

# On the birth and death of algebraic limit cycles in quadratic differential systems

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In 1958 started the study of the families of algebraic limit cycles in the class of planar quadratic polynomial differential systems. In the present we know one family of algebraic limit cycles of degree 2 and four families of algebraic limit cycles of degree 4, and that there are no limit cycles of degree 3. All the families of algebraic limit cycles of degree 2 and 4 are known, this is not the case for the families of degree higher than 4. We also know that there exist two families of algebraic limit cycles of degree 5 and one family of degree 6, but we do not know if these families are all the families of degree 5 and 6. Until today it is an open problem to know if there are algebraic limit cycles of degree higher than 6 inside the class of quadratic polynomial differential systems. Here we investigate the birth and death of all the known families of algebraic limit cycles of quadratic polynomial differential systems.

**Key words:** Algebraic limit cycles, Hopf bifurcation, homoclinic orbits, heteroclinic orbits, quadratic polynomial differential systems

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## 1 Introduction and statement of the main results

Consider the polynomial differential system in  $\mathbb{R}^2$  given by

$$\begin{aligned}\dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y),\end{aligned}\tag{1.1}$$

where the dot denotes derivative with respect to the *time*  $t$ . The number  $m = \max\{\deg P, \deg Q\}$  is the *degree* of system (1.1).

When  $m = 2$  system (1.1) is a *quadratic polynomial differential system* or simply a *quadratic system*. Associated with system (1.1) we have the polynomial vector field

$$\mathcal{X} = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}.$$

An isolated periodic orbit in the set of all periodic orbits of system (1.1) is a *limit cycle*. The notion of limit cycle goes back to Poincaré [25]. The study of limit cycles started to be very interesting for the mathematicians after the statement of the second part of the 16-th Hilbert problem [14] in 1900, and when they appear in a natural way in physical phenomena as in the Van der Pol equation [27], or in the Rayleigh equation [26]. But also they appear in differential equations of higher dimension as in [21].

Although there are hundreds of papers published on quadratic systems, there remains many open questions on the existence, birth and death of the limit cycles of quadratic systems. In recent years a variety of methods were used to investigate the existence of limit cycles in quadratic systems.

Let  $\mathbb{R}[x, y]$  be the ring of the real polynomials in the variables  $x$  and  $y$ . Given  $f \in \mathbb{R}[x, y]$  the algebraic curve  $f(x, y) = 0$  is an *invariant algebraic curve* of system (1.1) if there exists  $K \in \mathbb{R}[x, y]$  such that

$$P \frac{\partial f}{\partial x} + Q \frac{\partial f}{\partial y} = Kf.$$

The word invariant for the algebraic curve  $f(x, y) = 0$  comes from the fact that the curve is formed by orbits of the vector field  $\mathcal{X}$ , because on the points of this curve the condition (1) implies that the gradient of the curve is orthogonal to the vector field  $\mathcal{X}$ , in consequently the algebraic curve is formed by orbits of the system. An invariant algebraic curve  $f = 0$  is called *irreducible* if the polynomial  $f$  is irreducible

A limit cycle  $\gamma$  of system (1.1) is called an *algebraic limit cycle* if it is contained in some irreducible algebraic invariant curve  $f = 0$ . The degree of the algebraic limit cycle  $\gamma$  is the degree of  $f$ .

Until now we only know few different families of planar quadratic systems having algebraic limit cycles. More precisely, eight families of quadratic systems having algebraic limit cycles of degree  $n \leq 6$  are known. For degree 2 there is the family of quadratic systems known as Yuan-Xun's system, see [29]. It is known that there are no algebraic limit cycles of degree 3, see [9–11], or see Theorem 11 of [5] for a short proof. For degree 4 there are four families of quadratic systems having algebraic limit cycles [12, 28] and other two families described in [3], see also [4]. These five families mentioned before provide all the algebraic limit cycles of degrees 2 and 4 (up to affine change of variables and rescaling of the time). For degrees 5 and 6 we only know three one-parameter families of quadratic systems having algebraic limit cycles, and they are described in [1, 6]. It is unknown if these families are all the families of algebraic limit cycles of degrees 5 and 6 modulo changes of variables. The question of whether quadratic systems can have algebraic limit cycles of degree higher than 6 remains open. See details about this subject in [20].

It is known that when a quadratic system has an algebraic limit cycle of degree 2 or 4, this is the unique limit cycle of the system, see [3]. Since the algebraic limit cycles of degrees 5 and 6 have been obtained doing birational transformation of the quadratic systems having algebraic limit cycles of degree 4 (see [1, 6]), also the known quadratic systems having algebraic limit cycles of degrees 5 and 6 have a unique limit cycle. This fact simplifies the study of their phase portraits.

We note that the study of limit cycles is one of the more difficult problems of the qualitative theory of differential equation in the plane. See for example the 16-th Hilbert problem on the limit

cycles of polynomial differential systems [15, 17], or for instance for limit cycles in continuous and discontinuous piecewise differential systems [13, 18].

An isolated equilibrium point  $p$  of a quadratic system is a *centre* if there is a neighbourhood of  $p$  filled up of periodic solutions, with the exception of  $p$ .

The notion of *Hopf bifurcation* refers to the local birth or death of a periodic solution from an equilibrium point moving a parameter of the system. In a differential system a Hopf bifurcation typically occurs when a complex conjugate pair of eigenvalues of the linearised flow at an equilibrium point becomes purely imaginary. For more information about Hopf bifurcation, see for instance the book of Kuznetsov [16].

An orbit  $\gamma$  of system (1.1) is called a *homoclinic orbit* if there exists an equilibrium point  $p$  of the system such that  $p$  is the  $\alpha$ -limit and the  $\omega$ -limit of  $\gamma$ . Each homoclinic orbit together with its corresponding equilibrium point defines a *homoclinic loop*. A *heteroclinic orbit* is an orbit whose  $\alpha$ -limit and  $\omega$ -limit are two distinct equilibria. Several heteroclinic orbits connected forming a close curve define a *heteroclinic loop*.

The phase portraits of the quadratic systems are presented in the Poincaré disc, see Section 2 for an introduction to the Poincaré disc and its local charts  $U_k$  and  $V_k$  for  $k = 1, 2$ .

In this article we investigate where the known families of algebraic limit cycles of the quadratic systems are born and die. The main results of this article are the following four theorems.

**Theorem 1.1** (Degree 2) *Each planar quadratic system having an invariant ellipse after an affine change of variables and a rescaling of the time can be written as*

$$\begin{aligned} \dot{x} &= r^2 - x^2 - y^2 - y(c + ax + by), \\ \dot{y} &= x(c + ax + by), \end{aligned} \tag{1.2}$$

where  $r > 0$  and  $a^2 + b^2 + c^2 \neq 0$ . The invariant ellipse is

$$x^2 + y^2 - r^2 = 0.$$

For  $(a^2 + b^2)r^2 - c^2 < 0$ ,  $c^2 + 4(1 + b)r^2 > 0$  and  $a \neq 0$ , this ellipse is an algebraic limit cycle of degree 2, see Figure 1(b), (e) and (h), for  $b > -1$ ,  $b = -1$  and  $b < -1$ , respectively. Moreover, the family of these limit cycles born or die at a Hopf bifurcation when either  $r$  goes to zero, see Figure 1(a), (d) and (g) for  $b > -1$ ,  $b = -1$  and  $b < -1$ , respectively or in a homoclinic orbit formed by an ellipse having a saddle-node on it when  $(a^2 + b^2)r^2 - c^2 = 0$ , see Figure 1(c), (f) and (i) for  $b > -1$ ,  $b = -1$  and  $b < -1$ , respectively. We note that this algebraic limit cycle can be stable or unstable depending on the values of the parameters. Here we only drawn the phase portraits in the case that such a limit cycle is unstable.

