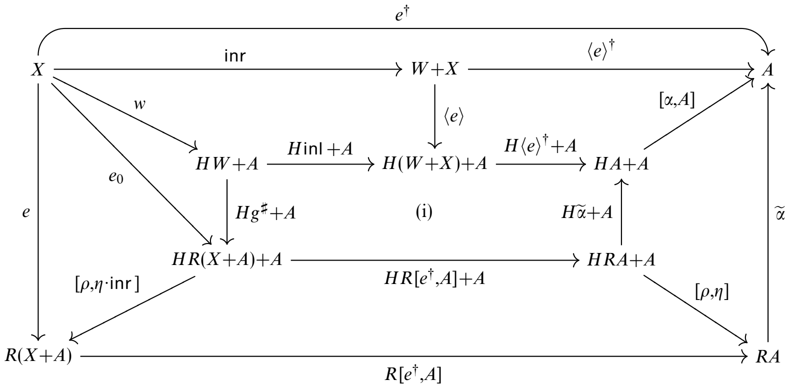
$$W + X \xrightarrow{[g,\mathsf{inm}]} HW + X + A \xrightarrow{[\mathsf{inl},w,\mathsf{inr}]} HW + A \xrightarrow{H\mathsf{inl}+A} H(W+X)+A \qquad (4.3)$$

where $\mathsf{inm} : X \longrightarrow HW + X + A$ is the middle coproduct injection. We obtain a unique solution $\langle e\rangle^\dagger : W + X \longrightarrow A$ and prove that the morphism

$$e^\dagger \equiv X \xrightarrow{\mathsf{inr}} W + X \xrightarrow{\langle e\rangle^\dagger} A \qquad (4.4)$$

is a solution of e.

Indeed, consider the diagram



(4.5)

All of its parts, except the square (i), clearly commute. The right-hand component of (i) is obvious. To prove the commutativity of the left-hand component of (i), we remove H and show that the equation

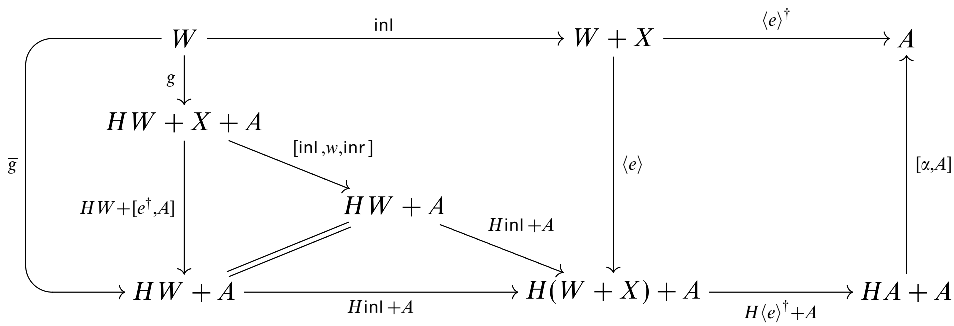
$$\langle e\rangle^{\dagger} \cdot \mathsf{inl} = \widetilde{\alpha} \cdot R[e^{\dagger}, A] \cdot g^{\sharp} \tag{4.6}$$

holds. To this end, observe first that $\widetilde{\alpha} \cdot R[e^{\dagger}, A] : R(X + A) \longrightarrow A$ is an H-algebra homomorphism between iterative algebras extending $[e^{\dagger}, A]$. An inspection of the proofs of Theorem 3.3 and Proposition 2.24 reveals that precomposing this homomorphism with the colimit injection $g^{\sharp} : W \longrightarrow R(X + A)$ yields the unique solution of the equation morphism

$$\overline{g} \equiv W \xrightarrow{g} HW + X + A \xrightarrow{HW+[e^{\dagger},A]} HW + A$$

in the iterative algebra A.

Thus, to establish (4.6) it suffices to show that $\langle e\rangle^{\dagger} \cdot \mathsf{inl}$ is a solution of $\overline{g}$. In fact, the outer square of the diagram



commutes. To prove this, observe that by (4.3) all parts except, perhaps, for the left-hand inner triangle, clearly commute. For that triangle, consider the components of the coproduct separately. The left- and right-hand components are obviously commutative. We do not claim this for the middle component. But this component commutes when extended to A in the upper right-hand corner. In fact, this yields the

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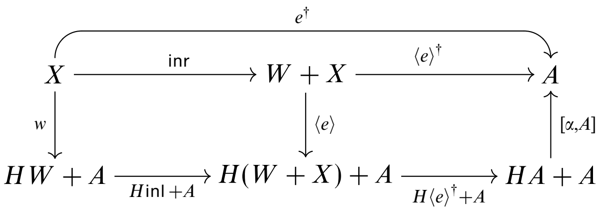
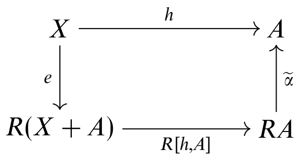
square



which commutes: see the upper part of Diagram (4.5).

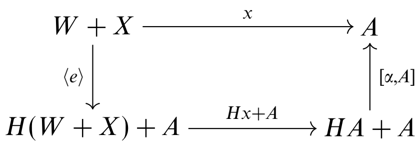
(2) *Uniqueness.*
Let h be any solution of e, that is, a morphism such that the square



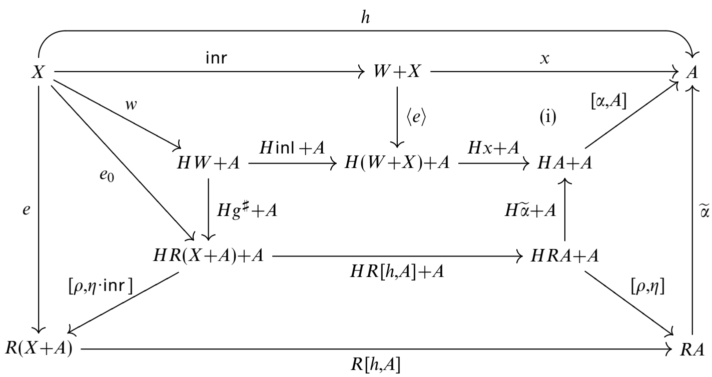
commutes. We shall show that

$$x \equiv W+X \xrightarrow{[\widetilde{\alpha}\cdot R[h,A]\cdot g^\sharp,h]} A$$

is a solution of $\langle e\rangle$ in A, and thus $h = \langle e\rangle^\dagger \cdot \mathsf{inr} = e^\dagger$, which completes the proof. Hence, we need to show that the square



commutes.
We will consider the components of the coproduct $W+X$ separately. For the right-hand component we obtain the following commutative diagram:



(4.7)

The outer square commutes, and it is clear that all the inner parts except (i) also commute, so the right-hand component of part (i) must also commute. (Note that this diagram is precisely (4.5) with h for $e^\dagger$ and x for $\langle e\rangle^\dagger$.)

