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GENERALIZATION OF THE PAIRWISE STOCHASTIC PRECEDENCE ORDER TO THE SEQUENCE OF RANDOM VARIABLES

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We discuss a new stochastic ordering for the sequence of independent random variables. It generalizes the stochastic precedence (SP) order that is defined for two random variables to the case $n > 2$. All conventional stochastic orders are transitive, whereas the SP order is not. Therefore, a new approach to compare the sequence of random variables had to be developed that resulted in the notion of the sequential precedence order. A sufficient condition for this order is derived and some examples are considered.

Keywords: hazard rate order, likelihood ratio order, nontransitivity, stochastic precedence order, usual stochastic order

1. INTRODUCTION

Stochastic orders are pairwise comparisons between two random variables defined over the same probability space (for brevity, we will omit the latter description in what follows). Numerous stochastic orders had been described and widely used in the literature including the most popular in reliability applications the usual stochastic order, the hazard rate order and the likelihood ratio order. The encyclopedic information on stochastic orders and their properties can be found in Shaked and Shantikumar [13]. For the sake of completeness, the definitions of orders that are used in our paper are given below. We also assume for simplicity and applicability reasons that the considered in this paper random variables are nonnegative, that is, lifetimes.

Let us first introduce the basic notation to be used throughout the paper. For an absolutely continuous random variable T, we denote the probability density function (pdf) by $f_T(\cdot)$, the cumulative distribution function (cdf) by $F_T(\cdot)$, the hazard rate function by $\lambda_T(\cdot)$ and the survival/reliability function by $\bar{F}_T(\cdot)$.

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Definition 1.1: *Let T_1 and T_2 be two random variables supported on $[0,\infty)$. Then, T_2 is said to be larger than T_1 in the*

(a) *likelihood ratio (lr) order denoted as $T_1 \leq_{\rm lr} T_2$, if*

$$f_{T_2}(t)/f_{T_1}(t) \text{ is increasing in } t \in [0,\infty);$$

(b) *hazard rate (hr) order denoted as $T_1 \leq_{\rm hr} T_2$, if*

$$\bar{F}_{T_2}(t)/\bar{F}_{T_1}(t) \text{ is increasing in } t \in [0,\infty);$$

(c) *usual stochastic (st) order denoted as $T_1 \leq_{\rm st} T_2$, if*

$$\bar{F}_{T_1}(t) \leq \bar{F}_{T_2}(t) \text{ for all } t \in [0,\infty).$$

Using pairwise comparisons, the sequence of n independent random variables T_i, $i = 1, 2, \ldots, n$ can be also ordered as

$$T_1 \leq T_2 \leq \cdots \leq T_n \tag{1.1}$$

in a suitable stochastic sense. For all mentioned basic stochastic orders, (1.1) is transitive meaning that $T_i \leq T_j$, for all $1 \leq i < j \leq n$.

We now define the stochastic precedence (SP) order that was not so extensively studied and applied as the foregoing orders (see [1, 4, 7–9, 12]).

Definition 1.2: *Let T_1 and T_2 be two independent random variables. Then, T_2 is said to be larger than T_1 in the SP order, denoted as $T_2 \geq_{\rm sp} T_1$, if*

$$P(T_2 \geq T_1) \geq P(T_1 \geq T_2). \tag{1.2}$$

For continuous random variables, (1.2) can be equivalently written as follows:

$$P(T_2 \geq T_1) \geq 0.5. \tag{1.3}$$

This order is relevant in numerous engineering applications when, for example, stress-strength [5] or peak over the threshold probabilities are considered.

Remark 1.1: *In fact, our usage of the word "order" is loose in this paper as formally the SP is not an order in the formal sense since it does not satisfy the transitive property (see below). On the other hand, (1.2) is still an ordering of two random variables.*

It is also widely used for comparisons of independent coherent systems with i.i.d components [10]. It was proved that in this case, the corresponding comparisons are distribution-free. The setting with not necessarily identical components was considered in Navarro and Rubio [9]. For other applications, see also Arcones et al. [1] and Hollander and Samaniego [7]. The idea of using (1.3) as a reasonable tool for comparing random variables probably goes to Savage [12, p. 245]. The SP order can be appropriate in some problems as it directly describes probabilities of interest (distinct from other popular stochastic orders). It can be easily shown that the stochastic precedence order for independent random variables follows from the usual stochastic order. Thus, it is weaker and more flexible and can describe random variables with crossing reliability functions.