**Theorem 1.2** (Degree 4) *We have the following four families of algebraic limit cycles of degree 4 in quadratic systems, modulo an affine change of variables and a rescaling of the time.*

(a) *The Yablonskii's [28] system*

$$\begin{aligned} \dot{x} &= -4abcx + 3(a + b)cx^2 - (a + b)y + 4xy, \\ \dot{y} &= ab(a + b)x + (4ab - \frac{3}{2}(a + b)^2 + 4abc^2)x^2 - 4abcy + 8(a + b)cy + 8y^2, \end{aligned} \tag{1.3}$$

has the irreducible invariant algebraic curve of degree 4

$$x^2(x - a)(x - b) + (cx^2 + y)^2 = 0. \tag{1.4}$$

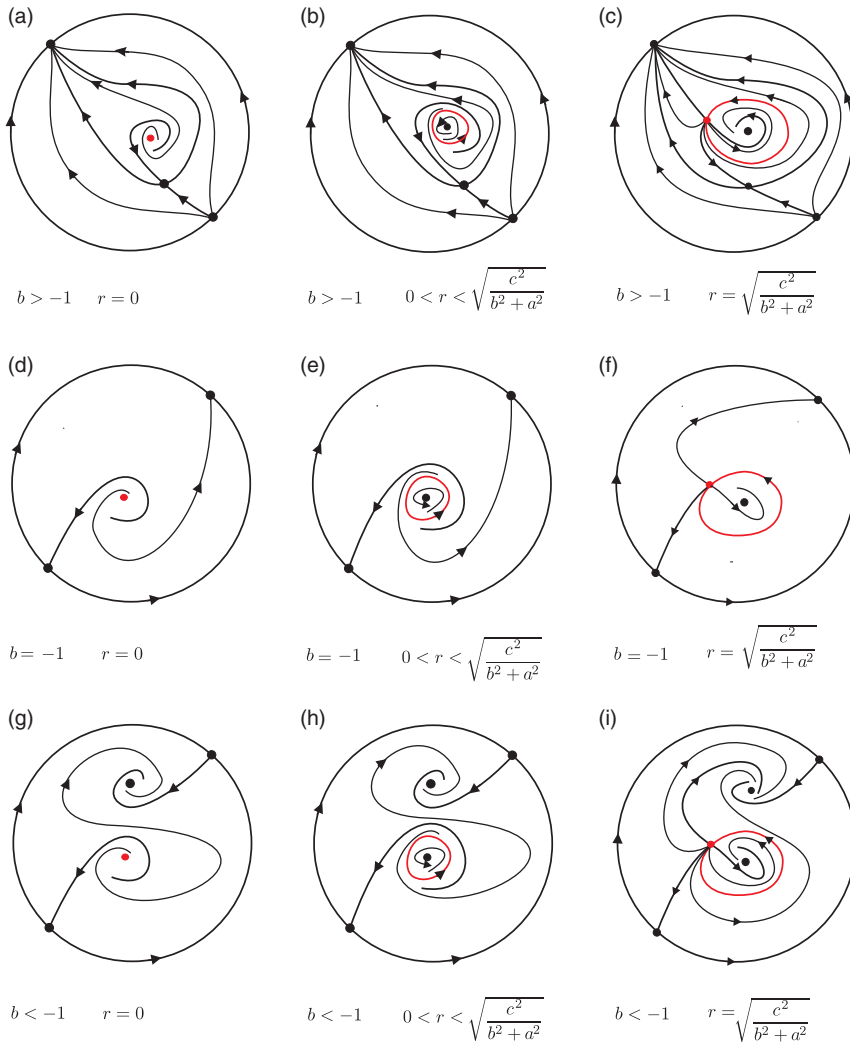


FIGURE 1. Birth and death of the algebraic limit cycles of degree 2. In Figures (a), (d) and (g) the algebraic limit cycle collapses to a singular point. In Figures (b), (e) and (h) we show the algebraic limit cycle. In Figures (c), (f) and (i) the algebraic limit cycles disappears in a homoclinic orbit.

For  $abc \neq 0$ ,  $a \neq b$ ,  $ab > 0$  and  $G = 4c^2(a - b)^2 + (3a - b)(a - 3b) < 0$ , this curve contains an algebraic limit cycle of degree 4, see Figure 2(a). When  $a \rightarrow b$ , this family of limit cycles born or die in a Hopf bifurcation, see Figure 2(b). When  $c \rightarrow 0$ , this family of limit cycles born or die in a periodic orbit of a centre, see Figure 2(c). When  $G \nearrow 0$ , this family of limit cycles born or die at a homoclinic orbit formed by an algebraic oval and a saddle-node on it, see Figure 2(d) if  $c \neq 0$  and Figure 2(e) if  $c \rightarrow 0$ . We note the algebraic limit cycle can be stable or unstable depending on the values of the parameters. Here we only drawn the phase portraits in the case that such a limit cycle is unstable.

(b) The Filippov's [12] system

$$\begin{aligned} \dot{x} &= 6(1 + a)x - 6(2 + a)x^2 + 2y + 12xy, \\ \dot{y} &= 3a(1 + a)x^2 + 15(1 + a)y - 2(9 + 5a)xy + 16y^2, \end{aligned} \tag{1.5}$$

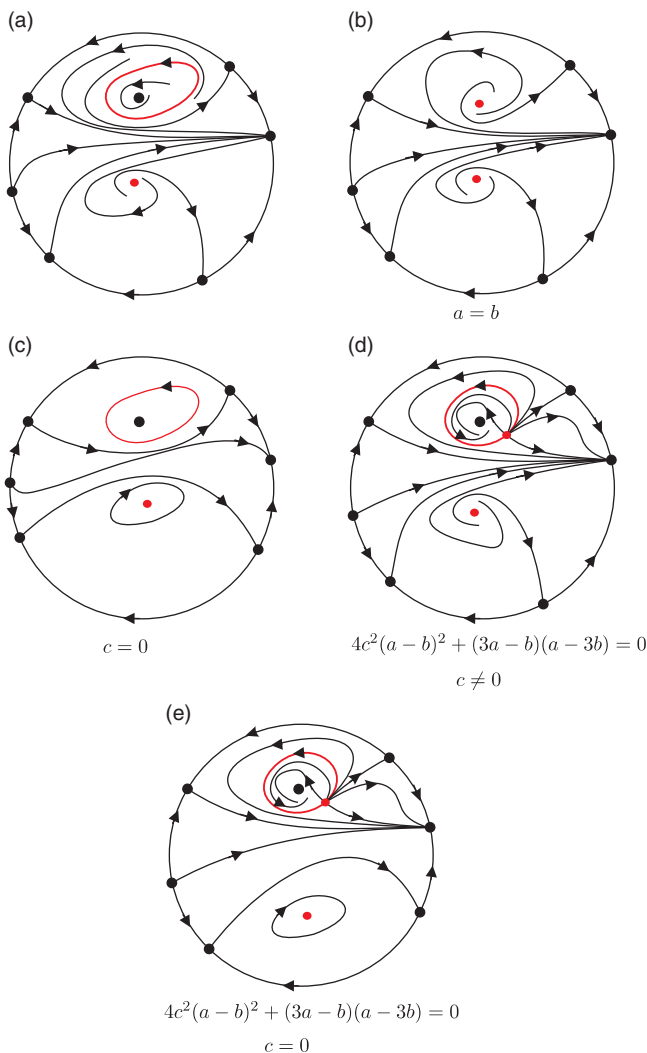


FIGURE 2. Birth and death of the algebraic limit cycles of the Yablonskii's [28] system.

has the irreducible invariant algebraic curve

$$3(1 + a)(ax^2 + y)^2 + 2y^2(2y - 3(1 + a)x) = 0. \tag{1.6}$$

For  $0 < a < 3/13$ , this curve contains an unstable algebraic limit cycle of degree 4 as the one shown in Figure 3(b). When  $a \searrow 0$ , this family of limit cycles tends to a heteroclinic cycle having some orbit at infinity, see Figure 3(a). When  $a \nearrow 3/13$ , the family of limit cycles born or die in a Hopf bifurcation, see Figure 3(c).

(c) Chavarriga et al's [5] system

$$\begin{aligned} \dot{x} &= 5x + 6x^2 + 4(1 + a)xy + ay^2, \\ \dot{y} &= x + 2y + 4xy + (2 + 3a)y^2, \end{aligned} \tag{1.7}$$

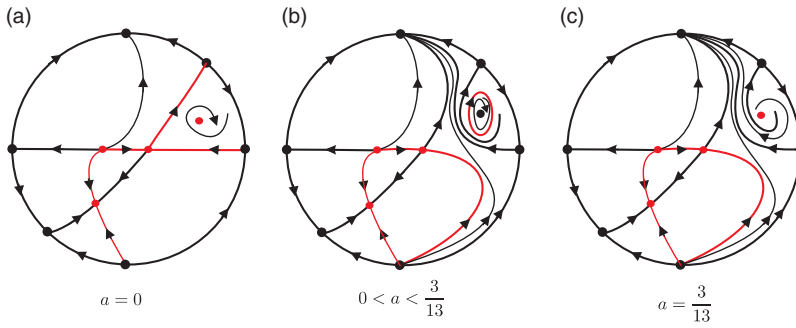


FIGURE 3. Birth and death of the algebraic limit cycles of the Filiptsov's [12] system.