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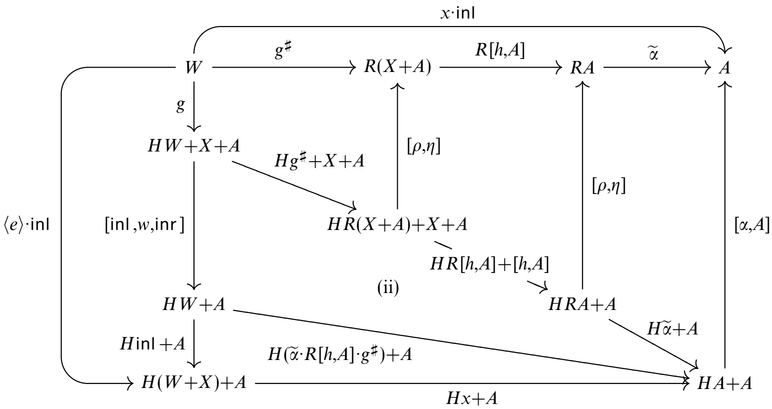
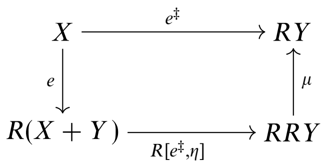
For the left-hand component of (i), consider the following diagram:



All of its parts commute, except possibly the middle component of (ii), which commutes when extended by $[\alpha, A]$ to A in the upper right-hand corner. In fact, this is easy to see by inspection of the upper three inner parts of Diagram (4.7). □

The rational solution theorem we have proved in previous work (Adámek *et al.* 2003a; Adámek *et al.* 2003b) is now an easy consequence of Theorem 4.6.

Corollary 4.7. Every rational guarded equation morphism $e : X \longrightarrow R(X+Y)$ has a unique solution in the algebra RY, that is, there exists a unique $e^\ddagger : X \longrightarrow RY$ such that the square

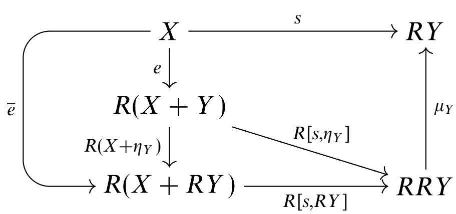


commutes.

Proof. Given a guarded rational equation morphism $e : X \longrightarrow R(X+Y)$, we form the equation morphism

$$\overline{e} \equiv X \xrightarrow{e} R(X+Y) \xrightarrow{R(X+\eta_Y)} R(X+RY).$$

This is a guarded equation morphism in the free iterative algebra RY. The result now follows from Theorem 4.6 applied to RY and to $\overline{e}$. In fact, there is a 1-1-correspondence between solutions of e and solutions of $\overline{e}$:



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Observe first that $\widetilde{\rho_Y} = \mu_Y : RRY \longrightarrow RY$. Now since s is a solution of e, the upper inner part commutes, and, equivalently, the outer square commutes, which is to say that s is a solution of $\overline{e}$. Since $\overline{e}$ has a unique solution, so does e. □

Proof of Theorem 4.4.

(a) *Sufficiency.*

Let A be an iterative algebra. We use

$$\gamma : F \longrightarrow R$$

to denote the natural transformation formed by the unique homomorphisms $\gamma_X : FX \longrightarrow RX$ of H-algebras with $\gamma_X \cdot \eta^0_X = \eta_X$. Observe that the square

$$\begin{array}{ccc} FX & \xrightarrow{\quad\gamma_X\quad} & RX \\ {\scriptstyle [\varphi_X,\eta^0_X]}\Big\uparrow & & \Big\uparrow{\scriptstyle [\rho_X,\eta_X]} \\ HFX+X & \xrightarrow[H\gamma_X+X]{} & HRX+X \end{array} \tag{4.8}$$

commutes.

Given a guarded finitary equation morphism $e : X \longrightarrow F(X+A)$, we prove that a unique solution $e^\dagger$ exists. To this end, we form the rational equation morphism

$$\overline{e} \equiv X \xrightarrow{\ e\ } F(X+A) \xrightarrow{\gamma_{X+A}} R(X+A)$$

and observe that it is guarded (use (4.8)). The unique solution $\overline{e}^\dagger$ solves e. In fact, in the diagram

$$\begin{array}{ccccc} X & & \xrightarrow{\qquad\overline{e}^\dagger\qquad} & & A \\ {\scriptstyle e}\Big\downarrow & & & \nearrow{\scriptstyle \widehat{\alpha}} & \Big\uparrow{\scriptstyle \widetilde{\alpha}} \\ F(X+A) & \xrightarrow{F[\overline{e}^\dagger,A]} & FA & & \\ {\scriptstyle \gamma_{X+A}}\Big\downarrow & & & \searrow{\scriptstyle \gamma_A} & \\ R(X+A) & & \xrightarrow[R[\overline{e}^\dagger,A]]{} & & RA \end{array}$$

