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Blow-up in a chemotaxis model without symmetry assumptions

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In this paper we prove the existence of solutions of the Keller–Segel model in chemotaxis, which blow up in finite or infinite time. This is done without assuming any symmetry properties of the solution.

1 Introduction

The collective behavior of the myxamoebae of the cellular slime mold *Dictyostelium discoideum* has astonished many scientists since Dictyostelium was found in 1935. During its lifecycle a Dictyostelium myxamoeba population grows by cell division as long as there is enough food. After the food resources are exhausted, the myxamoebae spread over the whole domain that they can reach. Then a so-called founder cell starts to exude cyclic Adenosine Monophosphate (cAMP) which attracts the starving myxamoebae. They start to move chemotactically positive in direction of the founder cell and are also stimulated to emit cAMP. During this process, the myxamoebae not only produce cAMP, but also consume it and secrete a phosphodisterase, which converts the cAMP into chemotactically inactive AMP. According to this chemotactically positive movement to the founder cell, the myxamoebae aggregate. At the end point of aggregation, the myamoebae form a pseudoplasmoid, where every myxamoeba maintains its individual integrity. This pseudoplasmoid moves phototactically positive towards light. Finally, a fruiting body is formed and spores are spread. When the spores become myxamoebae the lifecycle is closed.

Since 1970 when Keller & Segel [17] introduced their model for the aggregation of Dictyostelium discoideum, which is given in a simplified version by the equations

$$\left.\begin{array}{rclll} a_t &=& \nabla(\nabla a - \tilde{\chi} a \nabla c), & & x \in \Omega,\ t > 0 \\ c_t &=& k_c \Delta c - \gamma c + \tilde{\alpha} a, & & x \in \Omega,\ t > 0 \\ \frac{\partial a}{\partial n} &=& \frac{\partial c}{\partial n} \ = \ 0, & & x \in \partial\Omega,\ t > 0 \\ a(0,x) &=& a_0(x),\ c(0,x) \ = \ c_0(x), & & x \in \Omega, \end{array}\right\} \quad (1.1)$$

many authors have studied the possible blow-up of the solution of system (1.1). In (1.1) the function $a(x,t)$ represents the Dictyostelium myxamoeba density in point $x \in \Omega$ at time t, and the function $c(x,t)$ stands for the cAMP density, which attracts the myxamoebae to move positive chemotactically in the direction of a higher cAMP concentration. $\tilde{\alpha}$, $\tilde{\chi}$, k_c

and γ denote positive constants. Here and in the following sections, n denotes the outer normal vector field on $\partial\Omega$. For a detailed derivation of the equations, see, for instance, Horstmann [12], Keller & Segel [17] or Nanjundiah [23].

That there might exist solutions which blow up for $\Omega \subset \mathbb{R}^2$ has been expected in connection with the studies concerning the conjectures by Nanjundiah [23] and Childress & Percus [4, 5]. Nanjundiah [23] suggested in 1973 that "the end-point (in time) of aggregation is such that the cells are distributed in form of $\delta-$function concentration" [23, p. 102]. Childress & Percus [4, 5] formulated the following statement for space dimension $N = 2$:

- The myxamoeba density cannot form a $\delta-$function singularity, if the total myxamoeba density on $\Omega \subset \mathbb{R}^2$ is less than a critical number d_Ω.
- The myxamoeba density can form a $\delta-$function singularity, if the total myxamoeba density on Ω is larger than a critical value D_Ω.

In the following years, one was led to believe that the equality $d_\Omega = D_\Omega$ should hold for the critical values mentioned in the conjecture.

If one uses the transformation

$$A(t,x) = \frac{|\Omega| a(t,x)}{\int_\Omega a_0(x)dx}, \quad C(t,x) = \tilde{\chi}\left(c(t,x) - \frac{1}{|\Omega|}\int_\Omega c(t,x)dx\right) \tag{1.2}$$

(see also Gajewski & Zacharias [9], Horstmann [13], Jäger & Luckhaus [15] and Nanjundiah [23]), and the notation $\alpha\chi$ instead of $\tilde{\alpha}\tilde{\chi}\int_\Omega a(x,t)dx/|\Omega|$, we get a transformed version of the Keller–Segel model. This transformed system is given by

$$\left.\begin{array}{lll}
A_t &= \nabla\cdot(\nabla A - A\nabla C), & x\in\Omega,\ t>0\\
C_t &= k_c\Delta C - \gamma C + \alpha\chi(A-1), & x\in\Omega,\ t>0\\
\frac{\partial A}{\partial n} &= \frac{\partial C}{\partial n} = 0, & x\in\partial\Omega,\ t>0\\
A(0,x) &= A_0(x) > 0,\ C(0,x) = C_0(x), & x\in\Omega\\
\int_\Omega A_0(x)\,dx &= |\Omega|,\ \int_\Omega C(t,x) = 0, & t\geqslant 0.
\end{array}\right\} \tag{1.3}$$

In this paper, we study the possibility that solutions of system (1.3) might blow up (which would imply blow-up for solutions of system (1.1)). For clarity, we give the definition of solutions of system (1.3), which we will refer to as blow-up solutions.

Definition 1.1 A solution of (1.3) blows up or is a blow-up solution of (1.3), provided there is a time $T_{max} \leqslant \infty$ such that

$$\limsup_{t\to T_{max}} ||A(x,t)||_{L^\infty(\Omega)} = \infty \text{ or } \limsup_{t\to T_{max}} ||C^+(x,t)||_{L^\infty(\Omega)} = \infty,$$

where $C^+(x,t)$ denotes the positive part of the function $C(x,t)$. If $T_{max} < \infty$ we say that the solution of (1.3) blows up in finite time, and if $T_{max} = \infty$ we will call it blow-up in infinite time.

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Up to now, the existence of blow-up solutions of system (1.3) is only known under a radial symmetry assumption on the solution (see Herrero & Velázquez [10, 11] and Horstmann [14] for existence results of blow-up solutions of system (1.3) in the radially symmetric case). There are several blow-up results for simplified versions of (1.3) (see, for example, Jäger & Luckhaus [15], Nagai [20] and Nagai *et al.* [22]). However, most of them require a radial symmetry assumption for the solution. For those simplified versions of the Keller–Segel model, one can use techniques which are not applicable for (1.3). We will compare our results with some of those in the concluding section.

In this article, we prove the existence of blow-up solutions of (1.3) for a smooth domain $\Omega \subset \mathbb{R}^2$, provided $4\pi k_c < \alpha\chi|\Omega|$ and $\alpha\chi|\Omega|/k_c \neq 4\pi m$, where $m \in \mathbb{N}$. The proof will be based on the same idea that has been used in Horstmann [14, 12] to prove the existence of blow-up solutions in the radially symmetric setting of (1.3) with $\gamma = 0$ and a generalization of results by Brézis & Merle [1] and Li & Shafrir [19], which has been done by Wang & Wei [28].