However, if we want to order the sequence in (1.1) with respect to the SP order, that is,

$$T_1 \leq_{\mathrm{sp}} T_2 \leq_{\mathrm{sp}} \cdots \leq_{\mathrm{sp}} T_n, \tag{1.4}$$

then, not necessarily, $T_i \leq_{\mathrm{sp}} T_j$, $1 \leq i < j \leq n$ meaning that this order is nontransitive [11]. Let us call (1.4), for convenience, the chain stochastic precedence (CSP) order.

Thus, the SP order in general can be nonapplicable for ordering sequences of random variables. However, it can be generalized on the basis of definition (1.2) for natural ordering of sequences of random variables. The following definition describes our approach.

Definition 1.3: *The sequence of n independent random variables T_i, $i = 1, 2, \ldots, n$ is ordered in the sense of the stochastic sequential precedence (SSP) order if it gives the maximal probability, for example, to the event $T_1 \leq T_2 \leq \cdots \leq T_n$, as compared with probabilities of events for all other permutations in the sequence of events $\{\boldsymbol{J}\}$, that is,*

$$P_{1,2,\ldots,n} \equiv P(T_1 \leq T_2 \leq \cdots \leq T_n) \geq P_{\{\boldsymbol{J}\}}, \tag{1.5}$$

whereas the corresponding notation will be

$$(T_1 \leq T_2 \leq \cdots \leq T_n)_{\mathrm{SSP}}.$$

It is clear that for $n = 2$, (1.5) reduces to (1.2). We shall not be concerned that the absolute values in (1.5) can be very small, as we are interested in comparisons. It should be noted that the issue of nontransitivity does not arise in this setting, and therefore, the use of the term "order" is appropriate in this case.

2. MOTIVATING EXAMPLES

Example 2.1 (Nontransitivity): *For simplicity of illustration, consider the case of three independent discrete random variables with the following distributions [3]*

$$\begin{aligned} &P(T_1 = 3) = 1, \\ &P(T_2 = 1) = 0.4, \quad P(T_2 = 4) = 0.6, \\ &P(T_3 = 2) = 0.6, \quad P(T_3 = 5) = 0.4. \end{aligned}$$

Then, obviously,

$$P(T_1 < T_2) = 0.6, \quad P(T_2 < T_3) = 0.64 \quad \textit{and} \quad P(T_3 < T_1) = 0.6.$$

Hence, $T_i \nleq_{\mathrm{sp}} T_j$, $1 \leq i < j \leq 3$. Sometimes this nontransitivity for three random variables is called a voting paradox [3].

We had just outlined this new order in our recent paper while considering the problem of obtaining the optimal sequence of activation of components in the warm standby system. We shall briefly refer to this meaningful example and then study some initial properties of the SSP order.

Example 2.2 (Warm standby system): *Consider 1-out-of-n warm standby system when one of the components is activated at t = 0 (full load), and others are in the warm standby mode (reduced load). When the activated component fails, one of the operable standby components*

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is activated. The problem is to find the optimal activation sequence that maximizes the lifetime of the system in a suitable probabilistic sense. This open (for a general case) problem was solved in Finkelstein et al. , where it was proved that if the lifetimes of the components are ordered in the SSP sense, then this sequence of activation (starting with the shortest lifetime) results in a system's lifetime that is larger than a lifetime of a warm standby system for any other sequence of activation in the SP order sense (see also [14]).

An important feature of the developed approach is that it was shown that the ordering of the corresponding independent realizations of components' lifetimes, that is,

$$t_1 \le t_2 \le \cdots \le t_n \tag{2.1}$$

for the considered system, maximizes realization of its lifetime, that is $s_{1,2,\ldots,n} \ge s_{\{\boldsymbol{J}\}}$*, where* $s_{\{\boldsymbol{J}\}}$ *denotes this realization for the sequence of activation* $\{\boldsymbol{J}\}$*, whereas the corresponding lifetimes are denoted by* $S_{1,2,\ldots,n}$ *and* $S_{\{\boldsymbol{J}\}}$*, respectively. It follows from (1.5) that*

$$P(S_{1,2,\ldots,n} \ge S_{\{\boldsymbol{J}\}}) = \frac{P_{1,2,\ldots,n}}{P_{1,2,\ldots,n} + P_{\{\boldsymbol{J}\}}} \ge 0.5, \tag{2.2}$$

which is the SP order.