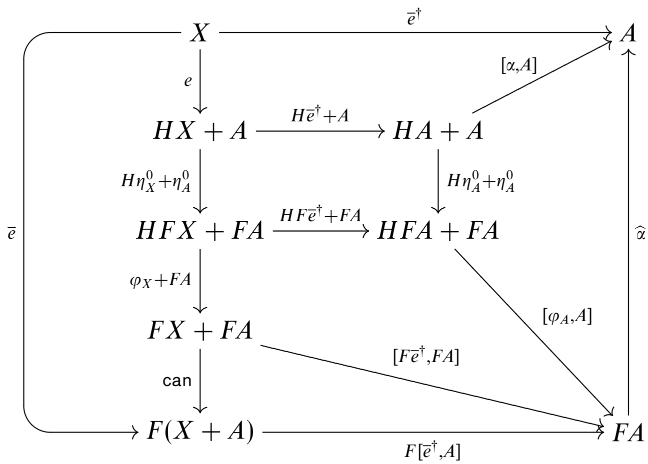
the outer square commutes (by definition of $\overline{e}^\dagger$), and the lower one does by the naturality of γ. The right-hand triangle commutes because both paths are homomorphisms extending id_A. Consequently, the upper square commutes, too. As for the uniqueness of solutions, suppose that in the above diagram $\overline{e}^\dagger : X \longrightarrow A$ denotes a solution of e. Then all inner parts of the diagram commute, thus, so does the outer square, whence $\overline{e}^\dagger$ is the unique solution of $\overline{e}$ (see Theorem 4.6).

(b) *Necessity.*

If $\alpha : HA \longrightarrow A$ is an H-algebra such that every finitary equation morphism e has a unique solution, then A is iterative. In fact, given a flat equation morphism $e : X \longrightarrow HX + A$, we use $\overline{e}$ to denote the following finitary equation morphism:

$$\overline{e} \equiv X \xrightarrow{\quad e\quad} HX + A \xrightarrow{H\eta^0_X+\eta^0_A} HFX + FA \xrightarrow{\varphi_X+FA} FX + FA \xrightarrow{\quad \mathsf{can}\quad} F(X+A).$$

It is easy to see that $\bar{e}$ is guarded. We will obtain a unique solution $\bar{e}^\dagger : X \longrightarrow A$, and prove that this solves e uniquely (in the sense of Definition 2.5). In other words, in the diagram



the upper square commutes. In fact, the outer square commutes by definition of $\bar{e}^\dagger$, the right-hand one commutes because $\widehat{\alpha} \cdot \eta_A^0 = id$, and since $\widehat{\alpha}$ is a homomorphism,

$$\widehat{\alpha} \cdot \varphi_A \cdot H\eta_A^0 = \alpha \cdot H\widehat{\alpha} \cdot H\eta_A^0 = \alpha.$$

Since the remaining inner parts commute (by naturality of η and φ), the commutativity of the upper square follows.

To prove that e has a unique solution, suppose that in the above diagram $\bar{e}^\dagger : X \longrightarrow A$ denotes a solution of e. Then all inner parts of the diagram commute, thus, the outer square does. This shows that $\bar{e}^\dagger$ is the unique solution of $\bar{e}$. □

5. Free iterative monads

In this section we present the main result of our paper, that for every finitary endofunctor H the rational monad is iterative in the sense of Calvin Elgot, and can be characterised as a free iterative monad on H. We first recall the concept of an iterative monad. Calvin Elgot's original definition (Elgot 1975) was formulated in Set in the language of Lawvere's algebraic theories; the present formulation is equivalent, as we proved in Aczel *et al.* (2003).

Remember our standing assumption that H denotes a finitary endofunctor of a locally finitely presentable category $\mathcal{A}$.

5.1. Iterative monads. For a monad $\mathbb{S} = (S, \eta, \mu)$ over Set we can form the complements of $\eta_X[X]$ in SX, say,

$$\sigma_X : S'X \longrightarrow SX$$

for all objects X. The monad $\mathbb{S}$ is called *ideal* if $\sigma : S' \longrightarrow S$ is a subfunctor of S, and the monad multiplication has a domain–codomain restriction $\mu' : S'S \longrightarrow S'$. For general base categories, instead of requiring a subfunctor S', we impose certain properties on μ'

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that are very similar to the monad laws for μ and η. The corresponding concept is given by the following definition.

Definition 5.2. An *ideal monad* is a six tuple

$$\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$$

consisting of a monad (S, η, μ) and natural transformations $\sigma : S' \longrightarrow S$ and $\mu' : S'S \longrightarrow S'$ such that

(1) $S = S' + Id$ with coproduct injections σ and η.
(2) The three diagrams

$$\begin{array}{ccc} S' & \xrightarrow{S'\eta} & S'S \\ & \searrow{\scriptstyle =} & \downarrow{\scriptstyle \mu'} \\ & & S' \end{array} \qquad \begin{array}{ccc} S'SS & \xrightarrow{\mu' S} & S'S \\ {\scriptstyle S\mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ S'S & \xrightarrow[\mu']{} & S' \end{array} \qquad \begin{array}{ccc} S'S & \xrightarrow{\mu'} & S' \\ {\scriptstyle \sigma S}\downarrow & & \downarrow{\scriptstyle \sigma} \\ SS & \xrightarrow[\mu]{} & S \end{array} \tag{5.1}$$

commute.

Remark 5.3. Note that the left-hand and middle diagrams in (5.1) express the fact that the pair (S', μ') is a right $\mathbb{S}$-module, and the right-hand diagram states that σ is a morphism of $\mathbb{S}$-modules from (S', μ') to (S, μ). The notion of a module appears for a monoidal category and a monoid in that category under the name action in Mac Lane (1998, VII.4). We chose the name module to remind readers of the classical example of Abelian groups; in this category, a monoid is precisely a ring R and an R-module is precisely a module of the ring R. Here we work in the monoidal category of endofunctors on $\mathcal{A}$ with composition as the tensor product and the identity functor as the tensor unit.

Examples 5.4.

(1) The rational monad is ideal. Recall from Remark 3.10 that $R = HR + Id$. Here we consider the natural transformation

$$\rho : HR \longrightarrow R$$

expressing the H-algebra structure $\rho_Z : HRZ \longrightarrow RZ$ of each RZ, see Definition 2.23. The 'restriction' of μ here is simply

$$\mu' = H\mu : HRR \longrightarrow HR.$$

In fact, we know from Remark 3.10 that $RZ = HRZ + Z$ with the coproduct injections ρ_Z and η_Z. Next, $(HR, H\mu)$ is an $\mathbb{R}$-module: the first two diagrams of (5.1) follow easily from the monad laws for μ and η, and the third square

$$\begin{array}{ccc} HRR & \xrightarrow{H\mu} & HR \\ {\scriptstyle H\rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ RR & \xrightarrow[\mu]{} & R \end{array}$$

commutes because each μ_Z is a homomorphism of H-algebras, see Definition 2.23.