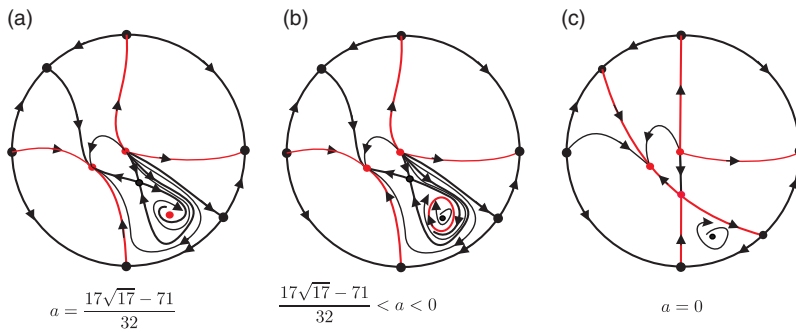


FIGURE 4. Birth and death of the algebraic limit cycles of the Chavarriga's system.

has the irreducible invariant algebraic curve

$$x^2 + x^3 + x^2y + 2axy^2 + 2axy^3 + a^2y^4 = 0. \tag{1.8}$$

When  $(17\sqrt{17} - 71)/32 < a < 0$ , this curve contains a stable algebraic limit cycle of degree 4, see Figure 4(b). When  $a \searrow (17\sqrt{17} - 71)/32$ , this family of limit cycles born or die at a Hopf bifurcation, see Figure 4(a); when  $a \nearrow 0$  this family of limit cycles born or die in a heteroclinic loop having some orbit at infinity, see Figure 4(c).

(d) The Chavarriga et al.'s [4] system

$$\begin{aligned} \dot{x} &= 2(1 + 2x - 2ax^2 + 6xy), \\ \dot{y} &= 8 - 3a - 14ax - 2axy - 8y^2. \end{aligned} \tag{1.9}$$

This system has the irreducible invariant algebraic curve

$$\frac{1}{4} + x - x^2 + ax^3 + xy + x^2y^2 = 0. \tag{1.10}$$

When  $0 < a < 1/4$ , this curve contains an unstable algebraic limit cycle of degree 4, see Figure 5(b). When  $a \searrow 0$ , the equilibrium point  $p_+$  tends to the infinite equilibrium  $(0, 0)$  of  $U_1$ , and the limit cycle tends to an orbit of the elliptic sector of that infinite equilibrium. Moreover, the equilibrium  $q_-$  tends to the infinite equilibrium  $(0, 0)$  of  $V_1$  (see the definitions

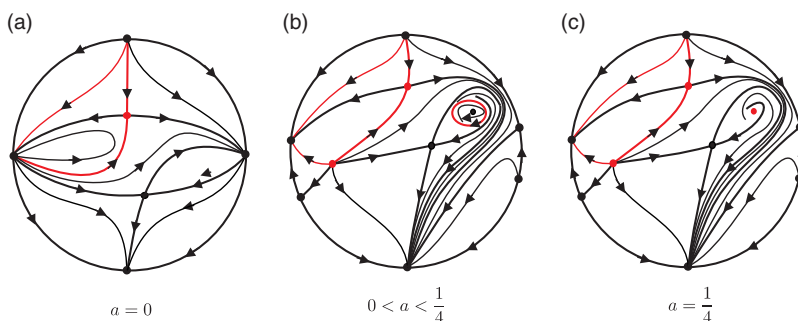


FIGURE 5. Birth and death of the algebraic limit cycles of the Chavarriga et al.'s [4] system.

of the equilibria  $p_+$  and  $q_-$  in the proof of this statement), see Figure 5(a). When  $a \nearrow 1/4$ , this family of limit cycles born or die at a Hopf bifurcation, see Figure 5(c).

We remark that in statement (d) of Theorem 1.2, we have shown the existence of a limit cycle bifurcating from an orbit of an elliptic sector of an infinite equilibrium point. This kind of bifurcation of limit cycles as far as we know until now was very unusual. This phenomenon will also happen for a family of algebraic limit cycles of degree 5 as we shall see in the next theorem.

In [6] the authors gave two one-parameter families of quadratic system with algebraic limit cycles of degrees 5 and 6. Recently in [1] the authors present a new one-parameter family of quadratic systems with a new algebraic limit cycle of degree 5.

**Theorem 1.3** (Degree 5) *We have the following two one-parameter families of algebraic limit cycles of degree 5 in quadratic systems.*

(a) *The Christopher et al.'s [6] system*

$$\begin{aligned} \dot{x} &= 28x - 12/(a + 4)y^2 - 2(a^2 - 16)(12 + a)x^2 + 6(3a - 4)xy, \\ \dot{y} &= (32 - 2a^2)x + 8y - (a + 12)(a^2 - 16)xy + (10a - 24)y^2, \end{aligned} \tag{1.11}$$

has the irreducible invariant algebraic curve

$$\begin{aligned} x^2 + (16 - a^2)x^3 + (a - 2)x^2y + \frac{1}{(4 + a)^2}y^4 - \frac{6}{(4 + a)^2}y^5 - \frac{2}{4 + a}xy^2 \\ + \frac{(a - 4)(12 + a)}{4}x^2y^2 + \frac{(12 + a)}{4 + a}xy^4 + \frac{8 - a}{4 + a}xy^3 = 0. \end{aligned} \tag{1.12}$$

When  $a \in (3\sqrt{7}/2, 4)$ , this curve contains a stable algebraic limit cycle of degree 5, see Figure 6(b). When  $a \searrow 3\sqrt{7}/2$ , this family of limit cycles born or die at a Hopf bifurcation, see Figure 6(a); and when  $a \nearrow 4$ , the equilibrium point  $p_1$  tends to the infinite equilibrium  $(0, 0)$  of  $U_1$ , the limit cycle tends to an orbit of the elliptic sector of that infinite equilibrium and the equilibrium  $p_+$  tends to the equilibrium  $(0, 0)$  of  $V_1$  (see the definitions of the equilibria  $p_1$  and  $p_+$  in the proof of this statement), see Figure 6(c).

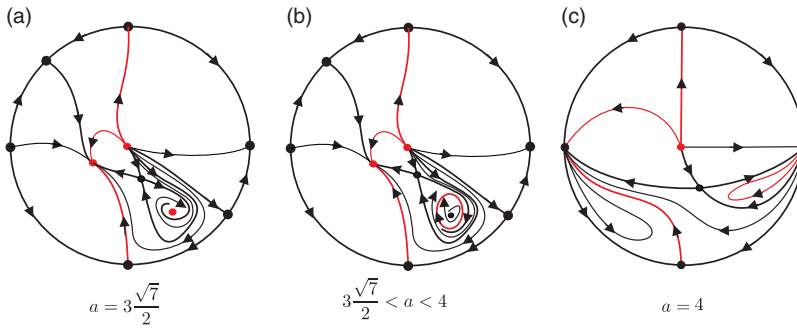


FIGURE 6. Birth and death of the algebraic limit cycles of the Christopher et al.'s [6] system of degree 5.