2 Summary

In 1998, Gajewski & Zacharias [9] proved the local existence of a weak solution of (1.3), where the definition of a weak solution is given as follows:

Definition 2.1 A pair of functions $(A(t,x), C(t,x))$ with

$$A \in L^\infty(0,T;L^\infty_+(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad A_t \in L^2(0,T;(H^1(\Omega))^*),$$

$$C \in L^\infty(0,T;L^\infty(\Omega)) \cap C(0,T;H^1(\Omega)), \quad C_t \in L^2(0,T;L^2(\Omega))$$

is called a weak solution of (1.3) if for all $h \in L^2(0,T;H^1(\Omega))$ the following identities hold:

$$0 = \int_0^T \langle A_t, h\rangle \, dt + \int_0^T \int_\Omega (\nabla A - A\nabla C)\cdot \nabla h \, dx \, dt,$$

$$0 = \int_0^T \int_\Omega C_t h \, dx \, dt + \int_0^T \int_\Omega (k_c \nabla C \cdot \nabla h + (\gamma C - \alpha\chi(A-1))\cdot h) \, dx \, dt.$$

(Here $\langle \cdot,\cdot \rangle$ denotes the dual product between $H^1(\Omega)$ and its dual space $(H^1(\Omega))^*$.)

Using the Lyapunov function

$$F(A(t),C(t)) = \int_\Omega \frac{1}{2\alpha\chi}\left(k_c|\nabla C(t)|^2 + \gamma C(t)^2\right) + A(t)(\log A(t) - 1) + 1dx - \int_\Omega (A(t)-1)C(t)dx$$

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and the lower estimate (see Gajewski & Zacharias [9, Lemma 4.7, p. 96])

$$F(A(t),C(t)) \geqslant \mathscr{F}(C(t)) = \frac{1}{2\alpha\chi}\int_\Omega k_c|\nabla C(t)|^2 + \gamma C(t)^2\,dx - |\Omega|\log\left(\frac{1}{|\Omega|}\int_\Omega e^{C(t)}dx\right) \tag{2.1}$$

for $t \geqslant 0$, it is possible to show for a smooth domain $\Omega \subset \mathbb{R}^2$ that the Lyapunov function is bounded from below, provided

$$\frac{\alpha\chi|\Omega|}{4k_c\pi} < 1.$$

This fact is a simple consequence from a Moser–Trudinger type inequality by Chang &Yang [2, Proposition 2.3].

Remark 2.2 Inequality (2.1) follows by minimizing

$$A(t)(\log A(t) - 1) + 1 - (A(t) - 1)C(t)$$

with respect to A for each C (see Gajewski & Zacharias [9, Proof of Lemma 4.7, pp.96–97] for details).

Remark 2.3 If the boundary of Ω is piecewise C^2, then one can bound the Lyapunov function from below provided

$$\frac{\alpha\chi|\Omega|}{4k_c\Theta} < 1,$$

where Θ denotes the smallest interior angle of $\partial\Omega$ (see, for example, Gajewski & Zacharias [9] or Horstmann [13]).

It results from the studies done by Nagai *et al* [21] that, in such a case, the L^∞-norm of $A(x,t)$ and $C(x,t)$ remains uniformly bounded for all $t \geqslant 0$. Gajewski & Zacharias [9] show that, in this case, the solution converges at least for subsequences $(t_k)_{k\in\mathbb{N}}$ with $t_k \to \infty$ to a stationary solution of (1.3). Horstmann [13] has shown that this statement is in fact true for $t \to \infty$. So we know from Gajewski & Zacharias [9] and Horstmann [13] that for $\alpha\chi|\Omega| < 4k_c\pi$:

$$A(t) \to A^*$$

in $L^2(\Omega)$ and

$$C(t) \rightharpoonup C^*$$

in $H^1(\Omega)$ as $t \to \infty$, where

$$A^* = \frac{|\Omega|e^{C^*}}{\int_\Omega e^{C^*}dx},$$

and C^* solves the nonlocal elliptic boundary value problem

$$\left.\begin{aligned} -k_c\Delta v + \gamma v &= \alpha\chi\left(\frac{|\Omega|e^v}{\int_\Omega e^v dx} - 1\right), && \text{in } \Omega \\ \frac{\partial v}{\partial n} &= 0, && \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.2}$$

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3 Existence of blow-up solutions for $\alpha\chi|\Omega| > 4k_c\pi$ and $\alpha\chi|\Omega|/k_c$ not equal to an integer multiple of 4π

Let us now assume in the following that $\alpha\chi|\Omega| > 4k_c\pi$. We set

$$v_\varepsilon(x) = \log\left(\frac{\varepsilon^2}{(\varepsilon^2 + \pi|x - x_0|^2)^2}\right) - \frac{1}{|\Omega|}\int_\Omega \log\left(\frac{\varepsilon^2}{(\varepsilon^2 + \pi|x - x_0|^2)^2}\right),$$

where x_0 is an arbitrary point on $\partial\Omega$. The sequence $(v_\varepsilon(x))_{\varepsilon>0}$ belongs to the set $\mathscr{D} \equiv \{v \in H^1(\Omega) \mid \int_\Omega v\ dx = 0\}$. One can easily check that

$$\mathscr{F}(v_\varepsilon) \to -\infty \tag{3.1}$$

and

$$||\nabla v_\varepsilon||_{L^2(\Omega)} \to \infty \tag{3.2}$$

as $\varepsilon \to 0$. For the explicit calculations we refer the interested reader elsewhere [13, Proof of Lemma 2] [28, Lemma 2.2]. A consequence of these observations is that the Lyapunov function $F(A(t), C(t))$ might become unbounded from below as $t \to T_{max}$. If we now can find a constant $\hat{K}$ such that

$$\mathscr{F}(v) > \hat{K} \tag{3.3}$$

holds, for all solutions of (2.2), we can construct initial data for (1.3), for which the corresponding solution of (1.3) has to blow up in finite or infinite time. For the radially symmetric case of (1.3) (with $\gamma = 0$) this was possible for $\alpha\chi|\Omega| > 8k_c\pi$ and $\alpha\chi|\Omega|/k_c$ not equal $8\pi m$ for $m \in \mathbb{N}$ (see Horstmann [12, 14] for details). That there are nontrivial solutions of (2.2) has been proved independently [13, 28] using techniques introduced by Struwe & Tarantello [27]. Now we use results similar to those used in Wang & Wei [28, Section 3] to show the existence of a constant $\hat{K}$ such that (3.3) holds for all solutions of (2.2), provided $\alpha\chi|\Omega|/k_c$ is not equal to $4\pi m$, $m \in \mathbb{N}$.

This claim will be shown by contradiction. Therefore let $\alpha\chi|\Omega|/k_c > 4\pi$ and not equal to $4\pi m$, $m \in \mathbb{N}$. If there is no constant $\hat{K}$ such that (3.3) holds, then there exists a sequence $(v_k)_{k\in\mathbb{N}}$ of solutions of (2.2) such that

$$||\nabla v_k||_{L^2(\Omega)} \to \infty, \tag{3.4}$$

$$\int_\Omega e^{v_k} dx \to \infty \tag{3.5}$$

and

$$\max_{x\in\overline{\Omega}} v_k(x) \ \to \ \infty \tag{3.6}$$

as $k \to \infty$. If (3.6) does not hold, we get a uniform L^∞-bound of the right-hand side of (2.2) for all k, which gives us the existence of the constant $\hat{K}$.