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(2) The free-algebra monad $\mathbb{F}$ of Section 4 is ideal. Here, analogously, we use $F = HF + Id$ and $\mu' = H\mu^0 : HFF \longrightarrow HF$, see (4.1).

(3) Classical algebraic theories (groups, lattices, and so on) are usually not ideal. For example, the equation $x \cdot x^{-1} = e$ in the algebraic theory S of groups means that we do not have $S = S' + Id$, more precisely, the complement of $\eta : Id \hookrightarrow S$ is not a subfunctor.

Definition 5.5. Let $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ be an ideal monad on $\mathcal{A}$.

(1) A *finitary equation morphism* is defined to be a morphism

$$e : X \longrightarrow S(X + Y)$$

in $\mathcal{A}$ where X is a finitely presentable object ('of variables') and Y is any object ('of parameters').

(2) A *solution* of e is defined to be a morphism

$$e^\dagger : X \longrightarrow SY$$

for which the square

$$\begin{array}{ccc} X & \xrightarrow{\;e^\dagger\;} & SY \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_Y} \\ S(X+Y) & \xrightarrow[S[e^\dagger, \eta_Y]]{} & SSY \end{array}$$

commutes.

(3) The equation morphism e is said to be *guarded* if it factors through the summand $S'(X + Y) + Y$ of $S(X + Y) = S'(X + Y) + X + Y$:

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & S(X+Y) \\ & \searrow & \uparrow{\scriptstyle [\sigma_{X+Y}, \eta_{X+Y}\,\mathsf{inr}]} \\ & & S'(X+Y)+Y \end{array}$$

(4) The ideal monad $\mathbb{S}$ is said to be *iterative* if every guarded finitary equation morphism has a unique solution.

Example 5.6. The rational monad of every finitary endofunctor is iterative, see Corollary 4.7.

Remark 5.7. Next we will define morphisms of ideal monads. Whenever our base category $\mathcal{A}$ has the (very common) property that coproduct injections are monomorphic, in an ideal monad $\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$ we automatically get a subfunctor $S' \hookrightarrow S$ and the module laws of (S', μ') follow automatically from the monad laws of S. This makes the definitions of morphisms easy and canonical. Let $\mathbb{T} = (T, \eta^T, \mu^T, T', \tau, \mu'^T)$ be another ideal monad. An ideal monad morphism is a monad morphism

$$m : (S, \eta, \mu) \longrightarrow (T, \eta^T, \mu^T)$$

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that has a restriction m' to the given subfunctors:

$$\begin{array}{ccc} S' & \xrightarrow{m'} & T' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ S & \xrightarrow[m]{} & S' \end{array}$$

However, we do not want to impose any side conditions on $\mathcal{A}$. The price is that ideal monad morphisms are defined as pairs (m, m').

Definition 5.8.

(1) An *ideal monad morphism* from an ideal monad $(S, \eta, \mu, S', \sigma, \mu')$ to another ideal monad $(T, \eta^T, \mu^T, T', \tau, \mu'^T)$ is a pair (m, m') that consists of a monad morphism $m : (S, \eta, \mu) \longrightarrow (T, \eta^T, \mu^T)$ and a natural transformation $m' : S' \longrightarrow T'$ such that the diagrams

$$\begin{array}{ccc} S'S & \xrightarrow{m' * m} & T'T \\ {\scriptstyle\mu'}\downarrow & & \downarrow{\scriptstyle\mu'^T} \\ S' & \xrightarrow[m']{} & T' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S' & \xrightarrow{m'} & T' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\tau} \\ S & \xrightarrow[m]{} & T \end{array} \qquad (5.2)$$

commute.

(2) Given a functor H, a natural transformation $\lambda : H \longrightarrow S$ is said to be *ideal* if it factors through $\sigma : S' \longrightarrow S$, that is, $\lambda = \sigma \cdot \lambda'$ for some natural transformation $\lambda' : H \longrightarrow S'$.

Remark 5.9. The left-hand square in Diagram (5.2) expresses the fact that $m' : S' \longrightarrow T'$ is a module morphism with change of base m. The right-hand square together with the preservation of the unit $m \cdot \eta = \eta^T$ expresses the fact that $m = m' + Id$. In fact, every ideal monad morphism is determined by its second component m'.

Example 5.10. For the rational monad $\mathbb{R}$, the natural transformation

$$\kappa \equiv H \xrightarrow{H\eta} HR \xrightarrow{\rho} R$$

is ideal.

Remark 5.11.

(1) We are going to prove that, for every finitary endofunctor H, the rational monad $\mathbb{R}$ is a free iterative monad on H. Since ideal monad morphisms are pairs, the freeness is expressed by a pair of equations. Notice, however, that under the assumption that coproduct injections in the base category $\mathcal{A}$ are monomorphic, see Remark 5.7, the freeness of $\mathbb{R}$ means, as expected, that for every iterative monad $\mathbb{S}$ and every ideal natural transformation $\lambda : H \longrightarrow S$ there exists a unique ideal monad morphism $\overline{\lambda} : \mathbb{R} \longrightarrow \mathbb{S}$ such that $\overline{\lambda} \cdot \kappa = \lambda$. The formulation below refrains from the assumption that coproduct injections are monomorphic.

(2) Parts of the following proof are identical to the corresponding parts of Theorem 5.14 of Milius (2005) as already mentioned in Section 1.5, *Related work*.

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Theorem 5.12 (Rational monad as a free iterative monad). For every iterative monad $\mathbb{S}$ and every ideal natural transformation $\lambda : H \longrightarrow S$ there exists a unique ideal monad morphism $(\overline{\lambda}, \overline{\lambda}'):\mathbb{R} \longrightarrow \mathbb{S}$ such that the diagrams

$$\begin{array}{ccc} H & \xrightarrow{H\eta} & HR \\ & {\scriptstyle \lambda'}\searrow & \downarrow{\scriptstyle \overline{\lambda}'} \\ & & S' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} H & \xrightarrow{\kappa} & R \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle \overline{\lambda}} \\ & & S \end{array} \tag{5.3}$$

commute.