(b) *The Alberich-Carraminana et al.'s [1] system*

$$\begin{aligned} \dot{x} &= 2(a + 12)(168a^2x + 3y) - 16a(28a^2 + 9a\gamma - 126a - 18\gamma - 252)x^2 \\ &\quad - (7a + 3\gamma - 42)xy, \\ \dot{y} &= (a + 12)(-1344a^3(a - 30)x - 72a(a - 2)y) - (64a^2(476a^3 + 3a^2(61\gamma + 910) \\ &\quad + 288a(3\gamma + 77) - 108\gamma + 28728)x^2 - 12a(77a^2 + 27a\gamma + 30\gamma + 1260)xy \\ &\quad - (7a + 3\gamma - 42)y^2, \end{aligned} \tag{1.13}$$

with  $\gamma = \sqrt{7(108 + a^2)}$  has the irreducible invariant algebraic curve of degree 5

$$\begin{aligned} &6912(a - 2)^2a^3(a + 12)^6(2a + 3) - (13824(a - 2)^2a^3(a + 12)^5(2a + 3)(-24 + 5a) \\ &+ 20736(a - 2)^2a^3(a + 12)^5(3 + 2a)\gamma)x - 864(a - 2)^2a^2(a + 12)^5(2a + 3)y \\ &+ (145152(-6 + a)(a - 2)^2a^3(a + 12)^4(2a + 3)\gamma + 1152(a - 2)^2a^3(a + 12)^4(63288 + 28044a \\ &- 3774a^2 + 685a^3))x^2 + (1728(a - 2)^2a^2(a + 12)^4(2a + 3)\gamma + 288(a - 2)^2a^2(a + 12)^4 \\ &(-468 - 36a + 37a^2))xy + 54(a - 2)^2a(a + 12)^5y^2 + (-6912(a - 9)(a - 2)^2a^3(a + 12)^3\gamma \\ &(-1188 - 408a + 109a^2) - 1792a^3(a + 12)^3(-4999968 + 4201632a + 1662552a^2 - 950004a^3 \\ &+ 148836a^4 - 18865a^5 + 1116a^6))x^3 + (-432(a - 2)^2a^2(a + 12)^3\gamma(-612 - 144a + 29a^2) \\ &- 336a^2(a + 12)^3(59616 - 134784a - 32688a^2 + 5580a^3 - 902a^4 + 99a^5))x^2y \\ &+ (-54(a - 2)^2a(a + 12)^4\gamma - 126(a - 2)a(a + 12)^4(108 + a^2))xy^2 - 14(a + 12)^5y^3 \\ &+ (2688(a - 30)a^3(a + 12)^2\gamma(71280 - 110808a - 43272a^2 + 5310a^3 - 2153a^4 + 252a^5) \\ &+ 1344(a - 30)a^3(a + 12)^2(-5520960 - 1314144a - 948456a^2 - 823284a^3 + 99942a^4 \\ &- 11421a^5 + 1334a^6))x^4 + (336(a - 30)a^2(a + 12)^2\gamma(-4320 + 3312a + 1848a^2 + 82a^3 + 39a^4) \\ &+ 336(a - 30)a^2(a + 12)^2(25920 + 57888a + 18504a^2 + 10596a^3 + 222a^4 + 103a^5))x^3y \\ &+ (21(a - 30)a(a + 12)^3\gamma(60 + 20a + 3a^2) + 42(a - 30)a(a + 12)^3(900 + 264a \\ &+ 27a^2 + 4a^3))x^2y^2 + (28224(a - 30)a^3(a + 12)\gamma(139968 + 194400a + 101448a^2 + 30420a^3 \\ &+ 2258a^4 + 477a^5) + 56448(a - 30)a^3(a + 12)(824256 + 3304800a + 1523232a^2 + 294840a^3 \\ &+ 74304a^4 + 2988a^5 + 631a^6))x^5 + (14112(a - 30)a^2(a + 12)\gamma(3240 + 2268a + 642a^2 \\ &+ 55a^3 + 12a^4) + 3528(a - 30)a^2(a + 12)(440640 + 222048a + 55512a^2 + 13644a^3 \\ &+ 582a^4 + 127a^5))x^4y = 0. \end{aligned} \tag{1.14}$$



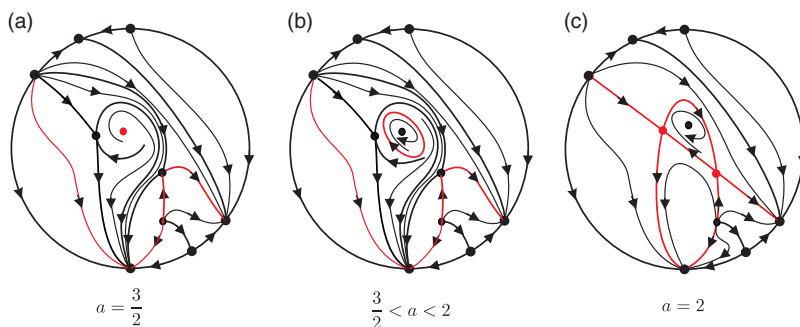


FIGURE 7. Birth and death of the algebraic limit cycles of a system of the Alberich-Carraminana et al.’s [1] system.

When  $a \in (3/2, 2)$ , this curve contains an unstable algebraic limit cycle of degree 5, see Figure 7(b). When  $a \searrow 3/2$ , this family of limit cycles born or die at a Hopf bifurcation, see Figure 7(a); and when  $a \nearrow 2$ , this family of limit cycles born or die at a heteroclinic loop formed by two pieces, one of a parabola and the other a straight line, and two saddles, see Figure 7(c).

**Theorem 1.4** (Degree 6) *The unique known family of algebraic limit cycle of degree 6 in quadratic systems is due to Christopher et al. [6], and it is given by the system*

$$\begin{aligned} \dot{x} &= 28(a - 30)ax + y + 168a^2x^2 + 3xy, \\ \dot{y} &= 16a(a - 30) (14(a - 30)ax + 5y + 84a^2x^2) + 24(17a - 6)axy + 6y^2. \end{aligned} \tag{1.15}$$

This system has the one-parameter family of irreducible algebraic invariant curves of degree 6

$$\begin{aligned} & -7y^3 + 3(a - 30)^2ay^2 + 18(a - 30)(-2 + a)axy^2 + 27(a - 2)^2ax^2y^2 \\ & + 24(a - 30)^3a^2xy + 144(a - 30)(a - 2)^2a^2x^3y + 48(a - 30)^4a^3x^2 \\ & + 576(a - 30)^2(-2 + a)^2a^3x^4 - 432(a - 2)^2a^2(3 + 2a)x^4y \\ & - 3456(a - 30)(-2 + a)^2a^3(3 + 2a)x^5 + 3456(a - 2)^2a^3(12 + a)(3 + 2a)x^6 \\ & + 24(a - 30)^2a^2(9a - 4)x^2y + 64(a - 30)^3a^3(9a - 4)x^3 = 0. \end{aligned} \tag{1.16}$$

When  $a \in (3/2, 2)$ , this curve contains a stable algebraic limit cycle of degree 6, see Figure 8(b). When  $a \searrow 3/2$ , this family of limit cycles born or die at a Hopf bifurcation, see Figure 8(a); and when  $a \nearrow 2$ , this family of limit cycles born or die in a heteroclinic loop having some orbits at infinity, see Figure 8(c).

The proof of Theorems 1.1–1.4 are in Sections 3–6, respectively.

## 2 Preliminaries

An equilibrium point  $p$  such that the eigenvalues of the linear part of the differential system at  $p$  have non-zero real part is called *hyperbolic equilibrium*. The local phase portraits of the hyperbolic equilibria are determined by their eigenvalues, see for instance Theorem 2.15 of [7]. If only

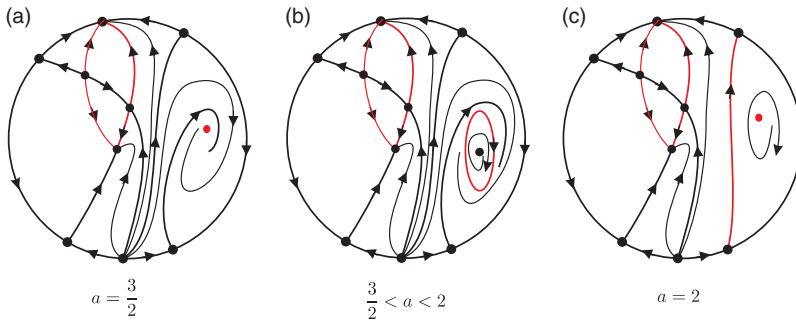


FIGURE 8. Birth and death of the algebraic limit cycles of the Christopher et al.'s [6] system of degree 6.