Thus, ordering of realization (2.1) results in the maximal realization of the lifetime of the system. However, the corresponding event has the maximal probability due to assumption (1.5). Finally, (2.2) defines the SP order for system's lifetimes.

The reasoning in this example prompts us that a similar logic can be followed for some optimization problems, where the corresponding results for realizations of relevant random variables can be derived. For instance, as in the following simple illustrative example.

Example 2.3 (Coherent system): *Consider a coherent system [2] of n independent components with lifetimes* T_i*,* $i = 1, 2, \ldots, n$*. Let their realizations be ordered as in (2.1). Assume that, based on the structure of the system, we know how to allocate these realizations to n "slots" of the system in order to maximize realization of the system's lifetime. For illustration, let* $n = 3$*, and denote*

$$P_{ijk} \equiv P(T_i \le T_j \le T_k), \quad i, j, k \in \{1, 2, 3\};\ i \ne j \ne k.$$

Thus, we have six permutations for three random variables with the corresponding probabilities $P_{123}, P_{132}, P_{213}, P_{231}, P_{312}$ *and* P_{321}*. Assume that the sequence of lifetimes is ordered in the SSP sense, that is,*

$$P_{123} \equiv P(T_1 \le T_2 \le T_3) \ge P_{\{\boldsymbol{J}\}}. \tag{2.3}$$

Let "1" in the set $\{1, 2, 3\}$ *denotes the single series part (slot) of a system, whereas "2" and "3" correspond to two slots in the parallel part. We can populate these slots with our independent components with lifetimes* T_i*,* $i = 1,2,3$*. Let notation* $\{T_1, T_2, T_3\}$ *means that the first lifetime is allocated to the slot "1," whereas the second and the third to slots "2" and "3," respectively. The same notation is used for realizations* $\{t_1, t_2, t_3\}$*. Assume that* $t_1 \le t_2 \le t_3$*. Let* $s_{\{t_i, t_j, t_k\}}$*,* $i, j, k \in \{1, 2, 3\}$*;* $i \ne j \ne k$ *denote the corresponding realizations of the lifetime of the system, whereas* $S_{\{t_i, t_j, t_k\}}$*,* $i, j, k \in \{1, 2, 3\}$*;* $i \ne j \ne k$ *are the corresponding random*

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lifetimes. It is easy to see that for this specific system, we have

$$s_{\{t_3,t_2,t_1\}} = s_{\{t_3,t_1,t_2\}} = t_2; \quad s_{\{t_2,t_3,t_1\}} = s_{\{t_2,t_1,t_3\}} = t_2; \quad s_{\{t_1,t_2,t_3\}} = s_{\{t_1,t_3,t_2\}} = t_1.$$

From the structure of the system, it almost surely follows that

$$S_{\{t_3,t_2,t_1\}} = S_{\{t_3,t_1,t_2\}}; \quad S_{\{t_2,t_3,t_1\}} = S_{\{t_2,t_1,t_3\}}; \quad S_{\{t_1,t_2,t_3\}} = S_{\{t_1,t_3,t_2\}}.$$

Let us compare, the first lifetime $S_{\{t_3,t_2,t_1\}}$ with any of the last four ones, for example, with $S_{\{t_1,t_2,t_3\}}$. From (2.3), and similar to (2.2),

$$P(S_{\{t_3,t_2,t_1\}} \geq S_{\{t_1,t_2,t_3\}}) = \frac{P_{123}}{P_{123} + P_{321}} \geq 0.5,$$

as P_{123} is the probability of the event $T_1 \leq T_2 \leq T_3$ meaning specifically that T_3 is the largest in the defined sense. Note that, in this case, we have larger realizations of system's lifetime (if T_3 is assigned to the slot "1") with larger probabilities of occurring of this event. Thus, we have the SSP order for lifetimes of components, which results in the SP order for variants of systems when comparing allocation $\{3, 2, 1\}$ (or $\{3, 1, 2\}$) with others.

On the other hand, it is well-known that the allocation $\{T_1, T_2, T_3\}$ for the considered system is also the best when the lifetimes and the variants of systems are compared in the sense of the usual stochastic order. Note that, as the lifetimes of the system for different variants of allocation are statistically dependent, we cannot say now that the usual stochastic order for these variants implies the SP order, which is true for the independent random variables.