We now use the transformation

$$u_k = v_k + \frac{\alpha\chi}{\gamma}.$$

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So each u_k solves the problem

$$\left.\begin{array}{rcll} -\Delta u_k + \frac{\gamma}{k_c} u_k & = & \mu_k e^{u_k}, & \text{in } \Omega \\ \frac{\partial u_k}{\partial n} & = & 0, & \text{on } \partial\Omega \\ \int\limits_{\Omega} u_k dx & = & \frac{\alpha\chi|\Omega|}{\gamma}, & \end{array}\right\} \tag{3.7}$$

where

$$\mu_k = \frac{\alpha\chi|\Omega|}{k_c \int\limits_{\Omega} e^{u_k} dx} \tag{3.8}$$

and $\mu_k \to 0$ as $k \to \infty$. According to the maximum principle for elliptic operators, we notice that $u_k > 0$ in Ω. In the following, we will show in the same way as in Wang & Wei [28] that the $(u_k)_{k\in\mathbb{N}}$ contain a subsequence (for the sake of simplicity, again denoted by $(u_k)_{k\in\mathbb{N}}$) such that

$$\mu_k \int\limits_{\Omega} e^{u_k} dx \to 4\pi m \tag{3.9}$$

for some integer m as $k \to \infty$. However, this would contradict the fact that $\alpha\chi|\Omega|/k_c \neq 4\pi m$, $m \in \mathbb{N}$. To show (3.9) we make use of the following lemma:

Lemma 3.1 [3] *Let* $L = \sum\limits_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ *be a uniformly elliptic operator, namely*

$$\nu_0 I \leqslant (a_{ij})_{1\leqslant i,j\leqslant 2} \leqslant \nu_1 I.$$

Then there exists a constant $\beta = \beta(\nu_0, \nu_1)$ *such that for any solution* u *of the following problem:*

$$Lu = f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

we have

$$\int\limits_{\Omega} \exp\left(\frac{\beta|u(x)|}{||f||_{L^1(\Omega)}}\right) dx \leqslant K.$$

Let us define the following set:

$$\mathcal{BS} \equiv \left\{ x \in \overline{\Omega} \;\middle|\; \begin{array}{l} \text{there exists a sequence } \mu_k \to 0, \text{ with corresponding} \\ \text{solutions } u_k \text{ of (3.7), and a sequence } (x_k)_{k\in\mathbb{N}} (x_k \in \overline{\Omega}), \\ \text{such that } u_k(x_k) \to \infty, \ x_k \to x \text{ as } k \to \infty \end{array} \right\}$$

By our assumption $\mathcal{BS} \neq \emptyset$ holds. We now set

$$\Sigma_k \equiv \int\limits_{\Omega} \mu_k e^{u_k} dx \left(= \frac{\alpha\chi|\Omega|}{k_c}\right).$$

Since

$$\int\limits_{\Omega} \frac{\mu_k e^{u_k}}{\Sigma_k} dx = 1$$

for all k, we can extract a subsequence of the u_k (still denoted by u_k as mentioned above) such that there exists a finite measure μ in the set of all real bounded Borel measures on

$\overline{\Omega}$ (denoted by $\mathscr{M}(\overline{\Omega})$), such that

$$\int_{\Omega} \frac{\mu_k e^{u_k}}{\Sigma_k} \varphi \, dx \to \int_{\Omega} \varphi \, d\mu \tag{3.10}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ as $k \to \infty$.

For each boundary point $x_0 \in \partial\Omega$ we can strengthen the boundary (see Wang & Wei [28] and Ni & Takagi [24] for more details) and the Laplacian becomes

$$L_{x_0} + \sum_{k=1}^{2} b_l \frac{\partial}{\partial x_l}$$

with a uniformly elliptic operator L_{x_0} and $|b_l| \leqslant C = const$. Using the compactness of the boundary, we can choose a uniform $\beta = \beta_0$ in Lemma 3.1 for all L_{x_0}, $x_0 \in \partial\Omega$. Now we define δ-regular points of $\overline{\Omega}$.

Definition 3.2 For any $\delta > 0$, we call $x_0 \in \overline{\Omega}$ a δ-regular point if there is a function $\varphi \in C_0^\infty(\mathbb{R}^2)$, $0 \leqslant \varphi \leqslant 1$, with $\varphi = 1$ in a neighborhood of x_0 such that

$$\int_{\Omega} \varphi \, d\mu < \frac{\beta_0}{1 + 3\delta}. \tag{3.11}$$

We also define the set $\Sigma(\delta)$ of all points in $\overline{\Omega}$ which are not δ-regular:

$$\Sigma(\delta) \equiv \{x_0 \in \overline{\Omega} \mid x_0 \text{ is not a } \delta\text{−regular point}\}. \tag{3.12}$$

We note the following:

Lemma 3.3 *For any* $1 < q < 2$*, there is a constant* C_q *independent of* k *such that* $\|\nabla u_k\|_q \leqslant C_q$.

Proof of Lemma 3.3 Let $q' = \frac{q}{q-1} > 2$. We know

$$\|\nabla u_k\|_q \leqslant \sup \left\{ \left| \int_{\Omega} \nabla u_k \cdot \nabla \varphi \, dx \right| \varphi \in L_1^{q'}(\Omega), \int_{\Omega} \varphi \, dx = 0, \|\varphi\|_{L_1^{q'}(\Omega)} = 1 \right\}.$$

By the Sobolev embedding theorem, we have

$$\|\varphi\|_{L^\infty(\Omega)} \leqslant C_1.$$

It is clear that

$$\begin{aligned} \left| \int_{\Omega} \nabla u_k \cdot \nabla \varphi \, dx \right| &= \left| \int_{\Omega} \Delta u_k \varphi \, dx \right| \\ &= \left| \int_{\Omega} \left(\frac{\gamma}{k_c} u_k - \mu_k e^{u_k} \right) \varphi \, dx \right| \end{aligned}$$

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$$\leqslant C_1 \int_\Omega \left(\frac{\gamma}{k_c} u_k + \mu_k e^{u_k} \right) dx$$
$$\leqslant C_2.$$

Here we have used the fact that $u_k > 0$ in Ω. □

Now we can use exactly the same arguments as in Wang & Wei [28, Proof of Lemmas 3.2 and 3.3] and Brézis & Merle [1] to see that

(1) If x_0 is a δ-regular point, then $(u_k)_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\overline{\Omega} \cap B_{R_0}(x_0))$, where $B_{R_0}(x_0)$ denotes a ball with radius R_0 centered in x_0.
(2) $\mathscr{BS} = \Sigma(\delta)$ for any $\delta > 0$.