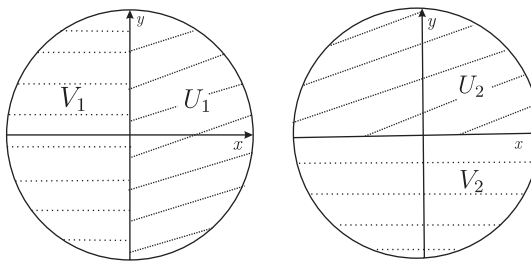


FIGURE 9. The local charts  $U_i$  and  $V_i$ , for  $i = 1, 2$  of the Poincaré disc  $\mathbb{D}$ .

one of the two eigenvalues of an equilibrium is zero, then we say that this is a *semi-hyperbolic equilibrium*, and the local phase portraits of such equilibria are well known, see for example Theorem 2.19 of [7].

In this paper all the phase portraits are drawn in the so-called Poincaré disc, which is roughly speaking, the closed disc centred at the origin of the coordinates and of radius one. The interior of this disc is identified with  $\mathbb{R}^2$ . Its boundary, the circle  $\mathbb{S}^1$ , is identified with the infinity of  $\mathbb{R}^2$ . In the plane  $\mathbb{R}^2$ , we can go to infinity in as many directions as points in the circle  $\mathbb{S}^1$ . For more details about the Poincaré disc, see Chapter 5 of [7].

In order to work with the Poincaré disc we need four local charts. They are represented in Figure 9 and they are defined as

$$\begin{aligned}
 U_1 &= \{(x, y) \in \mathbb{R}^2 : x > 0\}, & V_1 &= \{(x, y) \in \mathbb{R}^2 : x < 0\}, \\
 U_2 &= \{(x, y) \in \mathbb{R}^2 : y > 0\}, & V_2 &= \{(x, y) \in \mathbb{R}^2 : y < 0\}.
 \end{aligned}$$

We define  $\phi_k : U_k \rightarrow \mathbb{D}$  and  $\psi_k : V_k \rightarrow \mathbb{D}$  for  $k = 1, 2$  as follows:

$$\phi_1(x, y) = \left( \frac{y}{x}, \frac{1}{x} \right) = (u, v), \quad \phi_2(x, y) = \left( \frac{1}{y}, \frac{x}{y} \right) = (u, v),$$

and  $\psi_k = -\phi_k$ ,  $k = 1, 2$ . In Figure 10, these local coordinates are represented in the local charts  $U_1$  and  $U_2$ , respectively.

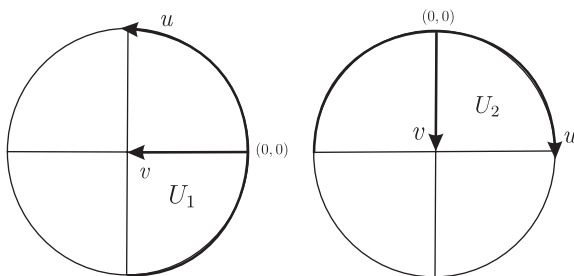


FIGURE 10. The coordinates  $(u, v)$  in the local charts  $U_1$  and  $U_2$ .

If  $\dot{x} = P(x, y)$ ,  $\dot{y} = Q(x, y)$  is a polynomial differential system and  $d$  is the maximum of the degrees of  $P$  and  $Q$ , then the expression of the extended flow on the local chart  $(U_1, \phi_1)$  is

$$\dot{u} = v^d \left[ -uP \left( \frac{1}{v}, \frac{u}{v} \right) + Q \left( \frac{1}{v}, \frac{u}{v} \right) \right], \quad \dot{v} = -v^{d+1} P \left( \frac{1}{v}, \frac{u}{v} \right).$$

The expression on the local chart  $(U_2, \phi_2)$  is

$$\dot{u} = v^d \left[ P \left( \frac{u}{v}, \frac{1}{v} \right) - uQ \left( \frac{u}{v}, \frac{1}{v} \right) \right], \quad \dot{v} = -v^{d+1} Q \left( \frac{u}{v}, \frac{1}{v} \right).$$

The expressions in the local charts  $(V_k, \psi_k)$ , for  $k = 1, 2$ , are the same than in the chart  $(U_k, \phi_k)$  multiplied by  $(-1)^{d-1}$ . We note that  $v = 0$  corresponds to the infinity  $\mathbb{S}^1$  in all the local charts.

Let  $\mathcal{X}_1$  and  $\mathcal{X}_2$  be two polynomial vector fields on  $\mathbb{R}^2$ , and let  $p(\mathcal{X}_1)$  and  $p(\mathcal{X}_2)$  be their respective polynomial vector fields on the Poincaré disc  $\mathbb{D}$ . We say that they are *topologically equivalent* if there exists a homeomorphism on the Poincaré disc  $\mathbb{D}$  which preserves the infinity  $\mathbb{S}^1$  and sends the orbits of  $p(\mathcal{X}_1)$  to orbits of  $p(\mathcal{X}_2)$ , preserving or reversing the orientation of all the orbits.

A *separatrix* of the Poincaré compactification  $p(\mathcal{X})$  is one of the following orbits: the finite equilibrium point, the limit cycles, all the orbits at the infinity  $\mathbb{S}^1$ , and the two orbits at the boundary of a hyperbolic sector at a finite or an infinite equilibrium point, see for more details on the separatrices [22, 23].

The set of all separatrices of  $p(\mathcal{X})$ , denoted by  $\Sigma_{\mathcal{X}}$ , is a closed set (see [23]). A *canonical region* of  $p(\mathcal{X})$  is an open connected component of  $\mathbb{D} \setminus \Sigma_{\mathcal{X}}$ .

A *separatrix configuration* of  $p(\mathcal{X})$ , denoted by  $\Sigma'_{\mathcal{X}}$ , is the union of the set  $\Sigma_{\mathcal{X}}$  with an orbit of each canonical region. Two separatrix configurations  $\Sigma'_{\mathcal{X}_1}$  and  $\Sigma'_{\mathcal{X}_2}$  are *topologically equivalent* if there is a homeomorphism  $h : \mathbb{D} \rightarrow \mathbb{D}$  such that  $h(\Sigma'_{\mathcal{X}_1}) = \Sigma'_{\mathcal{X}_2}$ .

According to the following theorem which was proved by Markus [22], Neumann [23] and Peixoto [24], it is sufficient to investigate the separatrix configuration of a polynomial differential system, for determining its global phase portrait.

**Theorem 2.1** *Two Poincaré compactified polynomial vector fields  $p(\mathcal{X}_1)$  and  $p(\mathcal{X}_2)$  with finitely many separatrices are topologically equivalent if and only if their separatrix configurations  $\Sigma'_{\mathcal{X}_1}$  and  $\Sigma'_{\mathcal{X}_2}$  are topologically equivalent.*

### 3 Proof of Theorem 1.1

The quadratic system (1.2) has the following equilibria:

$$p_{\pm} = \left( 0, -\frac{c \pm \sqrt{c^2 + 4(1+b)r^2}}{2(1+b)} \right),$$

$$q_{\pm} = \left( \frac{\pm b \sqrt{a^2 (r^2 (a^2 + b^2) - c^2)} - a^2 c}{a (a^2 + b^2)}, \mp \frac{\sqrt{a^2 (r^2 (a^2 + b^2) - c^2)} - bc}{a^2 + b^2} \right). \quad (3.1)$$

The quadratic system (1.2) has the invariant ellipse  $x^2 + y^2 - r^2 = 0$  for  $0 < r < \sqrt{\frac{c^2}{a^2 + b^2}}$  as a limit cycle because the unique equilibria on the ellipse are  $q_{\pm}$  which for these values of  $r$  are complex (i.e. their coordinates are not real), so the invariant ellipse is a periodic orbit, but since one of the points  $p_{\pm}$  is inside the region limited by the circle  $x^2 + y^2 - r^2 = 0$  and it is a focus the invariant circle is a limit cycle.

The fact that under the assumption of Theorem 1.1 the invariant ellipse  $x^2 + y^2 - r^2 = 0$  is an algebraic limit cycle was proved in [28], see also [3, 4, 20].

When  $r$  goes to zero the circle tends to an equilibrium point, the focus. So the limit cycle on the circle born or die in a Hopf bifurcation. On the other hand, when  $r = \sqrt{\frac{c^2}{a^2 + b^2}}$  the complex equilibria  $q_{\pm}$  becomes a saddle-node on the circle. In other words, the limit cycle on the circle ends in a homoclinic orbit to this saddle-node. See Figure 1.