3. SOME PROPERTIES OF THE SSP ORDER

While considering different stochastic orders for the pair of independent random variables, we are often interested in the relationships between them. It is well-known that for the simplest stochastic orders for two independent random variables, we have the following chain:

$$\text{LR} \implies \text{HR} \implies \text{ST} \implies \text{SP}.$$

It is interesting to obtain some relationships between the SSP order for n independent random variables and other orders for this sequence. This topic needs further investigation. Below we present some initial results. We start with the following example that will help us to formulate the sufficient condition for the SSP order.

EXAMPLE 3.1: *Let $n = 3$ and we want to establish the sufficient condition for $(T_1 \leq T_2 \leq T_3)_{\text{SSP}}$ to hold. There are six permutations. Let us consider the permutation $(1\,2\,3)$ and compare it with permutation $(1\,3\,2)$. For $P(T_1 \leq T_2 \leq T_3) \geq P(T_1 \leq T_3 \leq T_2)$ to hold, the following should hold:*

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} f_{T_1}(x) f_{T_2}(y)(1 - F_{T_3}(y))\,dy\,dx \geq \int_{x=0}^{\infty}\int_{y=x}^{\infty} f_{T_1}(x) f_{T_3}(y)(1 - F_{T_2}(y))\,dy\,dx.$$

For this, it suffices to show that

$$f_{T_2}(y)(1 - F_{T_3}(y)) \geq f_{T_3}(y)(1 - F_{T_2}(y)).$$

However, the last inequality means that $\lambda_{T_2}(y) \geq \lambda_{T_3}(y)$, which, obviously, defines the corresponding hazard rate order, that is, $T_2 \leq_{\text{hr}} T_3$.

Thus, the first guess would be that the hazard rate order is the sufficient condition for $(T_1 \leq T_2 \leq T_3)_{\mathrm{SSP}}$. However, considering comparisons with other permutation in the same manner as above does not lead to the hazard rate order $T_1 \leq_{\mathrm{hr}} T_2$.

REMARK 3.1: *Note that Example 3.1 does not mean that, in principle, the hazard rate order cannot be a sufficient condition for the SSP. It means only that in this way, we cannot prove it. We also tried (without success so far) to create a counterexample showing that the SSP holds, whereas the corresponding hazard rate ordering does not. Thus, whether the hazard rate ordering is the sufficient condition for the SSP or not, is still an open problem.*

On the other hand, we will prove in what follows that the stronger likelihood ratio ordering is the corresponding sufficient condition.

We begin with the following lemma.

LEMMA 3.1: *Let $\{T_i\}_{i=1}^n$ be a sequence of independent random variables such that $T_1 \leq_{\mathrm{lr}} T_2 \leq_{\mathrm{lr}} \cdots \leq_{\mathrm{lr}} T_n$. Further, let $\{i_1, i_2, \ldots, i_j, \ldots, i_k, \ldots, i_n\}$ be a permutation of $\{1, 2, \ldots, n\}$. Then, for $1 \leq i_j < i_k \leq n$,*

$$\begin{aligned} &P(T_{i_1} \leq \cdots \leq T_{i_{j-1}} \leq T_{i_j} \leq T_{i_{j+1}} \leq \cdots \leq T_{i_{k-1}} \leq T_{i_k} \leq T_{i_{k+1}} \leq \cdots \leq T_{i_n}) \\ &\quad \geq P(T_{i_1} \leq \cdots \leq T_{i_{j-1}} \leq T_{i_k} \leq T_{i_{j+1}} \leq \cdots \leq T_{i_{k-1}} \leq T_{i_j} \leq T_{i_{k+1}} \leq \cdots \leq T_{i_n}). \end{aligned}$$

PROOF: Note that

$$\begin{aligned} &P(T_{i_1} \leq \cdots \leq T_{i_{j-1}} \leq T_{i_j} \leq T_{i_{j+1}} \leq \cdots \leq T_{i_{k-1}} \leq T_{i_k} \leq T_{i_{k+1}} \leq \cdots \leq T_{i_n}) \\ &\quad = \int_{t_1=0}^{\infty} \cdots \int_{t_{j-1}=t_{j-2}}^{\infty} \int_{t_j=t_{j-1}}^{\infty} \int_{t_{j+1}=t_j}^{\infty} \cdots \\ &\qquad \int_{t_{k-1}=t_{k-2}}^{\infty} \int_{t_k=t_{k-1}}^{\infty} \int_{t_{k+1}=t_k}^{\infty} \cdots \int_{t_n=t_{n-1}}^{\infty} A(t_1, t_2, \ldots, t_n)\, dz, \end{aligned} \tag{3.1}$$