These two statements imply that $1 \leqslant card(\mathscr{BS}) < \infty$. Let $\mathscr{BS} = \{P_1, ..., P_M\}$. We decompose $\mathscr{BS}$ into a boundary blow-up set $\mathscr{BS}_{\partial\Omega} = \mathscr{BS} \cap \partial\Omega$ and an interior blow-up set $\mathscr{BS}_\Omega = \mathscr{BS} \cap \Omega$. For a small constant $r > 0$ we set

$$\sigma_j^k(r) = \int_{B_r(P_j)} \mu_k e^{u_k} dx.$$

We now see that for all small r the following equality holds:

$$\lim_{k\to\infty} \int_\Omega \mu_k e^{u_k} dx = \sum_{j=1}^N \lim_{k\to\infty} \sigma_j^k(r). \tag{3.13}$$

This implies that

$$\lim_{k\to\infty} \int_\Omega \mu_k e^{u_k} dx = \sum_{j=1}^N \lim_{r\to 0} \lim_{k\to\infty} \sigma_j^k(r), \tag{3.14}$$

which would give us (3.9), and thus a contradiction to the value of $\alpha\chi|\Omega|/k_c$, provided

$$\lim_{r\to 0} \lim_{k\to\infty} \sigma_j^k(r) = 4q\pi \tag{3.15}$$

for some $q \in \mathbb{N}$. However, this is true, as one can see in the following lemma, which is similar to Lemma 3.4 in Wang & Wei [28].

Lemma 3.4 *Suppose* $P_j \in \mathscr{BS}_{\partial\Omega}$, *then* $\lim_{r\to 0} \lim_{k\to\infty} \sigma_j^k(r) = 4\pi$. *If* $P_j \in \mathscr{BS}_\Omega$ *then* $\lim_{r\to 0} \lim_{k\to\infty} \sigma_j^k(r) = 8\pi$.

Proof of Lemma 3.4
We first prove the case when $P \in \mathscr{BS}_{\partial\Omega}$. Recall that the Pohozaev identity for a function u satisfying

$$\Delta u - \beta u + f(u) = 0, \text{ in } U \subset R^2$$

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is given by

$$\int_U (-\beta u^2 + 2F(u))dx = \int_{\partial U} \left[(x \cdot \nabla u)\frac{\partial u}{\partial n} - (x \cdot n)\frac{|\nabla u|^2}{2} \right] dS + \int_{\partial U} (x \cdot n) \left(-\beta \frac{u^2}{2} + F(u) \right) dS, \tag{3.16}$$

where $F(u) = \int_0^u f(s)ds$.

Let $f(u) = \mu_k e^u$ and $\beta = \frac{\gamma}{k_c}$. We may assume without loss of generality that $P = 0$. Now we set $U_r = B_r(0) \cap \bar{\Omega}$ and consider the function w_k which is a solution of the following problem:

$$\left. \begin{array}{rcll} \Delta w - \beta w & = & 0 & \text{in } U_r, \\ \frac{\partial w}{\partial n} & = & \frac{\partial u_k}{\partial n} & \text{on } \partial U_r. \end{array} \right\} \tag{3.17}$$

It is easy to see that $w_k = O(1)$ in $C^2(U_r)$, since $|\frac{\partial u_k}{\partial n}| \leqslant C$ on ∂U_r.

If we put $h_k = (u_k - w_k)/(\sigma_j^k(r))$, we have that $h_k \to G(\cdot, 0)$ in $C^2_{loc}(B_r(0) \cap \bar{\Omega} \backslash \{0\})$, where $G(\cdot, 0)$ satisfies

$$-\Delta G + \beta G = \delta_0 \ \text{ in } U_r, \frac{\partial G}{\partial n} = 0 \ \text{ on } \partial U_r.$$

(See a proof in Ding *et al.* [8] for this claim.) By potential theory, it is easy to see that for $|x|$ small

$$G(\cdot, 0) = -\frac{1}{\pi} \log |x| + O(1).$$

Hence we have

$$u_k = -\frac{\sigma_j^k(r)}{\pi} \log |x| + O(1)$$

in $C^1(\partial U_r)$ (here $O(1)$ may depend upon r, but is uniform in k).

By Pohozaev's identity, we have

$$\int_{U_r} (-\beta u_k^2 + 2\mu_k e^{u_k} - 2\mu_k)dx = \int_{\partial U_r} \left[(x \cdot \nabla u_k)\frac{\partial u_k}{\partial n} - (x \cdot n)\frac{|\nabla u_k|^2}{2} + (x \cdot n) \left(-\beta \frac{u_k^2}{2} + \mu_k e^{u_k} - \mu_k \right) \right] dS. \tag{3.18}$$

We now estimate each term on both sides of (3.18):

$$\int_{U_r} u_k^2 dx = O(r^{1/2} \|u_k\|_{L^4(U_r)}) = O(r^{1/2} \|u_k\|_{W^{1,3/2}(\Omega)}) = O(r^{1/2}),$$

$$\int_{U_r} 2\mu_k e^{u_k} dx = 2\sigma_j^k(r) + O(\mu_k),$$

$$\int_{U_r} 2\mu_k dx = O(\mu_k r^2),$$

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$$\int_{\partial U_r} (x \cdot \nabla u_k) \frac{\partial u_k}{\partial n} dS = \left(\frac{\sigma_j^k(r)}{\pi} \right)^2 \int_{\partial U_r} \left(\frac{(x \cdot n)}{|x|^2} + O(1) \right)$$

$$= \left(\frac{\sigma_j^k(r)}{\pi} \right)^2 (\pi + O(r)),$$

$$\int_{\partial U_r} (x \cdot n) \frac{|\nabla u_k|^2}{2} dS = \left(\frac{\sigma_j^k(r)}{\pi} \right)^2 \left(\frac{\pi}{2} + O(r) \right),$$

$$\int_{\partial U_r} u_k^2 dS = O(r),$$

$$\int_{\partial U_r} (x \cdot n) \mu_k e^{u_k} dS = O(\mu_k \max_{x \in \partial U_r} e^{u_k}) = O(\mu_k),$$

$$\int_{\partial U_r} (x \cdot n) \mu_k dS = O(\mu_k r).$$

Here we have used the statement of Lemma 3.3. Now let $k \to +\infty$ first, and then $r \to 0$. We see that

$$2 \lim_{r \to 0} \lim_{k \to \infty} \sigma_j^k(r) = \frac{1}{\pi^2} \frac{\pi}{2} \left(\lim_{r \to 0} \lim_{k \to +\infty} \sigma_j^k(r) \right)^2,$$

$$\lim_{r \to 0} \lim_{k \to +\infty} \sigma_j^k(r) = 4\pi.$$

The case when $P \in \mathscr{B}\mathscr{S}_\Omega$ can be proved similarly. For convenience, we give a sketch of the proof. Instead of (3.17), in this case we consider w_k satisfying

$$\left. \begin{array}{rcll} \Delta w - \beta w & = & 0 & \text{in } U_r, \\ w & = & u_k & \text{on } \partial U_r. \end{array} \right\} \tag{3.19}$$