Now we study the infinite equilibrium point in the Poincaré disc. System (1.2) in the local chart  $U_1$  writes

$$\begin{aligned} \dot{u} &= a + (1+b)u + cv + au^2 + (1+b)u^3 + cu^2v - r^2uv^2, \\ \dot{v} &= v(1 + au + (1+b)u^2 + cuv - r^2v^2), \end{aligned}$$

and in the local chart  $U_2$  is

$$\begin{aligned} \dot{u} &= -(1+b) - au - cv - (1+b)u^2 + r^2v^2 - au^3 - cu^2v, \\ \dot{v} &= -vu(b + au + cv). \end{aligned}$$

If  $b \neq -1$  the unique infinite equilibrium point in  $U_1$  is  $(-a/(1+b), 0)$ , which is a hyperbolic unstable node, and the origin of  $U_2$  is not an infinite equilibrium point. If  $b = -1$  there is no infinite equilibrium point in  $U_1$ , but then the origin of the local chart  $U_2$  becomes a singular point with a zero eigenvalue, using Theorem 2.19 from [7] we conclude that the origin of  $U_2$  is a saddle-node. We remark that in this case one finite singular point coalesces with the infinite one.

In summary, knowing the local phase portraits in all finite and infinite equilibria, together with the information about the invariant circle when  $0 \leq r \leq \sqrt{\frac{c^2}{a^2 + b^2}}$  we obtain the phase portraits in the Poincaré disc of system (1.2) described in Figure 1. This completes the proof of Theorem 1.1.

### 4 Proof of Theorem 1.2

The existence of the families of algebraic limit cycles of degree 4 under the conditions of the different statements of Theorem 1.2 was proved in [3, 4, 12, 28].

4.1 Proof of statement (a) of Theorem 1.2

The quadratic system (1.3) has the following equilibria:

$$p_0 = (0, 0), \quad p_1 = \left( \frac{2ab(a+b)}{3a^2 - 2ab + 3b^2}, -\frac{4a^2b^2c}{3a^2 - 2ab + 3b^2} \right),$$

$$p_{\pm} = \left( \frac{2c^2(a+b) + a + b \pm R}{4(c^2 + 1)}, -\frac{(2c^2(a+b) \pm R)(2c^2(a+b) + a + b \pm R)}{16c(c^2 + 1)} \right),$$

where  $R = \sqrt{c^2(4c^2(a-b)^2 + (3a-b)(a-3b))}$ . Note that the equilibria  $p_0$  and  $p_1$  always exist, while the points  $p_{\pm}$  are complex if  $4c^2(a-b)^2 + (3a-b)(a-3b) < 0$ .

Under the assumptions of statement (a) of Theorem 1.2, the invariant curve (1.4) is formed by an oval  $O$  surrounding the equilibrium  $p_1$ , and an isolated equilibrium  $p_0$ . So the oval  $O$  is a periodic orbit and since  $p_1$  is a hyperbolic stable focus the oval  $O$  is an unstable limit cycle, see Figure 2(a).

Note that  $ab \leq 0$  implies that  $4c^2(a-b)^2 + (3a-b)(a-3b) > 0$ . Hence the conditions  $abc \neq 0, a \neq b, ab > 0, 4c^2(a-b)^2 + (3a-b)(a-3b) < 0$  yield to  $c(a-b) \neq 0$  and  $4c^2(a-b)^2 + (3a-b)(a-3b) < 0$ .

Moreover, when  $a \rightarrow b$  the oval  $O$  tends to the focus  $p_1$ . Therefore when  $a = b$ , we have a Hopf bifurcation where the limit cycle  $O$  born or die, see Figure 2(b).

When  $c \rightarrow 0$  the point  $p_1$  tends to a centre and the oval  $O$  tends to a periodic orbit surrounding the centre, see Figure 2(c). This fact was proved in [19].

When  $4c^2(a-b)^2 + (3a-b)(a-3b) \nearrow 0$ , then complex equilibria  $p_{\pm}$  become a saddle-node on the oval  $O$ , so the limit cycle  $O$  ends in a homoclinic loop at this saddle-node. Additionally there are two distinct possibilities concerning to the isolated point  $p_0$ : if  $c \neq 0$  such point is a focus, see Figure 2(d), otherwise, it is a centre, see Figure 2(e).

In order to complete the phase portrait in the Poincaré disc, we must study the infinite equilibria. In the local chart  $U_1$  system (1.3) becomes

$$\begin{aligned} \dot{u} &= 4ab - 3/2(a+b)^2 + 4abc^2 + 5(a+b)cu + ab(a+b)v + 4u^2 + (a+b)u^2v, \\ \dot{v} &= v(-3c(a+b) - 4u + 4abcv + (a+b)uv), \end{aligned}$$

and in the local chart  $U_2$  is

$$\begin{aligned} \dot{u} &= \frac{-8u - 2(a+b)v - 10(a+b)cu^2 + (3a^2 - 2ab + 3b^2 - 8abc^2)u^3 - 2ab(a+b)u^2v}{2}, \\ \dot{v} &= -v(8(1+bc) + 8acu - 4abcv - (1/2)(3a^2 - 2ab + 3b^2 - 8abc^2)u^2 + ab(a+b)uv). \end{aligned}$$

Denote by  $S = \sqrt{-2ab(8 + 7c^2) + a^2(24 + 25c^2) + b^2(24 + 25c^2)}$ . The expression  $S$  is always real. Then the infinite equilibria in  $U_1$  are  $q_{\pm} = ((-5c(a+b) \pm S)/8, 0)$ . Note that  $-5c(a+b) - S < -5c(a+b) + S$  and that  $q_-$  is a hyperbolic saddle, and  $q_+$  is a hyperbolic stable node.

The origin of  $U_2$  is always a semi-hyperbolic infinite equilibrium with eigenvalues  $-4$  and  $0$ . Applying Theorem 2.19 of [7] this equilibrium is a saddle.

Taking into account the local phase portraits of the finite and infinite equilibria together with the invariant algebraic curve and the previous information, it follows the phase portraits in the Poincaré disc of system (1.3) described in Figure 2. This proves statement (a).

#### 4.2 Proof of statement (b) of Theorem 1.2

The quadratic system (1.5) when  $a \in [0, 3/13]$  has the following four equilibria:

$$p_0 = (0, 0), \quad p_1 = \left( \frac{5(a+1)}{2(1-a)}, \frac{15(a+1)^2}{8(1-a)} \right),$$

$$p_{\pm} = \left( \frac{a-R}{12(a+1)}, -\frac{a(16a+2R+33)+3(R+6)}{24(a+1)} \right),$$

where  $R = \sqrt{36 + a(72 + 37a)}$ .

For  $a \in (0, 3/13)$ , the invariant algebraic curve (1.6) is formed by an oval  $O$  surrounding the hyperbolic stable focus  $p_1$ , and a component homeomorphic to a straight line ending at the infinite equilibrium localised at the origin of  $V_2$  which contains the finite equilibria  $p_0$  and  $p_{\pm}$ , where  $p_0$  is a hyperbolic unstable node,  $p_+$  is a hyperbolic saddle and  $p_-$  is a hyperbolic stable node. So the oval  $O$  is an unstable limit cycle, see Figure 3(b).

When  $a \searrow 0$ , the oval  $O$  increases tending to a heteroclinic loop with an orbit contained at infinity, see Figure 3(a). When  $a \nearrow 3/13$ , the oval  $O$  tends to the equilibrium point  $p_1$ , see Figure 3(c), and consequently for  $a = 3/13$  we have a Hopf bifurcation.

In the local chart  $U_1$  system (1.5) becomes

$$\dot{u} = 3a(1+a) - 2u(3+2a) + 4u^2 + 9(1+a)uv - 2u^2v,$$

$$\dot{v} = 2v(3(2+a) - 6u - 3(1+a)v - uv),$$

and in the local chart  $U_2$  is

$$\dot{u} = -4u + 2v + 2(3+2a)u^2 - 9(1+a)uv - 3a(1+a)u^3,$$

$$\dot{v} = -v(16 + 2(-9+5a)u + 15(1+a)v + 3a(1+a)v^2).$$

In  $U_1$  we have the two equilibria  $q_{\pm} = ((3+2a \pm \sqrt{9-8a^2})/4, 0)$ , both are hyperbolic saddles. The origin of the local chart  $U_2$  is a hyperbolic stable node.