Remark 5.13. Consider the category $\mathsf{Fin}(\mathcal{A}, \mathcal{A})$ of all finitary endofunctors and natural transformations and the category

$$\mathsf{FIM}(\mathcal{A})$$

of all finitary iterative monads (that is, iterative monads $(S, \eta, \mu, S', \sigma, \mu')$ with S and S' finitary) and ideal monad morphisms. We have a forgetful functor

$$U : \mathsf{FIM}(\mathcal{A}) \longrightarrow \mathsf{Fin}(\mathcal{A}, \mathcal{A}), \qquad \mathbb{S} \longmapsto S'$$

The above theorem states that U has a left adjoint, *viz.*, the functor $H \longmapsto \mathbb{R}$.

Proof.

(1) For every object Z consider SZ as an H-algebra

$$HSZ \xrightarrow{\lambda_{SZ}} SSZ \xrightarrow{\mu_Z} SZ.$$

It is iterative. In fact, every equation morphism $e : X \longrightarrow HX + SZ$, with X in $\mathcal{A}_{fp}$, yields the following equation morphism with respect to $\mathbb{S}$:

$$\overline{e} \equiv X \xrightarrow{\quad e \quad} HX + SZ \xrightarrow{\lambda_X + SZ} SX + SZ \xrightarrow{\quad \mathsf{can} \quad} S(X + Z).$$

To verify that $\overline{e}$ is guarded, we use the restriction $\lambda' : H \longrightarrow S'$ of λ:

$$\begin{array}{ccccccc} X & \xrightarrow{e} & HX{+}SZ & \xrightarrow{\lambda_X+SZ} & SX{+}SZ & \xrightarrow{\qquad\qquad \mathsf{can} \qquad\qquad} & & S(X{+}Z) \\ & & & {\scriptstyle \lambda'_X+SZ}\searrow & \uparrow{\scriptstyle \sigma_Z+SZ} & & & \uparrow{\scriptstyle [\sigma_{X+Z}, \eta_{X+Z}\,\mathsf{inr}]} \\ & & & & S'X{+}SZ & \underset{S'X+[\sigma_Z,\eta_Z]}{\overset{S'X+[\sigma_Z,\eta_Z]^{-1}}{\rightleftarrows}} & S'X{+}S'Z{+}Z & \xrightarrow{\mathsf{can}+Z} S'(X{+}Z){+}Z \end{array}$$

To prove the commutativity of the square, consider the three components of $S'X + S'Z + Z$ separately, and use naturality of σ and η.
We will prove that a morphism $e^\dagger : X \longrightarrow SZ$ is a solution of e in the H-algebra SZ if and only if it is a solution of $\overline{e}$ with respect to the iterative monad $\mathbb{S}$. Thus, since $\overline{e}$ has a unique solution so does e.

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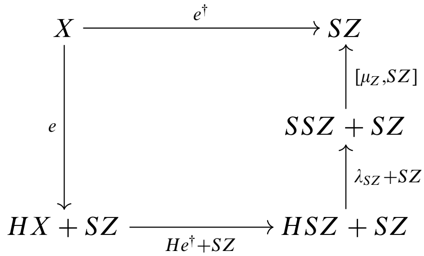
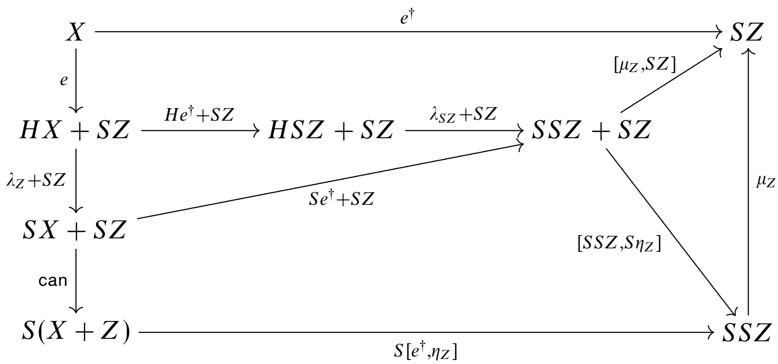
(1a) Let $e^\dagger$ be a solution of e in the algebra SZ, that is, let



(5.4)

commute. Then we see that the outer square in the diagram



(5.5)

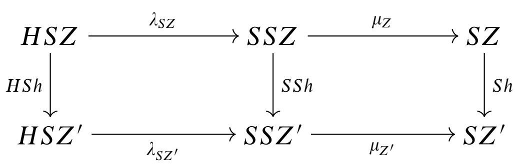
is commutative: the upper part is (5.4); the one directly below it uses the naturality of λ; the lower part is obviously commutative; and the right-hand one is also because of $\mu_Z \cdot S\eta_Z = id$. Hence, $e^\dagger$ is a solution of $\overline{e}$.

(1b) Conversely, let $e^\dagger$ be a solution of $\overline{e}$. Then the outer square of (5.5) commutes. Since the remaining three inner parts commute, so does the upper one, which is (5.4). Hence, $e^\dagger$ is a solution of e, as desired.

(2) *Existence of an ideal monad morphism $\overline{\lambda}$ such that (5.3) commutes.*
We use

$$\overline{\lambda}_Z : RZ \longrightarrow SZ$$

to denote the unique homomorphism of H-algebras with $\overline{\lambda}_Z \cdot \eta_Z = \eta^S_Z$. We first observe that $\overline{\lambda}$ is a natural transformation. Given a morphism $h : Z \longrightarrow Z'$, we have Sh is a homomorphism of H-algebras from SZ to SZ':



(5.6)

Thus, we have two parallel H-algebra homomorphisms from RZ to SZ':

$$Sh \cdot \overline{\lambda}_Z \qquad \text{and} \qquad \overline{\lambda}_{Z'} \cdot Rh.$$

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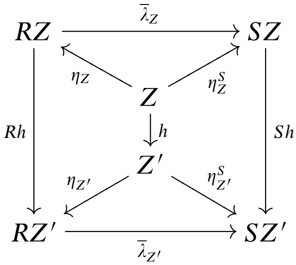
These agree when precomposed with η_Z:

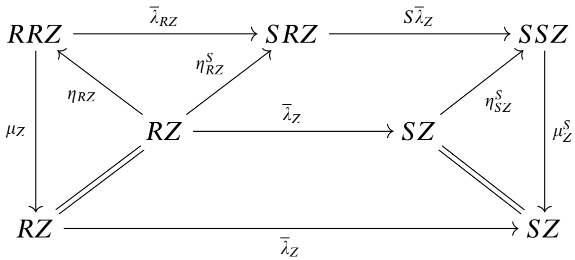


By the universal property of η_Z, and since SZ' is an iterative H-algebra, this proves that the above naturality square commutes.