We put $h_k = (u_k - w_k)/(\sigma_j^k(r))$ and assume that $P = 0 \in \Omega$. Similarly, $h_k \to G(\cdot, 0)$ in $C^2_{loc}(B_r(0)/\{0\})$, where G now is a Green function with Dirichlet boundary data:

$$-\Delta G + \beta G = \delta_0 \ \text{ in } B_r, \ G = 0 \ \text{ on } \partial U_r.$$

In this case, the Green's function has following expansion near 0:

$$G(\cdot, 0) = -\frac{1}{2\pi} \log |x| + O(1).$$

We obtain the same estimates as in the first case when $P \in \mathscr{B}\mathscr{S}_{\partial\Omega}$ except

$$\int_{\partial U_r} (x \cdot \nabla u_k) \frac{\partial u_k}{\partial n} dS = \left(\frac{\sigma_j^k(r)}{2\pi} \right)^2 \int_{\partial U_r} \left(\frac{(x \cdot n)}{|x|^2} + O(1) \right)$$

$$= \left(\frac{\sigma_j^k(r)}{2\pi} \right)^2 (2\pi + O(r)),$$

$$\int_{\partial U_r} (x \cdot n) \frac{|\nabla u_k|^2}{2} dS = \left(\frac{\sigma_j^k(r)}{2\pi} \right)^2 (\pi + O(r)).$$

Now applying Pohozaev's identity again, we have in this case

$$2\lim_{r\to 0}\lim_{k\to\infty}\sigma_j^k(r) = \frac{1}{4\pi^2}\pi\left(\lim_{r\to 0}\lim_{k\to+\infty}\sigma_j^k(r)\right)^2,$$

$$\lim_{r\to 0}\lim_{k\to+\infty}\sigma_j^k(r) = 8\pi \text{ for } P\in\mathscr{B}\mathscr{S}_\Omega.$$

This completes the proof. □

From Lemma 3.4, we get the following lemma:

Lemma 3.5 *Suppose* $\alpha\chi|\Omega|/4k_c\pi > 1$ *and* $\alpha\chi|\Omega|/k_c \neq 4\pi m$ *for* $m\in\mathbb{N}$, *then there exists a constant* $\hat{K}\in\mathbb{R}$ ($\hat{K}\leqslant 0$), *such that for all solutions* v *of* (2.2)

$$\mathscr{F}(v)\geqslant \hat{K} > -\infty$$

holds.

A direct consequence of this lemma, (3.1) and (3.2) is the following theorem, which also collects some known facts concerning blow-up solutions:

Theorem 3.6 *Let* $\Omega\subset\mathbb{R}^2$ *be a smooth domain, and let* $\hat{K}$ *denote the constant from Lemma* 3.5. *Furthermore, assume that* $4k_c\pi < \alpha\chi|\Omega|$, *and that*

$$\frac{\alpha\chi|\Omega|}{k_c}\neq 4\pi m$$

for $m\in\mathbb{N}$, *then there exist initial data* (A_0, C_0), *such that*

$$\hat{K} > F(A_0, C_0)$$

and the corresponding solution of (1.3) *blows up in finite or infinite time. For these blow-up solutions, the following statements hold:*

(1) $\lim_{t\to T_{max}} ||A(x,t)||_{L^2(\Omega)} = \infty$

(2) $\lim_{t\to T_{max}} \int_\Omega A(x,t)C(x,t)\,dx = \infty$

(3) $\lim_{t\to T_{max}} ||\nabla C(x,t)||_{L^2(\Omega)} = \infty$

(4) $\lim_{t\to T_{max}} \int_\Omega e^{C(x,t)}\,dx = \infty$

(5) $\lim_{t\to T_{max}} ||A(x,t)||_{L^\infty(\Omega)} = \lim_{t\to T_{max}} ||C(x,t)||_{L^\infty(\Omega)} = \infty$

(6) *If* $4\pi k_c < \alpha\chi|\Omega| < 8\pi k_c$ *and* Ω *is a simply connected domain, then*

$$\lim_{t\to T_{max}}\int_{\partial\Omega} e^{C(x,t)/2}dS = \infty.$$

Proof of Theorem 3.6

The existence of a blow-up solution follows from Lemma 3.5, (3.1) and (3.2). So let $\hat{K}$ be the constant from Lemma 3.5. We choose a ε_0 arbitrary but fixed and a fixed $x_0\in\partial\Omega$, such that

$$\hat{K} > \mathscr{F}(C_{\varepsilon_0}(x))$$

where

$$C_{\varepsilon_0}(x) = \log\left(\frac{\varepsilon_0^2}{(\varepsilon_0^2 + \pi|x - x_0|^2)^2}\right) - \frac{1}{|\Omega|}\int_\Omega \log\left(\frac{\varepsilon_0^2}{(\varepsilon_0^2 + \pi|x - x_0|^2)^2}\right).$$

The existence of an appropriate ε_0 is a direct consequence of (3.1) (resp. [13, Lemma 2] or [28, Lemma 2.2]). We see that

$$C_{\varepsilon_0}(x) \in W^{1,\infty}(\Omega).$$

Let us set

$$A_{\varepsilon_0}(x) = \frac{|\Omega| e^{C_{\varepsilon_0}(x)}}{\int_\Omega e^{C_{\varepsilon_0}(x)}}.$$

A_{ε_0} belongs to $L_+^\infty(\Omega)$ and

$$F(A_{\varepsilon_0}(x), C_{\varepsilon_0}(x)) = \mathscr{F}(C_{\varepsilon_0}(x)) < \hat{K}.$$

Choosing $A_0(x) = A_{\varepsilon_0}(x)$ and $C_0(x) = C_{\varepsilon_0}(x)$ as initial data, we see that the corresponding solution of the Keller–Segel model has to blow up in finite or infinite time.

The other statements of the theorem can be shown by using the Lyapunov function $F(A(x,t), C(x,t))$. Denoting the blow-up set of $A(t,x)$ with $\mathscr{BS}A$ and the blow-up set of the positive part of C with $\mathscr{BS}C^+$, we know from Horstmann [13, Proposition 2] that if there are initial data (A_0, C_0) such that the solution of (1.3) blows up, then

$$\mathscr{BS}A \cap \mathscr{BS}C^+ \neq \emptyset$$

and

$$\lim_{t\to T_{max}} \int_\Omega |\nabla C(t)|^2 dx = \infty, \text{ as well as } \lim_{t\to T_{max}} \int_\Omega e^{C(t)} dx = \infty.$$

Thus we see that statements (3) and (4) above are true for a blow-up solution of system (1.3). Furthermore, we see by the properties of $F(A, C)$ that

$$\begin{aligned} \hat{K} &\geqslant F(A_0(x), C_0(x)) \\ &\geqslant F(A(x,t), C(x,t)), \end{aligned}$$

and thus

$$\frac{1}{2\alpha\chi}\int_\Omega k_c|\nabla C(x,t)|^2 + \gamma C(x,t)^2 dx \leqslant \int_\Omega (A(x,t) - 1)C(x,t)dx + \hat{K} \tag{3.20}$$

holds. This inequality gives us statement (2), and using Cauchy's inequality we also derive statement (1). Statement (5) is a direct consequence of statements (1) and (4).