Taking into account the previous information, it follows the three phase portraits in the Poincaré disc of system (1.5) in Figure 3. This completes the proof of statement (b) of Theorem 1.2.

#### 4.3 Proof of statement (c) of Theorem 1.2

The quadratic system (1.7) when  $a \in [(17\sqrt{17}-71)/32, 0]$  has the following four equilibria:

$$p_0 = (0, 0), \quad p_1 = \left( -\frac{1}{2+a}, -\frac{1}{2+a} \right),$$

$$p_{\pm} = \left( \frac{(\pm R - 1 - 2a)(\pm 3R - 7 - 6a)}{24 - 36a}, \frac{\mp 3R + 7 + 6a}{6a - 4} \right),$$

where  $R = \sqrt{4a^2 + 16a + 1}$ .

For  $a \in ((17\sqrt{17}-71)/32, 0)$ , the invariant algebraic curve (1.8) is formed by an oval  $O$  surrounding the hyperbolic unstable focus  $p_-$  and two components homeomorphic to straight lines, one of these components contains the hyperbolic stable node  $p_1$  and the other contains the hyperbolic unstable node  $p_0$ . The remaining equilibrium  $p_+$  is a hyperbolic saddle. So the oval  $O$  is a stable limit cycle, see Figure 4(b).

When  $a \searrow (17\sqrt{17} - 71)/32$ , the oval  $O$  collapses to the equilibrium  $p_-$ . Hence at  $a = (17\sqrt{17} - 71)/32$  there is a Hopf bifurcation, see Figure 4(a). When  $a \nearrow 0$ , the limit cycle  $O$  ends in a heteroclinic loop with an orbit at infinity, see Figure 4(c).

In the local chart  $U_1$  system (1.7) becomes

$$\begin{aligned} \dot{u} &= -2u + v - (2 + a)u^2 - 3uv - au^3, \\ \dot{v} &= -v(6 + 4(1 + a)u + 5v + au^2), \end{aligned}$$

and in the local chart  $U_2$  is

$$\begin{aligned} \dot{u} &= a + (2 + a)u + 2u^2 + 3uv - u^2v, \\ \dot{v} &= -v(2 + 3a + 4u + 2v + uv). \end{aligned}$$

If  $a \in [(17\sqrt{17} - 71)/32, 0)$ , there are three infinite equilibria in  $U_1$ , namely  $(-1, 0)$  a hyperbolic saddle,  $(0, 0)$  a hyperbolic stable node and  $(-2/a, 0)$  a hyperbolic saddle. If  $a = 0$  only the two first infinite equilibria remain in  $U_1$ . The origin of  $U_2$  is a hyperbolic saddle only when  $a = 0$ .

In summary, with all the previous information we get the phase portraits in the Poincaré disc of system (1.7) in Figure 4. Hence statement (c) of Theorem 2 is proved.

#### 4.4 Proof of statement (d) of Theorem 1.2

The quadratic system (1.9) when  $a \in [0, 1/4]$  has the following four equilibria:

$$\begin{aligned} p_{\pm} &= \left( \frac{2 \pm \sqrt{4 - 7a}}{7a}, \frac{\pm 3\sqrt{4 - 7a} - 8}{14} \right), \\ q_{\pm} &= \left( \frac{\pm \sqrt{16 - a} - 4}{a}, \pm \frac{\sqrt{16 - a}}{2} - 1 \right). \end{aligned}$$

For  $a \in (0, 1/4)$ , the invariant algebraic curve (1.10) is formed by an oval  $O$  surrounding the hyperbolic stable focus  $p_+$  and two components homeomorphic to straight lines, one of these components contains the hyperbolic unstable node  $q_-$  and the hyperbolic saddle  $q_+$ . The remaining equilibrium  $p_-$  is a hyperbolic saddle. So the oval  $O$  is an unstable limit cycle, see Figure 5(b).

In the local chart  $U_1$  system (1.9) becomes

$$\begin{aligned} \dot{u} &= 2au - 14av - 20u^2 - 4uv + (8 - 3a)v^2 - 2uv^2, \\ \dot{v} &= 2v(2a - 6u - 2v - v^2), \end{aligned}$$

and in the local chart  $U_2$  is

$$\begin{aligned} \dot{u} &= 20u - 2au^2 + 4uv + 2v^2 + 14au^2v + (-8 + 3a)uv^2, \\ \dot{v} &= v(8 + 2au + 14auv - (8 - 3a)v^2). \end{aligned}$$

If  $a \in (0, 1/4]$  there are two infinite equilibria in  $U_1$ , namely  $(0, 0)$  a hyperbolic unstable node and  $(a/10, 0)$  a hyperbolic saddle. If  $a = 0$  only the  $(0, 0)$  is an infinite equilibrium in  $U_1$  which, using blow ups (see for a precise definition [2]), is formed by two hyperbolic sectors separated by two parabolic sectors. These parabolic sectors contain a piece of the straight line of the infinity, see Figure 5(a). The origin of  $U_2$  is a hyperbolic unstable node.

When  $a \nearrow 1/4$  the oval  $O$  collapses to the equilibrium  $p_+$ . Hence at  $a = 1/4$  there is a Hopf bifurcation, see Figure 5(c). When  $a \searrow 0$  the equilibrium  $p_+$  goes to the infinite equilibrium  $(0, 0)$  of the chart  $U_1$ , and the limit cycle on the oval  $O$  tends to an orbit of the elliptic sector of that infinite equilibrium; the equilibrium point  $q_-$  tends to the equilibrium  $(0, 0)$  of  $V_1$ , see again Figure 5(a).

In short, with all the previous information we obtain the phase portraits in the Poincaré disc of system (1.9) in Figure 5, and statement (d) of Theorem 2 is proved.

### 5 Proof of Theorem 1.3

#### 5.1 Proof of statement (a) of Theorem 1.3

The quadratic system (1.11) when  $a \in [3\sqrt{7}/2, 4]$  has the following four equilibria:

$$p_0 = (0, 0), \quad p_1 = \left( \frac{2}{a(a^2 - 16)}, -\frac{1}{a} \right),$$

$$p_{\pm} = \left( -\frac{2S}{(a^2 - 10a + 24)(a^2 + 16a + 48)^2}, \frac{S(26a^3 - 22a^2 - 360a \pm R + 576)}{3(a - 6)(a - 4)(a + 4)^2(a + 12)^2(13a - 24)} \right),$$

where  $R = \sqrt{(7a^2 - 108)(13a^2 + 28a - 96)^2}$  and  $S = -35a^3 - 86a^2 + 504a \pm R + 1152$ .

The proof of the existence of the algebraic limit cycle for system (1.11) under the conditions stated in statement (a) of Theorem 1.3 was given in [20].

For  $a \in (3\sqrt{7}/2, 4)$  the invariant algebraic curve (1.12) is formed by an oval  $O$  surrounding the hyperbolic unstable focus  $p_1$  and one component homeomorphic to straight line which contain the hyperbolic nodes  $p_{\pm}$ ,  $p_-$  is unstable and  $p_+$  is stable. The remaining equilibrium  $p_0$  is a hyperbolic saddle. So the oval  $O$  is a stable limit cycle, see Figure 6(b).

In the local chart  $U_1$  system (1.11) becomes

$$\dot{u} = \frac{(a - 4)(a + 4)^2(a + 12)u - 2(a - 4)(a + 4)^2v - 8a(a + 4)u^2 - 20(a + 4)uv + 12u^3}{a + 4},$$

$$\dot{v} = \frac{2v((a - 4)(a + 12)(a + 4)^2 - 3(3a - 4)(a + 4)u - 14(a + 4)v + 6u^2)}{a + 4},$$

and in the local chart  $U_2$  is

$$\dot{u} = -\frac{12 - 8a(a + 4)u + (a - 4)(a + 4)^2(a + 12)u^2 - 20(a + 4)uv - 2(a - 4)(a + 4)^2u^2v}{a + 4},$$

$$\dot{v} = v(2(12 - 5a) + (-4 + a)(4 + a)(12 + a)u - 8v + 2(-4 + a)(4 + a)uv).$$

If  $a \in [3\sqrt{7}/2, 2]$ , there are three infinite equilibria in  $U_1$ , namely  $((a^2 - 16)/2, 0)$  a hyperbolic saddle,  $(0, 0)$  a hyperbolic stable node and  $((48 + 16a + a^2)/6, 0)$  a hyperbolic saddle. If  $a = 4$ , using blow ups, the equilibrium point  $(0, 0)$  is formed by two hyperbolic sectors separated by two parabolic sectors and a piece of the straight line of the infinite is contained in both parabolic sectors, and the equilibrium  $((48 + 16a + a^2)/6, 0)$  is a hyperbolic saddle, see Figure 6(c). The origin of  $U_2$  is not an infinite equilibrium.