We now prove that $\overline{\lambda}$ is a monad morphism. Since $\overline{\lambda} \cdot \eta = \eta^S$ by definition, it remains to prove the commutativity of the diagram

$$\begin{array}{ccccc} RRZ & \xrightarrow{\overline{\lambda}_{RZ}} & SRZ & \xrightarrow{S\overline{\lambda}_Z} & SSZ \\ {\scriptstyle \mu_Z}\downarrow & & & & \downarrow{\scriptstyle \mu^S_Z} \\ RZ & & \xrightarrow[\overline{\lambda}_Z]{} & & SZ \end{array} \tag{5.7}$$

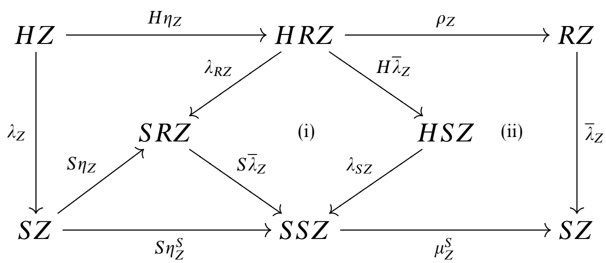
By (5.6), applied to $h = \overline{\lambda}_Z$, we see that $S\overline{\lambda}_Z$ is a homomorphism of iterative H-algebras. By the universal property of η_{RZ}, it is sufficient to prove that (5.7) commutes when precomposed with η_{RZ}:



The equation

$$\lambda = \overline{\lambda} \cdot \kappa = \overline{\lambda} \cdot \rho \cdot H\eta$$

follows from the commutativity of the following diagram



(5.8)

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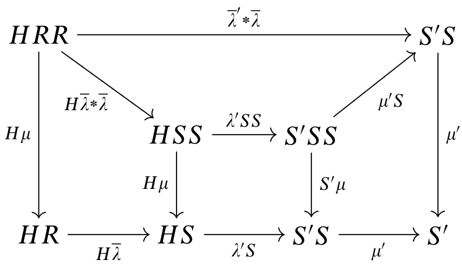
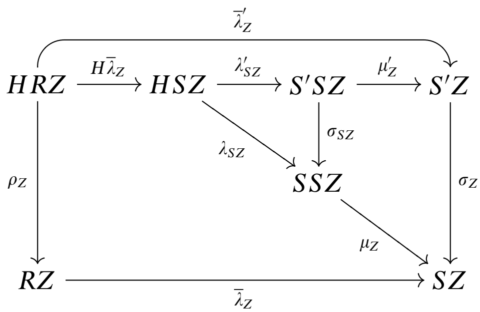
where (i) is naturality of λ and (ii) is clear since $\overline{\lambda}$ is a homomorphism. We now use the fact that $\mu^S_Z \cdot S\eta^S_Z = id$.
Thus, we have found a monad morphism $\overline{\lambda} : \mathbb{R} \longrightarrow \mathbb{S}$ with $\overline{\lambda} \cdot \kappa = \lambda$. It remains to verify that $\overline{\lambda}$ is part of an ideal monad morphism. Put

$$\overline{\lambda}' \equiv HR \xrightarrow{H\overline{\lambda}} HS \xrightarrow{\lambda' S} S'S \xrightarrow{\mu'} S'. \tag{5.9}$$

To see that the pair $(\overline{\lambda}, \overline{\lambda}')$ is an ideal monad morphism, we have to verify the commutativity of the diagrams (5.2) for this pair. For the left-hand diagram of (5.2), consider the diagram

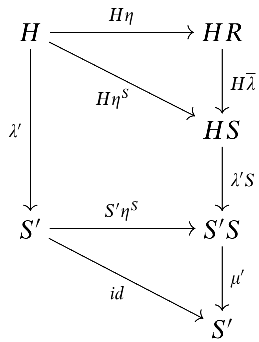


The upper part clearly commutes by the definition (5.9) of $\overline{\lambda}'$ and by the naturality of parallel composition. The other parts of the diagram are clear by invoking, from left to right, the fact that $\overline{\lambda}$ is a monad morphism, the naturality of λ' and the module laws of S'. To verify the right-hand square of (5.2), consider the diagram



This diagram commutes because its upper square is the definition (5.9) of $\overline{\lambda}'$, the lower left-hand part commutes since $\overline{\lambda}_Z$ is an H-algebra homomorphism, for the middle triangle we use the fact that λ is an ideal natural transformation, and the lower right-hand part commutes because of the right-hand square in (5.1)

Finally, we need to check the left-hand triangle of (5.3), that is, we need to show that $\overline{\lambda}' \cdot H\eta = \lambda'$. To see this, consider the diagram



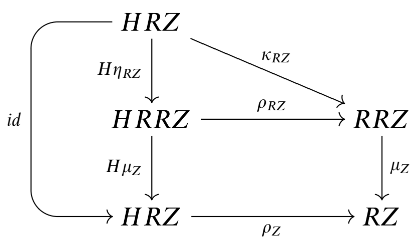
This diagram commutes because for the upper triangle we can use the fact that $\overline{\lambda}$ is a monad morphism, the middle part is naturality, and the lower triangle is the unit law of the $\mathbb{S}$-module S'.

(3) *Uniqueness of* $\overline{\lambda}$.