We still have to show the last statement of the theorem. Therefore, we note the following. In Horstmann [13, Lemma 3], it was shown that if $\alpha\chi|\Omega|/k_c < 8\pi$, $v \in H^1(\Omega)$

and $p \in (1, 8\pi k_c/\alpha\chi|\Omega|)$ be arbitrary but fixed, then

$$\log\left(\frac{1}{|\Omega|}\int_\Omega e^v\right) \leqslant \frac{p}{16\pi}\int_\Omega |\nabla v|^2\, dx + \frac{2}{q}\log\left(\int_{\partial\Omega} e^{qv/2} dS\right) + K(p, q, \alpha, \chi, k_c, |\Omega|) \tag{3.21}$$

where $q > 1$ is such that $1 = 1/p + 1/q$, and $K(p, q, \alpha, \chi, k_c, |\Omega|)$ denotes a constant depending on p, q, α, χ, k_c and $|\Omega|$. Using this inequality, we can estimate the Lyapunov function $F(A, C)$ for an arbitrary but fixed $p \in (1, 8\pi k_c/\alpha\chi|\Omega|)$ from below by

$$\begin{aligned} F(A(t), C(t)) &\geqslant \frac{1}{2\alpha\chi}\int_\Omega k_c|\nabla C(t)|^2 + \gamma C^2(t)\, dx - |\Omega|\log\left(\frac{1}{|\Omega|}\int_\Omega e^{C(t)} dx\right) \\ &\geqslant \int_\Omega \left(\frac{k_c}{2\alpha\chi} - \frac{p|\Omega|}{16\pi}\right)|\nabla C(t)|^2 + \frac{\gamma}{2\alpha\chi}C^2(t)\, dx \\ &\quad - \frac{2|\Omega|}{q}\log\left(\int_{\partial\Omega} e^{qC(t)/2}\, dS\right) + K_1(p, q, \alpha\chi, k_c, |\Omega|) \end{aligned}$$

where $q = p/(p-1)$ and $K_1(p, q, \alpha\chi, k_c, |\Omega|)$ is a constant depending on the parameters in the brackets. So in view of statement (3), we get that

$$\lim_{t \to T_{max}} \int_{\partial\Omega} e^{qC(x,t)/2} dS = \infty$$

for every $q \in (8\pi k_c/(8\pi k_c - \alpha\chi|\Omega|), \infty)$. However, it is possible to improve this result. Independently from Horstmann [13], Senba & Suzuki [26] improved the statement of Horstmann [13, Lemma 3], and showed [26, Proposition 2] that for $v \in H^1(\Omega)$ and a simply connected, smooth domain $\Omega \subset \mathbb{R}^2$, the following estimate holds:

$$\log\left(\frac{1}{|\Omega|}\int_\Omega e^v dx\right) \leqslant \frac{1}{16\pi}\int_\Omega |\nabla v|^2 dx + \frac{1}{2|\partial\Omega|}\int_{\partial\Omega} v\, dS + \log\left(\frac{1}{|\partial\Omega|}\int_{\partial\Omega} e^{v/2} dS\right) + K. \tag{3.22}$$

Here K is an absolute constant. Using this inequality instead of that of Horstmann [13, Lemma 3], we get the estimate

$$\begin{aligned} F(A(t), C(t)) &\geqslant \frac{1}{2\alpha\chi}\int_\Omega k_c|\nabla C(t)|^2 + \gamma C^2(t)\, dx - |\Omega|\log\left(\frac{1}{|\Omega|}\int_\Omega e^{C(t)} dx\right) \\ &\geqslant \int_\Omega \left(\frac{k_c}{2\alpha\chi} - \frac{|\Omega|}{16\pi}\right)|\nabla C(t)|^2 + \frac{\gamma}{2\alpha\chi}C^2(t)\, dx \\ &\quad - \frac{|\Omega|}{2|\partial\Omega|}\int_{\partial\Omega} C(t)\, dS - |\Omega|\log\left(\frac{1}{|\partial\Omega|}\int_{\partial\Omega} e^{C(t)/2} dS\right) - K|\Omega|, \end{aligned}$$

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which finally leads us to statement (6). □

Lemmas 3.4 and 3.5 also imply the following corollary for the radially symmetric case of system (1.3).

Corollary 3.7 *Suppose $\Omega \subset \mathbb{R}^2$ is a disc of radius R, which is centered in point $x_0 \in \mathbb{R}^2$. Further, assume that $\gamma > 0$. If*

$$\alpha\chi|\Omega| > 8\pi k_c,$$

then there exist radially symmetric blow-up solutions for system (1.3).

Remark 3.8 We know that $1 \leqslant card(\mathscr{BS}) < \infty$ holds. Since

$$\lim_{r\to 0}\lim_{k\to\infty}\sigma_j^k(r) = 8\pi,$$

we see in the radially symmetric case that the claim of Lemma 3.5 is true, provided $\alpha\chi|\Omega| > 8\pi k_c$. This implies the statement of Corollary 3.7.

Remark 3.9 If $\gamma = 0$ we still have to exclude in the radially symmetric case that $\alpha\chi|\Omega|/k_c$ is equal to an integer multiple of 8π. Let us briefly compare the present paper with the results and proofs in Horstmann [12, 14]. In this paper, we used the transformation $u_k = v_k + (\alpha\chi/\gamma)$. We concluded via maximum principle that $u_k > 0$. This property was then used several times in the present paper. However, in Horstmann [12, 14] we used the transformation

$$\tilde{u}_k = v_k - \log\left(\frac{1}{|\Omega|}\int_\Omega e^{v_k}\,dx\right) - \frac{\alpha\chi}{4k_c}|x|^2.$$

Unfortunately, we cannot apply the maximum principle in this case. We also get some problems with Pohozaev's identity for our transformed problem. Thus, we still have to exclude in the radially symmetric case of system (1.3) with $\gamma = 0$ that $\alpha\chi|\Omega|/k_c$ is equal to an integer multiple of 8π. See Horstmann [12, 14] for more details about this case.

Under the assumption that $T_{max} < \infty$, it is known that for the solution of system (1.1)

$$\lim_{t\to T_{max}} ||a(t)\log a(t)||_{L^1(\Omega)} = \infty$$

is true (see Senba & Suzuki [26, Theorem 1]). However, it is not clear if either $T_{max} < \infty$ or $T_{max} = \infty$ is true for a blow-up solution of system (1.3) (resp. system (1.1)). There is only one example of a blow-up solution known, which blows up in finite time. This has been constructed for the radially symmetric case by Herrero and Velázquez [11].