When  $a \searrow 3\sqrt{7}/2$ , the oval  $O$  collapses to the equilibrium  $p_1$ . Hence at  $a = 3\sqrt{7}/2$  there is a Hopf bifurcation, see Figure 6(a). When  $a \nearrow 4$  the equilibrium  $p_1$  tends to the infinite equilibrium  $(0, 0)$  of the chart  $U_1$ , and the limit cycle  $O$  tends to an orbit of the elliptic sector of that infinite equilibrium; the equilibrium  $p_+$  tends to the equilibrium  $(0, 0)$  of  $V_1$ , see again Figure 6(c).

In short, with all the previous information we obtain the phase portraits in the Poincaré disc of system (1.11) in Figure 6, and statement (a) of Theorem 3 is proved.

### 5.2 Proof of statement (b) of Theorem 1.3

The quadratic system (1.13) when  $a \in [3/2, 2]$  has the following four equilibria:

$$p_0 = (0, 0), \quad p_1 = \left( \frac{a + 12}{R}, -\frac{32a(a + 12)(2a + 3)}{R} \right),$$

$$p_{\pm} = \left( \frac{T \pm S}{U}, -\frac{4a(T \pm S)^2}{(a + 12)U} \right),$$

where  $R = \sqrt{7(108 + a^2)}$ ,  $S = \sqrt{7(a + 12)^3 (259a^3 - 336a^2 - 36aR^2 + 28476a + 75600)}$ ,  $T = 30240 + 1008a + 1302a^2 + 119a^3 + 504aR + 42a^2R$  and  $U = (7a + 3R - 42)(21a^2 + 8aR + 42a + 12R + 1008)$ .

The proof of the existence of the algebraic limit cycle of system (1.13) under the conditions of the statement (b) of Theorem 1.3 was given in [1].

For  $a \in (3/2, 2)$ , the invariant algebraic curve (1.14) is formed by an oval  $O$  surrounding the hyperbolic stable focus  $p_+$ , and two component homeomorphic to straight lines, one of them contains the equilibria  $p_-$  and  $p_1$ , where  $p_-$  is a hyperbolic unstable node and  $p_1$  is a hyperbolic saddle. The equilibrium  $p_0$  is a hyperbolic saddle. Therefore the oval  $O$  is an unstable limit cycle, see Figure 7(b).

In the local chart  $U_1$  system (1.13) becomes

$$\begin{aligned} \dot{u} = & -\frac{1}{2(12 + a)}(128a^2(28728 + 22176a + 2730a^2 + 476a^3 - 108R + 864aR + 183a^2R) \\ & + 56a(612 + 36a + 25a^2 + 18R + 9aR)u + 2688(a - 30)a^3(12 + a)v + (7a - 42 + 3R)u^2 \\ & + 96a(12 + a)(-3 + 5a)uv + 6(12 + a)u^2v), \\ \dot{v} = & -\frac{v}{2(12 + a)}(-16a(-252 - 126a + 28a^2 - 18R + 9aR) - (-42 + 7a + 3R)u \\ & + 336a^2(12 + a)v + 6(12 + a)uv), \end{aligned}$$

and in the local chart  $U_2$  is

$$\begin{aligned} \dot{u} = & \frac{1}{2(12 + a)}((-42 + 7a + 3R)u + 6(12 + a)v + 56a(612 + 36a + 25a^2 + 18R + 9aR)u^2 \\ & + 96a(12 + a)(-3 + 5a)uv + 128a^2(28728 + 22176a + 2730a^2 + 476a^3 - 108R \\ & + 864aR + 183a^2R)u^3 + 2688(-30 + a)a^3(12 + a)u^2v), \end{aligned}$$

$$\dot{v} = \frac{v}{2(12+a)}((-42+7a+3R)u - 6(12+a)v + 16a(-252-126a+28a^2-18R+9aR)u^2 - 336a^2(12+a)uv).$$

If  $a \in [3/2, 2]$ , there are two infinite equilibrium points in  $U_1$ , namely

$$q_{\pm} = \left( \frac{\pm 12V - 28a(25a^2 + 9a(R+4) + 18(R+34))}{7a + 3R - 42}, 0 \right),$$

where

$$V = \sqrt{a^2((2052 - a(47a + 540))R^2 + 42(a + 26)(a(a + 36) + 36)R + 441(a(a + 12) + 84)^2)}.$$

The equilibrium  $q_-$  is a hyperbolic saddle and  $q_+$  is a hyperbolic stable node. The origin of  $U_2$  is a hyperbolic unstable node.

When  $a \searrow 3/2$  the oval  $O$  collapses to the equilibrium  $p_+$ . Hence at  $a = 3/2$  there is a Hopf bifurcation, see Figure 7(a). When  $a \nearrow 2$  the limit cycle tends to a heteroclinic loop, see again Figure 7(c).

In short, with all the previous information we obtain the phase portraits in the Poincaré disc of system (1.13) in Figure 7. Therefore statement (b) of Theorem 3 is proved.

### 6 Proof of Theorem 1.4

The quadratic system (1.15) when  $a \in [3/2, 2]$  has the following four equilibria:

$$p_0 = (0, 0), \quad p_1 = \left( \frac{5}{a} - \frac{1}{6}, 0 \right),$$

$$p_{\pm} = \left( \frac{6 - 3a \mp R}{2a + 24}, \frac{4a(21a^2 + a(42 \pm 8R) + 12(84 \pm R))}{a + 12} \right),$$

where  $R = \sqrt{7(a^2 + 108)}$ .

The proof of the existence of the algebraic limit cycle for system (1.15) under the condition stated in Theorem 1.4 was given in [20].

For  $a \in (3/2, 2)$ , the invariant algebraic curve (1.16) is formed by an oval  $O$  surrounding the hyperbolic stable focus  $p_1$ , and one component homeomorphic to a straight line which contains the equilibria  $p_{\pm}$ , where  $p_+$  is a hyperbolic unstable node and  $p_-$  is a hyperbolic saddle. The equilibrium  $p_0$  is a hyperbolic stable node. Therefore the oval  $O$  is an unstable limit cycle, see Figure 8(b).

In the local chart  $U_1$  system (1.11) becomes

$$\begin{aligned} \dot{u} &= -1344(-a + 30)a^3 + 48(5a - 3)au + 224(a - 30)^2a^2v + 3u^2 + 52(a - 30)auv - u^2v, \\ \dot{v} &= -v(168a^2 + 3u + 28(a - 30)av + uv), \end{aligned}$$

and in the local chart  $U_2$  is

$$\begin{aligned} \dot{u} &= -3u + v - 48a(-3 + 5a)u^2 - 52(-30 + a)auv - 1344(-30 + a)a^3u^3 \\ &\quad - 224(-30 + a)^2a^2u^2v, \\ \dot{v} &= -2v(3 + 12a(-6 + 17a)u + 672(-30 + a)a^3u^2 + 40(-30 + a)av + 112(-30 + a)^2a^2uv). \end{aligned}$$

If  $a \in [3/2, 2]$ , there are two infinite equilibrium points in  $U_1$ , namely

$$q_{\pm} = \left( 8 \left( -5a^2 \pm 3\sqrt{a^2(2a^2 + 20a + 1) + 3a} \right), 0 \right).$$

Both equilibria are hyperbolic saddles. The origin of  $U_2$  is a hyperbolic stable node, see Figure 8.

When  $a \nearrow 3/2$ , the oval  $O$  collapses to the equilibrium point  $p_1$ . Hence at  $a = 3/2$  there is a Hopf bifurcation, see Figure 8(a). When  $a \nearrow 2$ , the limit cycle on the oval  $O$  tends to a heteroclinic cycle having an orbit at infinity, see Figure 8(c).

In summary, with all the previous information we obtain the phase portraits in the Poincaré disc of system (1.4) in Figure 8, and Theorem 4 is proved.

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### Conflict of Interest

None.

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