where

$$A(t_1, t_2, \ldots, t_n) = f_{T_{i_j}}(t_j) f_{T_{i_k}}(t_k) \left(\prod_{r=1}^{j-1} f_{T_{i_r}}(t_r)\right) \left(\prod_{s=j+1}^{k-1} f_{T_{i_s}}(t_s)\right) \left(\prod_{u=k+1}^{n} f_{T_{i_u}}(t_u)\right),$$

and

$$dz = dt_n \ldots dt_{k+1} dt_k dt_{k-1} \ldots dt_{j+1} dt_j dt_{j-1} \ldots dt_1$$

and

$$t_1 \leq \cdots \leq t_{j-1} \leq t_j \leq t_{j+1} \leq \cdots \leq t_{k-1} \leq t_k \leq t_{k+1} \leq \cdots \leq t_n.$$

Similarly,

$$\begin{aligned} &P(T_{i_1} \leq \cdots \leq T_{i_{j-1}} \leq T_{i_k} \leq T_{i_{j+1}} \leq \cdots \leq T_{i_{k-1}} \leq T_{i_j} \leq T_{i_{k+1}} \leq \cdots \leq T_{i_n}) \\ &\quad = \int_{t_1=0}^{\infty} \cdots \int_{t_{j-1}=t_{j-2}}^{\infty} \int_{t_k=t_{j-1}}^{\infty} \int_{t_{j+1}=t_k}^{\infty} \cdots \\ &\qquad \int_{t_{k-1}=t_{k-2}}^{\infty} \int_{t_j=t_{k-1}}^{\infty} \int_{t_{k+1}=t_j}^{\infty} \cdots \int_{t_n=t_{n-1}}^{\infty} A(t_1, t_2, \ldots, t_n)\, dw, \end{aligned}$$

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where

$$dw = dt_n \ldots dt_{k+1} dt_j dt_{k-1} \ldots dt_{j+1} dt_k dt_{j-1} \ldots dt_1$$

and

$$t_1 \leq \cdots \leq t_{j-1} \leq t_k \leq t_{j+1} \leq \cdots \leq t_{k-1} \leq t_j \leq t_{k+1} \leq \cdots \leq t_n.$$

Since t_j and t_k are dummy variables, we interchange them in the above probability expression. Then, we get

$$\begin{aligned} &P(T_{i_1} \leq \cdots \leq T_{i_{j-1}} \leq T_{i_k} \leq T_{i_{j+1}} \leq \cdots \leq T_{i_{k-1}} \leq T_{i_j} \leq T_{i_{k+1}} \leq \cdots \leq T_{i_n}) \\ &\quad = \int_{t_1=0}^{\infty} \cdots \int_{t_{j-1}=t_{j-2}}^{\infty} \int_{t_j=t_{j-1}}^{\infty} \int_{t_{j+1}=t_j}^{\infty} \cdots \\ &\quad \int_{t_{k-1}=t_{k-2}}^{\infty} \int_{t_k=t_{k-1}}^{\infty} \int_{t_{k+1}=t_k}^{\infty} \cdots \int_{t_n=t_{n-1}}^{\infty} B(t_1, t_2, \ldots, t_n)\, dz, \end{aligned} \quad \textbf{(3.2)}$$

where

$$B(t_1, t_2, \ldots, t_n) = f_{T_{i_j}}(t_k) f_{T_{i_k}}(t_j) \left(\prod_{r=1}^{j-1} f_{T_{i_r}}(t_r) \right) \left(\prod_{s=j+1}^{k-1} f_{T_{i_s}}(t_s) \right) \left(\prod_{u=k+1}^{n} f_{T_{i_u}}(t_u) \right)$$

and

$$t_1 \leq \cdots \leq t_{j-1} \leq t_j \leq t_{j+1} \leq \cdots \leq t_{k-1} \leq t_k \leq t_{k+1} \leq \cdots \leq t_n.$$

Further,

$$A(t_1, t_2, \ldots, t_n) - B(t_1, t_2, \ldots, t_n) \geq 0 \quad \textbf{(3.3)}$$

holds if, for $t_j \leq t_k$,

$$f_{T_{i_j}}(t_j) f_{T_{i_k}}(t_k) \geq f_{T_{i_j}}(t_k) f_{T_{i_k}}(t_j,),$$

or equivalently,

$$T_{i_j} \leq_{lr} T_{i_k}, \quad \text{for } 1 \leq i_j < i_k \leq n,$$

which follows from the hypothesis that $T_1 \leq_{\mathrm{lr}} T_2 \leq_{\mathrm{lr}} \cdots \leq_{\mathrm{lr}} T_n$. Thus, on using (3.3), the result follows from the expressions given in (3.1) and (3.2). ■