Suppose $T_{max} < \infty$, then we also do not know if either

(1) $\inf_{0\leqslant t<T_{max}} F(A(t), C(t)) > -\infty$ or
(2) $\lim_{t\to T_{max}} F(A(t), C(t)) = -\infty$.

In fact, a numerical example for a blow-up solution of system (1.3) given by Gajewski &

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Zacharias [9] behaves in such a way that

$$\lim_{t\to T_{max}} F(A(t), C(t)) = -\infty$$

(see Gajewski & Zacharias [9, Remark 4.5, pp. 94, 95]), while one can also think about the possibility that

$$\inf_{0\leqslant t<T_{max}} F(A(t), C(t)) > -\infty,$$

since we are talking about finite time blow up. Nevertheless, we can formulate the following lemma, which gives us another result for a blow-up solution that is independent from the questions mentioned above. (The statements from Theorem 3.6 are also independent from these facts, as one can easily see from the proof given in the present paper.)

Lemma 3.10 *Suppose the solution* $(A(t), C(t))$ *of system* (1.3) *blows up. Then*

$$\lim_{t\to T_{max}} ||A(t)\log A(t)||_{L^1(\Omega)} = \infty. \tag{3.23}$$

The proof of Lemma 3.10 is similar to that in Post [25, Proof of Proposition 3.2].

Proof of Lemma 3.10 Let $(A(t), C(t))$ denote a blow-up solution of system (1.3). Since

$$\int_\Omega A(t)\log A(t)dx \geqslant -\frac{|\Omega|}{e}$$

we see that

$$F(A(t), C(t)) \geqslant -\frac{|\Omega|}{e} - \int_\Omega (A(t)-1)C(t)dx + \frac{k_c}{2\alpha\chi}||\nabla C(t)||^2_{L^2(\Omega)}.$$

Furthermore, we know from Cianchi [7, Theorems 2, 3] that

$$||C(t)||^2_{L^\Phi(\Omega)} \leqslant \tilde{K}||\nabla C(t)||^2_{L^2(\Omega)},$$

where $\Phi(s) \equiv e^s - s - 1$ (remember

$$\int_\Omega C(t)\ dx = 0$$

for all $t \geqslant 0$). Here and in the following, $L^\Phi(\Omega)$ denotes the Orlicz space which corresponds to the Young function $\Phi(s)$, and $||\cdot||_{L^\Phi(\Omega)}$ its norm. With Ψ we denote the Young function complementary to Φ, and consequently, with $L^\Psi(\Omega)$ the Orlicz space with norm $||\cdot||_{L^\Psi(\Omega)}$ which corresponds to the Young function Ψ. It is known that $\Psi(s) \equiv (s+1)\log(s+1) - s$ (see Kufner *et al.* [18, Example 3.3.5(iii)]). For more details on Orlicz spaces, we refer once again to Kufner *et al.* [18].

Using Hölder's inequality for Orlicz spaces [18, Theorem 3.7.5, p. 152], we see that

$$\begin{aligned} F(A(t), C(t)) &\geqslant -\frac{|\Omega|}{e} - \int_\Omega (A(t)-1)C(t)dx + \frac{k_c}{2\alpha\chi}||\nabla C(t)||^2_{L^2(\Omega)} \\ &\geqslant -\frac{|\Omega|}{e} - ||C(t)||_{L^\Phi(\Omega)}||A(t)-1||_{L^\Psi(\Omega)} + \frac{k_c}{2\alpha\chi}||\nabla C(t)||^2_{L^2(\Omega)} \end{aligned}$$

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$$\geqslant -\frac{|\Omega|}{e} - \frac{\tilde{K}}{4\varepsilon}||A(t)-1||^2_{L^\Psi(\Omega)} + \left(\frac{k_c}{2\alpha\chi} - \varepsilon\right)||\nabla C(t)||^2_{L^2(\Omega)},$$

where $\varepsilon < \frac{k_c}{2\alpha\chi}$. However, this gives us – together with Gajewski & Zacharias [9, Lemma 6.3] and

$$\int_\Omega A(t)\,dx = |\Omega| \text{ (for all } t \geqslant 0),$$

the claim of Lemma 3.10. □

Remark 3.11 We obtain

$$||A(t)\log A(t)||_{L^1(\Omega)} \leqslant K||A(t)||_{L^p(\Omega)} \tag{3.24}$$

for every $1 < p$ (see Kufner *et al.* [18, Theorem 3.17.1, p. 185]), and consequently,

$$\lim_{t\to T_{max}} ||A(t)||_{L^p(\Omega)} = \infty$$

for a blow-up solution of system (1.3).

Remark 3.12 All statements (statement (6) excepted) of Theorem 3.6 and Lemma 3.10 are also true for blow-up solutions of (1.3) if $\Omega \subset \mathbb{R}^2$ has a boundary, which is piecewise C^2.

4 Conclusions

After submitting the present paper, we became aware of a paper by Kabeya & Ni [16], where they prove the existence of nontrivial solutions of the equation

$$\left.\begin{array}{rcll} 0 & = & \Delta v - \gamma v + \tilde{\alpha}\lambda e^{\tilde{\chi}v}, & \text{in } \Omega, \\ v & > & 0, & \text{in } \Omega, \\ \frac{\partial v}{\partial n} & = & 0, & \text{on } \partial\Omega \end{array}\right\} \tag{4.1}$$

using a mountain pass type argument. To prove their results [16], they assume that the constants $\tilde{\alpha}\lambda$ and γ are of the same order, that γ is sufficiently large and

$$t = \frac{\tilde{\alpha}\lambda e^{\tilde{\chi}t}}{\gamma}$$

has two positive solutions, where λ denotes a positive constant. Kabeya and Ni [16] also prove that these solutions (which correspond to those nontrivial stationary solutions of (2.2) found by Horstmann [13] and Wang & Wei [28]) have only one single maximum which lies on the boundary of Ω, provided that γ is large enough. Kabeya's and Ni's results, and Theorem 3.6, indicate that for $\alpha\chi|\Omega|/4k_c\pi > 1$ and sufficiently large γ, there exist solutions of (1.3) that blow up in finite or infinite time, provided γ and $\alpha\chi$ are of the same order. We believe that this is also true for small γ, but we have no proof for this conjecture. However, this would also correspond to the biological situation where a certain threshold for the myxamoeba density is desired to initiate the aggregation of the myxamoebae. In Cohen & Robertson [6], a threshold value of $5 \cdot 10^4$ myxamoebae per cm^2 is given for Dictyostelium discoideum. From the biological point of view, it makes no sense to exclude certain values beyond that threshold. However, we have to exclude the integer multiples of 4π for technical reasons. As we have mentioned above, we believe that there are blow-up solutions of (1.3) if $\alpha\chi|\Omega|/4k_c\pi > 1$, without excluding other values for $\alpha\chi|\Omega|/k_c$. However, to prove this claim one has to show that the solutions of (2.2) have only one single maximum which lies on the boundary of Ω for all possible values of γ.