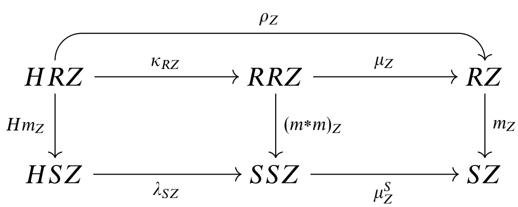
Suppose that (m, m') is an ideal monad morphism from $\mathbb{R}$ to $\mathbb{S}$ such that Diagrams (5.3) commute with (m, m') instead of $(\overline{\lambda}, \overline{\lambda}')$. We are going to show that for any object Z, m_Z is an H-algebra homomorphism extending η^S_Z. Then the freeness of RZ as an iterative H-algebra implies that $m = \overline{\lambda}$, which then leads us to conclude that $m' = \overline{\lambda}'$. First, note that for any object Z we have the equation

$$\rho_Z = \mu_Z \cdot \kappa_{RZ}. \tag{5.10}$$

Indeed, the diagram



commutes. Consequently, the diagram



commutes since the upper part is (5.10), the right-hand square commutes because m is a monad morphism, and the left-hand one does because $m \cdot \kappa = \lambda$ and by naturality.

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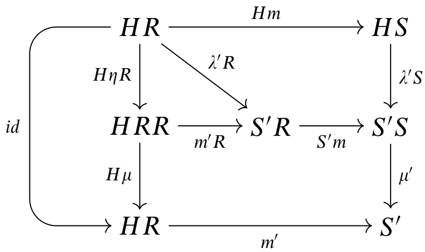
Thus, $m_Z : RZ \longrightarrow SZ$ is an H-algebra homomorphism between iterative H-algebras such that $m_Z \cdot \eta_Z = \eta^S_Z$. This implies that $m = \overline{\lambda}$ and from this it follows that

$$m' = \mu' \cdot \lambda' S \cdot Hm = \mu' \cdot \lambda' S \cdot H\overline{\lambda}$$

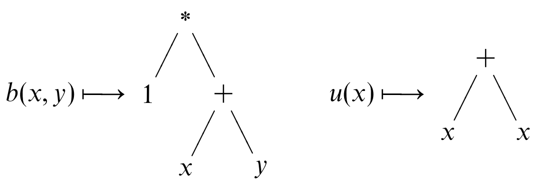
where the first equation holds due to the following diagram:



The lower square commutes since m' is a module homomorphism with change of base m, the left-hand part by the unit law of the monad $\mathbb{R}$, the upper triangle by (5.3) and the upper right-hand part by naturality. This completes the proof. □

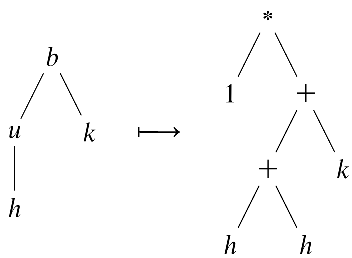
Example 5.14. For polynomial endofunctors on Set, the freeness of $\mathbb{R}$ specialises to *second-order substitution*, see Courcelle (1983), that is, substitution of rational trees for operation symbols.

For example, consider a signature Σ with binary and unary operation symbols b and u, respectively, and another signature Γ with two binary operation symbols $+$ and $*$ and a constant symbol 1. The assignment



(5.11)

of operation symbols in Σ to rational trees over Γ gives rise to a natural transformation $\lambda : H_\Sigma \longrightarrow R_\Gamma$. The induced monad morphism $\overline{\lambda} : \mathbb{R}_\Sigma \longrightarrow \mathbb{R}_\Gamma$ substitutes, for any set of variables X, the operation symbols in trees of $R_\Sigma X$ by trees of $R_\Gamma X$ according to λ. For example, for $X = \{h, k\}$, we get that $\overline{\lambda}_X$ performs the assignment



The requirement that λ be an ideal transformation means that no operation symbol of Σ is replaced by a single variable, that is, λ is a so-called *non-erasing* substitution.

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Remark 5.15. We have defined a rational monad for every finitary endofunctor of a locally finitely presentable category. One may ask what happens if we 'raise the index of presentability' to an uncountable regular cardinal λ. That is, what is the 'λ-rational' monad of a λ-accessible endofunctor H (that is, one, preserving λ-filtered colimits)?

It is easy to see that the main results above remain true if we systematically replace 'finitely presentable' by 'λ-presentable' and 'finitary' by 'λ-accessible'. An H-algebra A might be called *λ-iterative* if every equation morphism $e : X \longrightarrow HX + A$ with X λ-presentable has a unique solution $e^\dagger : X \longrightarrow A$.

Then, for a λ-accessible endofunctor $H : \mathcal{A} \longrightarrow \mathcal{A}$, one can prove the following:

(1) The category of all λ-iterative H-algebras is reflective in $\mathsf{Alg}\, H$.

(2) The resulting λ-accessible monad $\mathbb{R}^\lambda$ on $\mathcal{A}$ is a free λ-iterative monad on H. Again, *λ-iterative* means unique solvability of equations $X \longrightarrow R^\lambda(X+Z)$ with X λ-presentable. Moreover, $R^\lambda Z = \operatorname{colim} \mathrm{Eq}_Z^\lambda$, where Eq_Z^λ is the obvious modification of the diagram Eq_Z from Corollary 3.9.

However, in the case of uncountable λ such a monad $\mathbb{R}$ coincides with the *completely iterative monad* $\mathbb{T}$ of H, which has been described in Aczel *et al.* (2003) and Milius (2005). This monad $\mathbb{T}$ is given object-wise by final coalgebras for the endofunctor $H(-)+Z$: $\mathcal{A} \longrightarrow \mathcal{A}$. To show that $\mathbb{T} \cong \mathbb{R}^\lambda$, it therefore suffices to prove the following proposition.

Proposition 5.16. For uncountable λ, the object $R^\lambda Z$ is a final coalgebra for $H(-)+Z$. More precisely, the isomorphism $i_Z : R^\lambda Z \longrightarrow HR^\lambda Z + Z$ is a final coalgebra for $H(-)+Z$.