Theorem 3.1: *Let $\{T_i\}_{i=1}^n$ be a sequence of independent random variables such that $T_1 \leq_{\mathrm{lr}} T_2 \leq_{\mathrm{lr}} \cdots \leq_{\mathrm{lr}} T_n$. Then $(T_1 \leq T_2 \leq \cdots T_n)_{\mathrm{SSP}}$.*

Proof: Note that a set of n random variables could be arranged in $n!$ different ways by interchanging any two of them. Thus, the proof follows from repetitive use of Lemma 3.1. For instance, let us consider $n = 3$. Then, from Lemma 3.1, we have

$$P(T_1 \leq T_2 \leq T_3) \geq P(T_1 \leq T_3 \leq T_2) \geq P(T_3 \leq T_1 \leq T_2). \quad \textbf{(3.4)}$$

Again,

$$P(T_1 \leq T_2 \leq T_3) \geq P(T_3 \leq T_2 \leq T_1) \quad \textbf{(3.5)}$$

and

$$P(T_1 \leq T_2 \leq T_3) \geq P(T_2 \leq T_1 \leq T_3) \geq P(T_2 \leq T_3 \leq T_1). \quad \textbf{(3.6)}$$

On using (3.4), (3.5) and (3.6), we get $(T_1 \leq T_2 \leq T_3)_{\mathrm{SSP}}$. ■

In the following intuitively clear theorem, we show that the SSP order is stronger than the CSP (1.4).

THEOREM 3.2: *Let $\{T_i\}_{i=1}^n$ be a sequence of independent random variables. Then, the SSP order (1.5) implies the CSP order (1.4), that is,*

$$SSP \Rightarrow CSP.$$

PROOF: Let N be a "sufficiently large" number of trials for the sequence $\{T_i\}_{i=1}^n$, whereas $N_{1,2,\ldots,n}$ denote the number of realizations (out of N), that result in (2.1).

Select T_i and T_{i+1}, $i = 1, 2, \ldots, n-1$. Inequalities (2.1) correspond to the case when their realizations are ordered as $t_i \le t_{i+1}$. Now we consider realizations where $t_i \ge t_{i+1}$ with all other realizations of other random variables being the same as for the previous case. Denote the overall number of realizations of the latter kind by $N_{1,2,\ldots,i-1,i+1,i,i+2,\ldots,n}$. From our assumption $(T_1 \le T_2 \le \cdots \le T_n)_{\mathrm{SSP}}$, it follows that

$$N_{1,2,\ldots,i-1,i+1,i,i+2,\ldots,n} \le N_{1,2,\ldots,n}.$$

But this means that

$$N_{i+1,i} \le N_{i,i+1},$$

where $N_{i,i+1}$ and $N_{i+1,i}$ are the numbers of realizations for the pair of random variables T_i and T_{i+1} for which $T_i \le T_{i+1}$ and $T_i \ge T_{i+1}$, respectively. But this, in fact, means the SP order for this pair by definition. We can perform this reasoning for all adjacent pairs and arrive at (1.4). ■

4. CONCLUDING REMARKS

The SP order is natural in various engineering applications when, for example, stress-strength or peak over the threshold probabilities are considered. It can be attractive for probabilistic description of real-life problems as it directly describes probabilities of interest (distinct from other popular stochastic orders). It is well-known that the usual stochastic order implies the SP order, which gives the corresponding sufficient condition.

However, distinct from the conventional stochastic orders (e.g., the usual stochastic order, the hazard rate order and the likelihood ratio order) that are transitive when ordering the sequence of random variables, the SP order in this case, can be nontransitive.

Therefore, in this note, we discuss the new stochastic ordering for the sequence of independent random variables that is called the SSP order. It generalizes the SP order that is defined for two random variables to the case $n > 2$.

We show that the likelihood ratio ordering is the sufficient condition for the SSP ordering of the sequence of the independent random variables. Moreover, the SSP order implies the SP in this sequence (CSP). It is interesting to consider the following open problem: either to prove that the hazard rate ordering can be a "better" sufficient condition for the SSP, or to construct the corresponding counterexample.

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