Furthermore we believe that the blow-up time of a blow-up solution is finite. This has been conjectured by several authors, and was first shown for a simplified version of the Keller–Segel model by Jäger & Luckhaus [15] under a radial symmetry assumption on the solution. Jäger & Luckhaus [15] studied the asymptotic behavior of the solution of

$$\left.\begin{array}{rcll} A_t &=& \nabla\cdot(\nabla A - A\nabla C), & x\in\Omega,\ t>0 \\ 0 &=& \Delta C + \frac{\alpha\chi}{k_c}(A-1), & x\in\Omega,\ t>0 \\ \frac{\partial A}{\partial n} &=& \frac{\partial C}{\partial n} = 0, & x\in\partial\Omega,\ t>0 \\ A(0,x) &=& A_0(x) > 0,\ C(0,x) = C_0(x), & x\in\Omega \\ \int_\Omega A_0(x)\,dx &=& |\Omega|,\ \int_\Omega C(t,x) = 0, & t\geqslant 0. \end{array}\right\} \qquad (4.2)$$

They prove that radially symmetric solutions of (4.2) can blow up in finite time for suitable initial data. Nagai [20] shows that there exist radially symmetric initial data such that the solution of

$$\left.\begin{array}{rcll} a_t &=& \nabla(\nabla a - \tilde{\chi} a\nabla c), & x\in\Omega,\ t>0 \\ 0 &=& k_c\Delta c - \gamma c + \tilde{\alpha} a, & x\in\Omega,\ t>0 \\ \frac{\partial a}{\partial n} &=& \frac{\partial c}{\partial n} = 0, & x\in\partial\Omega,\ t>0 \\ a(0,x) &=& a_0(x),\ c(0,x) = c_0(x), & x\in\Omega, \end{array}\right\} \qquad (4.3)$$

blows up in finite time, if

$$\int_\Omega a_0(x)dx > \frac{8\pi k_c}{\tilde{\alpha}\tilde{\chi}}.$$

In a recent work Nagai *et al.* [22] study (4.3) without assuming any symmetry of the solution. However, to prove their results [22] they have to assume that the solution blows up in finite time. There is no proof that such solutions do really exist.

The fact that the second equation of (4.2) and (4.3) is elliptic allows us to decouple the system, which seems to be impossible in the case of two parabolic equations. As mentioned in the introduction, there are only the results by Herrero & Velázquez [10, 11] where finite time blow-up of a radially symmetric solution of (1.3) is shown. Thus our results presented in the present paper are new, and show that there are nonsymmetric initial data such that the solution of (1.3) blows up in finite or infinite time.

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References

[1] Brézis, H. & Merle, F. (1991) Uniform estimates and blow-up behaviour for solutions of $-\Delta u = V(x)e^u$ in two dimensions. *Comm. P.D.E.* **16**, 1223–1253.

[2] Chang, S.-Y. A. & Yang, P. (1988) Conformal deformation of metrics on S^2. *J. Differential Geometry*, **27**, 259–296

[3] Chanillo, S. & Li, Y. (1992) Continuity of solutions of uniformly elliptic equation in R^2. *Manuscripta Math.* **77**, 415–433.

[4] Childress, S. (1984) Chemotactic collapse in two dimensions. *Lecture Notes in Biomathematics*, **55**, Springer-Verlag, pp. 61–66.

[5] Childress, S. & Percus, J. K. (1981) Nonlinear aspects of chemotaxis. *Math. Biosc.* **56**, 217–237.

[6] Cohen, M. H. & Robertson, A. (1971) Wave propagation in the early stages of aggregation of cellular slime molds. *J. Theor. Biol.* **31**, 101–118.

[7] Cianchi, A. (1996) A sharp embedding theorem for Orlicz–Sobolev spaces. *Indiana Univ. Math. J.* **45**, 39–65.

[8] Ding, W., Jost, J., Li, J. & Wang, G. (1997) The differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface. *Asian J. Math.* **1**, 230–248.

[9] Gajewski, H. & Zacharias, K. (1998) Global behavior of a reaction-diffusion system modelling chemotaxis. *Math. Nachr.* **195**, 77–114.

[10] Herrero, M. A. & Velázquez, J. J. L. (1996) Chemotactic collapse for the Keller-Segel model. *J. Math. Biol.* **35**, 583–623.

[11] Herrero, M. A. & Velázquez, J. J. L. (1997) A blow-up mechanism for a chemotaxis model. *Ann. Scuola Normale Superiore*, **24**, 633–683.

[12] Horstmann, D. (1999 *Aspekte positiver Chemotaxis*. Universität zu Köln, Dissertation.

[13] Horstmann, D. The nonsymmetric case of the Keller-Segel model in chemotaxis: some recent results. *Nonlinear Differential Equations and Applications (NoDEA)* (to appear).

[14] Horstmann, D. On the existence of radially symmetric blowup-solutions for the so-called Keller–Segel model. Preprint.

[15] Jäger, W. & Luckhaus, S. (1992) On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Trans. Amer. Math. Soc.* **329**, 819–824.

[16] Kabeya, Y. & Ni, W.-M. (1998) Stationary Keller-Segel model with the linear sensitivity. *RIMS Kokyuroku*, **1025**, 44–65.

[17] Keller, E. F. & Segel, L. A. (1970) Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.* **26**, 399–415.

[18] Kufner, A., John, O. & Fucik, S. (1977) *Function spaces.* Noordhoff.

[19] Li, Y. & Shafrir, I. (1994) Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two. *Indiana Univ. Math. J.* **43**, 1255–1270.

[20] Nagai, T. (1995) Blow-up of radially symmetric solutions to a chemotaxis system. *Adv. Math. Sci. Appl.* **5**, 581–601.

[21] Nagai, T., Senba, T. & Yoshida, K. (1997) Application of the Moser-Trudinger inequality to a parabolic system of chemotaxis. *Funkc. Ekvacioj, Ser.Int.* **40**, 411–433.

[22] Nagai, T., Senba, T. & Suzuki, T. (1998) Concentration behavior of blow-up solutions for a simplified system of chemotaxis. Preprint.

[23] Nanjundiah, V. (1973) Chemotaxis, signal relaying and aggregation morphology. *J. Theor. Biol.* **42**, 63–105.

[24] Ni, W.-M. & Takagi, I. (1991) On the shape of least energy solution to a semilinear Neumann problem. *Comm. Pure Appl. Math.* **41**, 819–851.

[25] Post, K. (1999) *A system of non-linear partial differential equations modeling chemotaxis with sensitivity functions.* Humboldt-Universität zu Berlin, Dissertation.

[26] Senba, T. & Suzuki, T. (1999) Local and Norm behavior of blowup solutions to a parabolic system of chemotaxis. Preprint.

[27] Struwe, M. & Tarantello, G. (1998) On multivortex solutions in Chern-Simons gauge theory. *Bolletino U. M. I.* **8**, 109–121.

[28] Wang, G. & Wei, J. Steady state solutions of a reaction-diffusion system modeling chemotaxis. *Math. Nachr.* (to appear).