Proof. We use the fact, which was proved in Adámek and Porst (2004), that since λ is uncountable, the category EQ_Z^λ is a dense full subcategory of the locally λ-presentable category of all coalgebras for $H(-)+Z$. Thus, it suffices to prove that for every

$$e : X \longrightarrow HX + Z$$

in EQ_Z^λ there exists a unique homomorphism into $i_Z : R^\lambda Z \longrightarrow HR^\lambda Z + Z$. Since the colimit injection $e^\sharp : X \longrightarrow R^\lambda Z$ is such a homomorphism, it only remains to verify uniqueness. This is done analogously to Proposition 3.2. □

6. Conclusions and future work

We have proved that all finitary endofunctors H generate a free iterative monad $\mathbb{R}$. All we needed in our proof was the assumption that the base category is locally finitely presentable. This is the 'real McCoy' that we had tried to achieve in Adámek *et al.* (2003a) and Adámek *et al.* (2003b): there we obtained the same result, but only in the base category Set, and the proof was much more complicated. The reason was that when writing those papers we did not follow in the footsteps of Evelyn Nelson and Jerzy Tiuryn, who realised long ago that iterative algebras are more basic than Elgot's iterative theories.

The results of the present paper are analogous to results on completely iterative algebras and completely iterative theories. The latter were introduced in Elgot *et al.* (1978) in analogy to iterative theories by dropping the finiteness restriction on the objects of variables: one studies equation morphisms with arbitrary objects X of variables, and

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requires unique solutions of these more general equations. Stefan Milius (Milius 2005) defines completely iterative algebras for an endofunctor H on a category $\mathcal{A}$ with binary coproducts, and he relates them to completely iterative monads: H has free completely iterative algebras TX if and only if H generates a free completely iterative monad $\mathbb{T}$ if and only if H has 'enough final coalgebras', that is, every functor $H(-)+X$ has a final coalgebra TX.

It is then natural to ask whether there is a monad in between the free iterative monad $\mathbb{R}$ and the free completely iterative one $\mathbb{T}$: for example, we could consider, for an accessible functor and some uncountable cardinal λ, all equation morphisms with a λ-presentable object X of variables. However, we have shown that the answer is no: one gets the monad $\mathbb{T}$, see Remark 5.15.

The main technical result of our paper is a description of an initial iterative algebra as a colimit of all H-coalgebras carried by finitely presentable objects. From this result we showed that the algebraic theory formed by all free iterative H-algebras is iterative in the sense of Calvin Elgot. In fact, that theory can be characterised as a free iterative theory on H. The freeness of the rational monad can be used to formulate clearly the 'second-order substitution' described for rational Σ-trees by Bruno Courcelle (Courcelle 1983), see Example 5.14.

Our result can be applied to arbitrary base categories that are locally finitely presentable. For example, to the category of all finitary endofunctors of Set. In the future we intend to use this in an attempt to describe the monad of algebraic trees (Courcelle 1983), categorically.

References

Aczel, P., Adámek, J., Milius, S. and Velebil, J. (2003) Infinite Trees and Completely Iterative Theories: A Coalgebraic View. *Theoretical Computer Science* **300** 1–45.

Adámek, J. (1974) Free Algebras and Automata Realization in the Language of Categories. *Comment. Math. Univ. Carolinae* **15** 589–602.

Adámek, J. and Milius, S. (2006) Terminal Coalgebras and Free Iterative Theories. In *Information and Computation* **204** 1139–1172.

Adámek, J., Milius, S. and Velebil, J. (2003a) Free iterative theories – a coalgebraic view. *Mathematical Structures in Computer Science* **13** 259–320.

Adámek, J., Milius, S. and Velebil, J. (2003b) On Rational Monads and Free Iterative Theories. *Electronic Notes in Theoretical Computer Science* **69**.

Adámek, J., Milius, S. and Velebil, J. (2006) Elgot Algebras. To appear in *Logical Methods in Computer Science*.

Adámek, J. and Porst, H.-E. (2004) On Tree Coalgebras and Coalgebra Presentations. *Theoretical Computer Science* **311** 257–283.

Adámek, J. and Rosický, J. (1994) *Locally presentable and accessible categories*, Cambridge University Press.

Arbib, M. A. and Manes, E. G. (1986) *Algebraic approaches to program semantics*, Texts and Monographs in Computer Science, Springer-Verlag.

Bloom, S. and Elgot, C. C. (1974) The Existence and Construction of Free Iterative Theories. *Journal of Computer and System Sciences* **12** 305–318.

Courcelle, B. (1983) Fundamental Properties of Infinite Trees. *Theoretical Computer Science* **25** 95–169.

Elgot, C. C. (1975) Monadic Computation and Iterative Algebraic Theories. In: Rose, H. E. and Shepherdson, J. C. (eds.) *Logic Colloquium '73*, North-Holland 175–230.

Elgot, C. C., Bloom, S. L. and Tindell, R. (1978) On the Algebraic Structure of Rooted Trees. *Journal of Computer and System Sciences* **16** 361–399.

Gabriel, P. and Ulmer, F. (1971) Lokal präsentierbare Kategorien. *Springer-Verlag Lecture Notes in Mathematics* **221**.

Ghani, N., Lüth, C. and De Marchi, F. (2002) Coalgebraic Monads. *Electronic Notes in Theoretical Computer Science* **65** (1).

Ginali, S. (1979) Regular trees and the free iterative theory. *Journal of Computer and System Sciences* **18** 228–242.

Lambek, J. (1968) A fixpoint theorem for complete categories. *Mathematische Zeitung* **103** 151–161.

Lane, S. M. (1998) *Categories for the Working Mathematician*, 2nd edition, Springer-Verlag.

Milius, S. (2005) Completely Iterative Algebras and Completely Iterative Monads. *Information and Computation* **196** 1–41.

Nelson, E. (1983) Iterative Algebras. *Theoretical Computer Science* **25** 67–94.

Rutten, J. J. M. M. (2000) Universal coalgebra, a theory of systems. *Theoretical Computer Science* **249** (1) 3–80.

Tiuryn, J. (1980) Unique Fixed Points vs. Least Fixed Points. *Theoretical Computer Science* **12** 229–254.

Worrell, J. (2005) On the final sequence of a finitary set functor. *Theoretical Computer Science* **338** 184–199.